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# QUASIMÖBIUS MAPS, WEAKLY QUASIMÖBIUS MAPS AND UNIFORM PERFECTNESS IN QUASI-METRIC SPACES

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**Abstract.** In this paper, first, we define the weakly quasimöbius maps in quasi-metric spaces and obtain a series of elementary properties of these maps. Then we find conditions under which a weakly quasimöbius map is quasimöbius in quasi-metric spaces. With the aid of uniform perfectness, three related results are proved, and some applications are also given.

## 1. Introduction

This paper is continuation to [32]. In [32], we investigated the uniform perfectness in quasi-metric spaces. First, we established the equivalence of uniform perfectness with homogeneous density, $\sigma$-density etc. Based on the obtained equivalence, the invariant property of uniform perfectness under quasisymmetric or quasimöbius maps was proved, and the relationships among uniform perfectness, (power) quasimöbius maps and (power) quasisymmetric maps were discussed.

The main aim of the present paper is to define the weakly quasimöbius maps in quasi-metric spaces, establish a series of elementary properties of these maps, and then discuss the relations between weakly quasimöbius maps and quasimöbius maps with the aid of uniform perfectness. We start with the definition of quasi-metric spaces.

**Definition 1.1.** For a given set $Z$ and a constant $K \geq 1$,

(1) a function $\rho\colon Z \times Z \to [0, +\infty)$ is said to be $K$-*quasi-metric* if
   (a) for all $x$ and $y$ in $Z$, $\rho(x, y) \geq 0$, and $\rho(x, y) = 0$ if and only if $x = y$;
   (b) $\rho(x, y) = \rho(y, x)$ for all $x$, $y \in Z$;
   (c) $\rho(x, z) \leq K(\rho(x, y) \vee \rho(y, z))$ for all $x$, $y$, $z \in Z$,
   where the notations: $r \vee s$ and $r \wedge s$ for numbers $r$, $s$ in $\mathbf{R}$ mean $r \vee s = \max\{r, s\}$ and $r \wedge s = \min\{r, s\}$.
(2) the pair $(Z, \rho)$ is said to be a $K$-quasi-metric space if $\rho$ is $K$-quasi-metric. Also, we say that $K$ is the quasi-metric coefficient of $(Z, \rho)$.

In the following, we always assume that $(Z, \rho)$ contains at least four points.

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Obviously, if $(Z,\rho)$ is $K_1$-quasi-metric, it must be $K_2$-quasi-metric for any $K_2 \geq K_1$. Hence, for convenience, in the following, we assume that the quasi-metric coefficients of all quasi-metric spaces are the same, denoted by $K$, and also, we assume that $K > 1$.

The following result easily follows from [7, Proposition 2.2.5].

**Lemma 1.1.** *Let $(X,\rho)$ be a $K$-quasi-metric space. If there exists some constant $0 < \varepsilon \leq 1$ such that $K^\varepsilon \leq 2$, then there is a metric $d_\varepsilon$ on $X$ such that*

$$\frac{1}{4}\rho^\varepsilon(z_1,z_2) \leq d_\varepsilon(z_1,z_2) \leq \rho^\varepsilon(z_1,z_2)$$

*for all $z_1, z_2 \in X$.*

For more properties concerning quasi-metric spaces, see [2, 3, 6, 7, 8, 9, 10, 12, 19, 22, 23, 26, 30, 31] etc.

**Definition 1.2.** Suppose $\eta$ is a homeomorphism from $[0,\infty)$ to $[0,\infty)$. A homeomorphism $f\colon (Z_1,\rho_1) \to (Z_2,\rho_2)$ between two quasi-metric spaces $(Z_1,\rho_1)$ and $(Z_2,\rho_2)$ is said to be

(1) *$\eta$-quasisymmetric* if $\rho_1(x,a) \leq t\rho_1(x,b)$ implies

$$\rho_2(x',a') \leq \eta(t)\rho_2(x',b')$$

for all $a$, $b$, $x$ in $(Z_1,\rho_1)$ and $t \geq 0$;

(2) *weakly $(h,H)$-quasisymmetric* if $\rho_1(x,a) \leq h\rho_1(x,b)$ implies

$$\rho_2(x',a') \leq H\rho_2(x',b')$$

for all $a$, $b$, $x$ in $(Z_1,\rho_1)$, where the constants $h > 0$ and $H \geq 1$ are called the weak quasisymmetry coefficients of $f$.

Here and in what follows, primes always denote the images of points under $f$, for example, $x' = f(x)$ etc.

We remark that, in general, the weak quasisymmetry means the weak $H$-quasisymmetry (cf. [14]). Obviously, the weak $(h,H)$-quasisymmetry is a generalization of the weak $H$-quasisymmetry since the weak $H$-quasisymmetry coincides with the weak $(1,H)$-quasisymmetry.

It is known that every Möbius transformation in $\mathbf{R}^n$ leaves the cross ratio invariant. As a generalization of Möbius transformations in metric spaces, in [26], Väisälä introduced a class of maps, i.e. quasimöbius maps, under which the cross ratio is in a certain sense quasi-invariant, and got the close connections with quasisymmetric maps and quasiconformal maps. The introduction of quasimöbius maps has provided a handy tool when studying the quasisymmetric maps and the quasiconformal maps. Many references related to the relationships among quasimöbius maps, quasisymmetric maps and quasiconformal maps have been in literature; see [1, 5, 6, 7, 13, 15, 16, 17, 18, 19, 20, 21, 25, 29, 30] etc. The precise definition for quasimöbius maps is as follows.

**Definition 1.3.** Suppose $\theta$ is a homeomorphism from $[0,\infty)$ to $[0,\infty)$. A homeomorphism $f\colon (Z_1,\rho_1) \to (Z_2,\rho_2)$ between two quasi-metric spaces is said to be *$\theta$-quasimöbius* if $r(a,b,c,d) \leq t$ implies

$$r(a',b',c',d') \leq \theta(t)$$

for all $a$, $b$, $c$, $d$ in $(Z_1, \rho_1)$ and $t \geq 0$, where

$$r(a,b,c,d) = \frac{\rho_1(a,c)\rho_1(b,d)}{\rho_1(a,b)\rho_1(c,d)}$$

denotes the *cross ratio* of the quadruple $(a, b, c, d)$.

Similar to the definition of weak quasisymmetric maps, we introduce the following definition of weakly quasimöbius maps.

**Definition 1.4.** A homeomorphism $f\colon (Z_1, \rho_1) \to (Z_2, \rho_2)$ between two quasi-metric spaces is said to be *weakly* $(h, H)$*-quasimöbius* if $r(a,b,c,d) \leq h$ implies

$$r(a',b',c',d') \leq H$$

for each quadruple $(a,b,c,d)$ in $(Z_1, \rho_1)$, where the constants $h > 0$ and $H \geq 1$ are called the weakly quasimöbius coefficients of $f$.

Clearly, every quasimöbius map is weakly quasimöbius. But the converse is not always true. To find the conditions under which weakly quasimöbius maps are quasimöbius is the main aim of this paper. We shall study this problem with the aid of uniform perfectness. Our main results are Theorems 3.1, 5.1 and 5.2 below.

The organization of this paper is as follows. In Section 2, we will introduce some necessary notations and concepts, recall some known results, and prove a series of basic and useful results. In Section 3, we shall show that for a weakly quasimöbius map in a uniformly perfect quasi-metric space, it is quasimöbius if and only if the image space is also uniformly perfect and the inverse of the map is weakly quasimöbius too. The goal of Section 4 is to check that between two uniformly perfect and $\kappa$-HTB quasi-metric spaces, every weakly quasisymmetric map must be quasisymmetric. This result is useful for the discussions in Section 5. But the result itself is independently significant (See Remark 4.1 below). Based on the main results in Sections 3 and 4, we mainly demonstrate in Section 5 that between two uniformly perfect and homogeneous quasi-metric spaces, every weakly quasimöbius map is quasimöbius. Some applications of the main results in Section 5 will be given in the last section, Section 6.

## 2. Basic terminology and results

In this section, we shall introduce necessary notations and concepts, recall some known results and obtain a series of basic results which will be used later on. Most results stated in this section are from the case of metric spaces. The methods of proofs are also similar. We give the proofs just for completeness. This section consists of six subsections.

**2.1. Cross ratios.** The following result is obvious (see [32, Proposition 2.1]): For a quasi-metric space $(Z, \rho)$ and points $a$, $b$, $c$, $d$, $z$ in $(Z, \rho)$,

$$r(a,b,c,d) = \frac{1}{r(a,c,b,d)} \quad \text{and} \quad r(a,b,c,d) = r(a,b,z,d)r(a,z,c,d). \tag{2.1}$$

In [4], Bonk and Kleiner introduce the following useful notation:

$$\langle a,b,c,d \rangle = \frac{\rho(a,c) \wedge \rho(b,d)}{\rho(a,b) \wedge \rho(c,d)}.$$

Bonk and Kleiner established a relation between $r(a,b,c,d)$ and $\langle a,b,c,d \rangle$ in the setting of metric spaces [4, Lemma 3.3]. As shown in the following result from [32], this useful property is also valid in quasi-metric spaces.

**Lemma A.** [32, Lemma 2.1] *Suppose $(Z,\rho)$ is quasi-metric. Then for any $a$, $b$, $c$, $d$ in $(Z,\rho)$, we have*

(1) $\frac{1}{\theta_K\left(\frac{1}{r(a,b,c,d)}\right)} \le \langle a,b,c,d\rangle \le \theta_K(r(a,b,c,d))$;

(2) $\theta_K^{-1}(\langle a,b,c,d\rangle) \le r(a,b,c,d) \le \frac{1}{\theta_K^{-1}\left(\frac{1}{\langle a,b,c,d\rangle}\right)}$,

*where $\theta_K(t)=K^2(t\vee\sqrt{t})$ and $K$ denotes the quasi-metric coefficient of $(Z,\rho)$.*

As a consequence of Lemma A, we have

**Lemma 2.1.** *Suppose that $f\colon (Z_1,\rho_1)\to(Z_2,\rho_2)$ is a homeomorphism, where each $(Z_i,\rho_i)$ is a quasi-metric space ($i=1,2$), and that $h>1$ and $H\ge 1$ are constants. For any $a$, $b$, $c$ and $d$ in $(Z_1,\rho_1)$, if*

$$\langle a,b,c,d\rangle > \frac{1}{hK^2} \ \ \text{implies}\ \ \langle a',b',c',d'\rangle > \frac{1}{H},$$

*then*

$$r(a,b,c,d) > \frac{1}{h} \ \ \text{implies}\ \ r(a',b',c',d') > \frac{1}{H^2K^4}.$$

*Proof.* If $r(a,b,c,d)>\frac{1}{h}$, then by Lemma A, we have

$$\langle a,b,c,d\rangle > \frac{1}{hK^2},$$

which implies

$$\langle a',b',c',d'\rangle > \frac{1}{H}.$$

Again by Lemma A, we obtain that

$$r(a',b',c',d') > \frac{1}{H^2K^4}.$$

Hence the proof of the lemma is complete. □

**2.2. (Weakly) quasisymmetric maps and (weakly) quasimöbius maps.** Obviously, if $f$ is weakly $(h,H)$-quasisymmetric (resp. weakly $(h,H)$-quasimöbius), then it is weakly $(h_1,H_1)$-quasisymmetric (resp. weakly $(h_1,H_1)$-quasimöbius) for any $h_1\le h$ and $H_1\ge H$. Hence, in the following, we always assume that $H>1$.

In the rest of this subsection, we always assume that $(Z_1,\rho_1)$, $(Z_2,\rho_2)$ and $(Z_3,\rho_3)$ are quasi-metric. First, let us recall a result from [32] concerning quasimöbius maps.

**Lemma B.**

(1) [32, Lemma 2.2(1)] *If $f\colon (Z_1,\rho_1)\to(Z_2,\rho_2)$ is $\eta$-quasisymmetric, then it is $\theta$-quasimöbius, where $\theta(t)=\frac{1}{\theta_K^{-1}\left(\frac{1}{\eta\circ\theta_K(t)}\right)}$.*

(2) [32, Lemma 2.3(1)] *If $f\colon (Z_1,\rho_1)\to(Z_2,\rho_2)$ and $g\colon(Z_2,\rho_2)\to(Z_3,\rho_3)$ are $\theta_1$-quasimöbius and $\theta_2$-quasimöbius, respectively, then $g\circ f$ is $\theta$-quasimöbius with $\theta=\theta_2\circ\theta_1$.*

Then we have the following lemmas.

