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MULTIPLE POSITIVE SOLUTIONS FOR THE NONLINEAR SCHRÖDINGER–POISSON SYSTEM

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Abstract. We consider the following Schrödinger–Poisson system in $\mathbf{R}^3$

$$\begin{cases} -\Delta u + u + \alpha K(|x|)\Phi(x)u = |u|^{p-2}u, & x \in \mathbf{R}^3, \\ -\Delta\Phi = K(|x|)u^2, & x \in \mathbf{R}^3, \end{cases} \tag{0.1}$$

where $2 < p < 6$, α can be regarded as a parameter and $K(r)(r = |x|)$ is a positive continuous function. There are constants $a \in \mathbf{R}$ and $b \in (0, \frac{1}{2}]$, such that $K(r) \sim r^a e^{-br}$, as $r \to +\infty$. Then, (0.1) possesses a non-radial positive solution with exactly m maximum points for suitable range of α.

1. Introduction and main result

The following Schrödinger–Poisson system

$$\begin{cases} -\Delta u + V(x)u + K(x)\Phi(x)u = |u|^{p-2}u, & x \in \mathbf{R}^3, \\ -\Delta\Phi = K(x)u^2, & x \in \mathbf{R}^3, \end{cases} \tag{1.1}$$

has been studied extensively by many researchers, where $2 < p < 6$. This system has been first introduced in [5] as a physical model describing a charged wave interacting with its own electrostatic field in quantum mechanic. The unknowns u and Φ represent the wave functions associated to the particle and electric potential, and functions V and K are respectively an external potential and nonnegative density charge. We refer to [5] and the references therein for more physical background.

In recent years, there has been increasing interest in studying problem (1.1). In the case of $V(x) \equiv 1$, $K(x) \equiv \lambda > 0$, the existence of radially symmetric positive solutions of system (1.1) was obtained by D'Aprile and Mugnai in [9] and Ruiz in [20] for $p \in (3, 6)$. Azzollini and Pomponio in [4] established the existence of ground state solutions for $p \in (3, 6)$. For $p \in (2, 3)$, $\lambda = 1$, Ruiz in [20] proved that (1.1) does not admit any nontrivial solution. When $K(x) \equiv 1$ and $V(x)$ is not a constant, the authors proved that there exist radially symmetric solutions concentrate on the spheres in [12, 14] and a positive bound state solution concentrates on the local minimum of the potential V in [13]. By using constrained minimization on the sign-changing Nehari manifold and the Brouwer degree theory, Wang and Zhou in [23]

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studied the existence of sign-changing solutions for (1.1). Ruiz and Vaira in [21] constructed the multi-bump solutions whose bumps concentrated around the local minimum of the potential V. The proofs explored in [21] are based on a singular perturbation, essentially a Lyapunov–Schmidt reduction method. In [11], He and Li concerned with the problem (1.1) with critical nonlinearity. For the more general nonlinearity $f(u)$, by applying the method of penalized functions, Sun et al. in [22] proved the system (1.1) has one nontrivial solution in the case $3 < p \leq 4$. For more related results, one can refer to [3, 1, 5, 7, 10, 15, 24, 25] and the references therein.

Our motivation to study (1.1) mainly comes from the results: In [18], Lin et al. proved the single Schrödinger equation

$$-\Delta u + (1 + \epsilon K(x))u = |u|^{p-2}u, \quad x \in \mathbf{R}^N,$$

has multi-bump solutions with the condition:

$$K(x) \in C(\mathbf{R}^N, \mathbf{R}^+), \quad \lim_{|x|\to\infty} K(x) = 0, \quad \lim_{|x|\to\infty} \frac{\ln(K(x))}{|x|} = 0. \tag{1.2}$$

In [17], Li et al. consider the infinitely many positive solutions of the following Schrödinger–Poisson system

$$\begin{cases} -\Delta u + u + K(|x|)\Phi(x)u = Q(|x|)|u|^{p-2}u, & x \in \mathbf{R}^3, \\ -\Delta \Phi = K(|x|)u^2, & x \in \mathbf{R}^3, \end{cases} \tag{1.3}$$

where $K(|x|)$ and $Q(|x|)$ are bounded and positive functions, $2 < p < 6$. $K(|x|)$ and $Q(|x|)$ have the following expansions:

$$K(r) = \frac{a}{r^m} + O\left(\frac{1}{r^{m+\theta}}\right), \quad (a > 0,\ m > \frac{1}{2},\ \theta > 0), \quad r = |x| \to +\infty, \tag{1.4}$$

$$Q(r) = Q_0 + \frac{b}{r^n} + O\left(\frac{1}{r^{n+\vartheta}}\right), \ (Q_0 > 0,\ b \in \mathbf{R},\ n > 1,\ \vartheta > 0),\ r = |x| \to +\infty. \tag{1.5}$$

As we can see, the expansion (1.4) is a special form of (1.2). Motivated by all results mentioned above, it is very natural for us to pose an interesting question: if the condition $\lim_{|x|\to\infty} \frac{\ln(K(x))}{|x|} < 0$ holds, can we obtain the multiple positive solutions for the problem (1.1)? Especially, in [19], Long and Peng studied the existence of multiple positive solutions for the single Schrödinger equation under the above codition. Now, our answer is positive. We consider the following condition (K) which satisfies $\lim_{|x|\to\infty} \frac{\ln(K(x))}{|x|} < 0$.

In this paper, we studied the following Schrödinger–Poisson system

$$\begin{cases} -\Delta u + u + \alpha K(|x|)\Phi(x)u = |u|^{p-2}u, & x \in \mathbf{R}^3, \\ -\Delta \Phi = K(|x|)u^2, & x \in \mathbf{R}^3, \end{cases} \tag{1.6}$$

where $K(r)(r = |x|)$ is a positive continuous function, $2 < p < 6$. We assume that $K(r)$ satisfies the following condition:

(K) There are constants $a \in \mathbf{R}$ and $b \in (0, \frac{1}{2}]$, such that

$$K(r) \sim r^a e^{-br},$$

as $r \to +\infty$.

We summarize our main results as follows.

Theorem 1.1. *If $K(r)$ satisfies (K), for a fixed integer $m > 1$, provided one of the following conditions holds:*

(1) *If* $b > \sin\frac{\pi}{m}$, *then* $\alpha > \alpha_1$ *for suitable large* $\alpha_1 > 0$;
(2) *If* $b < \sin\frac{\pi}{m}$, *then* $0 < \alpha < \alpha_2$ *for suitable small* $\alpha_2 > 0$, *where* α_1 *and* α_2 *are only dependent on* b *and* m.

Then problem (1.6) *possesses a non-radial positive solution with exactly m maximum points.*

In the following, we sketch the main idea in the proof of Theorem 1.1. The Sobolev space $H^1(\mathbf{R}^3)$ is endowed with the standard norm

$$\|u\| = \sqrt{\langle u, u\rangle}, \quad u \in H^1(\mathbf{R}^3),$$

which is induced by the inner product

$$\langle v_1, v_2\rangle = \int_{\mathbf{R}^3} \nabla v_1 \nabla v_2 + \int_{\mathbf{R}^3} v_1 v_2.$$

The homogeneous Sobolev space

$$D^{1,2}(\mathbf{R}^3) = \{u \in L^{2^*}(\mathbf{R}^3) \colon \nabla u \in L^2(\mathbf{R}^3)\},$$

with the norm

$$\|u\|_{D^{1,2}} = \left(\int_{\mathbf{R}^3} |\nabla u|^2\right)^{\frac{1}{2}}.$$

For every $u \in H^1(\mathbf{R}^3)$, the Riesz theorem implies that there exists a unique $\Phi_u \in D^{1,2}(\mathbf{R}^3)$ such that $-\Delta\Phi_u = K(x)u^2$ and Φ_u can be represented by

$$\Phi_u(x) = \frac{1}{4\pi}\int_{\mathbf{R}^3} \frac{K(y)u^2(y)}{|x-y|}\, dy. \tag{1.7}$$

Furthermore, one has

$$\|\Phi_u\|_{D^{1,2}} \le c\|u\|^2,$$

where $c > 0$. Thus system (1.6) can be reduced into equation

$$-\Delta u + u + K(|x|)\Phi_u u = |u|^{p-2}u. \tag{1.8}$$

The solution of (1.6) can be approximated by using the solution U of the following problem

$$\begin{cases} -\Delta u + u = u^{p-1}, \quad u > 0 \text{ in } \mathbf{R}^3, \\ u(0) = \max\limits_{x\in\mathbf{R}^3} u(x). \end{cases} \tag{1.9}$$

It is well-known that the unique solution U of (1.9) satisfies $U(x) = U(|x|)$ and $U' < 0$ (see [16]).

