GENERAL MAXIMAL OPERATORS AND THE REVERSE HÖLDER CLASSES

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Abstract. By a basis in $\mathbf{R}^n$ we mean a collection of open and bounded sets $\mathfrak{B}$. In this paper we show that, if the general maximal operator $M_{\mathfrak{B}}$ is bounded on $L^p(\mathbf{R}^n)$ for $p > 1$ and the weight w belongs to the reverse Hölder $\mathrm{RH}_{\infty,\mathfrak{B}}$ class, then the weighted maximal operator $M_{\mathfrak{B},w}$ is bounded on $L^p(\mathbf{R}^n, w)$ for $p > 1$. When the general basis $\mathfrak{B}$ has dyadic substructure with the Stein property, we investigate the equivalence between the Muckenhoupt class $A_{\infty,\mathfrak{B}}$ and the reverse Hölder class $\mathrm{RH}_{1,\mathfrak{B}}$. We also discuss equivalent ways of defining the reverse Hölder class $\mathrm{RH}_{1,\mathfrak{B}}$.

1. Introduction

The purpose of this paper is to develop a theory of weights for general maximal operators. We first fix some notations. By weights we will always mean non-negative, locally integrable functions on $\mathbf{R}^n$ which are positive on a set of positive measure. Given a measurable set E and a weight w, $w(E) = \int_E w(x)\,dx$, $|E|$ denotes the Lebesgue measure of E and $\mathbf{1}_E$ denotes the characteristic function of E. Given $1 < p < \infty$, $p' = \frac{p}{p-1}$ will denote the conjugate exponent number of p. Let $0 < p \le \infty$ and w be a weight. We define the weighted Lebesgue space $L^p(\mathbf{R}^n, w) = L^p(w)$ to be a Banach space equipped with the norm (or quasi norm)

$$\|f\|_{L^p(w)} = \left(\int_{\mathbf{R}^n} |f(x)|^p w(x)\,dx \right)^{1/p}.$$

In their paper [2], Alfonseca, Soria and Vargas proposed that directional maximal operators enjoy an almost-orthogonality principle in $L^2(\mathbf{R}^2)$. Motivated by an extension of this interesting property to the setting of radial weights, we considered in [18] the following weights: The weight w is of the form $w(x) = w_0(|x|)$, $x \in \mathbf{R}^2$, where $w_0 \colon [0, \infty) \to [0, \infty)$ satisfies

$$(1.1) \qquad \sup_{r_1 < r < r_2} w_0(r) \le \frac{C}{r_2 - r_1} \int_{r_1}^{r_2} w_0(r)\,dr$$

for all $0 < r_1 < r_2 < \infty$. Notice that r^a with $a > 0$ satisfies this condition. For this weight w we proved the following geometrical fact which was observed in [1] when $w \equiv 1$.

Let $0 < \theta_1, \theta_2 < \pi/4$. Let

$$\omega_0 = (1, 0), \quad \omega_1 = (\cos\theta_1, \sin\theta_1) \quad \text{and} \quad \omega_2 = (\cos(-\theta_2), \sin(-\theta_2)).$$

Let B be a rectangle whose longest side is parallel to ω_1 and let R be a rectangle whose longest side is parallel to ω_2. Suppose that $B \cap R \neq \emptyset$ and that the long side length of B is bigger than that of R. Then there exists a rectangle $\widetilde{R} \supset R$ whose longest side is parallel to ω_0 such that

$$\frac{w(R \cap B)}{w(R)} \leq C \frac{w(\widetilde{R} \cap B)}{w(\widetilde{R})},$$

where the constant C is independent of θ_1, θ_2, B and R.

One of the main ingredients for the proof of this somehow complicated fact is that, w fulfills a quite nice formula:

$$(1.2) \qquad \frac{w(R)}{|R|} \approx \frac{1}{\operatorname{rad}(R)} \int_{r_1(R)}^{r_2(R)} w_0(r)\, dr$$

for any rectangle $R \subset \mathbf{R}^2$, where

$$r_1(R) = \inf_{x \in R} |x|, \quad r_2(R) = \sup_{x \in R} |x| \quad \text{and} \quad \operatorname{rad}(R) = r_2(R) - r_1(R).$$

By the use of these relations (1.1) and (1.2), in [17], the first author established the logarithmic boundedness of weighted small Kakeya maximal operators on $L^2(\mathbf{R}^2, w)$.