**Lemma 2.2.** (1) *If $f\colon (Z_1,\rho_1)\to(Z_2,\rho_2)$ is an $\eta$-quasisymmetric map, then it is weakly $(h,H)$-quasisymmetric, where $H=\eta(h)$ for any $h>0$;*

(2) *Suppose $f\colon (Z_1,\rho_1)\to(Z_2,\rho_2)$ is $\eta$-quasisymmetric. Then its inverse $f^{-1}\colon (Z_2,\rho_2)\to(Z_1,\rho_1)$ is $\eta_1$-quasisymmetric with $\eta_1(t)=\left(\eta^{-1}(t^{-1})\right)^{-1}$ for $t>0$ (cf.* [24, Theorem 2.2]*);*

(3) *If* $f\colon (Z_1,\rho_1)\to(Z_2,\rho_2)$ *and* $g\colon (Z_2,\rho_2)\to(Z_3,\rho_3)$ *are* $\eta_1$*-quasisymmetric and* $\eta_2$*-quasisymmetric, respectively, then* $g\circ f$ *is* $\eta$*-quasisymmetric with* $\eta=\eta_2\circ\eta_1$ *(cf.* [24, Theorem 2.2]*);*

(4) *Suppose* $f\colon (Z_1,\rho_1)\to(Z_2,\rho_2)$ *is a* $\theta$*-quasimöbius map. Then its inverse* $f^{-1}\colon (Z_2,\rho_2)\to(Z_1,\rho_1)$ *is* $\theta_1$*-quasimöbius with* $\theta_1(t)=\big(\theta^{-1}(t^{-1})\big)^{-1}$ *for* $t>0$ *(cf.* [28, Theorem 6.23]*).*

*Proof.* The assertions (1) and (3) are obvious. In order to prove (2), we let $a$, $b$, $c\in Z_1$, and let $t>0$ be such that

$$\rho_2(a',b')\le t\rho_2(a',c').$$

Suppose $\rho_1(a,b)>\eta_1(t)\rho_1(a,c)$. Then

$$\rho_2(a',c')<\eta\left(\frac{1}{\eta_1(t)}\right)\rho(a',b')=\frac{1}{t}\rho_2(a',b'),$$

which is a contradiction. Hence (2) is true.

The proof of (4) is similar, and so we prove this lemma. □

**Lemma 2.3.** *Suppose* $f\colon (Z_1,\rho_1)\to(Z_2,\rho_2)$ *is a homeomorphism.*

(1) *If* $f$ *is weakly* $(h,H)$*-quasimöbius, then for any* $a$*,* $b$*,* $c$ *and* $d$ *in* $(Z_1,\rho_1)$*,*

$$\langle a,b,c,d\rangle\le h_1 \ \text{ implies } \ \langle a',b',c',d'\rangle\le H_1,$$

*where* $h_1=\frac{1}{\theta_K(\frac{1}{h})}$ *and* $H_1=K^2H$*.*

(2) *If for any* $a$*,* $b$*,* $c$ *and* $d$ *in* $(Z_1,\rho_1)$*,*

$$\langle a,b,c,d\rangle\le h \ \text{ implies } \ \langle a',b',c',d'\rangle\le H,$$

*then* $f$ *is weakly* $(h_2,H_2)$*-quasimöbius, where* $h_2=\theta_K^{-1}(h)$ *and* $H_2=\frac{1}{\theta_K^{-1}(\frac{1}{H})}$*.*

*Proof.* We only need to prove the first statement in the lemma since the proof of the second one is similar. Since $\langle a,b,c,d\rangle\le h_1$, we have $r(a,b,c,d)\le h$, because otherwise, it follows from Lemma A that $\langle a,b,c,d\rangle\ge\frac{1}{\theta_K\left(\frac{1}{r(a,b,c,d)}\right)}>h_1$. Then the assumption "$f$ being weakly $(h,H)$-quasimöbius" leads to

$$r(a',b',c',d')\le H.$$

Again, Lemma A guarantees that

$$\langle a',b',c',d'\rangle\le H_1.$$

Hence the proof of the lemma is finished. □

**Lemma 2.4.** *If* $f\colon (Z_1,\rho_1)\to(Z_2,\rho_2)$ *is weakly* $(h,H)$*-quasisymmetric, then it is weakly* $(\theta_K^{-1}(h),\frac{1}{\theta_K^{-1}(\frac{1}{H})})$*-quasimöbius.*

*Proof.* To prove this lemma, by Lemma 2.3, it suffices to check that for any points $z_1,z_2,z_3,z_4\in Z$, if $\langle z_1,z_2,z_3,z_4\rangle\le h$, then

$$\langle z_1',z_2',z_3',z_4'\rangle\le H.$$

To this end, we assume that $\rho_1(z_1,z_3)\le\rho_1(z_2,z_4)$. Then it follows from $\langle z_1,z_2,z_3,z_4\rangle\le h$ that

$$\rho_1(z_1,z_3)\le h(\rho_1(z_1,z_2)\wedge\rho_1(z_3,z_4)).$$

Hence

$$\rho_2(z_1',z_3')\le H(\rho_2(z_1',z_2')\wedge\rho_2(z_3',z_4')),$$

since $f$ is $(h,H)$-weakly quasisymmetric, and thus

$$\langle z_1', z_2', z_3', z_4'\rangle \le \frac{\rho_2(z_1',z_3')}{\rho_2(z_1',z_2')\wedge\rho_2(z_3',z_4')} \le H,$$

as required. □

**Lemma 2.5.** *Suppose $f\colon (Z_1,\rho_1)\to(Z_2,\rho_2)$ and $g\colon (Z_2,\rho_2)\to(Z_3,\rho_3)$ are homeomorphisms.*

(1) *If $f$ is weakly $(h,H)$-quasimöbius and $g$ is $\theta$-quasimöbius, then $g\circ f$ is weakly $(h,\theta(H))$-quasimöbius;*
(2) *If $f$ is $\theta$-quasimöbius and $g$ is weakly $(h,H)$-quasimöbius, then $g\circ f$ is weakly $(\theta^{-1}(h),H)$-quasimöbius;*
(3) *If $f$ is weakly $(h,H)$-quasimöbius and $g$ is weakly $(H,H_1)$-quasimöbius, then $g\circ f$ is weakly $(h,H_1)$-quasimöbius.*

*Proof.* Obviously, we only need to demonstrate the first statement in the lemma since the proofs for the remaining two are similar. For the proof, let $z_1$, $z_2$, $z_3$, $z_4$ be four points in $Z_1$. Assume that $\langle z_1,z_2,z_3,z_4\rangle\le h$. Then it follows from the assumption "$f$ being weakly $(h,H)$-quasimöbius" that

$$\langle z_1', z_2', z_3', z_4'\rangle \le H.$$

Since $g$ is $\theta$-quasimöbius, we see that

$$\langle z_1'', z_2'', z_3'', z_4''\rangle \le \theta(H),$$

which is what we want, where $z_i''$ denotes the image of $z_i'$ in $Z_3$ under $g$ for $i\in\{1,\dots,4\}$. □

The following corollary is a direct consequence of Lemmas B, 2.2, 2.4 and 2.5.

**Corollary 2.1.** *Suppose $f\colon (Z_1,\rho_1)\to(Z_2,\rho_2)$ and $g\colon (Z_2,\rho_2)\to(Z_3,\rho_3)$ are homeomorphisms.*

(1) [32, Lemma 2.3(2)] *If $f$ is $\theta$-quasimöbius and $g$ is $\eta$-quasisymmetric, then $g\circ f$ is $\theta_1$-quasimöbius, where $\theta_1(t)=\frac{1}{\theta_K^{-1}\big(\frac{1}{\eta\circ\theta_K\circ\theta(t)}\big)}$;*
(2) [32, Lemma 2.2(1)] *If $f$ is $\eta$-quasisymmetric and $g$ is $\theta$-quasimöbius, then $g\circ f$ is $\theta_2$-quasimöbius, where $\theta_2(t)=\theta\big(\frac{1}{\theta_K^{-1}\big(\eta\circ\theta_K(t)\big)}\big)$;*
(3) *If $f$ is $\theta$-quasimöbius and $g$ is weakly $(h,H)$-quasisymmetric, then $g\circ f$ is weakly $\big(\theta^{-1}\circ\theta_K^{-1}(h),\frac{1}{\theta_K^{-1}(\frac{1}{H})}\big)$-quasimöbius;*
(4) *If $f$ is weakly $(h,H)$-quasimöbius and $g$ is $\eta$-quasisymmetric, then $g\circ f$ is weakly $\big(h,\frac{1}{\theta_K^{-1}(\frac{1}{\eta\circ\theta_K(H)})}\big)$-quasimöbius;*
(5) *If $f$ is weakly $(h,H)$-quasimöbius and $g$ is weakly $(K^2H,H_2)$-quasisymmetric, then $g\circ f$ is weakly $\big(h,\frac{1}{\theta_K^{-1}(\frac{1}{H_2})}\big)$-quasimöbius;*
(6) *If $f$ is $\eta$-quasisymmetric and $g$ is weakly $(h,H)$-quasimöbius, then $g\circ f$ is weakly $\big(\theta_K^{-1}\circ\eta^{-1}(\frac{1}{\theta_K(\frac{1}{h})}),H\big)$-quasimöbius;*
(7) *If $f$ is weakly $(h,H)$-quasisymmetric and $g$ is $\theta$-quasimöbius, then $g\circ f$ is weakly $\big(\theta_K^{-1}(h),\theta(\frac{1}{\theta_K^{-1}(\frac{1}{H})})\big)$-quasimöbius;*
(8) *If $f$ is weakly $(h,H)$-quasisymmetric and $g$ is weakly $\big(\frac{1}{\theta_K^{-1}(\frac{1}{H})},H_3\big)$-quasimöbius, then $g\circ f$ is weakly $(\theta_K^{-1}(h),H_3)$-quasimöbius.*

### 2.3. Uniform perfectness, homogeneous density and $\sigma$-density.

**Definition 2.1.** A quasi-metric space $(Z, \rho)$ is called *uniformly perfect* if there is a constant $\tau \in (0, 1)$ such that for each $x$ in $(Z, \rho)$ and every $r > 0$, $\mathbf{B}(x, r) \setminus \mathbf{B}(x, \mu r) \neq \emptyset$ provided that $Z \setminus \mathbf{B}(x, r) \neq \emptyset$. Also, we say that $(Z, \rho)$ is uniformly $\tau$-perfect.

Let us recall the following useful result from [32] concerning the invariant of uniform perfectness under the quasisymmetric or quasimöbius maps.

(In the rest of this paper, we make the following notational convention: Suppose $A$ denotes a condition with data $v$ and $A'$ another condition with data $v'$. We say that $A$ implies $B$ quantitatively if $A$ implies $B$ so that $v'$ depends only on $v$. If $A$ and $A'$ imply each other quantitatively, then we say that they are quantitatively equivalent.)

**Lemma C.** [32, Theorem 1.2 and Lemma 3.2] *Let $f\colon (Z_1, \rho_1) \to (Z_2, \rho_2)$ be $\eta$-quasisymmetric or $\theta$-quasimöbius, where both $(Z_i, \rho_i)$ $(i = 1, 2)$ are quasi-metric. Then $(Z_1, \rho_1)$ is uniformly $\tau$-perfect if and only if $(Z_2, \rho_2)$ is uniformly $\tau'$-perfect, quantitatively.*

**Definition 2.2.** Suppose $\{x_i\}_{i \in \mathbf{Z}}$ denotes a sequence of points in a quasi-metric space $(Z, \rho)$ with $a \neq x_i \neq b$.

(1) If $x_i \to a$ as $i \to -\infty$ and $x_i \to b$ as $i \to +\infty$, then $\{x_i\}$ is called a *chain* joining $a$ and $b$; Further, if there is a constant $\sigma > 1$ such that for all $i$,
$$|\log r(a, x_i, x_{i+1}, b)| \leq \log \sigma,$$
then $\{x_i\}$ is called a *$\sigma$-chain*.