For any positive integer $m > 1$, let us define

$$y_j = \left(r\cos\frac{2(j-1)\pi}{m}, r\sin\frac{2(j-1)\pi}{m}, 0\right) := (y_j', 0), \quad j = 1, 2, \ldots, m, \tag{1.10}$$

where $r \in \left[\frac{(1-\tau)\ln\alpha}{b-\sin\frac{\pi}{m}}, \frac{(1+\tau)\ln\alpha}{b-\sin\frac{\pi}{m}}\right]$ for some $\tau > 0$ small enough. Define

$$\begin{aligned} H = \Big\{u \colon u \in H^1(\mathbf{R}^3),\ u \text{ is even in } x_h,\ h = 2,3;\ u(r\cos\theta, r\sin\theta, x_3) \\ = u\Big(r\cos\Big(\theta + \frac{2\pi j}{m}\Big), r\sin\Big(\theta + \frac{2\pi j}{m}\Big), x_3\Big),\ j = 1, 2, \ldots, m\Big\}. \end{aligned}$$

Let

$$U_{\mathbf{y}}(x) = \sum_{j=1}^{m} U_{y_j}(x), \tag{1.11}$$

where $U_{y_j}(\cdot) = U(\cdot - y_j)$.

Theorem 1.1 is a direct consequence of the following result.

Theorem 1.2. *Suppose $K(r)$ satisfies (K), $b \neq \sin\frac{\pi}{m}$ and provided one of following conditions holds:*

(1) *If $b > \sin\frac{\pi}{m}$, then $\alpha > \alpha_1$ for suitable large $\alpha_1 > 0$;*
(2) *If $b < \sin\frac{\pi}{m}$, then $0 < \alpha < \alpha_2$ for suitable small $\alpha_2 > 0$, where α_1 and α_2 are only dependent on b and m.*

Then (1.6) has a positive solution u_m of the form

$$u_m = U_{\mathbf{y}_m}(x) + w_m,$$

where $w_m \in H$, $r_m \in \left[\frac{(1-\tau)\ln\alpha}{b-\sin\frac{\pi}{m}}, \frac{(1+\tau)\ln\alpha}{b-\sin\frac{\pi}{m}}\right]$, $\tau > 0$ small enough and as $\alpha \to +\infty$(or 0^+), $\|w_m\|_{H^1} \to 0$.

Our paper is organized as follows. In section 2, we will carry out Lyapunov–Schmidt reduction. Then, we will study the reduced finite-dimensional problem and prove our mian result in section 3. Some technical estimates are left in the appendix.

2. Finite-dimensional reduction

We begin the Lyapunov–Schmidt for the proof of Theorem 1.2. Assume

$$r \in \Lambda_m := \left[\frac{(1-\tau)\ln\alpha}{b-\sin\frac{\pi}{m}}, \frac{(1+\tau)\ln\alpha}{b-\sin\frac{\pi}{m}}\right], \tag{2.1}$$

where $\tau > 0$ small enough. Define

$$I(u) = \frac{1}{2}\int_{\mathbf{R}^3} |\nabla u|^2 + \frac{1}{2}\int_{\mathbf{R}^3} u^2 + \frac{\alpha}{4}\int_{\mathbf{R}^3} K(|x|)\Phi_u u^2 - \frac{1}{p}\int_{\mathbf{R}^3}|u|^p, \quad \forall u \in H.$$

It is easy to check that

$$\begin{aligned}&\int_{\mathbf{R}^3} \nabla u_1 \nabla u_2 + \int_{\mathbf{R}^3} u_1 u_2 - (p-1)\int_{\mathbf{R}^3} U_{\mathbf{y}}^{p-2} u_1 u_2 + \alpha\int_{\mathbf{R}^3} K(|x|)\Phi_{U_{\mathbf{y}}} u_1 u_2 \\ &+ 2\alpha\int_{\mathbf{R}^3} K(|x|)\left(\int_{\mathbf{R}^3}\frac{K(|y|)}{|x-y|}U_{\mathbf{y}} u_1\,dy\right)U_{\mathbf{y}} u_2, \quad u_1 u_2 \in H,\end{aligned}$$

is a bounded bi-linear functional in H. Hence, by Lax–Milgram Theorem there is a bounded linear operator $\mathcal{L}$ from H to H, such that

$$\begin{aligned}\langle \mathcal{L}u_1, u_2\rangle = &\int_{\mathbf{R}^3} \nabla u_1 \nabla u_2 + \int_{\mathbf{R}^3} u_1 u_2 - (p-1)\int_{\mathbf{R}^3} U_{\mathbf{y}}^{p-2} u_1 u_2 + \alpha\int_{\mathbf{R}^3} K(|x|)\Phi_{U_{\mathbf{y}}} u_1 u_2 \\ &+ 2\alpha\int_{\mathbf{R}^3} K(|x|)\left(\int_{\mathbf{R}^3}\frac{K(|y|)}{|x-y|}U_{\mathbf{y}} u_1 dy\right)U_{\mathbf{y}} u_2, \quad u_1 u_2 \in H.\end{aligned}$$

The following result implies that $\mathcal{L}$ is invertible in H.

Lemma 2.1. *There are positive constants C, small α_2 and α_1 sufficiently large such that for $\alpha > \alpha_1$ when $b > \sin\frac{\pi}{m}$ or $0 < \alpha < \alpha_2$ when $b < \sin\frac{\pi}{m}$,*

$$\|\mathcal{L}u\| \geq C\|u\|, \quad u \in H.$$

Proof. We only prove the lemma for the case $b > \sin\frac{\pi}{m}$. Here we prove it by a contradiction argument. Suppose to the contrary that there exist $\alpha_k \to +\infty$ and $u_k \in H$ with

$$\|\mathcal{L}u_k\| = o(1)\|u_k\|.$$

Then we have

$$\langle \mathcal{L}u_k, \varphi\rangle = o(1)\|u_k\|\|\varphi\|, \ \forall \varphi \in H. \tag{2.2}$$

We may assume that $\|u_k\|^2 = 1$.

Denote

$$\Omega_j = \left\{ x = (x', x_3) \in \mathbf{R}^2 \times \mathbf{R} \colon \left\langle \frac{x'}{|x'|}, \frac{y'_j}{|y'_j|} \right\rangle \geq \cos\frac{\pi}{m} \right\}, \quad j = 1, 2, \cdots, m.$$

By symmetry, we see from (2.2),

$$\begin{aligned} &\int_{\Omega_1} \nabla u_k \nabla \varphi + \int_{\Omega_1} u_k \varphi - (p-1)\int_{\Omega_1} U_{\mathbf{y}}^{p-2} u_k \varphi + \alpha \int_{\Omega_1} K(|x|)\Phi_{U_{\mathbf{y}}} u_k \varphi \\ &+ 2\alpha \int_{\Omega_1} K(|x|) \left(\int_{\mathbf{R}^3} \frac{K(|y|)}{|x-y|} U_{\mathbf{y}} u_k \, dy \right) U_{\mathbf{y}} \varphi = o(1)\|\varphi\|, \quad \varphi \in H. \end{aligned} \tag{2.3}$$

Particularly, choosing $\varphi = u_k$ we get

$$\begin{aligned} &\int_{\Omega_1} |\nabla u_k|^2 + \int_{\Omega_1} u_k^2 - (p-1)\int_{\Omega_1} U_{\mathbf{y}}^{p-2} u_k^2 + \alpha \int_{\Omega_1} K(|x|)\Phi_{U_{\mathbf{y}}} u_k^2 \\ &+ 2\alpha \int_{\Omega_1} K(|x|) \left(\int_{\mathbf{R}^3} \frac{K(|y|)}{|x-y|} U_{\mathbf{y}} u_k \, dy \right) U_{\mathbf{y}} u_k = o(1) \end{aligned} \tag{2.4}$$

and

$$\int_{\mathbf{R}^3} |\nabla u_k|^2 + \int_{\mathbf{R}^3} u_k^2 = 1. \tag{2.5}$$

Let $\tilde{u}_k(x) = u_k(x - y_1)$. It is easy to check that for any $R > 0$, we can choose k large enough such that $B_R(y_1) \subset \Omega_1$. Consequently, (2.5) yields that

$$\int_{B_R(0)} |\nabla \tilde{u}_k|^2 + \int_{B_R(0)} \tilde{u}_k^2 \leq 1.$$

Thus we may assume that there exists a $u \in H^1(\mathbf{R}^3)$ such that as $k \to +\infty$,

$$\tilde{u}_k \rightharpoonup u, \quad \text{weakly in } H^1(\mathbf{R}^3)$$

and

$$\tilde{u}_k \to u, \quad \text{strongly in } L^2_{\text{loc}}(\mathbf{R}^3).$$

Noting that $\tilde{u}_k$ is even in x_h, $h = 2, 3$, then u is even in $x_h, h = 2, 3$. On the other hand, from

$$\int_{\mathbf{R}^3} \frac{\partial U_{y_1}}{\partial r} U_{y_1}^{p-2} u_k = 0,$$

we obtain

$$\int_{\mathbf{R}^3} \frac{\partial U}{\partial x_1} U^{p-2} \tilde{u}_k = 0.$$

So u satisfies

$$\int_{\mathbf{R}^3} \frac{\partial U}{\partial x_1} U^{p-2} u = 0. \tag{2.6}$$

Now we prove that u satisfies

$$-\Delta u + u - (p-1)U^{p-2}u = 0, \quad \text{in } \mathbf{R}^3.$$

Define

$$\tilde{H} = \left\{ \varphi \colon \varphi \in H^1(\mathbf{R}^3), \int_{\mathbf{R}^3} \frac{\partial U}{\partial x_1} U^{p-2} \varphi = 0 \right\}.$$