Fix $N \gg 1$. For a real number $a > 0$ let $\mathfrak{K}_{N,a}$ be the family of all tubes in $\mathbf{R}^n$, $n \geq 2$, which are congruent to the tubes with height Na and width a, but with arbitrary directions and centers. Let $\mathfrak{K}_N = \bigcup_{a>0} \mathfrak{K}_{N,a}$. For an $f \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ the small Kakeya maximal operator $M_{\mathfrak{K}_{N,a}}$ is defined by

$$M_{\mathfrak{K}_{N,a}} f(x) = \sup_{T \in \mathfrak{K}_{N,a}} \mathbf{1}_T(x) \frac{1}{|T|} \int_T |f(y)|\, dy$$

and the Kakeya maximal operator $M_{\mathfrak{K}_N}$ is defined by

$$M_{\mathfrak{K}_N} f(x) = \sup_{T \in \mathfrak{K}_N} \mathbf{1}_T(x) \frac{1}{|T|} \int_T |f(y)|\, dy.$$

It is conjectured that $M_{\mathfrak{K}_N}$ is bounded on $L^n(\mathbf{R}^n)$ with the operator norm which grows no faster than $O((\log N)^{\alpha_n})$ for some $\alpha_n > 0$ as $N \to \infty$. In the case $n = 2$, this conjecture was solved affirmatively by Córdoba [3] with the exponent $\alpha_2 = 2$ and reproved by Strömberg [16] with $\alpha_2 = 1$. In the higher dimensional case, $n > 2$, these estimates were proved so far only for some restricted class of functions; see for example the references in [18].

For a weight w the corresponding weighted maximal operators are defined by

$$M_{\mathfrak{K}_{N,a},w} f(x) = \sup_{T \in \mathfrak{K}_{N,a}} \mathbf{1}_T(x) \frac{1}{w(T)} \int_T |f(y)| w(y)\, dy,$$

$$M_{\mathfrak{K}_N,w} f(x) = \sup_{T \in \mathfrak{K}_N} \mathbf{1}_T(x) \frac{1}{w(T)} \int_T |f(y)| w(y)\, dy.$$

It is shown in [18] that, if the weight w is of the form $w(x) = |x|^a$, $a > 0$, then

$$(1.3) \qquad \|M_{\mathfrak{K}_N,w} f\|_{L^2(\mathbf{R}^2,w)} \leq C \log N \|f\|_{L^2(\mathbf{R}^2,w)}$$

and is also shown in [17] that, if w satisfies (1.1), then for any $a > 0$

$$(1.4) \qquad \|M_{\mathfrak{K}_{N,a},w} f\|_{L^2(\mathbf{R}^2,w)} \leq C \sqrt{\log N} \|f\|_{L^2(\mathbf{R}^2,w)}.$$

In this paper we shall prove a theorem concerning general maximal operator which implies those two estimates (1.3) and (1.4) as a corollary; see Remark 2.8.

When the case $\mathfrak{B} = \mathfrak{Q}$, the collection of all cubes in $\mathbf{R}^n$ with sides parallel to the coordinate axes, it is well known that the Muckenhoupt A_p condition characterizes the weighted L^p estimate for several important operators; see for example [6]. The reverse Hölder class $\mathrm{RH}_{1,\mathfrak{Q}}$ is closely related to the Muckenhoupt class $A_{\infty,\mathfrak{Q}}$. However, it seems that further much more is known about the Muckenhoupt class $A_{\infty,\mathfrak{Q}}$ than about the reverse Hölder class $\mathrm{RH}_{1,\mathfrak{Q}}$. In [4], Cruz-Uribe and Neugebauer investigated deeply the structure of the reverse Hölder class $\mathrm{RH}_{1,\mathfrak{Q}}$. In this paper we study the structure of the reverse Hölder classes with general basis $\mathfrak{B}$ and prove the equivalence between the Muckenhoupt class $A_{\infty,\mathfrak{B}}$ and the reverse Hölder class $\mathrm{RH}_{1,\mathfrak{B}}$. Our essential tools are the Calderón–Zygmund decomposition and the dyadic structure of the basis which is assumed a priori.