(2) $(Z, \rho)$ is said to be *$\sigma$-dense* $(\sigma > 1)$ if any pair of points in $(Z, \rho)$ can be joined by a $\sigma$-chain.

We remark that (1) a $\sigma$-dense space does not contain any isolated point, and (2) a $\sigma$-dense space is $\sigma'$-dense for any $\sigma' \geq \sigma$.

**Definition 2.3.** A quasi-metric space $(Z, \rho)$ is said to be *homogeneously dense*, abbreviated *HD*, if for each pair of points $a, b \in Z$, there is a point $x$ in $Z$ such that
$$\lambda_1 \rho(a, b) \leq \rho(a, x) \leq \lambda_2 \rho(a, b),$$
where $\lambda_1$ and $\lambda_2$ are constants with $0 < \lambda_1 \leq \lambda_2 < 1$. To emphasize the parameters, we also say that $(Z, \rho)$ is $(\lambda_1, \lambda_2)$-*HD*.

The following results from [32] will be applied several times later in this paper.

**Lemma D.** [32, Lemma 3.1(1)] *If a quasi-metric space is $(\lambda_1, \lambda_2)$-HD, then it must be $(\lambda_1^n, \lambda_2^n)$-HD for any positive integer $n$.*

**Lemma E.** [32, Theorem 1.1] *Let $(Z, \rho)$ be a quasi-metric space. Then the following are quantitatively equivalent:*

(1) *$Z$ is uniformly $\tau$-perfect;*
(2) *$Z$ is $(\lambda_1, \lambda_2)$-HD;*
(3) *$Z$ is $\sigma$-dense;*
(4) *there are numbers $\mu_1$ and $\mu_2$ with $0 < \mu_1 \leq \mu_2 < 1$ such that for any triple $(a, c, d)$ in $(Z, \rho)$, there is a point $x \in Z$ satisfying $\mu_1 \leq r(a, x, c, d) \leq \mu_2$.*

### 2.4. Doubling and $\kappa$-$HTB$ quasi-metric spaces.

**Definition 2.4.** A quasi-metric space $(Z,\rho)$ is called *$C$-doubling* if there is a constant $C$ such that every ball $\mathbf{B}$ in $(Z,\rho)$ can be covered with at most $C$ balls of half the radius of $\mathbf{B}$.

**Definition 2.5.** A quasi-metric space $(Z,\rho)$ is *$\kappa$-homogeneously totally bounded*, abbreviated $\kappa$-$HTB$, if there is an increasing function $\kappa\colon [\frac{1}{2},\infty)\to[1,\infty)$ such that for each $\alpha\geq\frac{1}{2}$, every closed ball $\overline{\mathbf{B}}(x,r)$ in $(Z,\rho)$ can be covered with sets $A_1,\cdots,A_s$ in $(Z,\rho)$ such that $s\leq\kappa(\alpha)$ and $\operatorname{diam}(A_i)<r/\alpha$ for all $i$.

The following result concerning $\kappa$-$HTB$ in the setting of metric spaces is from [24] or [27] (see [24, Remarks 2.8] or [27, Section 2.8]).

**Lemma 2.6.** *If $(Z,\rho)$ is a $\kappa$-$HTB$ quasi-metric space and if $a_1,\ldots,a_s$ are points in the closed ball $\overline{\mathbf{B}}(x,r)$ with $\rho_1(a_i,a_j)\geq t>0$ whenever $i\neq j$, then $s\leq\kappa\big(\frac{r}{t}\big)$.*

*Proof.* Since $(Z,\rho)$ is $\kappa$-HTB, by definition, we see that for $\alpha=\frac{r}{t}$, the closed ball $\overline{\mathbf{B}}(x,r)$ can be covered by $A_1,\ A_2,\ \cdots,\ A_{s_1}$ with

$$s_1\leq\kappa(\alpha)=\kappa\left(\frac{r}{t}\right)\ \text{ and }\ \operatorname{diam}(A_i)<\frac{r}{\alpha}=t\ \text{ for all }\ i.$$

Then we claim that each $A_i$ contains at most one element from $\{a_1,\ldots,a_s\}$. Suppose not. Then there are an $i\in\{1,\ldots,s\}$ and two points $a_{j_1}\neq a_{j_2}\in\{a_1,\ldots,a_s\}$ such that

$$a_{j_1},a_{j_2}\in A_i,$$

which implies that

$$\operatorname{diam}(A_i)\geq\rho(a_{j_1},a_{j_2})\geq t.$$

It is impossible, and hence

$$s\leq s_1\leq\kappa\left(\frac{r}{t}\right),$$

as required. □

The following equivalence between doubling quasi-metric spaces and $\kappa$-$HTB$ quasi-metric spaces is needed in the proof of Lemma 2.8 below and the discussions in Section 5.

**Lemma 2.7.** *A $K$-quasi-metric space is doubling if and only if it is $\kappa$-$HTB$, quantitatively.*

*Proof.* Assume that $(Z,\rho)$ is a quasi-metric space. If it is $\kappa$-$HTB$, by letting $\alpha=2$, then we see from the definition that for any $z\in Z$ and $r>0$, there are $s\leq\kappa(2)$ sets $A_i$ in $(Z,\rho)$ with $\operatorname{diam}(A_i)<\frac{r}{2}$ such that

$$\overline{\mathbf{B}}(z,r)\subset\bigcup_{i=1}^{s}A_i.$$

Obviously, for each $i$, there is a point $z_i\in Z$ such that $A_i\subset\mathbf{B}(z_i,\frac{r}{2})$. Hence

$$\overline{\mathbf{B}}(z,r)\subset\bigcup_{i=1}^{s}\mathbf{B}\left(z_i,\frac{r}{2}\right),$$

which shows that $(Z,\rho)$ is $\kappa(2)$-doubling. Hence the sufficiency is true.

In the following, we prove the necessity. For any $\alpha\geq\frac{1}{2}$, $r>0$ and $z\in Z$, consider the closed ball $\overline{\mathbf{B}}(z,r)$ in $(Z,\rho)$. Without loss of generality, we may assume that $K\alpha > 1$. Then there is a unique integer $N \geq 1$ such that

$$2^{N-1} \leq K\alpha < 2^N.$$

Since $(Z, \rho)$ is $C$-doubling, we see that there are at most $C$ balls $\mathbf{B}(z_i, \frac{r}{2})$ such that

$$\overline{\mathbf{B}}(z, r) \subset \bigcup_{i=1}^{C} \mathbf{B}\left(z_i, \frac{r}{2}\right),$$

where $z_i \in (Z, \rho)$. By induction, we can know that there are at most $C^N$ balls $\mathbf{B}(z_j, \frac{r}{2^N})$ which cover $\overline{\mathbf{B}}(z, r)$. Let $A_j = \overline{\mathbf{B}}(z_j, \frac{r}{2^N})$. Then

$$\operatorname{diam}(A_j) \leq \frac{Kr}{2^N} < \frac{r}{\alpha}.$$

Clearly, $\overline{\mathbf{B}}(z, r)$ is covered by $A_1, \ldots, A_s$ with

$$s = C^N \leq C^{\frac{\log(K\alpha)}{\log 2} + 1}.$$

Let

$$\kappa(\alpha) = C^{\frac{\log(K\alpha)}{\log 2} + 1}.$$

Then we see that $(Z, \rho)$ is $\kappa$-$HTB$, and so, the lemma is proved. □

**Lemma 2.8.** *Suppose that $f \colon (Z_1, \rho_1) \to (Z_2, \rho_2)$ is $\eta$-quasisymmetric between two quasi-metric spaces. Then $(Z_1, \rho_1)$ is $\kappa$-$HTB$ if and only if $(Z_2, \rho_2)$ is $\kappa'$-$HTB$, quantitatively.*

*Proof.* By Lemma 2.2, it is sufficient to prove the necessity. For any $y' \in Z_2$ and $r > 0$, let $\{y'_i\}_{i \in \Lambda}$ be a maximal $\frac{r}{2}$-separated subset of the ball $\mathbf{B}(y', r) \subset Z_2$, where "$\frac{r}{2}$-separated" means that $\rho(z_i, z_j) \geq \frac{r}{2}$ for all $i \neq j \in \Lambda$. The existence of this maximal subset in $\mathbf{B}(z_0, r)$ is guaranteed by Zorn's lemma. Obviously, the union of the balls $\mathbf{B}(y'_i, \frac{r}{2})$ covers $\mathbf{B}(y', r)$. Hence, to prove that $(Z_2, \rho_2)$ is $\kappa'$-$HTB$, by Lemma 2.7, it suffices to show that there is a constant $M > 0$ such that the number $\operatorname{card}(\Lambda)$ of elements in $\Lambda$ satisfies

$$\operatorname{card}(\Lambda) \leq M,$$

where $M$ is independent of $y'$ and $r$.

Let

$$r_0 = \inf\{s \leq r \colon \mathbf{B}(y', s) = \mathbf{B}(y', r)\}.$$

If $r_0 = 0$, then $\mathbf{B}(y', r) = \{y'\}$, and clearly, $\operatorname{card}(\Lambda) = 1$.

In the following, we assume that $0 < r_0 \leq r$. By the choice of $r_0$, there must be a point $y'_0 \in \mathbf{B}(y', r)$ such that

$$\frac{r_0}{2} < \rho_2(y', y'_0) < r.$$

Moreover, we have the following assertion.

**Claim 2.1.** *For all $i \neq j \in \Lambda$,*

$$x_i \in \mathbf{B}(x, \eta'(2)\rho_1(x, x_0)) \quad \text{and} \quad \rho_1(x_i, x_j) \geq \frac{\rho_1(x, x_0)}{K\eta'(2K)},$$

*where $x_i = f^{-1}(y'_i), x = f^{-1}(y'), x_0 = f^{-1}(y'_0)$ and $\eta'(t) = \frac{1}{\eta^{-1}(\frac{1}{t})}$.*

Note first that the inverse $f^{-1}$ of $f$ is $\eta'$-quasisymmetric with $\eta'(t) = \frac{1}{\eta^{-1}(\frac{1}{t})}$ (cf. Lemma 2.2). Since $\rho_2(y'_i, y') \le r_0 < 2\rho_2(y', y'_0)$, we have

$$\rho_1(x_i, x) < \eta'(2)\rho_1(x_0, x),$$

and so $x_i \in \mathbf{B}(x, \eta'(2)\rho_1(x, x_0))$.

Similarly, since $\rho_2(y'_i, y') < r \le 2\rho_2(y'_i, y'_j)$ for $i \ne j \in \Lambda$, we know that

$$\rho_1(x_i, x) < \eta'(2)\rho_1(x_i, x_j).$$

On the other hand, it follows from

$$\rho_2(y'_i, y'_0) \le K(\rho_2(y'_i, y') \vee \rho_2(y', y'_0)) < Kr < 2K\rho_2(y'_i, y'_j)$$

that

$$\rho_1(x_i, x_0) < \eta'(2K)\rho_1(x_i, x_j).$$

Furthermore, we obtain that

$$\rho_1(x, x_0) \le K(\rho_1(x, x_i) \vee \rho_1(x_i, x_0)) \le K\eta'(2K)\rho_1(x_i, x_j),$$

as desired.

Now, let us continue the proof. Since $(Z_1, \rho_1)$ is $\kappa$-*HTB*, by Claim 2.1 and Lemma 2.6, we see that

$$\mathrm{card}(\Lambda) \le \kappa(K\eta'(2K)^2),$$

and hence the proof of this lemma is complete. □

We remark that, in the metric spaces, Lemma 2.8 coincides with Theorem 2.10 in [24]. But our method of proof is different.

**2.5. Doubling measure spaces, homogeneous spaces and Ahlfors regular spaces.** In the following, we assume that $(Z, \rho)$ is a quasi-metric space and a positive measure $\mu$ is defined on a $\sigma$-algebra of subsets of $(Z, \rho)$, which contains the balls $\mathbf{B}(x, r)$ in $(Z, \rho)$.