For any $R>0$, let $\varphi \in C_0^\infty(B_R(0))\cap \tilde{H}$ be any function, satisfying that φ is even in $x_h, h=2,3$. Then $\varphi_1(x) =: \varphi(x-y_1) \in C_0^\infty(B_R(y_1))$. By using (2.4), we find

$$\int_{\mathbf{R}^3} \nabla u \nabla \varphi + \int_{\mathbf{R}^3} u\varphi - (p-1)\int_{\mathbf{R}^3} U^{p-2}u\varphi = 0. \tag{2.7}$$

Furthermore, since u is even in $x_h,\ h=2,3$, (2.7) is true for any function $\varphi \in C_0^\infty(\mathbf{R}^3)$, which is odd in $x_h,\ h=2,3$. Therefore, (2.7) is true for any $\varphi \in C_0^\infty(B_R(0))\cap \tilde{H}$. Since $C_0^\infty(\mathbf{R}^3)$ is dense in $H^1(\mathbf{R}^3)$, it is easy to prove that

$$\int_{\mathbf{R}^3} \nabla u \nabla \varphi + \int_{\mathbf{R}^3} u\varphi - (p-1)\int_{\mathbf{R}^3} U^{p-2}u\varphi = 0, \quad \forall \varphi \in \tilde{H}. \tag{2.8}$$

But (2.8) is true for $\varphi = \frac{\partial U}{\partial x_1}$. Thus (2.8) is true for any $\varphi \in H^1(\mathbf{R}^3)$, and hence $u = c\frac{\partial U}{\partial x_1}$ because u is even in $x_h,\ h=2,3$. By (2.6), we find $u=0$. Consequently,

$$\int_{B_R(y_1)} u_k^2 = o(1), \quad \forall R>0.$$

By the Hölder inequality and Lemma A.2, we have

$$\begin{aligned}
&\left|\alpha\int_{\Omega_1} K(|x|)\Phi_{U_{\mathbf{y}}} u_k^2 + 2\alpha \int_{\Omega_1} K(|x|)\left(\int_{\mathbf{R}^3} \frac{K(|y|)}{|x-y|} U_{\mathbf{y}} u_k\, dy\right) U_{\mathbf{y}} u_k\right| \\
&\le \alpha r^a e^{-br} C \int_{\mathbf{R}^3} u_k^2 + \alpha \int_{\mathbf{R}^3} K(|x|) \big|\Phi_{U_{\mathbf{y}}}\big|^{\frac12} \big|\Phi_{u_k}\big|^{\frac12} U_{\mathbf{y}} u_k \\
&\le \alpha r^a e^{-br} C + \alpha r^a e^{-br} r^{\frac{a}{2}} e^{-\frac{b}{2}r} C \int_{\mathbf{R}^3} \big|\Phi_{u_k}\big|^{\frac12} U_{\mathbf{y}} u_k \\
&\le \alpha r^a e^{-br} C + \alpha r^a e^{-br} C \|u_k\|^2 \left(\int_{\mathbf{R}^3} U_{\mathbf{y}}^2\right)^{\frac12} \to 0.
\end{aligned}$$

Thus, by (2.2) and (K), we have

$$\begin{aligned}
o(1) &= \int_{\Omega_1} |\nabla u_k|^2 + \int_{\Omega_1} u_k^2 - (p-1)\int_{\Omega_1} U_{\mathbf{y}}^{p-2} u_k^2 + \alpha \int_{\Omega_1} K(|x|)\Phi_{U_{\mathbf{y}}} u_k^2 \\
&\quad + 2\alpha \int_{\Omega_1} K(|x|)\left(\int_{\mathbf{R}^3} \frac{K(|y|)}{|x-y|} U_{\mathbf{y}} u_k dy\right) U_{\mathbf{y}} u_k \\
&\ge \int_{\Omega_1} |\nabla u_k|^2 + \int_{\Omega_1} u_k^2 + o(1) + o_R(1)\int_{\Omega_1} u_k^2 \ge \frac12 > 0,
\end{aligned}$$

which is impossible for large R. □

Now, we prove the following proposition.

Proposition 2.2. *Under the assumptions of Lemma 2.1, there exists a C^1 map with respect to r from Λ_m to H: $\varphi = \varphi(r)$, satisfying $\varphi \in H$, and*

$$\Big\langle \frac{\partial J(\varphi)}{\partial \varphi}, v \Big\rangle = 0, \quad \forall v \in H.$$

Moreover, there is a small $\tau > 0$, such that

$$\|\varphi\| \le C\left(\alpha e^{-(1-\tau)2br} + e^{-\min\{p-1-\sigma, 2-\sigma\} r \sin\frac{\pi}{m}}\right). \tag{2.9}$$

Proof. Denote

$$J(\varphi) = I(U_{\mathbf{y}} + \varphi), \quad \varphi \in H.$$

By direct computation, we have

$$\begin{aligned}
J(\varphi) = I\left(U_{\mathbf{y}}+\varphi\right) &= \frac{1}{2}\int_{\mathbf{R}^3} |\nabla U_{\mathbf{y}} + \nabla\varphi|^2 + \frac{1}{2}\int_{\mathbf{R}^3}(U_{\mathbf{y}}+\varphi)^2 \\
&\quad + \frac{\alpha}{4}\int_{\mathbf{R}^3} K(|x|)\Phi_{U_{\mathbf{y}}+\varphi}(U_{\mathbf{y}}+\varphi)^2 - \frac{1}{p}\int_{\mathbf{R}^3}|U_{\mathbf{y}}+\varphi|^p \\
&= \frac{1}{2}\int_{\mathbf{R}^3}|\nabla U_{\mathbf{y}}|^2 + \frac{1}{2}\int_{\mathbf{R}^3} U_{\mathbf{y}}^2 + \frac{\alpha}{4}\int_{\mathbf{R}^3} K(|x|)\Phi_{U_{\mathbf{y}}}U_{\mathbf{y}}^2 - \frac{1}{p}\int_{\mathbf{R}^3}|U_{\mathbf{y}}|^p \\
&\quad + \int_{\mathbf{R}^3}\left(\sum_{j=1}^m U_{y_j}^{p-1} - U_{\mathbf{y}}^{p-1}\right)\varphi + \alpha\int_{\mathbf{R}^3} K(|x|)\Phi_{U_{\mathbf{y}}}U_{\mathbf{y}}\varphi \\
&\quad + \frac{1}{2}\int_{\mathbf{R}^3}|\nabla\varphi|^2 + \frac{1}{2}\int_{\mathbf{R}^3}\varphi^2 - \frac{p-1}{2}\int_{\mathbf{R}^3}|U_{\mathbf{y}}|^{p-2}\varphi^2 + \frac{\alpha}{2}\int_{\mathbf{R}^3}K(|x|)\Phi_{U_{\mathbf{y}}}\varphi^2 \\
&\quad + \alpha\int_{\mathbf{R}^3}K(|x|)\left(\int_{\mathbf{R}^3}\frac{K(|y|)}{|x-y|}U_{\mathbf{y}}\varphi dy\right)U_{\mathbf{y}}\varphi \\
&\quad + \alpha\int_{\mathbf{R}^3}K(|x|)\Phi_{\varphi}U_{\mathbf{y}}\varphi + \frac{\alpha}{4}\int_{\mathbf{R}^3}K(|x|)\Phi_{\varphi}\varphi^2 - \frac{1}{p}\int_{\mathbf{R}^3}|U_{\mathbf{y}}+\varphi|^p + \frac{1}{p}\int_{\mathbf{R}^3}|U_{\mathbf{y}}|^p \\
&\quad + \int_{\mathbf{R}^3}|U_{\mathbf{y}}|^{p-1}\varphi + \frac{p-1}{2}\int_{\mathbf{R}^3}|U_{\mathbf{y}}|^{p-2}\varphi^2.
\end{aligned}$$

Hence,

$$J(\varphi) = J(0) + f(\varphi) + \frac{1}{2}\langle\mathcal{L}\varphi,\varphi\rangle + R(\varphi),$$

where

$$f(\varphi) = \int_{\mathbf{R}^3}\left(\sum_{j=1}^m U_{y_j}^{p-1} - U_{\mathbf{y}}^{p-1}\right)\varphi + \alpha\int_{\mathbf{R}^3}K(|x|)\Phi_{U_{\mathbf{y}}}U_{\mathbf{y}}\varphi. \tag{2.10}$$