It is known that there are several characterizations of the Muckenhoupt class $A_{\infty,\mathfrak{Q}}$, which are defined by many authors such as Muckenhoupt, Coifman and Fefferman, Fujii and so on. However, these characterizations are not valid for a general basis $\mathfrak{B}$ instead of a basis $\mathfrak{Q}$. In the recent paper [5], Duoandikoetxea, Martín-Reyes and Ombrosi compared several characterizations of $A_{\infty,\mathfrak{B}}$ on a σ-finite measure space (X, μ), a basis is a collection of μ-measurable subsets B of X such that $0 < \mu(B) < \infty$. They established several implications among such conditions without further assumptions on the basis (or, for example, assuming the boundedness of the maximal operator associated with $\mathfrak{B}$), but their assumptions could not decide whether the weights $w \in A_{\infty,\mathfrak{B}}$ belong to $\mathrm{RH}_{1,\mathfrak{B}}$.

The paper is organized as follows: In Section 2 we state and prove two theorems (Theorems 2.6 and 2.10). For a general basis $\mathfrak{B}$, we consider the weight w belonging to the reverse Hölder classes $\mathrm{RH}_{s,\mathfrak{B}}$, $1 < s \leq \infty$, and, for the weighted general maximal operator $M_{\mathfrak{B},w}$, we introduce a sufficient condition for one-weight norm inequalities to hold. As a corollary (Corollary 2.7), we show that, if the general maximal operator $M_{\mathfrak{B}}$ is bounded on $L^p(\mathbf{R}^n)$ for $p > 1$ and the weight w belongs to the reverse Hölder class $\mathrm{RH}_{\infty,\mathfrak{B}}$, then the weighted general maximal operator $M_{\mathfrak{B},w}$ is bounded on $L^p(\mathbf{R}^n, w)$ for $p > 1$. In Section 3 we investigate some properties of $\mathrm{RH}_{\infty,\mathfrak{B}}$ weights and show how they are related to the Muckenhoupt class $A_{1,\mathfrak{B}}$. In Section 4, under the assumption that the general basis $\mathfrak{B}$ has dyadic substructure with the Stein property, we introduce the Gehring lemma (Theorem 4.6) and investigate the equivalence between the Muckenhoupt class $A_{\infty,\mathfrak{B}}$ and the reverse Hölder class $\mathrm{RH}_{1,\mathfrak{B}}$ (Theorems 4.8 and 4.10). We also discuss an equivalent way of defining the reverse Hölder class $\mathrm{RH}_{1,\mathfrak{B}}$ (Theorem 4.12) following their papers [9] and [15]. In the final section (Section 5) we extend the nice formula (1.2) to the higher dimensions (Lemma 5.1. This formula enables us that the weighted estimates for strong maximal operators and the Kakeya maximal operators in the higher dimensions (Proposition 5.3).

Throughout this paper all the notations are standard or will be defined as needed. The letter C will be used for constants that may change from one occurrence to another. Constants with subscripts, such as C_1, C_2, do not change in different occurrences. By $A \approx B$ we mean that $c^{-1}B \leq A \leq cB$ with some positive constant c independent of appropriate quantities.

2. General maximal operators and weight classes

By a basis in $\mathbf{R}^n$ we mean a collection of open and bounded sets $\mathfrak{B}$. For a basis $\mathfrak{B}$ we consider associated weights w such that $0 < w(B) < \infty$ for every $B \in \mathfrak{B}$. Given such a $\mathfrak{B}$ and w, the corresponding maximal operator $M_{\mathfrak{B},w}$ is defined by

$$M_{\mathfrak{B},w}f(x) = \sup_{B \in \mathfrak{B}} \mathbf{1}_B(x) \frac{1}{w(B)} \int_B |f(y)| w(y)\, dy.$$

If $w \equiv 1$, we just write $M_{\mathfrak{B}}f(x)$. Following the notation in [4], we define $B(w) = \frac{w(B)}{|B|}$.