**Definition 2.6.** Let $(Z, \rho)$ be a quasi-metric space. A positive Borel measure $\mu$ is said to be $C$-*doubling* if there is a constant $C$ such that

$$\mu(2\mathbf{B}) \le C\mu(\mathbf{B}) < \infty$$

for all balls $\mathbf{B}$ in $(Z, \rho)$.

**Definition 2.7.** If a quasi-metric space $(Z, \rho)$ carries a $C$-doubling measure $\mu$, then it is called a *homogeneous space*, which is denoted by $(Z, \rho, \mu)$. Also, we say that $(Z, \rho, \mu)$ is $(K, C)$-*HS*, where we recall that $K$ denotes the quasi-metric coefficient of $(Z, \rho)$.

**Lemma 2.9.** *Every homogeneous space is doubling.*

*Proof.* Assume that $(Z, \rho, \mu)$ is $(K, C)$-*HS*. For any $z_0 \in Z$ and $r > 0$, let $\{z_i\}_{i=0}^n$ be a maximal $\frac{r}{2}$-separated subset of $\mathbf{B}(z_0, r)$. Obviously, the union of the balls $\mathbf{B}(z_i, \frac{r}{2})$ covers the ball $\mathbf{B}(z_0, r)$. Hence, to prove that $(Z, \rho)$ is doubling as a quasi-metric space, it suffices to show the number $n$ is independent of $z_0$ and $r$.

First, we know that there is a unique integer $m$ such that

$$2^m \le K < 2^{m+1}.$$

Here, we assume that $m \ge 1$.

Second, by the choice of points $z_i$, one easily sees that the balls $\mathbf{B}(z_i, \frac{r}{2K})$ are disjoint subsets of $\mathbf{B}(z_0, Kr)$, and also $\mathbf{B}(z_0, Kr) \subset \mathbf{B}(z_i, K^2 r)$ for $i = 0, 1, \dots, n$. Hence the assumption in the lemma implies

$$\mu(\mathbf{B}(z_0, Kr)) \leq \mu(\mathbf{B}(z_i, K^2 r)) \leq C^{3m+4} \mu\left(\mathbf{B}\left(z_i, \frac{K^2 r}{2^{3m+4}}\right)\right) \leq C^{3m+4} \mu\left(\mathbf{B}\left(z_i, \frac{r}{2K}\right)\right),$$

and so

$$\frac{n+1}{C^{3m+4}} \mu(\mathbf{B}(z_0, Kr)) \leq \sum_{i=0}^{n} \mu\left(\mathbf{B}\left(z_i, \frac{r}{2K}\right)\right) \leq \mu(\mathbf{B}(z_0, Kr)).$$

Hence

$$n \leq C^{3m+4} \leq C^{\frac{3 \log K}{\log 2} + 4},$$

which is what we need. □

**Definition 2.8.** A quasi-metric space $(Z, \rho)$ is said to be *Ahlfors $Q$-regular* if $(Z, \rho)$ admits a positive Borel measure $\mu$ such that

$$C^{-1} R^Q \leq \mu(\mathbf{B}(x, R)) \leq C R^Q$$

for all $x \in Z$ and $0 < R < \operatorname{diam}(Z)$ (possibly, $\operatorname{diam}(Z) = \infty$), where the constants $C \geq 1$ and $Q > 0$ are called the Ahlfors regularity coefficients of $(Z, \rho)$.

**Lemma 2.10.** *Every Ahlfors regular space is uniformly perfect and homogeneous. Also it is doubling.*

*Proof.* Assume that the quasi-metric space $(Z, \rho)$ is Ahlfors regular. That is, $(Z, \rho)$ admits a positive Borel measure $\mu$ such that for all $x \in Z$ and $0 < R < \operatorname{diam}(Z)$,

$$C^{-1} R^Q \leq \mu(\mathbf{B}(x, R)) \leq C R^Q, \tag{2.2}$$

where $C \geq 1$ and $Q > 0$ are the Ahlfors regularity coefficients of $(Z, \rho)$. Obviously, we have

$$\mu(\mathbf{B}(x, 2R)) \leq C^2 2^Q \mu(\mathbf{B}(x, R)).$$

Hence $(Z, \rho, \mu)$ is $(K, C^2 2^Q)$-$HS$.

To finish the proof, by Lemma 2.9, it remains to show the uniform perfectness of $(Z, \rho)$. For this, we set

$$\tau = \frac{1}{2 C^{2/Q}}.$$

Then for any $x \in Z$ and $r > 0$ with $Z \setminus \mathbf{B}(x, r) \neq \emptyset$, it follows from (2.2) that

$$\mu(\mathbf{B}(x, \tau r)) \leq C (\tau r)^Q = \frac{1}{2^Q C} r^Q < \mu(\mathbf{B}(x, r)).$$

Hence

$$\mathbf{B}(x, r) \setminus \mathbf{B}(x, \tau r) \neq \emptyset,$$

which guarantees that $(Z, \rho)$ is uniformly $\tau$-perfect. □

We remark that the assertions in Lemma 2.10 in the setting of metric spaces were stated by David and Semmes in [9] (see [9, §5.4] and [9, Lemma 16.3]).

## 3. Weakly quasimöbius maps, quasimöbius maps and uniform perfectness

This section is devoted to the proof of Theorem 3.1 below. Before this proof, we shall establish a result concerning quasisymmetric maps and weakly quasisymmetric maps, i.e. Lemma 3.1 below. Based on this result, Theorem 3.1 will be proved. Also, Lemma 3.1 plays a key role in the proof of the main result in Section 4.

### 3.1. The main result.

**Theorem 3.1.** *Suppose $f\colon (Z_1,\rho_1) \to (Z_2,\rho_2)$ is a homeomorphism, where $(Z_1,\rho_1)$ is a uniformly $\tau$-perfect quasi-metric space and $(Z_2,\rho_2)$ is a quasi-metric space. Then the following statements are quantitatively equivalent.*

(1) *If $f$ is weakly $(h_1,H_1)$-quasimöbius with $h_1>1$ and $H_1\ge 1$, then it is $\theta$-quasimöbius;*
(2) *$(Z_2,\rho_2)$ is uniformly $\tau'$-perfect and $f^{-1}$ is weakly $(h_2,H_2)$-quasimöbius with $h_2>1$ and $H_2\ge 1$.*

### 3.2. An auxiliary result.

**Lemma 3.1.** *Under the assumptions of Theorem 3.1, the following statements are quantitatively equivalent.*

(1) *The weak $(h_1,H_1)$-quasisymmetry of $f$ implies its $\eta$-quasisymmetry, where $h_1>1$ and $H_1\ge 1$;*
(2) *$(Z_2,\rho_2)$ is uniformly $\tau'$-perfect and $f^{-1}$ is weakly $(h_2,H_2)$-quasisymmetric with $h_2>1$ and $H_2\ge 1$.*

*Proof.* By Lemmas 2.2 and C, we see that the implication from (1) to (2) is obvious. So we only need to show the one from (2) to (1).

We assume that $f$ is weakly $(h_1,H_1)$-quasisymmetric, $f^{-1}$ is weakly $(h_2,H_2)$-quasisymmetric and $(Z_2,\rho_2)$ is uniformly $\tau'$-perfect. To prove that $f$ is $\eta$-quasisymmetric, we need some preparation.

By Lemma E, there are constants $\lambda_i$ and $\mu_i$ $(i=1,2)$ such that $(Z_1,\rho_1)$ is $(\lambda_1,\lambda_2)$-$HD$ and $(Z_2,\rho_2)$ is $(\mu_1,\mu_2)$-$HD$, where $\lambda_i=\lambda_i(K,\tau)\in(0,1)$ and $\mu_i=\mu_i(K,\tau')\in(0,1)$. (Here and in what follows, the notation $\lambda_i=\lambda_i(K,\tau)$ means that the constant $\lambda_i$ depends only on $K$ and $\tau$.) Without loss of generality, we may assume that

$$h_1=h_2=h>1,\quad H_1=H_2=H \quad\text{and}\quad H=\frac{1}{\lambda_2}\vee\frac{1}{\mu_2}\vee h.$$

Let $a$, $b$, $x$ be distinct points in $(Z_1,\rho_1)$, and set

$$t=\frac{\rho_1(x,a)}{\rho_1(x,b)}\quad\text{and}\quad t'=\frac{\rho_2(x',a')}{\rho_2(x',b')}.$$

To find the needed homeomorphism $\eta$, we need to obtain a relation between $t$ and $t'$. For this, we divide the discussion into two cases.

**Case 3.1.** $t\le 1$.

It follows from the assumption "$H\ge\frac{1}{\lambda_2}$" that there is an integer $n\ge 2$ such that

$$\lambda_2^n<\frac{1}{H}\le\lambda_2^{n-1}.$$

Then we have the following claim.

**Claim 3.1.** *There exist an integer $s \geq 1$ and a finite sequence $\{b_0 = b, \cdots, b_s\}$ in $(Z_1, \rho_1)$ such that*

(1) *for each $i \in \{0, \ldots, s-1\}$, $\lambda_1^n \rho_1(b_i, x) \leq \rho_1(b_{i+1}, x) \leq \lambda_2^n \rho_1(b_i, x)$;*
(2) $\lambda_1^n \rho_1(b_s, x) \leq \rho_1(a, x) \leq \rho_1(b_s, x)$.

Let $b_0 = b$. Since $(Z_1, \rho_1)$ is $(\lambda_1^n, \lambda_2^n)$-$HD$, we see that there is a point $b_1 \in Z_1$ such that

$$\lambda_1^n \rho_1(b_0, x) \leq \rho_1(b_1, x) \leq \lambda_2^n \rho_1(b_0, x) < \frac{1}{H} \rho_1(b, x).$$

If

$$\rho_1(b_1, x) < \rho_1(a, x) \leq \rho_1(b_0, x),$$

then we take $s = 1$. Otherwise, there must exist a point $b_2 \in Z_1$ such that

$$\lambda_1^n \rho_1(b_1, x) \leq \rho_1(b_2, x) \leq \lambda_2^n \rho_1(b_1, x) < \frac{1}{H^2} \rho_1(b, x).$$

If

$$\rho_1(b_2, x) < \rho_1(a, x) \leq \rho_1(b_1, x),$$

then we take $s = 2$. Otherwise, there must exist a point $b_3 \in Z_1$ such that

$$\lambda_1^n \rho_1(b_2, x) \leq \rho_1(b_3, x) \leq \lambda_2^n \rho_1(b_2, x) < \frac{1}{H^3} \rho_1(b, x).$$

By repeating this procedure, we easily see that there is an integer $s$ such that

$$\lambda_1^n \rho_1(b_s, x) \leq \rho_1(b_{s+1}, x) \leq \lambda_2^n \rho_1(b_s, x)$$

and

$$\rho_1(b_{s+1}, x) < \rho_1(a, x) \leq \rho_1(b_s, x).$$

Obviously, this $s$ and the finite sequence $\{b_0 = b, \cdots, b_s\}$ are the required.

Next, we find a relation between $t$ and $t'$ by two steps. First, we get a relation between $t'$ and $s$. It follows from the first assertion in Claim 3.1 that

$$\rho_1(b_{i+1}, x) \leq \lambda_2^n \rho_1(b_i, x) < \frac{1}{H} \rho_1(b_i, x).$$

By the assumption "$f^{-1}$ being weakly $(h, H)$-quasisymmetric", necessarily, we obtain

$$\rho_2(b'_{i+1}, x') \leq \frac{1}{h} \rho_2(b'_i, x'),$$

and thus we have

$$\rho_2(b'_s, x') \leq \frac{1}{h^s} \rho_2(b', x').$$

Meanwhile, the second assertion in Claim 3.1 implies

$$\rho_1(a, x) \leq \rho_1(b_s, x) < h\rho_1(b_s, x),$$

and so it follows from the assumption "$f$ being weakly $(h, H)$-quasisymmetric" that

$$\rho_2(a', x') \leq H\rho_2(b'_s, x') \leq Hh^{-s} \rho_2(b', x'),$$

which leads to

$$t' \leq Hh^{-s}. \tag{3.1}$$

Second, we find a relation between $t$ and $s$ as follows. Again, it follows from the second assertion in Claim 3.1 that

$$\rho_1(a, x) \geq \lambda_1^n \rho_1(b_s, x) \geq \cdots \geq \lambda_1^{n(s+1)} \rho_1(b, x),$$

and thus

$$t \geq \lambda_1^{n(s+1)}. \tag{3.2}$$

Now, by (3.1) and (3.2), we can easily get a relation between $t$ and $t'$, which is as follows.

$$t' \leq Hht^{\frac{\log h \log \lambda_2}{\log \lambda_1 (\log H - \log \lambda_2)}}. \tag{3.3}$$

**Case 3.2.** $t > 1$.

If $t' \leq 1$, then

$$t' < t. \tag{3.4}$$

In the following, we assume that $t' > 1$. It follows from the assumption "$H \geq \frac{1}{\mu_2}$" that there is an integer $m \geq 2$ such that

$$\mu_2^m < \frac{1}{H} \leq \mu_2^{m-1}.$$

Since $(Z_2, \rho_2)$ is $(\mu_1^m, \mu_2^m)$-$HD$, the similar reasoning as in Claim 3.1 guarantees that the following claim holds.