$\mathcal{L}$ is the bounded linear map from H to H in Lemma 2.1, and

$$\begin{aligned}
R(\varphi) &= \alpha\int_{\mathbf{R}^3}K(|x|)\Phi_{\varphi}U_{\mathbf{y}}\varphi + \frac{\alpha}{4}\int_{\mathbf{R}^3}K(|x|)\Phi_{\varphi}\varphi^2 - \frac{1}{p}\int_{\mathbf{R}^3}|U_{\mathbf{y}}+\varphi|^p + \frac{1}{p}\int_{\mathbf{R}^3}|U_{\mathbf{y}}|^p \\
&\quad + \int_{\mathbf{R}^3}|U_{\mathbf{y}}|^{p-1}\varphi + \frac{p-1}{2}\int_{\mathbf{R}^3}|U_{\mathbf{y}}|^{p-2}\varphi^2.
\end{aligned}$$

It is not difficult to verify that $f(\varphi)$ is a bounded linear functional in H, so there exists an $f_m \in H$ such that

$$f(\varphi) = \langle f_m, \varphi\rangle.$$

Thus, to find a critical point for $J(\varphi)$, we only need to solve

$$f_m + \mathcal{L}\varphi + R'(\varphi) = 0. \tag{2.11}$$

From Lemma 2.1 we know that $\mathcal{L}$ is invertible. Therefore, (2.11) can be rewritten as

$$\varphi = \mathcal{A}(\varphi) =: -\mathcal{L}^{-1}f_m - \mathcal{L}^{-1}R'(\varphi).$$

Set

$$\mathcal{N} = \left\{\varphi\colon \varphi\in H, \|\varphi\| \le \alpha e^{-(1-\tau-\tau_1)2br} + e^{-\min\{p-1-\sigma-\tau_1, 2-\sigma-\tau_1\}r\sin\frac{\pi}{m}}\right\}.$$

where $\tau_1 > 0$ sufficiently small. When $2 < p \le 3$, we can verify that

$$\|R'(\varphi)\| \le C\|\varphi\|^{p-1}.$$

Hence Lemma 2.3 below implies

$$
\begin{aligned}
\|\mathcal{A}(\varphi)\| &\le C\|f_m\| + C\|\varphi\|^{p-1} \\
&\le C\alpha e^{-(1-\tau)2br} + Ce^{-\min\{p-1-\sigma,2-\sigma\}r\sin\frac{\pi}{m}} + C\Big(\alpha e^{-(1-\tau-\tau_1)2br} \\
&\quad + e^{-\min\{p-1-\sigma-\tau_1,2-\sigma-\tau_1\}r\sin\frac{\pi}{m}}\Big)^{p-1} \\
&\le \alpha e^{-(1-\tau-\tau_1)2br} + e^{-\min\{p-1-\sigma-\tau_1,2-\sigma-\tau_1\}r\sin\frac{\pi}{m}}.
\end{aligned}
\tag{2.12}
$$

Thus, $\mathcal{A}$ maps $\mathcal{N}$ into $\mathcal{N}$ when $2 < p \le 3$.

Meanwhile, when $2 < p \le 3$, we see

$$\|R''(\varphi)\| \le C\|\varphi\|^{p-2}.$$

Thus,

$$
\begin{aligned}
&\|\mathcal{A}(\varphi_1) - \mathcal{A}(\varphi_2)\| = \|\mathcal{L}^{-1}R'(\varphi_1) - \mathcal{L}^{-1}R'(\varphi_2)\| \\
&\le C\|R'(\varphi_1) - R'(\varphi_2)\| \le C\|R''(\varepsilon\varphi_1 + (1-\varepsilon)\varphi_2)\|\|\varphi_1 - \varphi_2\| \\
&\le C(\|\varphi_1\|^{p-2} + \|\varphi_2\|^{p-2})\|\varphi_1 - \varphi_2\| \le \frac{1}{2}\|\varphi_1 - \varphi_2\|,
\end{aligned}
$$

where $\varepsilon \in (0,1)$. Thus, we have proved that when $2 < p \le 3$, $\mathcal{A}$ is a contraction map.

When $p > 3$, by the Hölder inequality and the Sobolev inequality, we get

$$
\begin{aligned}
&|\langle R'(\varphi), \xi\rangle| \\
&= \Big|2\alpha\int_{\mathbf{R}^3} K(|x|)\left(\int_{\mathbf{R}^3}\frac{K(|y|)}{|x-y|}\varphi\xi\,dy\right)U_{\mathbf{y}}\varphi + \alpha\int_{\mathbf{R}^3} K(|x|)\Phi_\varphi U_{\mathbf{y}}\xi + \alpha\int_{\mathbf{R}^3} K(|x|)\Phi_\varphi\varphi\xi \\
&\quad - \int_{\mathbf{R}^3}|U_{\mathbf{y}}+\varphi|^{p-1}\xi + \int_{\mathbf{R}^3}|U_{\mathbf{y}}|^{p-1}\xi + (p-1)\int_{\mathbf{R}^3}|U_{\mathbf{y}}|^{p-2}\varphi\xi\Big| \\
&\le \Big|2\alpha\int_{\mathbf{R}^3} K(|x|)\left(\int_{\mathbf{R}^3}\frac{K(|y|)}{|x-y|}\varphi\xi\,dy\right)U_{\mathbf{y}}\varphi + \alpha\int_{\mathbf{R}^3} K(|x|)\Phi_\varphi U_{\mathbf{y}}\xi + \alpha\int_{\mathbf{R}^3} K(|x|)\Phi_\varphi\varphi\xi\Big| \\
&\quad + \Big|\int_{\mathbf{R}^3}|U_{\mathbf{y}}+\varphi|^{p-1}\xi - \int_{\mathbf{R}^3}|U_{\mathbf{y}}|^{p-1}\xi - (p-1)\int_{\mathbf{R}^3}|U_{\mathbf{y}}|^{p-2}\varphi\xi\Big| \\
&\le C\alpha r^a e^{-br}\int_{\mathbf{R}^3}|\Phi_\varphi|^{\frac12}|\Phi_\xi|^{\frac12}U_{\mathbf{y}}|\varphi| + C\alpha r^a e^{-br}\left(\int_{\mathbf{R}^3}|\Phi_\varphi|^6\right)^{\frac16}\left(\int_{\mathbf{R}^3}|\xi|^{\frac{12}{5}}\right)^{\frac{5}{12}}\left(\int_{\mathbf{R}^3}|U_{\mathbf{y}}|^{\frac{12}{5}}\right)^{\frac{5}{12}} \\
&\quad + C\alpha r^a e^{-br}\left(\int_{\mathbf{R}^3}|\Phi_\varphi|^6\right)^{\frac16}\left(\int_{\mathbf{R}^3}|\xi|^{\frac{12}{5}}\right)^{\frac{5}{12}}\left(\int_{\mathbf{R}^3}|\varphi|^{\frac{12}{5}}\right)^{\frac{5}{12}} + C\int_{\mathbf{R}^3}|U_{\mathbf{y}}|^{p-3}|\varphi|^2|\xi| \\
&\le C\alpha^{\frac{b\tau-\sin\frac{\pi}{m}}{b-\sin\frac{\pi}{m}}}\|\varphi\|^2\|\xi\| + C\alpha^{\frac{b\tau-\sin\frac{\pi}{m}}{b-\sin\frac{\pi}{m}}}\|\Phi_\varphi\|_{D^{1,2}}\|\xi\| + C\alpha^{\frac{b\tau-\sin\frac{\pi}{m}}{b-\sin\frac{\pi}{m}}}\|\Phi_\varphi\|_{D^{1,2}}\|\xi\|\|\varphi\| \\
&\quad + C\left(\int_{\mathbf{R}^3}\left(|U_{\mathbf{y}}|^{p-3}|\varphi|^2\right)^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}\|\xi\| \\
&\le C\|\varphi\|^2\|\xi\| + C\|\varphi\|^2\|\xi\| + C\|\varphi\|^3\|\xi\| + C\left(\int_{\mathbf{R}^3}|\varphi|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{p}}\|\xi\|.
\end{aligned}
$$