Definition 2.1. Let $1 \le p < \infty$. A weight w belongs to the Muckenhoupt class $A_{p,\mathfrak{B}}$ whenever its Muckenhoupt constant $[w]_{A_{p,\mathfrak{B}}}$ is finite. That is,

$$[w]_{A_{p,\mathfrak{B}}} = \sup_{B \in \mathfrak{B}} B(w) B(w^{-1/(p-1)})^{p-1} < \infty, \quad 1 < p < \infty,$$

$$[w]_{A_{1,\mathfrak{B}}} = \sup_{B \in \mathfrak{B}} \frac{B(w)}{\operatorname{ess\,inf}_{x \in B} w(x)} < \infty.$$

Notice that by Hölder's inequality, for $1 < p < q < \infty$,

$$1 \le [w]_{A_{q,\mathfrak{B}}} \le [w]_{A_{p,\mathfrak{B}}} < \infty$$

and hence the inclusion $A_{p,\mathfrak{B}} \subset A_{q,\mathfrak{B}}$ holds. So, for $1 < p < \infty$, the Muckenhoupt classes $A_{p,\mathfrak{B}}$ form an increasing chain and thus we define

$$A_{\infty,\mathfrak{B}} = \bigcup_{p>1} A_{p,\mathfrak{B}}.$$

Definition 2.2. Let $1 < p \le \infty$. A weight w belongs to the reverse Hölder class $\mathrm{RH}_{p,\mathfrak{B}}$ if and only if

$$[w]_{\mathrm{RH}_{p,\mathfrak{B}}} = \sup_{B \in \mathfrak{B}} \frac{B(w^p)^{1/p}}{B(w)} < \infty, \quad 1 < p < \infty,$$

$$[w]_{\mathrm{RH}_{\infty,\mathfrak{B}}} = \sup_{B \in \mathfrak{B}} \frac{\operatorname{ess\,sup}_{x \in B} w(x)}{B(w)} < \infty.$$

It follows from Hölder's inequality that, for $1 < p < q < \infty$,

$$1 \le [w]_{\mathrm{RH}_{p,\mathfrak{B}}} \le [w]_{\mathrm{RH}_{q,\mathfrak{B}}} < \infty$$

and hence that the inclusion $\mathrm{RH}_{p,\mathfrak{B}} \supset \mathrm{RH}_{q,\mathfrak{B}}$ holds. So, for $1 < p < \infty$, the reverse Hölder classes $\mathrm{RH}_{p,\mathfrak{B}}$ form a decreasing chain and thus we define

$$\mathrm{RH}_{1,\mathfrak{B}} = \bigcup_{p>1} \mathrm{RH}_{p,\mathfrak{B}}.$$

For the general bases $\mathfrak{B}$ and associated weights w, very little is known concerning the boundedness of $M_{\mathfrak{B}}$ and $M_{\mathfrak{B},w}$ on $L^p(w)$. The most important result of one weight theory for general maximal operators is theorem due to Jawerth [10]. This theorem was reproved by Lerner [11] with better understanding of the dependency of the constants.

Theorem 2.3. (Jawerth [10]) *Let $1 < p < \infty$. For the general bases $\mathfrak{B}$ and associated weights w, let $\sigma = w^{-1/(p-1)}$. Then $M_{\mathfrak{B}}$ is bounded on $L^p(w)$ and on $L^{p'}(\sigma)$ if and only if $w \in A_{p,\mathfrak{B}}$, $M_{\mathfrak{B},\sigma}$ is bounded on $L^p(\sigma)$ and $M_{\mathfrak{B},w}$ is bounded on $L^{p'}(w)$.*

The following abstract theorem from [14], which is proved by the use of Theorem 2.3, gives a necessary and sufficient condition for the boundedness of the $M_{\mathfrak{B},w}$, in terms of the unweighted maximal function $M_{\mathfrak{B}}$ in the special case $w \in A_{\infty,\mathfrak{B}}$.