**Claim 3.2.** *There are an integer $k \geq 1$ and a finite sequence $\{a'_0 = a', a'_1, \cdots, a'_k\}$ in $(Z_2, \rho_2)$ such that*

(1) *for each $i \in \{0, \cdots, k-1\}$, $\mu_1^m \rho_2(a'_i, x') \leq \rho_2(a'_{i+1}, x') \leq \mu_2^m \rho_2(a'_i, x')$;*
(2) $\mu_1^m \rho_2(a'_k, x') \leq \rho_2(b', x') \leq \rho_2(a'_k, x')$.

Next, we are going to find a relation between $t$ and $k$. It follows from Claim 3.2(1) that

$$\rho_2(a'_{i+1}, x') \leq \mu_2^m \rho_2(a'_i, x') < \frac{1}{H}\rho_2(a'_i, x'),$$

and so by the assumption "$f$ being weakly $(h, H)$-quasisymmetric", necessarily, we get that

$$\rho_1(a_{i+1}, x) \leq \frac{1}{h}\rho_1(a_i, x),$$

which leads to

$$\rho_1(a_k, x) \leq \frac{1}{h^k}\rho_1(a, x).$$

Since Claim 3.2(2) shows that

$$\rho_2(b', x') \leq \rho_2(a'_k, x') < h\rho_2(a'_k, x'),$$

from the assumption "$f^{-1}$ being weakly $(h, H)$-quasisymmetric", we deduce that

$$\rho_1(b, x) \leq H\rho_1(a_k, x),$$

which implies

$$\rho_1(b, x) \leq Hh^{-k}\rho_1(a, x).$$

Hence

$$t \geq H^{-1}h^k, \tag{3.5}$$

as required.

It follows from the fact

$$\rho_2(b', x') \geq \mu_1^{m(k+1)}\rho_2(a', x')$$

that

$$t' \leq \mu_1^{-m(k+1)},$$

and then (3.5) leads to

$$t' \le \mu_1^{\frac{\log H - \log \mu_2}{\log h \log \mu_2} \log(Hh)} t^{\frac{\log H - \log \mu_2}{\log h \log \mu_2} \log \mu_1}. \tag{3.6}$$

Now, we are ready to construct the needed homeomorphism. Let

$$\eta(t) = \begin{cases} Mt^{\frac{\log h \log \lambda_2}{\log H - \log \lambda_2) \log \lambda_1}}, & \text{if } 0 < t \le 1, \\ Mt^{\frac{\log H - \log \mu_2}{\log h \log \mu_2} \log \mu_1}, & \text{if } t > 1, \end{cases}$$

where $M = Hh \vee \mu_1^{\frac{\log H - \log \mu_2}{\log h \log \mu_2} \log(Hh)}$. Then it follows from (3.3), (3.4) and (3.6) that $f$ is $\eta$-quasisymmetric, and thus, the proof of the lemma is complete. □

**3.3. The proof of Theorem 3.1.** It follows from Lemmas 2.2 and C that the implication from (1) to (2) is obvious. So, we only need to prove the one from (2) to (1).

We assume that $f$ is weakly $(h_1, H_1)$-quasimöbius, $f^{-1}$ is weakly $(h_2, H_2)$-quasimöbius and $(Z_2, \rho_2)$ is uniformly $\tau'$-perfect. To finish the proof, we need to construct a desired homeomorphism $\theta\colon [0, +\infty) \to [0, +\infty)$ which depends only on the given data. We start with some preparation.

By Lemma E, there exist constants $\sigma = \sigma(\tau) > 1$ and $\sigma' = \sigma'(\tau') > 1$ such that $(Z_1, \rho_1)$ and $(Z_2, \rho_2)$ are $\sigma$-dense and $\sigma'$-dense, respectively.

For simplicity, in the following, we assume that

$$h_1 = h_2 = h > 1, \quad H_1 = H_2 = H \ge 1 \quad \text{and} \quad K \ge \max\{\sigma, \sigma', h, H\}.$$

**Claim 3.3.** *For $a$, $d$ in $(Z_1, \rho_1)$, there is a chain $\{x_i\}_{i\in\mathbf{Z}}$ in $(Z_1, \rho_1)$ joining $a$ and $d$ such that for all $i$,*

$$\frac{1}{K^2} \le r(a, x_{i+1}, x_i, d) \le \frac{1}{K}. \tag{3.7}$$

In order to prove this claim, we let $\{w_j\}_{j\in\mathbf{Z}}$ be a $\sigma$-chain in $(Z_1, \rho_1)$ joining $a$ and $d$, i.e.

$$\frac{1}{\sigma} \le r(a, w_j, w_{j+1}, d) \le \sigma,$$
$$w_j \to a \ \text{ as } \ j \to -\infty \ \text{ and } \ w_j \to d \ \text{ as } \ j \to +\infty.$$

We find the required chain from $\{w_j\}_{j\in\mathbf{Z}}$ in the following way.

Since $r(a, w_0, w_1, d) \le K$ and $r(a, w_0, w_j, d) \to +\infty$ as $j \to +\infty$, we see that there is a $j > 1$ such that $r(a, w_0, w_j, d) > K$. Let

$$j_0 = \min\{j > 1\colon r(a, w_0, w_j, d) > K\}.$$

Similarly, it follows from $r(a, w_{j_0}, w_{j_0+1}, d) \le K$ and $r(a, w_{j_0}, w_j, d) \to +\infty$ as $j \to +\infty$ that there is a $j > j_0 + 1$ such that $r(a, w_{j_0}, w_j, d) > K$. Let

$$j_1 = \min\{j > j_0 + 1\colon r(a, w_{j_0}, w_j, d) > K\}.$$

Then we have

$$\frac{1}{K} \ge r(a, w_{j_1}, w_{j_0}, d) = r(a, w_{j_1-1}, w_{j_0}, d) r(a, w_{j_1}, w_{j_1-1}, d) \ge \frac{1}{\sigma K} \ge \frac{1}{K^2}.$$

Also, $r(a, w_{j_0-1}, w_{j_0}, d) \le K$ and $r(a, w_j, w_{j_0}, d) \to +\infty$ as $j \to -\infty$. Then there is a $j < j_0 - 1$ such that $r(a, w_j, w_{j_0}, d) > K$. Let

$$j_{-1} = \max\{j < j_0 - 1\colon r(a, w_j, w_{j_0}, d) > K\}.$$

Hence we have

$$\frac{1}{K} \geq r(a, w_{j_0}, w_{j_{-1}}, d) = r(a, w_{j_0}, w_{j_{-1}+1}, d)r(a, w_{j_{-1}+1}, w_{j_{-1}}, d) \geq \frac{1}{K^2}.$$

By repeating this procedure, we can find a subsequence $\{w_{j_i}\}_{i\in\mathbf{Z}}$ of $\{w_j\}_{j\in\mathbf{Z}}$ such that

$$\frac{1}{K^2} \leq r(a, w_{j_{i+1}}, w_{j_i}, d) \leq \frac{1}{K}.$$

Since $w_{j_i} \to d$ as $i \to +\infty$ and $w_{j_i} \to a$ as $i \to -\infty$, by letting $x_i = w_{j_i}$ for each $i$, we easily know that the claim is true.

Since $(Z_2, \rho_2)$ is $\sigma'$-dense, a similar argument as in the proof of Claim 3.3 guarantees that the following claim holds.

**Claim 3.4.** *For $a'$, $d'$ in $(Z_2, \rho_2)$, there is a chain $\{y'_j\}_{j\in\mathbf{Z}}$ in $(Z_2, \rho_2)$ joining $a'$ and $d'$ such that for all $j$, we have*

$$\frac{1}{K^2} \leq r(a', y'_{j+1}, y'_j, d') \leq \frac{1}{K}.$$

For $a$, $b$, $c$ and $d$ in $(Z_1, \rho_1)$, let

$$T = r(a, b, c, d) \quad \text{and} \quad T' = r(a', b', c', d').$$

The next thing we want to do is to find a relation between $T$ and $T'$. For this, we divide the discussion into two cases.

**Case 3.3.** $T = r(a, b, c, d) \leq \frac{h^2}{K^2}$.

By Claim 3.3, there exists a chain $\{x_i\}_{i\in\mathbf{Z}}$ in $(Z_1, \rho_1)$ joining $a$ and $d$ such that for all $i$,

$$\frac{1}{K^2} \leq r(a, x_{i+1}, x_i, d) \leq \frac{1}{K}.$$

Let

$$p = \min\left\{i\colon r(a, b, x_i, d) \geq \frac{h}{K^2}\right\} \quad \text{and} \quad q = \min\left\{i\colon r(a, c, x_i, d) \geq \frac{1}{h}\right\}.$$

Since

$$\min\{r(a, b, x_i, d), r(a, c, x_i, d)\} \to +\infty \ \text{ as } \ i \to +\infty$$

and

$$\max\{r(a, b, x_i, d), r(a, c, x_i, d)\} \to 0 \ \text{ as } \ i \to -\infty,$$

we see that both $p$ and $q$ exist. Next, we prove $p \geq q$. It follows from

$$\frac{h^2}{K^2} \geq T = \frac{r(a, b, x_p, d)}{r(a, c, x_p, d)} \geq \frac{h}{K^2 r(a, c, x_p, d)}$$

that

$$r(a, c, x_p, d) \geq \frac{1}{h}.$$

Obviously, $p \geq q$.

The following estimates on the cross ratios $r(a, b, x_p, d)$ etc are useful. By Claim 3.3, one can easily get the following estimates on $r(a, b, x_p, d)$ and $r(a, x_q, c, d)$, respectively:

$$\frac{h}{K^2} \leq r(a, b, x_p, d) = r(a, x_{p-1}, x_p, d)r(a, b, x_{p-1}, d) \leq h \tag{3.8}$$

and

$$\frac{h}{K^2} \leq r(a, x_q, x_{q-1}, d) r(a, x_{q-1}, c, d) = r(a, x_q, c, d) \leq h, \tag{3.9}$$

and so we infer from the assumption "$f$ being weakly $(h, H)$-quasimöbius" that

$$r(a', b', x'_p, d') \leq H \quad \text{and} \quad r(a', x'_q, c', d') \leq H. \tag{3.10}$$

To get a relation between $T$ and $T'$, we need to consider two possibilities. The first possibilities is when $p = q$. Under this assumption, by (3.8) and (3.9), we obtain that

$$T = r(a, b, x_p, d) r(a, x_p, c, d) \geq h^2 K^{-4}.$$

Moreover, (3.10) implies

$$T' = r(a', b', x'_p, d') r(a', x'_p, c', d') \leq H^2.$$

Hence we get

$$T' \leq h^{-2} H^2 K^4 T. \tag{3.11}$$

Now, we consider the remaining possibility, that is, $p > q$. Under this assumption, we need a lower bound for $p - q$ in terms of $T$. It follows from Claim 3.3 that

$$T = r(a, b, x_p, d) r(a, x_p, x_{p-1}, d) \cdots r(a, x_q, x_{q-1}, d) r(a, x_{q-1}, c, d) \geq h^2 K^{2(q-p-2)},$$

which implies

$$p - q \geq \frac{2 \log h - \log T}{2 \log K} - 2,$$

as required.