Hence, we deduce that

$$\|R'(\varphi)\| \le C(\|\varphi\|^2 + \|\varphi\|^3).$$

For the estimate of $\|R''(\varphi)\|$, we have

$$
\begin{aligned}
&|R''(\varphi)(\xi,\eta)|\\
&= \Big|2\alpha\int_{\mathbf{R}^3} K(|x|)\Big(\int_{\mathbf{R}^3}\frac{K(|y|)}{|x-y|}\eta\xi\,dy\Big)U_{\mathbf{y}}\varphi + 2\alpha\int_{\mathbf{R}^3} K(|x|)\Big(\int_{\mathbf{R}^3}\frac{K(|y|)}{|x-y|}\varphi\xi\,dy\Big)U_{\mathbf{y}}\eta\\
&\quad + 2\alpha\int_{\mathbf{R}^3} K(|x|)\Big(\int_{\mathbf{R}^3}\frac{K(|y|)}{|x-y|}\varphi\eta\,dy\Big)U_{\mathbf{y}}\xi + 2\alpha\int_{\mathbf{R}^3} K(|x|)\Big(\int_{\mathbf{R}^3}\frac{K(|y|)}{|x-y|}\varphi\xi\,dy\Big)\varphi\eta\\
&\quad + \alpha\int_{\mathbf{R}^3} K(|x|)\Phi_\varphi\xi\eta - (p-1)\int_{\mathbf{R}^3}(U_{\mathbf{y}}+\varphi)^{p-2}\xi\eta + (p-1)\int_{\mathbf{R}^3} U_{\mathbf{y}}^{p-2}\xi\eta\Big|\\
&\le C(\|\varphi\|+\|\varphi\|^2)\|\xi\|\|\eta\|,
\end{aligned}
$$

which implies

$$\|R''(\varphi)\| \le C(\|\varphi\|+\|\varphi\|^2).$$

Thus, we have

$$
\begin{aligned}
\|\mathcal{A}(\varphi)\| &\le C\|f_m\| + C\|\varphi\|^2\\
&\le C\alpha e^{-(1-\tau)2br} + Ce^{-\min\{p-1-\sigma,2-\sigma\}r\sin\frac{\pi}{m}}\\
&\quad + C\Big(\alpha e^{-(1-\tau-\tau_1)2br} + e^{-\min\{p-1-\sigma-\tau_1,2-\sigma-\tau_1\}r\sin\frac{\pi}{m}}\Big)^2\\
&\le \alpha e^{-(1-\tau-\tau_1)2br} + e^{-\min\{p-1-\sigma-\tau_1,2-\sigma-\tau_1\}r\sin\frac{\pi}{m}}
\end{aligned}
\tag{2.13}
$$

and

$$
\begin{aligned}
&\|\mathcal{A}(\varphi_1)-\mathcal{A}(\varphi_2)\| = \|\mathcal{L}^{-1}R'(\varphi_1)-\mathcal{L}^{-1}R'(\varphi_2)\|\\
&\le C\|R'(\varphi_1)-R'(\varphi_2)\| \le C\|R''(\varepsilon\varphi_1+(1-\varepsilon)\varphi_2)\|\|\varphi_1-\varphi_2\| \le \frac{1}{2}\|\varphi_1-\varphi_2\|,
\end{aligned}
$$

where $\varepsilon\in(0,1)$. Hence, $\mathcal{A}$ is also a contraction map from $\mathcal{N}$ to $\mathcal{N}$.

By the contraction mapping theorem, we see that there is a unique φ such that (2.11) holds. Moreover, it follows from (2.12) and (2.13) that (2.9) holds. □

Lemma 2.3. *There exists a constant $C>0$, such that*

$$\|f_m\| \le C(\alpha e^{-(1-\tau)2br} + e^{-\min\{p-1-\sigma,2-\sigma\}r\sin\frac{\pi}{m}}),$$

where $\tau>0$ is an arbitrary small constant.

Proof. We recall

$$f(\varphi) = \int_{\mathbf{R}^3}\Big(\sum_{j=1}^m U_{y_j}^{p-1} - U_{\mathbf{y}}^{p-1}\Big)\varphi + \alpha\int_{\mathbf{R}^3} K(|x|)\Phi_{U_{\mathbf{y}}}U_{\mathbf{y}}\varphi. \tag{2.14}$$

Since it follows from (3.18) and (3.19) in [8], by Lemma A.1, we have

$$
\begin{aligned}
\Big|\int_{\mathbf{R}^3} U_{\mathbf{y}}^{p-1}\varphi - \sum_{j=1}^m\int_{\mathbf{R}^3} U_{y_j}^{p-1}\varphi\Big| &\le \begin{cases} C\displaystyle\sum_{k\ne j}\int_{\mathbf{R}^3} U_{y_k}^{\frac{p-1}{2}}U_{y_j}^{\frac{p-1}{2}}|\varphi|, & \text{if } 2<p\le 3,\\ C\displaystyle\sum_{k\ne j}\int_{\mathbf{R}^3} U_{y_k}^{p-2}U_{y_j}|\varphi|, & \text{if } p>3,\end{cases}\\
&\le Ce^{-\min\{p-1-\tau,2-\tau\}r\sin\frac{\pi}{m}}\|\varphi\|,
\end{aligned}
$$

where $\tau>0$ small enough.

By the Hölder inequality, we have

$$
\begin{aligned}
\Big|\int_{\mathbf{R}^3} K(|x|)\Phi_{U_{\mathbf{y}}}U_{\mathbf{y}}\varphi\Big| &= \Big|\sum_{j=1}^{m}\int_{\mathbf{R}^3} K(|x|)\Phi_{U_{\mathbf{y}}}U_{y_j}\varphi\Big| \\
&\le \sum_{j=1}^{m}\left(\int_{\mathbf{R}^3}\big(K(|x|)\Phi_{U_{\mathbf{y}}}U_{y_j}\big)^2\right)^{\frac12}\left(\int_{\mathbf{R}^3}\varphi^2\right)^{\frac12} \\
&\le \|\Phi_{U_{\mathbf{y}}}\|_{D^{1,2}}\sum_{j=1}^{m}\left(\int_{\mathbf{R}^3}\big(K(|x|)U_{y_j}\big)^3\right)^{\frac13}\|\varphi\|.
\end{aligned} \tag{2.15}
$$

Let us evaluate $\|\Phi_{U_{\mathbf{y}}}\|_{D^{1,2}}$ and $\int_{\mathbf{R}^3}\big(K(|x|)U_{y_j}\big)^3$.

$$
\begin{aligned}
\|\Phi_{U_{\mathbf{y}}}\|^2_{D^{1,2}} &= \int_{\mathbf{R}^3}|\nabla\Phi_{U_{\mathbf{y}}}|^2 = \int_{\mathbf{R}^3}K(|x|)\Phi_{U_{\mathbf{y}}}U_{\mathbf{y}}^2 \\
&= \sum_{j=1}^{m}\int_{\mathbf{R}^3}K(|x|)\Phi_{U_{\mathbf{y}}}U_{y_j}^2 + \sum_{i\ne j}\int_{\mathbf{R}^3}K(|x|)\Phi_{U_{\mathbf{y}}}U_{y_i}U_{y_j} \\
&\le C\|\Phi_{U_{\mathbf{y}}}\|_{D^{1,2}}\sum_{j=1}^{m}\left(\int_{\mathbf{R}^3}\big(K(|x|)U_{y_j}^2\big)^{\frac65}\right)^{\frac56} \\
&\quad + C\|\Phi_{U_{\mathbf{y}}}\|_{D^{1,2}}\sum_{i\ne j}\left(\int_{\mathbf{R}^3}\big(K(|x|)U_{y_i}U_{y_j}\big)^{\frac65}\right)^{\frac56}
\end{aligned} \tag{2.16}
$$

and

$$
\begin{aligned}
&\int_{\mathbf{R}^3}\big(K(|x|)U_{y_j}\big)^3 = \int_{\mathbf{R}^3}\big(K(|x+y_j|)U\big)^3 \\
&= \int_{B_{(1-\kappa)r}(0)}\big(K(|x+y_j|)U\big)^3 + \int_{\mathbf{R}^3\setminus B_{(1-\kappa)r}(0)}\big(K(|x+y_j|)U\big)^3 \\
&\le \int_{B_{(1-\kappa)r}(0)}|x+y_j|^{3a}e^{-3b|x+y_j|}U^3 + \int_{\mathbf{R}^3\setminus B_{(1-\kappa)r}(0)}K^3(|x+y_j|)e^{-3(1-\kappa)r} \\
&= O(r^{3a}e^{-3br}) + O(e^{-3(1-\kappa)r}).
\end{aligned} \tag{2.17}
$$

Similarly,

$$
\int_{\mathbf{R}^3}\big(K(|x|)U_{y_j}^2\big)^{\frac65} = O(r^{\frac65 a}e^{-\frac65 br}) + O(e^{-\frac{12}{5}(1-\kappa)r}). \tag{2.18}
$$

Denote

$$
\Omega_j = \left\{x=(x',x'')\in\mathbf{R}^2\times\mathbf{R}\colon \left\langle\frac{x'}{|x'|},\frac{y_j'}{|y_j'|}\right\rangle \ge \cos\frac{\pi}{m}\right\}, j=1,2,\dots,m.
$$

For any $x\in\Omega_j$, we have

$$
|x-y_i|\ge|x-y_j|,\quad \forall x\in\Omega_j,\ i\ne j,
$$

then

$$
|x-y_i|\ge\tfrac12|y_i-y_j|,\quad \forall x\in\Omega_j,\ i\ne j.
$$

So, for all arbitrary constant $\eta\in(0,1)$, we find

$$
\begin{aligned}
U_{y_i} &\le Ce^{-\eta|x-y_i|}e^{-(1-\eta)|x-y_i|} \le Ce^{-\frac{\eta}{2}|y_i-y_j|}e^{-(1-\eta)|x-y_j|} \\
&\le Ce^{-\eta r\sin\frac{\pi}{m}}e^{-(1-\eta)|x-y_j|},\quad \forall x\in\Omega_j,\ i\ne j.
\end{aligned} \tag{2.19}
$$