Theorem 2.4. (Pérez [14]) *The following are equivalent:*

(i) *For every $1 < p < \infty$ and every $w \in A_{p,\mathfrak{B}}$, we have*

$$M_{\mathfrak{B}} \colon L^p(w) \to L^p(w);$$

(ii) *For every $1 < p < \infty$ and every $w \in A_{\infty,\mathfrak{B}}$, we have*

$$M_{\mathfrak{B},w} \colon L^p(w) \to L^p(w).$$

Theorem 2.4 and some important examples (cf. [14]) motivate the following definition.

Definition 2.5. A basis $\mathfrak{B}$ is a Muckenhoupt basis if for every $1 < p < \infty$ and every $w \in A_{p,\mathfrak{B}}$, we have

$$M_{\mathfrak{B}} \colon L^p(w) \to L^p(w).$$

With this definition Theorem 2.4 states that $\mathfrak{B}$ is a Muckenhoupt basis if and only if the weighted maximal function satisfies $M_{\mathfrak{B},w} \colon L^p(w) \to L^p(w)$ for every $1 < p < \infty$ and every $w \in A_{\infty,\mathfrak{B}}$.

We are interested in the theory of weights for the Kakeya maximal operators. The main interest of this theory is to determine the factor N appearing in its operator norms. From this point of view, Theorem 2.4 is quite abstract and thus it cannot apply to our situation. Attempting one weighted estimate for the Kakeya maximal operator $M_{\mathfrak{K}_N}$ in Wolff's range (cf. [21]), we found the following theorem concerning general maximal operators which is our first theorem of this paper.

Theorem 2.6. *Let $\mathfrak{B}$ be a basis and $1 < p, s \leq \infty$. Suppose that $w \in \mathrm{RH}_{s,\mathfrak{B}}$ with $[w]_{\mathrm{RH}_{s,\mathfrak{B}}} = C_1$, $\|M_{\mathfrak{B}}\|_{L^t(\mathbf{R}^n)} = C_2$ with $t = p - (p-1)/s > 1$. Suppose further that there exists a constant $C_3 > 0$ such that*

$$(2.1) \qquad \sum_j \frac{w(E(B_j))}{B_j(w)} \leq C_3 \left| \bigcup_j B_j \right|$$

holds for any countable subfamily $\{B_j\} \subset \mathfrak{B}$ and any pairwise disjoint subset $E(B_j) \subset B_j$ for varying j. Then we have

$$\|M_{\mathfrak{B},w}\|_{L^p(w)} \leq 2 C_1^{1/p'} C_2^{1/(p's)'} C_3^{1/p}.$$

Proof. Let $f \in L^p(w)$. For every integer k, we shall consider the set

$$S_k = \{x \in \mathbf{R}^n \colon 2^k < M_{\mathfrak{B},w}f(x) \leq 2^{k+1}\}.$$

Using a limiting argument, we may assume that S_k are compact sets. From the definition of $M_{\mathfrak{B},w}$, $S_k \subset \bigcup_j B_{k,j}$, where $B_{k,j} \in \mathfrak{B}$ satisfies

$$\frac{1}{w(B_{k,j})} \int_{B_{k,j}} |f(y)| w(y) \, dy > 2^k.$$

Define $E(B_{k,1}) = B_{k,1} \cap S_k$ and, for $j > 1$,

$$E(B_{k,j}) = \left(B_{k,j} \setminus \bigcup_{i<j} B_{k,i} \right) \cap S_k.$$

The sets S_k form a disjoint collection and each S_k is the disjoint union of the sets $E(B_{k,j})$ for varying j. Thus,

$$\int_{\mathbf{R}^n} M_{\mathfrak{B},w} f(x)^p w(x)\, dx = \sum_{k,j} \int_{E(B_{k,j})} M_{\mathfrak{B},w} f(x)^p w(x)\, dx \leq \sum_{k,j} 2^{(k+1)p} w(E(B_{k,j}))$$

$$\leq 2^p \sum_{k,j} w(E(B_{k,j})) \left(\frac{1}{w(B_{k,j})} \int_{B_{k,j}} |f(y)| w(y)\, dy \right)^p.$$