Now, we can establish a relation between $T$ and $T'$ as follows. Since

$$r(a, x_{i+1}, x_i, d) \leq \frac{1}{K} \leq \frac{1}{H},$$

by the assumption "$f^{-1}$ being weakly $(h, H)$-quasimöbius", necessarily, we have

$$r(a', x'_{i+1}, x'_i, d') \leq \frac{1}{h},$$

and so,

$$T' = r(a', b', x'_p, d') r(a', x'_p, x'_{p-1}, d') \cdots r(a', x'_{q+1}, x'_q, d') r(a', x'_q, c', d') \leq H^2 h^{q-p}.$$

Hence we easily get

$$T' \leq h^2 H^2 h^{-\frac{\log h}{\log K}} T^{\frac{\log h}{2 \log K}}. \tag{3.12}$$

**Case 3.4.** $T = r(a, b, c, d) > \frac{h^2}{K^2}$.

By Claim 3.4, there is a chain $\{y'_j\}_{j \in \mathbf{Z}}$ in $(Z_2, \rho_2)$ joining $a'$ and $d'$ such that for all $j$,

$$\frac{1}{K^2} \leq r(a', y'_{j+1}, y'_j, d') \leq \frac{1}{K}. \tag{3.13}$$

Let

$$m = \min\{j \colon r(a', b', y'_j, d') \geq H\} \quad \text{and} \quad n = \min\{j \colon r(a', c', y'_j, d') \geq \frac{1}{HK^2}\}.$$

Since

$$r(a', b', y'_j, d') \to +\infty \text{ as } j \to +\infty \text{ and } r(a', b', y'_j, d') \to 0 \text{ as } j \to -\infty,$$

we see that both $m$ and $n$ exist. Then we have

$$H \le r(a', b', y'_m, d') = r(a', y'_{m-1}, y'_m, d') r(a', b', y'_{m-1}, d') \le HK^2 \tag{3.14}$$

and

$$\frac{1}{HK^2} \le r(a', c', y'_n, d') = r(a', y'_{n-1}, y'_n, d') r(a', c', y'_{n-1}, d') \le \frac{1}{H}, \tag{3.15}$$

and, necessarily, we have

$$r(a, b, y_m, d) \ge h \text{ and } r(a, y_n, c, d) \ge h, \tag{3.16}$$

since $f$ is weakly $(h, H)$-quasimöbius.

To get a relation between $T$ and $T'$, we need to consider three possibilities: $n = m$, $n > m$ and $n < m$. We first consider the possibility when $n = m$. It follows from (3.16) that

$$T = r(a, b, y_n, d) r(a, y_n, c, d) \ge h^2.$$

Moreover, (3.14), together with (3.15), implies

$$T' = r(a', b', y'_n, d') r(a', y'_n, c', d') \le H^2 K^4.$$

Hence we get

$$T' \le h^{-2} H^2 K^4 T. \tag{3.17}$$

For the remaining possibilities when $n > m$ and when $n < m$, since the discussions are similar, obviously, we only need to consider the possibility when $n > m$. Under this assumption, we still need an upper bound of $n - m$ in terms of $T$. Since, necessarily, it follows from (3.13) that

$$r(a, y_j, y_{j+1}, d) \ge h,$$

we see that

$$T = r(a, b, y_m, d) r(a, y_m, y_{m+1}, d) \cdots r(a, y_{n-1}, y_n, d) r(a, y_n, c, d) \ge h^{n-m+2}.$$

So we can get

$$n - m \le \frac{\log T}{\log h} - 2.$$

It is the right time for us to get a relation between $T$ and $T'$ in this possibility. Since (3.13) leads to

$$\begin{aligned} T' &= r(a', b', y'_{m-1}, d') r(a', y'_{m-1}, y'_m, d') \cdots r(a', y'_{n-1}, y'_n, d') r(a', y'_n, c', d') \\ &\le H^2 K^{2(n-m+2)}, \end{aligned}$$

we have

$$T' \le H^2 T^{\frac{2 \log K}{\log h}}. \tag{3.18}$$

Now, we are ready to complete the proof of the implication from (2) to (1). Let

$$\theta(t) = h^{-2} H^2 K^4 (t^\alpha \vee t^{\frac{1}{\alpha}}),$$

where $\alpha = \frac{\log h}{2 \log K}$. By (3.11), (3.12), (3.17) and (3.18), obviously, we see that $f$ is $\theta$-quasimöbius. □

## 4. Weak quasisymmtry and quasisymmetry of homeomorphisms in certain quasi-metric spaces

It is known that the weak $(1, H)$-quasisymmetry of homeomorphisms between two $\kappa$-HTB metric or doubling spaces implies the quasisymmetry provided that the preimage space is path-connected (see [27, Theorem 2.9] or [12, Theorem 10.19]). Consequently, by Lemma C, the image space must be uniformly perfect. In this section, we consider the case when the preimage space need not be connected. With the aid of the uniform perfectness, we get the following result, Theorem 4.1, which will be useful in the proofs of the main results in the next section.

### 4.1. The main result.

**Theorem 4.1.** *Suppose*

(1) $f\colon (Z_1, \rho_1) \to (Z_2, \rho_2)$ *is a homeomorphism between two quasi-metric spaces;*
(2) $(Z_2, \rho_2)$ *is uniformly* $\tau$*-perfect and* $\kappa$*-HTB.*

*Then the following statements are quantitatively equivalent.*

(1) *If* $f$ *is weakly* $(h, H)$*-quasisymmetric with* $h > 0$ *and* $H \geq 1$*, then* $f$ *is* $\eta$*-quasisymmetric;*
(2) $(Z_1, \rho_1)$ *is uniformly* $\tau'$*-perfect and* $\kappa'$*-HTB.*

**Remark 4.1.** By Lemmas C and E, we easily see that Theorem 4.1 is a generalization of [27, Theorem 2.9] and [12, Theorem 10.19] in two aspects: (1) The condition “the preimage space being a connected $\kappa$-HTB space” in [27, Theorem 2.9] and [12, Theorem 10.19] is replaced by the one “the preimage space being a uniformly perfect and $\kappa$-HTB space”. Note that the connectedness implies the uniform perfectness; (2) The condition “$f$ being weakly $(1, H)$-quasisymmetric with $H \geq 1$ in [27, Theorem 2.9] and [12, Theorem 10.19]” is replaced by the one “$f$ being weakly $(h, H)$-quasisymmetric with $h > 0$ and $H \geq 1$”.

**4.2. An auxiliary result.** The following result plays a key role in the proof of Theorem 4.1.

**Lemma 4.1.** *Suppose*

(1) $f\colon (Z_1, \rho_1) \to (Z_2, \rho_2)$ *is weakly* $(h, H)$*-quasisymmetric between two quasi-metric spaces with* $h > 0$ *and* $H \geq 1$*;*
(2) $(Z_1, \rho_1)$ *is uniformly* $\tau$*-perfect, and*
(3) $(Z_2, \rho_2)$ *is* $\kappa$*-HTB.*

*Then* $f^{-1}$ *is weakly* $(h_1, H_1)$*-quasisymmetric, where* $h_1 = H$*,* $H_1 = \lambda_1^{-\kappa(KH^3)-1}$ *and* $\lambda_1 = \lambda_1(\tau, K)$*.*

*Proof.* We start with some preparation. Without loss of generality, we assume that $h < K$. (Here, we recall that $K$ denotes the coefficient of the quasi-metric spaces.) From Lemmas D and E, together with the assumption that $(Z_1, \rho_1)$ is uniform $\tau$-perfect, we deduce that there are two constants $\lambda_1 = \lambda_1(\tau, K)$ and $\lambda_2 = \lambda_2(\tau, K)$ such that

(1) $0 < \lambda_1 \leq \lambda_2 < \frac{h}{K^2} < \frac{h}{K} < 1$; and
(2) $(Z_1, \rho_1)$ is $(\lambda_1, \lambda_2)$-$HD$.

With this preparation, obviously, to prove Lemma 4.1, it is sufficient to demonstrate the following.

**Claim 4.1.** *For points $x$, $a$, $b$ in $(Z_1, \rho_1)$, if $\rho_1(x,a) < \frac{1}{H_1}\rho_1(x,b)$, then*

$$\rho_2(x',a') < \frac{\rho_2(x',b')}{h_1}.$$

Now, we prove the claim. Since $H_1 = \lambda_1^{-\kappa(KH^3)-1}$, we see that there is a unique integer $n \geq 2$ such that

$$\lambda_1^n \leq \frac{1}{H_1} < \lambda_1^{n-1},$$

and so

$$-\frac{\log H_1}{\log \lambda_1} \leq n < 1 - \frac{\log H_1}{\log \lambda_1}.$$

Let $b_0 = b$. Since $(Z_1, \rho_1)$ is $(\lambda_1, \lambda_2)$-$HD$, we know that there is a finite sequence $\{b_i\}_{i=1}^n$ such that for each $i \in \{1, \ldots, n\}$,

$$\lambda_1 \rho_1(x, b_{i-1}) \leq \rho_1(x, b_i) \leq \lambda_2 \rho_1(x, b_{i-1}).$$

We assert that

$$\rho_1(x, b_j) \vee \rho_1(b_j, a) \leq h\rho_1(b_j, b_i) \tag{4.1}$$

for $0 \leq i < j < n$.

By the choice of $b_i$ and the fact $\lambda_2 < \frac{h}{K^2} < \frac{h}{K} < 1$, we have

$$\rho_1(x, b_j) \leq \lambda_2 \rho_1(x, b_i) < \frac{h}{K}(\rho_1(x, b_j) \vee \rho_1(b_j, b_i)) = \frac{h}{K}\rho_1(b_j, b_i), \tag{4.2}$$

and so

$$\rho_1(b_j, a) \leq K(\rho_1(x, b_j) \vee \rho_1(x, a)) = K\rho_1(x, b_j) \leq h\rho_1(b_j, b_i), \tag{4.3}$$

since the assumption in the claim guarantees that

$$\rho_1(x, a) < \frac{1}{H_1}\rho_1(x, b) < \lambda_1^{n-1}\rho_1(x, b) \leq \rho_1(x, b_j).$$

Hence we conclude from (4.2) and (4.3) that the assertion (4.1) is true.

We continue the proof of Claim 4.1. We shall finish the proof by applying Lemma 2.6. For this, we do some preparation. Since $f$ is weakly $(h, H)$-quasisymmetric, from (4.1), it follows that

$$\rho_2(x', b'_j) \vee \rho_2(b'_j, a') \leq H\rho_2(b'_j, b'_i),$$

and so

$$\rho_2(b'_j, b'_i) \geq \frac{1}{KH}\rho_2(x', a').$$

Meanwhile, since

$$\rho_1(x, b_j) \leq \lambda_2 \rho_1(b, x) < \frac{h}{K^2}\rho_1(b, x) < h\rho_1(b, x)$$

for $1 \leq j < n$, we also have

$$\rho_2(b'_j, x') \leq H\rho_2(b', x'),$$

which implies that the points $b'_j$ lie inside $\overline{\mathbf{B}}(x', H\rho_2(b', x'))$ for all $j$ in $\{0, 1, \ldots, n-1\}$.

Now, we are ready for the application of Lemma 2.6. Since $(Z_2, \rho_2)$ is $\kappa$-$HTB$, Lemma 2.6 guarantees that

$$-\frac{\log H_1}{\log \lambda_1} \leq n \leq \kappa\Big(\frac{KH^2\rho_2(b', x')}{\rho_2(x', a')}\Big),$$

from which, necessarily,

$$\rho_2(x', a') < \frac{\rho_2(x', b')}{h_1}.$$

Hence Claim 4.1 is proved, and thus, the proof of Lemma 4.1 is complete. □

**4.3. The proof of Theorem 4.1.** The necessity easily follows from Lemmas 2.2, C and 2.8. To prove the sufficiency, we first check the weak quasisymmetry of $f^{-1}$. Since $f\colon (Z_1, \rho_1) \to (Z_2, \rho_2)$ is weakly $(h, H)$-quasisymmetric, $(Z_1, \rho_1)$ is uniformly $\tau'$-perfect and $(Z_2, \rho_2)$ is $\kappa$-$HTB$, we infer from Lemma 4.1 that $f^{-1}$ is weakly $(h_1, H_1)$-quasisymmetric, where $h_1 = H$, $H_1 = \lambda_1^{-\kappa(KH^3)-1}$ and $\lambda_1 = \lambda_1(\tau', K) \in (0, 1)$.