Using (2.19), we can obtain

$$
\begin{aligned}
&\int_{\mathbf{R}^3}\big(K(|x|)U_{y_i}U_{y_j}\big)^{\frac{6}{5}}\le Cm\int_{\Omega_j}\big(K(|x|)U_{y_i}U_{y_j}\big)^{\frac{6}{5}}\\
&\le Cm\int_{\Omega_j}\big(K(|x|)e^{-\eta r\sin\frac{\pi}{m}}e^{-(1-\eta)|x-y_j|}U_{y_j}\big)^{\frac{6}{5}}\\
&\le Cme^{-\frac{6}{5}\eta r\sin\frac{\pi}{m}}\int_{\mathbf{R}^3}\big(K(|x+y_j|)e^{-(1-\eta)|x|}U\big)^{\frac{6}{5}}\\
&= Cme^{-\frac{6}{5}\eta r\sin\frac{\pi}{m}}\int_{B_{(1-\kappa)r}(0)}\big(K(|x+y_j|)e^{-(1-\eta)|x|}U\big)^{\frac{6}{5}}\\
&\quad + Cme^{-\frac{6}{5}\eta r\sin\frac{\pi}{m}}\int_{\mathbf{R}^3\setminus B_{(1-\kappa)r}(0)}\big(K(|x+y_j|)e^{-(1-\eta)|x|}U\big)^{\frac{6}{5}}\\
&\le Cme^{-\frac{6}{5}\eta r\sin\frac{\pi}{m}}\int_{B_{(1-\kappa)r}(0)}\big(|x+y_j|^a e^{-b|x+y_j|}e^{-(1-\eta)|x|}U\big)^{\frac{6}{5}}\\
&\quad + Cme^{-\frac{6}{5}\eta r\sin\frac{\pi}{m}}\int_{\mathbf{R}^3\setminus B_{(1-\kappa)r}(0)}\big(K(|x+y_j|)e^{-(2-\eta)|x|}\big)^{\frac{6}{5}}\\
&\le Cme^{-\frac{6}{5}\eta r\sin\frac{\pi}{m}}\int_{\mathbf{R}^3}\big(|x+y_j|^a e^{-b|x+y_j|}e^{-(1-\eta)|x|}U\big)^{\frac{6}{5}}\\
&\quad + Cme^{-\frac{6}{5}\eta r\sin\frac{\pi}{m}}e^{-\frac{6}{5}(2-\eta)(1-\kappa)r}\int_{\mathbf{R}^3}\big(K(|x+y_j|)\big)^{\frac{6}{5}}\\
&\le Cme^{-\frac{6}{5}\eta r\sin\frac{\pi}{m}}\big(r^{\frac{6}{5}a}e^{-\frac{6}{5}br}+e^{-\frac{6}{5}(2-\eta)(1-\kappa)r}\big).
\end{aligned}
\tag{2.20}
$$

Inserting (2.16)–(2.20) into (2.15), we have

$$
\begin{aligned}
&\Big|\int_{\mathbf{R}^3}K(|x|)\Phi_{U_{\mathbf{y}}}U_{\mathbf{y}}\varphi\Big|\le\|\Phi_{U_{\mathbf{y}}}\|_{D^{1,2}}\sum_{j=1}^m\bigg(\int_{\mathbf{R}^3}\big(K(|x|)U_{y_j}\big)^3\bigg)^{\frac{1}{3}}\|\varphi\|\\
&\le C\left(\sum_{j=1}^m\bigg(\int_{\mathbf{R}^3}\big(K(|x|)U_{y_j}^2\big)^{\frac{6}{5}}\bigg)^{\frac{5}{6}}\right.\\
&\quad\left.+\sum_{i\neq j}\bigg(\int_{\mathbf{R}^3}\big(K(|x|)U_{y_i}U_{y_j}\big)^{\frac{6}{5}}\bigg)^{\frac{5}{6}}\right)\sum_{j=1}^m\bigg(\int_{\mathbf{R}^3}\big(K(|x|)U_{y_j}\big)^3\bigg)^{\frac{1}{3}}\|\varphi\|\\
&\le C\Big(r^a e^{-br}+e^{-2(1-\kappa)r}+e^{-\eta r\sin\frac{\pi}{m}}r^a e^{-br}\\
&\quad+e^{-\eta r\sin\frac{\pi}{m}}e^{-(2-\eta)(1-\kappa)r}\Big)\big(r^a e^{-br}+e^{-(1-\kappa)r}\big)\|\varphi\|\\
&\le Ce^{-(1-\tau)2br}\|\varphi\|,
\end{aligned}
\tag{2.21}
$$

where we choose $(2-\eta)(1-\kappa)>b$, $\tau>0$ sufficiently small. Thus, the result follows. □

3. Proof of the main result

In this section we will prove Theorem 1.2.

Proof of Theorem 1.2. Let $\varphi(r)$ be the map obtained in Proposition 2.2. Define

$$\mathcal{F}(r)=I(U_{\mathbf{y}}+\varphi(r)),\quad \forall r\in\Lambda_m.$$

It is well-known that if r is a critical point of $\mathcal{F}(r)$, then $U_{\mathbf{y}}+\varphi(r)$ is a solution of (1.6) (see [6]). As a consequence, in order to complete the proof of the Theorem, we only need to prove that $\mathcal{F}(r)$ has a critical point in Λ_m.

Hence, by Proposition 2.2 and Lemma A.3, we have

$$\begin{aligned}
&\mathcal{F}(r) = I(U_{\mathbf{y}}) + f(\varphi) + \frac{1}{2}\langle \mathcal{L}\varphi,\varphi\rangle + R(\varphi) = I(U_{\mathbf{y}}) + O(\|f_m\|\|\varphi\| + \|\varphi\|^2)\\
&= A + \alpha B r^{2a} e^{-2br} - B' r^{-1} e^{-2r\sin\frac{\pi}{m}} + D\alpha r^{2a} e^{-2br}\sum_{j=2}^{m} V(y_j - y_1) + \alpha O\Big(e^{-2b(1+\tau)r}\Big)\\
&\quad + O\Big(e^{-(1+\tau)2r\sin\frac{\pi}{m}}\Big) + O\Big(\alpha e^{-(1-\tau)2br} + e^{-\min\{p-1-\sigma,2-\sigma\}r\sin\frac{\pi}{m}}\Big)^2\\
&= A + \alpha B r^{2a} e^{-2br} - B' r^{-1} e^{-2r\sin\frac{\pi}{m}} + D\alpha r^{2a} e^{-2br}\sum_{j=2}^{m} V(y_j - y_1) + \alpha O\Big(e^{-2b(1+\tau)r}\Big)\\
&\quad + O\Big(e^{-(1+\tau)2r\sin\frac{\pi}{m}}\Big),
\end{aligned}$$

where A, B, B' and D are defined in Lemma A.3. We consider its minimum respect to r:

$$\min\{\mathcal{F}(r)\colon r\in\Lambda_m\}. \tag{3.1}$$

Assume that (3.1) is achieved by some r_m in Λ_m, we will prove that r_m is an interior point of Λ_m. Here, we only consider the case $b > \sin\frac{\pi}{m}$, using the same method, we can obtain the result in the case $b < \sin\frac{\pi}{m}$.

Investigating the following smooth function in Λ_m,

$$g(r) := \alpha B r^{2a} e^{-2br} - B' r^{-1} e^{-2r\sin\frac{\pi}{m}}.$$

It is easy to check that $g(r)$ has a minimum point $\tilde{r}_m$, satisfying

$$e^{-(2b-2\sin\frac{\pi}{m})\tilde{r}_m} = \frac{1}{\alpha}\frac{B'}{2B}\frac{\tilde{r}_m^{-1-2a}(1+2\tilde{r}_m\sin\frac{\pi}{m})}{b\tilde{r}_m - a}.$$

Thus

$$\tilde{r}_m = \left(\frac{1}{b-\sin\frac{\pi}{m}} + o(1)\right)\ln\alpha,$$

with

$$\begin{aligned}
g(\tilde{r}_m) &= \alpha B\tilde{r}_m^{2a} e^{-2b\tilde{r}_m} - B'\tilde{r}_m^{-1} e^{-2\tilde{r}_m\sin\frac{\pi}{m}}\\
&= \alpha B\tilde{r}_m^{2a} e^{-2\tilde{r}_m\sin\frac{\pi}{m}}\frac{1}{\alpha}\frac{B'}{2B}\frac{\tilde{r}_m^{-1-2a}(1+2\tilde{r}_m\sin\frac{\pi}{m})}{b\tilde{r}_m - a} - B'\tilde{r}_m^{-1} e^{-2\tilde{r}_m\sin\frac{\pi}{m}}\\
&= B'\tilde{r}_m^{-1} e^{-2\tilde{r}_m\sin\frac{\pi}{m}}\left(\frac{\sin\frac{\pi}{m}}{b} - 1 + o\left(\frac{1}{\tilde{r}_m}\right)\right) < 0.
\end{aligned} \tag{3.2}$$

By direct computation, we deduce that

$$\mathcal{F}(r_m) \le \mathcal{F}(\tilde{r}_m) \le A + g(\tilde{r}_m) + O\left(\alpha^{\frac{\sin\frac{\pi}{m}+\delta}{\sin\frac{\pi}{m}-b}}\right) < A. \tag{3.3}$$