Notice that

$$\left(\frac{1}{w(B_{k,j})} \int_{B_{k,j}} |f(y)| w(y)\, dy \right)^p = B_{k,j}(w)^{-p} \left(\frac{1}{|B_{k,j}|} \int_{B_{k,j}} |f(y)| w(y)\, dy \right)^p$$

and by Hölder's inequality that

$$\left(\frac{1}{|B_{k,j}|} \int_{B_{k,j}} |f(y)| w(y)\, dy \right)^p = \left(\frac{1}{|B_{k,j}|} \int_{B_{k,j}} |f(y)| w(y)^{1/p} \cdot w(y)^{1/p'}\, dy \right)^p$$

$$\leq \left(\frac{1}{|B_{k,j}|} \int_{B_{k,j}} |f(y)|^{p/t} w(y)^{1/t}\, dy \right)^t \left(\frac{1}{|B_{k,j}|} \int_{B_{k,j}} w(y)^s\, dy \right)^{p/(p's)},$$

where we have used

$$\frac{p}{t} = \frac{p}{p - (p-1)/s} = \frac{1}{1 - 1/(p's)} = \frac{p's}{p's - 1} = (p's)'.$$

These yield

$$\int_{\mathbf{R}^n} M_{\mathfrak{B},w} f(x)^p w(x)\, dx$$

$$\leq 2^p \sum_{k,j} \frac{w(E(B_{k,j}))}{B_{k,j}(w)} B_{k,j}(w)^{1-p} \left(B_{k,j}(w^s)^{1/s} \right)^{p-1} \left(\frac{1}{|B_{k,j}|} \int_{B_{k,j}} |f(y)|^{p/t} w(y)^{1/t}\, dy \right)^t$$

$$\leq 2^p C_1^{p-1} \sum_{k,j} \mu_{k,j} g_{k,j},$$

where

$$\mu_{k,j} = \frac{w(E(B_{k,j}))}{B_{k,j}(w)}, \quad g_{k,j} = \left(\frac{1}{|B_{k,j}|} \int_{B_{k,j}} |f(y)|^{p/t} w(y)^{1/t}\, dy \right)^t.$$

We view the sum $\sum_{k,j} \mu_{k,j} g_{k,j}$ as an integral on a measure space (X, μ) built over the set $X = \{(k,j)\}$ by assigning to each (k,j) the measure $\mu_{k,j}$. For $\lambda > 0$ call

$$\Gamma(\lambda) = \{(k,j) \in X \colon g_{k,j} > \lambda\}, \quad G(\lambda) = \bigcup_{(k,j) \in \Gamma(\lambda)} B_{k,j}.$$

Then $\sum_{k,j} \mu_{k,j} g_{k,j} = \int_0^\infty \mu(\Gamma(\lambda))\, d\lambda$. Observe that

$$\mu(\Gamma(\lambda)) = \sum_{(k,j) \in \Gamma(\lambda)} \frac{w(E(B_{k,j}))}{B_{k,j}(w)} \leq C_3 |G(\lambda)|$$

and that

$$G(\lambda) \subset \{x \in \mathbf{R}^n \colon M_{\mathfrak{B}}[|f|^{p/t} w^{1/t}](x)^t > \lambda\}.$$

These entail

$$\int_{\mathbf{R}^n} M_{\mathfrak{B},w}f(x)^p w(x)\,dx \leq 2^p C_1^{p-1} C_3 \int_{\mathbf{R}^n} M_{\mathfrak{B}}[|f|^{p/t} w^{1/t}](x)^t\,dx$$

$$\leq 2^p C_1^{p-1} C_3 C_2^t \int_{\mathbf{R}^n} |f(x)|^p w(x)\,dx.$$

Consequently,

$$\|M_{\mathfrak{B},w}\|_{L^p(w)} \leq 2 C_1^{1/p'} C_2^{1/(p's)'} C_3^{1/p}.$$

This completes the proof. $\qquad\square$

The next corollary leads the relations (1.3) and (1.4).