Further, we prove that $f$ is weakly $(h_2, H_2)$-quasisymmetric with $h_1 > 1$ and $H_2 > 1$. Since $f^{-1}$ is weakly $(h_1, H_1)$-quasisymmetric, $(Z_2, \rho_2)$ is uniformly $\tau$-perfect and $(Z_1, \rho_1)$ is $\kappa'$-$HTB$, again, we deduce from Lemma 4.1 that $f$ is weakly $(h_2, H_2)$-quasisymmetric, where $h_2 = H_1$, $H_2 = \lambda_2^{-\kappa'(KH_1^3)-1}$ and $\lambda_2 = \lambda_2(\tau, K) \in (0, 1)$. Then the quasisymmetry of $f$ easily follows from Theorem 3.1, and so, the proof of Theorem 4.1 is finished. □

## 5. Weakly quasimöbius maps and quasimöbius maps in uniformly perfect and homogeneous spaces

In this section, we shall show that in uniformly perfect and homogeneous quasi-metric spaces, every weakly quasimöbius map must be quasimöbius map (Theorem 5.1 below), and also, a generalized form of this result (Theorem 5.2 below) will be proved.

**5.1. The main results.**

**Theorem 5.1.** *Suppose $f\colon (Z_1, \rho_1) \to (Z_2, \rho_2)$ is a homeomorphism between two uniformly perfect and homogeneous quasi-metric spaces. Then the following statements are quantitatively equivalent.*

(1) *$f$ is weakly $(h, H)$-quasimöbius with $h > 0$ and $H \geq 1$;*
(2) *it is $\theta$-quasimöbius.*

By Lemma 2.10, the following result is a direct consequence of Theorem 5.1.

**Corollary 5.1.** *Every weakly quasimöbius homeomorphism between two Ahlfors regular spaces is quasimöbius.*

**Theorem 5.2.** *Suppose*

(1) *$(Z_1, \rho_1)$ is a quasi-metric space;*
(2) *$(Z_2, \rho_2)$ is a uniformly perfect and homogeneous space.*

*Then the following statements are quantitatively equivalent.*

(1) *If $f\colon (Z_1, \rho_1) \to (Z_2, \rho_2)$ is weakly $(h, H)$-quasimöbius with $h > 1$ and $H \geq 1$, then $f$ is $\theta$-quasimöbius;*
(2) *$(Z_1, \rho_1)$ is uniformly perfect.*

**Remark 5.1.** By comparing Theorem 5.2 with Theorem 5.1, we see that there is no assumption "$(Z_1, \rho_1)$ being homogeneous" in Theorem 5.2. But the assumption "$h > 0$" in Theorem 5.1 is replaced by a stronger one "$h > 1$" in Theorem 5.2. We do not know if Theorem 5.2 still holds when $h > 0$.

**5.2. The proof of Theorem 5.1.** The implication from (2) to (1) is clear, it suffices to show the reverse implication. Assume that $f\colon (Z_1, \rho_1) \to (Z_2, \rho_2)$ is weakly $(h,H)$-quasimöbius between two uniformly perfect and homogeneous spaces. We start the proof with the following claim.

**Claim 5.1.** *There is an $\eta$-quasisymmetric embedding $\psi$ from $(Z_1,\rho_1)$ to $\mathbf{R}^N$, where the control function $\eta$ and the dimension $N\in\mathbf{N}$ depend only on the given data and some constants in $(0,1)$.*

By Lemma 2.9, we know that $(Z_1,\rho_1)$ is $C_3$-doubling as a quasi-metric space. Also, Lemma 1.1 guarantees that there exists a constant $\delta\in(0,1)$ such that the identity map $\mathrm{id}_1\colon (Z_1,\rho_1)\to(Z_1,d)$ is $\eta_1$-quasisymmetric with $\eta_1(t)=4t^\delta$, where $(Z_1,d)$ is a metric space. It follows from Lemmas 2.7 and 2.8 that $(Z_1,d)$ is $C_4$-doubling, and thus we infer from [9, §5.4] or [2] that for any $\epsilon\in(0,1)$, there is a bilipschitz embedding $\varphi\colon (Z_1,d^\varepsilon)\to\mathbf{R}^N$, where $N\in\mathbf{N}$ and the bilipschitz constant $L$ depend only on $\varepsilon$ and $C_4$. Note that, in $\mathbf{R}^n$, we take the Euclidean metric. Obviously, $\varphi$ is $\eta_2$-quasisymmetric with $\eta_2(t)=16t$.

Since the identity map $\mathrm{id}_2\colon (Z_1,d)\to(Z_1,d^\epsilon)$ is $\eta_3$-quasisymmetric, where $\eta_3(t)=t^\epsilon$, it follows from Lemma 2.2 that the composed map $\varphi\circ\mathrm{id}_2\circ\mathrm{id}_1$ is the desired.

Now, let us prove Theorem 5.1. By Claim 5.1, we know that there are quasisymmetric embeddings $\psi_1$ and $\psi_2$ such that

$$\psi_1\colon (Z_1,\rho_1)\to W_1 \quad\text{and}\quad \psi_2\colon (Z_2,\rho_2)\to W_2,$$

where $W_1\subset\mathbf{R}^{N_1}$, $W_2\subset\mathbf{R}^{N_2}$ with $N_1$, $N_2\in\mathbf{N}$, and the control function of $\psi_1$ (resp. $\psi_2$) is $\eta_1$ (resp. $\eta_2$). Without loss of generality, in the following, we assume that $N_1=N_2=N$. Since the uniform perfectness is an invariant property with respect to quasisymmetric maps (cf. Lemma C), we know from Lemma 2.5 that

$$\widetilde{f}=\psi_2\circ f\circ\psi_1^{-1}\colon W_1\to W_2$$

is weakly $(h_1,H_1)$-quasimöbius between $W_1$ and $W_2$ which are uniformly perfect subsets of $\mathbf{R}^N$.

By using translations $q_i$ $(i=1,2)$ in $\overline{\mathbf{R}}^N$, where $\overline{\mathbf{R}}^N=\mathbf{R}^N\cup\{\infty\}$, we assume that the origin $0\in q_1(W_1)\cap q_2(W_2)\subset\overline{\mathbf{R}}^N$ and $q_2\circ\psi_2\circ f\circ\psi_1^{-1}\circ q_1^{-1}(0)=0$. Let

$$u(x)=\frac{x}{|x|^2}$$

be the reflection about the unit sphere centered at the origin in $\overline{\mathbf{R}}^N$, and let

$$\widehat{f}=u\circ q_2\circ\widetilde{f}\circ q_1^{-1}\circ u^{-1}\colon u\circ q_1(W_1)\to u\circ q_2(W_2).$$

Since each translation in $\mathbf{R}^n$ is $\theta_1$-quasimöbius with $\theta_1(t)=t$ and $u$ is $\theta_2$-quasimöbius, where $\theta_2(t)=81t$ (cf. [28, Theorem 6.22]), we get from Lemma 2.5 that $\widehat{f}$ is weakly $(h_2,H_2)$-quasimöbius. Furthermore, $u\circ q_1(W_1)$ and $u\circ q_2(W_2)$ are two uniformly perfect subsets of $\overline{\mathbf{R}}^N$, since uniform perfectness is preserved by quasimöbius maps (cf. Lemma C). Obviously, $\widehat{f}(\infty)=\infty$ and $r(x,a,b,\infty)=|x-b|/|x-a|$ for all $x$, $a$, $b\in Z$. These facts show that $\widehat{f}$ is weakly $(h_2,H_2)$-quasisymmetric.

We are now in a position to complete the proof by means of Theorem 4.1. Note that any subset of $\overline{\mathbf{R}}^N$ is $\kappa$-*HTB* since $\overline{\mathbf{R}}^N$ itself is $\kappa$-*HTB*. It follows from Theorem 4.1 that $\widehat{f}$ is $\eta$-quasisymmetric, and so Lemmas 2.2 and 2.5 guarantee that

$$f=\psi_2^{-1}\circ q_2^{-1}\circ u^{-1}\circ\widehat{f}\circ u\circ q_1\circ\psi_1$$

is $\theta$-quasimöbius, where the control function $\theta$ depends only on the given data. □

**5.3. The proof of Theorem 5.2.** Since the inverse of a quasimöbius map is still quasimöbius, we see that the necessity easily follows from Lemmas 2.2 and C. In the following, we prove the sufficiency.

Under the assumptions in the theorem, to prove that $f$ is quasimöbius, by Theorem 3.1, it is enough to show that $f^{-1}$ is weakly $(H, H_1^2K^4)$-quasimöbius, where $H_1 \geq 1$ is a constant depending only on the given data. (Here we assume that $H > 1$. Otherwise, we can replace $H$ by $H+1$.) For this, by Lemma 2.1, it is enough to demonstrate that for distinct points $x_1$, $x_2$, $x_3$, and $x_4$ in $(Z_1, \rho_1)$, if

$$t = \langle x_1', x_2', x_3', x_4' \rangle > \frac{1}{HK^2}, \tag{5.1}$$

then there is a constant $H_1 \geq 1$ such that

$$s = \langle x_1, x_2, x_3, x_4 \rangle > \frac{1}{H_1}.$$

Without loss of generality, we assume that $r_1 = \rho_1(x_1, x_3) \leq \rho_1(x_2, x_4)$ and both $(Z_1, \rho_1$ and $(Z_2, \rho_2)$ are uniformly $\tau$-perfect with $0 < \tau < 1$. Under this convention, to find the needed $H_1$, we divide the discussions into two cases.

**Case 5.1.** $s > \frac{\tau}{2K^5}$.

In this case, we take

$$H_1' = \frac{2}{\tau} K^5. \tag{5.2}$$

**Case 5.2.** $s \leq \frac{\tau}{2K^5}$.

In this case, we start with some preparation. Since

$$\frac{r_1}{\rho_1(x_1, x_2) \wedge \rho_1(x_3, x_4)} = \frac{\rho_1(x_1, x_3) \wedge \rho_1(x_2, x_4)}{\rho_1(x_1, x_2) \wedge \rho_1(x_3, x_4)} = s,$$

we obtain that

$$\min\{\rho_1(x_i, x_j) \colon i \in \{1,3\},\ j \in \{2,4\}\} \vee \rho_1(x_1, x_3) \geq \frac{1}{K}(\rho_1(x_1, x_2) \wedge \rho_1(x_3, x_4)) = \frac{r_1}{Ks},$$

and thus it follows from the case assumption $s \leq \frac{\tau}{2K^5}$ that

$$\min\{\rho_1(x_i, x_j) \colon i \in \{1,3\},\ j \in \{2,4\}\} \geq \frac{r_1}{Ks}. \tag{5.3}$$

Since $\frac{1}{Ks} > \frac{2K^4}{\tau}$, we see that there must be a unique integer $m \geq 1$ such that

$$\left(\frac{K^4}{\tau}\right)^m \leq \frac{1}{Ks} < \left(\frac{K^4}{\tau}\right)^{m+1}. \tag{5.4}$$

Next, we need to find some special points in $(Z_1, \rho_1)$. For each $i \in \{1, \ldots, m\}$, it follows from (5.3) that

$$Z_1 \setminus \mathbf{B}\left(x_1, \left(\frac{K^4}{\tau}\right)^i r_1\right) \neq \emptyset.$$

Then the assumption that $(Z_1, \rho_1)$ is uniformly $\tau$-perfect guarantees that there exists a point $z_i$ in $(Z_1, \rho_1)$ such that

$$z_i \in \mathbf{B}\left(x_1, \left(\frac{K^4}{\tau}\right)^i r_1\right) \setminus \mathbf{B}\left(x_1, \left(\frac{K^4}{\tau}\right)^i \tau r_1\right).$$

In this way, we have got a finite sequence $\{z_i\}_{i=1}^m$. For this finite sequence, we have the following lower bound of $\rho_1(z_i,z_j)$ in terms of $r_1$.