On the other hand, we suppose that $r_m = \left(\frac{1+\tau}{b-\sin\frac{\pi}{m}}\right)\ln\alpha$, then

(3.4)

$$\begin{aligned}
\mathcal{F}(r_m) &= A + \alpha B r_m^{2a} e^{-2br_m} - B' r_m^{-1} e^{-2r_m \sin\frac{\pi}{m}} + D\alpha r_m^{2a} e^{-2br_m} \sum_{j=2}^{m} V(y_j - y_1) \\
&\quad + \alpha O\left(e^{-b(1+\tau)r_m}\right) + O\left(e^{-(1+\tau)2r_m \sin\frac{\pi}{m}}\right), \\
&= A + \alpha B \left(\frac{(1+\tau)\ln\alpha}{b-\sin\frac{\pi}{m}}\right)^{2a} e^{-2b\left(\frac{1+\tau}{b-\sin\frac{\pi}{m}}\right)\ln\alpha} \\
&\quad - B'\left(\frac{(1+\tau)\ln\alpha}{b-\sin\frac{\pi}{m}}\right)^{-1} e^{-2\left(\frac{1+\tau}{b-\sin\frac{\pi}{m}}\right)\ln\alpha\sin\frac{\pi}{m}} \\
&\quad + D\alpha\left(\frac{(1+\tau)\ln\alpha}{b-\sin\frac{\pi}{m}}\right)^{2a} e^{-2b\left(\frac{1+\tau}{b-\sin\frac{\pi}{m}}\right)\ln\alpha} \sum_{j=2}^{m} V(y_j - y_1) + O\left(\alpha^{\frac{\sin\frac{\pi}{m}+\delta}{\sin\frac{\pi}{m}-b}}\right) \\
&> A.
\end{aligned}$$

This is a contradiction to (3.3).

Similarly

$$\mathcal{F}\left(\left(\frac{1-\tau}{b-\sin\frac{\pi}{m}}\right)\ln\alpha\right) > A.$$

Hence we can check that (3.1) is achieved by some r_m, which is in the interior of Λ_m. As a result, r_m is a critical point of $\mathcal{F}(r)$. Therefore

$$U_{r_m} + \varphi(r_m)$$

is a solution of (1.6). □

Appendix A. Some technical estimates

In this section, we will give the energy expansion for the approximate solutions. Recall

$$y_j = \left(r\cos\frac{2(j-1)\pi}{m}, r\sin\frac{2(j-1)\pi}{m}, 0\right), \quad j = 1, \ldots, m,$$

$$\Omega_j = \left\{x = (x', x_3) \in \mathbf{R}^2 \times \mathbf{R}\colon \left\langle \frac{x'}{|x'|}, \frac{y_j'}{|y_j'|}\right\rangle \geq \cos\frac{\pi}{m}\right\}, j = 1, 2, \ldots, m$$

and

$$I(u) = \frac{1}{2}\int_{\mathbf{R}^3} |\nabla u|^2 + \frac{1}{2}\int_{\mathbf{R}^3} u^2 + \frac{\alpha}{4}\int_{\mathbf{R}^3} K(|x|)\Phi_u u^2 - \frac{1}{p}\int_{\mathbf{R}^3} |u|^p,$$

where Φ_u is the solution of $-\Delta\Phi_u = K(|x|)u^2$.

Recall that U is the unique solution of

$$\begin{cases} -\Delta u + u = u^{p-1}, & u > 0 \text{ in } \mathbf{R}^3, \\ u(0) = \max\limits_{x\in\mathbf{R}^3} u(x). \end{cases}$$

Let V be the solution of

$$\begin{cases} -\Delta v = U^2, & \text{in } \mathbf{R}^3, \\ v \in D^{1,2}(\mathbf{R}^3). \end{cases}$$

Then, V is radial, and $rV(r) \to V_0 > 0$, as $r \to +\infty$.

Now, we give the following Lemma:

Lemma A.1. [2, Lemma 3.7] *Given $u, u' \colon \mathbf{R}^3 \to \mathbf{R}$ two positive continuous radial functions such that:*

$$u(x) \sim |x|^a e^{-b|x|},\ u'(x) \sim |x|^{a'} e^{-b'|x|}\ (x \to \infty)$$

where $a, a' \in \mathbf{R}$, $b > 0, b' > 0$. Let $\xi \in \mathbf{R}^3$ tend to infinity. Then, the following asymptotic estimates hold:

(1) *If $b < b'$,*

$$\int_{\mathbf{R}^3} u_\xi u' \sim |\xi|^a e^{-b|\xi|}.$$

(2) *If $b = b'$, suppose, for simplicity, that $a \geq a'$, then*

$$\int_{\mathbf{R}^3} u_\xi u' \sim \begin{cases} |\xi|^{a+a'+2} e^{-b|\xi|}, & a' > -2, \\ |\xi|^a e^{-b|\xi|} \log|\xi|, & a' = -2, \\ |\xi|^a e^{-b|\xi|}, & a' < -2. \end{cases}$$

Lemma A.2. *For a suitable $\sigma > 0$, we have*

$$\Phi_{U_{\mathbf{y}}}(y) = r^a e^{-br} \sum_{j=1}^m V(y - y_j) + O\left(\sum_{j=1}^m r^a e^{-(b+\sigma)r} \frac{1}{1 + |y - y_j|} \right).$$

Proof. Firstly,

$$U_{\mathbf{y}}^2 = U_{y_1}^2 + O\left(U_{y_1} \sum_{j=2}^m U_{y_j} + \left(\sum_{j=2}^m U_{y_j} \right)^2 \right).$$

For any $x \in \Omega_1$, we have

$$|x - y_i| \geq \frac{1}{2} |y_i - y_1|$$

and for any $\beta > 0$,

$$\sum_{j=2}^m U_{y_j}^\beta(x) \leq C \sum_{j=2}^m e^{-\beta|x-y_j|} \leq C \sum_{j=2}^m e^{-\frac{\beta}{2}|y_1 - y_j|} \leq C e^{-\beta \frac{\pi r}{m}}, \quad x \in \Omega_1.$$

As a result,

$$U_{y_1} \sum_{j=2}^m U_{y_j} \leq U_{y_1}^{\frac{3}{2}} \sum_{j=2}^m U_{y_j}^{\frac{1}{2}} \leq C e^{-\frac{1}{2}|x-y_1|} e^{-\frac{\pi r}{2m}}, \quad x \in \Omega_1,$$

and

$$\left(\sum_{j=2}^m U_{y_j} \right)^2 \leq U_{y_1} \left(\sum_{j=2}^m U_{y_j}^{\frac{1}{2}} \right)^2 \leq C e^{-\frac{1}{2}|x-y_1|} e^{-\frac{\pi r}{m}}, \quad x \in \Omega_1.$$

So,

$$U_{\mathbf{y}}^2 = U_{y_1}^2 + O\Big(e^{-\frac{1}{2}|x-y_1|} e^{-\frac{\pi r}{2m}} \Big).$$

By Lemma A.1 and the above result, we are led to

$$
\begin{aligned}
&\int_{\Omega_j} \frac{K(|y|)}{|x-y|} U_{\mathbf{y}}^2(y)\,dy = \int_{\Omega_j} \frac{K(|y|)}{|x-y|} \left(U_{y_j}^2 + O\left(e^{-\frac{1}{2}|y-y_j|} e^{-\frac{\pi r}{2m}} \right) \right) dy \\
&= \int_{\Omega_j \bigcap B_{\frac{r}{2}}(0)} \frac{K(|y+y_j|)}{|x-y-y_j|} U^2 dy + O\left(\int_{\mathbf{R}^3 \setminus B_{\frac{r}{2}}(0)} \frac{K(|y+y_j|)}{|x-y-y_j|} U^2\,dy \right) \\
&\quad + O\left(r^a e^{-(b+\sigma)r} \frac{1}{1+|y-y_j|} \right) \\
&= \int_{\Omega_j \bigcap B_{\frac{r}{2}}(0)} (1+o(1))|y+y_j|^a e^{-b|y+y_j|} \frac{U^2}{|x-y-y_j|}\,dy \\
&\quad + O\left(\int_{\mathbf{R}^3 \setminus B_{\frac{r}{2}}(0)} \frac{K(|y+y_j|)}{|x-y-y_j|} U^2\,dy \right) + O\left(r^a e^{-(b+\sigma)r} \frac{1}{1+|y-y_j|} \right) \\
&= r^a e^{-br} V(y-y_j) + O\left(r^a e^{-(b+\sigma)r} \frac{1}{1+|y-y_j|} \right).
\end{aligned} \tag{A.1}
$$

So

$$
\Phi_{U_{\mathbf{y}}}(y) = r^a e^{-br} \sum_{j=1}^m V(y-y_j) + O\left(\sum_{j=1}^m r^a e^{-(b+\sigma)r} \frac{1}{1+|y-y_j|} \right). \qquad \square
$$