Corollary 2.7. *Let $\mathfrak{B}$ be a basis and $1 < p \leq \infty$. Suppose that the weight w is in* $\mathrm{RH}_{\infty,\mathfrak{B}}$ *with* $[w]_{\mathrm{RH}_{\infty,\mathfrak{B}}} = C_1$ *and that* $\|M_{\mathfrak{B}}\|_{L^p(\mathbf{R}^n)} = C_2$. *Then we have*

$$\|M_{\mathfrak{B},w}\|_{L^p(w)} \leq 2 C_1 C_2.$$

Proof. Since $t = p$ and $(p's)' = 1$, we merely check the condition (2.1) with $C_3 = C_1$. There holds

$$\sum_j \frac{w(E(B_j))}{B_j(w)} \leq \sum_j |E(B_j)| \frac{\operatorname{ess\,sup}_{x \in B_j} w(x)}{B_j(w)} \leq C_1 \sum_j |E(B_j)| \leq C_1 \left| \bigcup_j B_j \right|,$$

where we have used that the sets $E(B_j) \subset B_j$ are pairwise disjoint. $\qquad\square$

Remark 2.8. Let $w(x) = w_0(|x|)$, $x \in \mathbf{R}^2$, and w_0 satisfy (1.1). Then by (1.2) we see that w belongs to $\mathrm{RH}_{\infty,\mathfrak{K}_N}$. Thus, Corollary 2.7 yields (1.3) and (1.4) together with well-known unweighted estimates; see [3, 16].

When $s = \infty$ the sufficient condition (2.1) in Theorem 2.6 can be checked quite easily. But, when $1 < s < \infty$, we cannot say any more. The next theorem (Theorem 2.10) characterizes the boundedness of $M_{\mathfrak{B},w}$ on $L^p(w)$ in terms of a so-called Tauberian condition.

Definition 2.9. We say that a weight w associated to the basis $\mathfrak{B}$ satisfies the Tauberian condition (A) if there are constants $0 < \lambda < 1$, $0 < c = c(\lambda) < \infty$ such that for all measurable sets E

$$(\mathrm{A}) \qquad w(\{x \in \mathbf{R}^n \colon M_{\mathfrak{B}}(\mathbf{1}_E)(x) > \lambda\}) \leq cw(E).$$

Theorem 2.10. *Let $\mathfrak{B}$ be a basis. Let $w \in \mathrm{RH}_{s,\mathfrak{B}}$, $1 < s \leq \infty$, and w satisfy the condition (A). Suppose that $M_{\mathfrak{B}} \colon L^p(\mathbf{R}^n) \to L^p(\mathbf{R}^n)$ for all $p > 1$. Then $M_{\mathfrak{B},w} \colon L^p(w) \to L^p(w)$ holds for all $p > 1$.*

Proof. Let $f \in L^p(w)$. In the same manner as in the proof of Theorem 2.6, for every integer k, we consider the set

$$S_k = \{x \in \mathbf{R}^n \colon 2^k < M_{\mathfrak{B},w}f(x) \leq 2^{k+1}\}.$$

We choose compact sets $K_k \subset S_k$ and, for each k, we choose a finite cover $\{B_{k,j}\} \subset \mathfrak{B}$ such that

$$(2.2) \qquad K_k \subset \bigcup_j B_{k,j} \quad \text{and} \quad \frac{1}{w(B_{k,j})} \int_{B_{k,j}} |f(y)| w(y)\,dy > 2^k.$$

Now, we have

$$\int_{\bigcup_k K_k} M_{\mathfrak{B},w}f(x)^p w(x)\,dx \leq 2^p \sum_k 2^{kp} w\left(\bigcup_j B_{k,j} \right).$$

We claim that then there exists a subfamily $\{\overline{B}_{k,j}\} \subset \{B_{k,j}\}$ satisfying

$$(2.3) \qquad \sum_k 2^{kp} w\left(\bigcup_j B_{k,j}\right) \le C \sum_k 2^{kp} w\left(\bigcup_j \overline{B}_{k,j}\right)$$

and, for each (k, j), there exists a subset $E(\overline{B}_{k,j}) \subset \overline{B}_{k,j}$ such that

$$(2.4) \qquad (1 - \lambda)|\overline{B}_{k,j}| \le |E(\overline{B}_{k,j})|$$

and the sets $E(\overline{B}_{k,j})$ are pairwise disjoint for varying (k, j).