**Claim 5.2.** *For all* $i\neq j\in\{1,\dots,m\}$, $\rho_1(z_i,z_j)\geq \frac{1}{K}\rho_1(z_j,x_1)\geq \left(\frac{K^4}{\tau}\right)^j\frac{\tau}{K}r_1$.

Since
$$\rho_1(z_i,x_1)<\left(\frac{K^4}{\tau}\right)^i r_1,$$
we easily see that for $1\leq i<j\leq m$,
$$\rho_1(z_j,x_1)\geq \left(\frac{K^4}{\tau}\right)^j\tau r_1\geq \left(\frac{K^4}{\tau}\right)^{i+1}\tau r_1>K\rho_1(z_i,x_1),$$
and thus
$$\rho_1(z_i,z_j)\vee\rho_1(z_i,x_1)\geq\frac{1}{K}\rho_1(z_j,x_1)>\rho_1(z_i,x_1).$$
Hence
$$\rho_1(z_i,z_j)=\rho_1(z_i,z_j)\vee\rho_1(z_i,x_1)\geq\frac{1}{K}\rho_1(z_j,x_1)\geq\left(\frac{K^4}{\tau}\right)^j\frac{\tau}{K}r_1,$$
as required.

Further, we need an analogous result for the images $z_i'$. First, we establish the following lower bound for $\langle z_i',a',z_j',b'\rangle$.

**Claim 5.3.** *For* $1\leq i<j\leq m$,
$$\langle z_i',a',z_j',b'\rangle\geq\frac{1}{HK^2},$$
*where* $a\in\{x_1,x_3\}$ *and* $b\in\{x_2,x_4\}$.

Obviously, for $i\in\{1,\dots,m\}$, one easily has
$$\rho_1(z_i,a)\leq\rho_1(z_i,x_1)\vee\rho_1(z_i,x_3)\leq K(\rho_1(z_i,x_1)\vee\rho_1(x_1,x_3))\leq\left(\frac{K^4}{\tau}\right)^i Kr_1,$$
and so (5.3), (5.4) and Claim 5.2 lead to
$$\langle z_i,a,z_j,b\rangle=\frac{\rho_1(z_i,z_j)\wedge\rho_1(a,b)}{\rho_1(z_i,a)\wedge\rho_1(z_j,b)}\geq\frac{\left(\left(\frac{K^4}{\tau}\right)^j\frac{\tau r_1}{K}\right)\wedge\frac{r_1}{Ks}}{\left(\frac{K^4}{\tau}\right)^{j-1}Kr_1}=K^2.$$
Hence it follows from Lemma A that
$$r(z_i,a,z_j,b)\geq\theta_K^{-1}(\langle z_i,a,z_j,b\rangle)\geq\theta_K^{-1}(K^2)=1>\frac{1}{h},$$
and so
$$r(z_i,z_j,a,b)<h.$$
Here we recall that $\theta_K(t)=K^2(t\vee\sqrt{t})$ for $t>0$.

Since $f$ is weakly $(h,H)$-quasimöbius, we obtain that
$$\langle z_i',a',z_j',b'\rangle=\frac{1}{\langle z_i',z_j',a',b'\rangle}\geq\frac{1}{\theta_K(r(z_i',z_j',a',b'))}\geq\frac{1}{\theta_K(H)}=\frac{1}{HK^2},$$
from which the claim follows.

Set
$$r_2=\min\{\rho_2(x_i',x_j')\colon i\in\{1,3\},\ j\in\{2,4\}\}.$$

Then we can reach the following lower bound for $\rho_2(z'_i, z'_j)$ in terms of $r_2$ by applying Claim 5.3.

**Claim 5.4.** *For any* $i \neq j \in \{1, \dots, m\}$,

$$\rho_2(z'_i, z'_j) \geq \frac{t_1}{2HK^4} r_2,$$

*where* $t_1 = t \wedge 1$.

Suppose on the contrary that there are two points $z_k$ and $z_l$ with $k \neq l \in \{1, \dots, m\}$ such that

$$\rho_2(z'_k, z'_l) < \frac{t_1}{2HK^4} r_2.$$

Without loss of generality, we may assume $k < l$. Since

$$t = \frac{\rho_2(x'_1, x'_3) \wedge \rho_2(x'_2, x'_4)}{\rho_2(x'_1, x'_2) \wedge \rho_2(x'_3, x'_4)} \leq \frac{\rho_2(x'_1, x'_3) \wedge \rho_2(x'_2, x'_4)}{r_2},$$

we have for $i \neq j \in \{1, 2, 3, 4\}$,

$$\rho_2(x'_i, x'_j) \geq t_1 r_2, \tag{5.5}$$

where $t_1 = 1 \wedge t$. Then it is easy to see that $\mathbf{B}(x'_i, \frac{t_1 r_2}{2K^2})$ and $\mathbf{B}(x'_j, \frac{t_1 r_2}{2K^2})$ are disjoint balls for all $i \neq j \in \{1, 2, 3, 4\}$.

Now, we assert that if $z'_k \in \mathbf{B}(x'_i, \frac{t_1 r_2}{2K^2})$, then $z'_l \notin \cup_{j \in \{1,2,3,4\} \setminus \{i\}} \mathbf{B}(x'_j, \frac{t_1 r_2}{2K^2})$.

Suppose on the contrary that $z'_l \in \mathbf{B}(x'_j, \frac{t_1 r_2}{2K^2})$ for some $j \in \{1, 2, 3, 4\} \setminus \{i\}$. Then it follows from the contrary assumption that

$$\rho_2(x'_i, x'_j) \leq K^2(\rho_2(x'_i, z'_k) \vee \rho_2(z'_k, z'_l) \vee \rho_2(z'_l, x'_j)) < \frac{1}{2} t_1 r_2,$$

which contradicts with (5.5).

Easily, it follows from the assertion above that there exist at least two points $a_0 \in \{x_1, x_3\}$ and $b_0 \in \{x_2, x_4\}$ such that $z'_k$ and $z'_l$ are out of the union of the two balls $\mathbf{B}(a'_0, \frac{t_1 r_2}{2K^2})$ and $\mathbf{B}(b'_0, \frac{t_1 r_2}{2K^2})$, and so

$$\rho_2(a'_0, z'_k) \wedge \rho_2(z'_l, b'_0) \geq \frac{t_1 r_2}{2K^2}.$$

Again, it follows from the contrary assumption that

$$\langle z'_k, a'_0, z'_l, b'_0 \rangle = \frac{\rho_2(z'_k, z'_l) \wedge \rho_2(a'_0, b'_0)}{\rho_2(a'_0, z'_k) \wedge \rho_2(z'_l, b'_0)} < \frac{1}{HK^2},$$

which contradicts with Claim 5.3, and thus the claim is proved.

Without loss of generality, we assume that

$$r_2 = \rho_2(a'_1, b'_1),$$

where $a'_1 \in \{x'_1, x'_3\}$ and $b'_1 \in \{x'_2, x'_4\}$.

To prove the existence of the constant in this case, further, we need an upper bound for $m$. We shall apply Lemma 2.6 to reach this goal. For this, we still need a relation between $a'_1$ and $z'_i$ for each $i$ as follows.

**Claim 5.5.** *There is at most one element in* $\{z'_i\}_{i=1}^m$ *which is out of the closed ball* $\overline{\mathbf{B}}(a'_1, HK^3 r_2)$.

Suppose on the contrary that there are at least two elements in $\{z'_i\}_{i=1}^m$ which are out of the closed ball $\overline{\mathbf{B}}(a'_1, HK^3r_2)$. Without loss of generality, we may assume $z'_1$ and $z'_2 \in Z_2 \setminus \overline{\mathbf{B}}(a'_1, HK^3r_2)$.

Since

$$\rho_2(z'_2, b'_1) \vee \rho_2(a'_1, b'_1) \geq \frac{1}{K}\rho_2(z'_2, a'_1) > HK^2r_2,$$

we have

$$\rho_2(z'_2, b'_1) > HK^2r_2,$$

and so

$$\langle z'_1, a'_1, z'_2, b'_1\rangle = \frac{\rho_2(z'_1, z'_2) \wedge \rho_2(a'_1, b'_1)}{\rho_2(z'_1, a'_1) \wedge \rho_2(z'_2, b'_1)} < \frac{1}{HK^2},$$

which contradicts with Claim 5.3, and thus the proof of Claim 5.5 is complete.

Now, we are ready to get an upper bound for $m$ by exploiting Lemma 2.6. First, Lemmas 2.7 and 2.9 guarantee that $(Z_2, \rho_2)$ is $\kappa$-$HTB$, where $\kappa = \kappa(K, C)$, and then it follows from Lemma 2.6, Claims 5.4 and 5.5 that

$$m \leq \kappa\left(\frac{2H^2K^7}{t_1}\right) + 1.$$

It is the right time for us to find the needed constant in this case. We infer from (5.4) that

$$\frac{1}{Ks} < \left(\frac{K^4}{\tau}\right)^{m+1} \leq \left(\frac{K^4}{\tau}\right)^{\kappa\left(\frac{2H^2K^7}{t_1}\right)+2},$$

and since by (5.1), $t_1 = 1 \wedge t \geq \frac{1}{HK^2}$, we obtain

$$s \geq \frac{1}{K}\left(\frac{K^4}{\tau}\right)^{-\kappa\left(2H^3K^9\right)-2}.$$

Now, we can take

$$H''_1 = K\left(\frac{K^4}{\tau}\right)^{\kappa(2H^3K^9)+2}, \tag{5.6}$$

as required.

Now, we are in a position to obtain the needed constant $H_1$. Let

$$H_1 = \max\{H'_1, H''_1\}.$$

Then we conclude from (5.2) and (5.6) that this $H_1$ is our needed, and so the theorem is proved. □

## 6. Applications

The aim of this section is to give two applications of the main results in Section 5.

Let $\operatorname{Hyp} Z$ denote the hyperbolic approximation of a quasi-metric space $(Z, \rho)$ (see [19, Section 3] for the precise statement of the definition). Then we have

**Theorem 6.1.** *Every weakly quasimöbius map between two uniformly perfect and homogeneous spaces is the boundary map of a quasiisometric map between their corresponding hyperbolic approximation spaces.*

*Proof.* First, by Theorem 5.1, we know that every weakly quasimöbius map between two uniformly perfect and homogeneous spaces is quasimöbius. Then it follows from [19, Corollary 4] that there is a quasiisometric map $F\colon \operatorname{Hyp} Z_1 \to \operatorname{Hyp} Z_2$ with $\partial_\infty F = f$, where $\partial_\infty F$ denotes the restriction $F$ to the boundary of $\operatorname{Hyp} Z_1$. Hence the proof is complete. □

We remark that by Theorem 5.2, the assumption "$(Z_1, \rho_1)$ being homogeneous" in Theorem 6.1 is reluctant. To state Theorem 6.1 in this way is just for simplicity.

In [11], Gehring and Martio established the following result (see [11] for the related definitions).

**Theorem F.** [11, Theorem 3.1] *Suppose $D$ is an $M$-QED doamin and $D'$ is a $b$-locally connected domain in $\overline{\mathbf{R}}^n$, and suppose $f$ is a $K$-quasiconformal map of $D$ onto $D'$. Then*

(1) *$f$ has an extension to $\overline{D}$.*
(2) *for distinct points $x_1$, $x_2$, $x_3$, $x_4$ in $\overline{D}$,*

$$r(x_1, x_2, x_3, x_4) \le c \ \textit{ implies } \ r(x_1', x_2', x_3', x_4') \le c',$$

*where $c'$ is a constant depending only on $b$, $c$, $n$, $K$ and $M$.*

As the second application of Theorem 5.1 or Theorem 5.2, we have

**Theorem 6.2.** *Under the assumptions of Theorem F, the extension of $f$ in $\overline{D}$ is $\theta$-quasimöbius with $\theta$ depending only on $b$, $n$, $K$ and $M$.*

*Proof.* Since every domain in $\overline{\mathbf{R}}^n$ is homogeneous and uniformly perfect, we can easily know from Theorem F together with Theorem 5.1 or Theorem 5.2 that Theorem 6.2 is true. □

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