Lemma A.3. *We have*

$$
\begin{aligned}
I(U_r) = {} & A + \alpha B r^{2a} e^{-2br} - B' r^{-1} e^{-2r\sin\frac{\pi}{m}} + D\alpha r^{2a} e^{-2br} \sum_{j=2}^m V(y_j - y_1) \\
& + \alpha O\left(e^{-2b(1+\tau)r} \right) + O\left(e^{-(1+\tau)2r\sin\frac{\pi}{m}} \right),
\end{aligned}
$$

where $A = m\left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathbf{R}^3} U^p$, $B = \frac{m}{4} \int_{\mathbf{R}^3} VU^2$ *and* $D = \frac{m}{4} \int_{\mathbf{R}^3} U^2$, $\tau > 0$ *sufficiently small.*

Proof. Recall that

$$
I(U_{\mathbf{y}}) = \frac{1}{2} \int_{\mathbf{R}^3} |\nabla U_{\mathbf{y}}|^2 + \frac{1}{2} \int_{\mathbf{R}^3} U_{\mathbf{y}}^2 + \frac{\alpha}{4} \int_{\mathbf{R}^3} K(|x|) \Phi_{U_{\mathbf{y}}} U_{\mathbf{y}}^2 - \frac{1}{p} \int_{\mathbf{R}^3} |U_{\mathbf{y}}|^p,
$$

By direct computation, we have

$$
\frac{1}{2} \int_{\mathbf{R}^3} |\nabla U_{\mathbf{y}}|^2 + \frac{1}{2} \int_{\mathbf{R}^3} U_{\mathbf{y}}^2 = \frac{m}{2} \int_{\mathbf{R}^3} U^p + \frac{m}{2} \sum_{j=2}^m \int_{\mathbf{R}^3} U_{y_1}^{p-1} U_{y_j}. \tag{A.2}
$$

We also have

$$
\begin{aligned}
\int_{\mathbf{R}^3} |U_{\mathbf{y}}|^p &= m \int_{\mathbf{R}^3} U_{y_1}^p + mp \int_{\mathbf{R}^3} \sum_{k=2}^m U_{y_1}^{p-1} U_{y_k} \\
&\quad + \begin{cases} O\Big(\sum_{k \neq j} \int_{\mathbf{R}^3} U_{y_k}^{\frac{p}{2}} U_{y_j}^{\frac{p}{2}} \Big), & \text{if } 2 < p \leq 3, \\ O\Big(\sum_{k \neq j} \int_{\mathbf{R}^3} U_{y_k}^{p-2} U_{y_j}^2 \Big), & \text{if } p > 3, \end{cases} \\
&= m \int_{\mathbf{R}^3} U^p + mp \int_{\mathbf{R}^3} |U_{y_1}|^{p-1} \sum_{j=2}^m U_{y_j} + mO\Big(e^{-(1+\tau)|y_2 - y_1|} \Big),
\end{aligned} \tag{A.3}
$$

where $\tau > 0$ sufficiently small.

Using Lemma A.2, we see

$$
\begin{aligned}
\int_{\mathbf{R}^3} K(|x|)\Phi_{U_{\mathbf{y}}} U_{\mathbf{y}}^2 &= m\int_{\Omega_1} K(|x|)\Phi_{U_{\mathbf{y}}} U_{\mathbf{y}}^2 \\
&= m\int_{\Omega_1} K(|x|)\Bigg(r^a e^{-br}\sum_{j=1}^m V(y-y_j) \\
&\quad + O\left(\sum_{j=1}^m r^a e^{-(b+\sigma)r}\frac{1}{1+|y-y_j|}\right)\Bigg)\left(U_{y_1}^2 + O\left(e^{-\frac{1}{2}|x-y_1|}e^{-\frac{\pi r}{2m}}\right)\right) \\
&= m\int_{\Omega_1} K(|x|) r^a e^{-br}\sum_{j=1}^m V(y-y_j) U_{y_1}^2 \\
&\quad + m\int_{\Omega_1} K(|x|) r^a e^{-br}\sum_{j=1}^m V(y-y_j) O\left(e^{-\frac{1}{2}|x-y_1|}e^{-\frac{\pi r}{2m}}\right) \\
&\quad + m\int_{\Omega_1} K(|x|) O\left(\sum_{j=1}^m r^a e^{-(b+\sigma)r}\frac{1}{1+|y-y_j|}\right) U_{y_1}^2 \\
&\quad + m\int_{\Omega_1} K(|x|) O\left(\sum_{j=1}^m r^a e^{-(b+\sigma)r}\frac{1}{1+|y-y_j|}\right) O\left(e^{-\frac{1}{2}|x-y_1|}e^{-\frac{\pi r}{2m}}\right) \\
&= m\int_{\Omega_1} K(|x|) r^a e^{-br}\sum_{j=1}^m V(y-y_j) U_{y_1}^2 \\
&\quad + m\int_{\Omega_1} K(|x|) O\left(\sum_{j=1}^m r^a e^{-(b+\sigma)r}\frac{1}{1+|y-y_j|}\right) U_{y_1}^2 \\
&\quad + O\left(k r^a e^{-b(1+\delta)r}\int_{\mathbf{R}^3} K(|x|)\sum_{j=1}^m \frac{1}{1+|y-y_j|} e^{-\frac{1}{2}|x-y_1|}\right) \\
&= m r^{2a} e^{-2br}\int_{\mathbf{R}^3} VU^2 + m r^{2a} e^{-2br}\sum_{j=2}^m V(y_j-y_1)\int_{\mathbf{R}^3} U^2 \\
&\quad + mO\left(r^{2a} e^{-2b(1+\delta)r}\sum_{j=2}^m \frac{1}{|y_1-y_j|}\right) \quad + mO\left(r^{2a}e^{-2b(1+\delta)r}\right),
\end{aligned} \tag{A.4}
$$

where $\delta > 0$ is a suitable constant.

Above all, we deduce

$$
\begin{aligned}
I(U_{\mathbf{y}}) &= \frac{1}{2}\int_{\mathbf{R}^3} |\nabla U_{\mathbf{y}}|^2 + \frac{1}{2}\int_{\mathbf{R}^3} U_{\mathbf{y}}^2 + \frac{\alpha}{4}\int_{\mathbf{R}^3} K(|x|)\Phi_{U_{\mathbf{y}}} U_{\mathbf{y}}^2 - \frac{1}{p}\int_{\mathbf{R}^3} |U_{\mathbf{y}}|^p \\
&= \frac{m}{2}\int_{\mathbf{R}^3} U^p + \frac{m}{2}\sum_{j=2}^m \int_{\mathbf{R}^3} U_{y_1}^{p-1} U_{y_j} - \frac{1}{p} m\int_{\mathbf{R}^3} U^p - m\int_{\mathbf{R}^3} U_{y_1}^{p-1}\sum_{j=2}^m U_{y_j} \\
&\quad + mO\left(e^{-(1+\tau)|y_2-y_1|}\right) \\
&\quad + \frac{\alpha}{4} m r^{2a} e^{-2br}\int_{\mathbf{R}^3} VU^2 + \frac{\alpha}{4} m r^{2a} e^{-2br}\sum_{j=2}^m V(y_j-y_1)\int_{\mathbf{R}^3} U^2
\end{aligned}
$$

$$
\begin{aligned}
&+ m\alpha O\left(r^{2a}e^{-2b(1+\delta)r}\sum_{j=2}^{m}\frac{1}{|y_1-y_j|}\right) + m\alpha O\left(r^{2a}e^{-2b(1+\delta)r}\right)\\
\text{(A.5)}\qquad &= m\left(\frac{1}{2}-\frac{1}{p}\right)\int_{\mathbf{R}^3}U^p - \frac{m}{2}\int_{\mathbf{R}^3}\left(U_{y_1}^{p-1}U_{y_2}+U_{y_1}^{p-1}U_{y_m}\right)\\
&+ \frac{\alpha}{4}mr^{2a}e^{-2br}\int_{\mathbf{R}^3}VU^2\\
&+ \frac{\alpha}{4}mr^{2a}e^{-2br}\sum_{j=2}^{m}V(y_j-y_1)\int_{\mathbf{R}^3}U^2 + mO\left(e^{-(1+\tau)|y_2-y_1|}\right)\\
&+ O\left(r^{2a}e^{-2b(1+\delta)r}\sum_{j=2}^{m}\frac{1}{|y_1-y_j|}\right) + m\alpha O\left(r^{2a}e^{-2b(1+\delta)r}\right)\\
&= m\left(\frac{1}{2}-\frac{1}{p}\right)\int_{\mathbf{R}^3}U^p + \frac{\alpha}{4}mr^{2a}e^{-2br}\int_{\mathbf{R}^3}VU^2\\
&+ \frac{\alpha}{4}mr^{2a}e^{-2br}\sum_{j=2}^{m}V(y_j-y_1)\int_{\mathbf{R}^3}U^2\\
&- B'r^{-1}e^{-2r\sin\frac{\pi}{m}} + \alpha O\left(e^{-2b(1+\tau)r}\right) + O\left(e^{-(1+\tau)2r\sin\frac{\pi}{m}}\right),
\end{aligned}
$$

where $\tau > 0$ sufficiengly small. □

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