We shall verify this claim. Without loss of generality we may assume that $B_{k,j} = \emptyset$ except for a finite number of (k, j). Let $N = \max\{k \colon B_{k,j} \ne \emptyset\}$. We follow a well-known selecting procedure argument (cf. [6, p. 463] for instance).

Step 1. Let $\overline{B}_{N,1} = B_{N,1}$ and, once $\overline{B}_{N,1}, \dots, \overline{B}_{N,j-1}$ have been selected, we choose $\overline{B}_{N,j}$ to be the first set in $\{B_{N,.}\}$ (if any) such that

$$(2.5) \qquad \left|\overline{B}_{N,j} \cap \left(\bigcup_{s<j} \overline{B}_{N,s}\right)\right| < \lambda|\overline{B}_{N,j}|.$$

Let $E_N = \bigcup_j \overline{B}_{N,j}$. Now we verify that

$$(2.6) \qquad \bigcup_j B_{N,j} \subset \{x \in \mathbf{R}^n \colon M_{\mathfrak{B}}[\mathbf{1}_{E_N}](x) \ge \lambda\}.$$

Let $x \in \bigcup_j B_{N,j}$. If $x \in E_N$, then it is obvious that it is contained on the set to the right since $\lambda < 1$. If $x \notin E_N$, then x is contained in some discarded B in the selection process and hence $|B \cap E_N| \ge \lambda|B|$, which means that $M_{\mathfrak{B}}[\mathbf{1}_{E_N}](x) \ge \lambda$. Since w satisfies the condition (A), (2.6) yields

$$w\left(\bigcup_j B_{N,j}\right) \le Cw(E_N).$$

If we let $E(\overline{B}_{N,j}) = \overline{B}_{N,j} \setminus \bigcup_{s<j} \overline{B}_{N,s}$, then we see that, by (2.5), $\{E(\overline{B}_{N,j})\}$ is a disjoint family with $E(\overline{B}_{N,j}) \subset \overline{B}_{N,j}$ and $(1 - \lambda)|\overline{B}_{N,j}| < |E(\overline{B}_{N,j})|$.

Step 2. Let $\overline{B}_{N-1,1}$ be the first set in $\{B_{N-1,.}\}$ such that

$$|\overline{B}_{N-1,1} \cap E_N| < \lambda|\overline{B}_{N-1,1}|.$$

Once $\overline{B}_{N-1,1}, \dots, \overline{B}_{N-1,j-1}$ have been selected, we choose $\overline{B}_{N-1,j}$ to be the first set in $\{B_{N-1,.}\}$ (if any) such that

$$(2.7) \qquad \left|\overline{B}_{N-1,j} \cap \left\{E_N \cup \left(\bigcup_{s<j} \overline{B}_{N-1,s}\right)\right\}\right| < \lambda|\overline{B}_{N-1,j}|.$$

Let $E_{N-1} = \bigcup_j \overline{B}_{N-1,j}$. By the same observation as the above, we can verify that

$$\bigcup_j B_{N-1,j} \subset \{x \in \mathbf{R}^n \colon M_{\mathfrak{B}}[\mathbf{1}_{E_{N-1} \cup E_N}](x) \ge \lambda\}$$

and see that

$$w\left(\bigcup_j B_{N-1,j}\right) \le Cw(E_{N-1} \cup E_N).$$

If we let

$$E(\overline{B}_{N-1,j}) = \overline{B}_{N,j} \setminus \left\{ E_N \cup \left(\bigcup_{s<j} \overline{B}_{N-1,s} \right) \right\},$$

then we have, by (2.7), $\{E(\overline{B}_{N-1,j})\} \cup \{E(\overline{B}_{N,j})\}$ is a disjoint family with $E(\overline{B}_{N-1,j}) \subset \overline{B}_{N-1,j}$ and $(1-\lambda)|\overline{B}_{N-1,j}| < |E(\overline{B}_{N-1,j})|$.

Continuing these steps, We obtain a subfamily $\{\overline{B}_{k,j}\} \subset \{B_{k,j}\}$ satisfying (2.4) and

$$w \left(\bigcup_j B_{k,j} \right) \le C w \left(\bigcup_{\ell=k}^{N} E_\ell \right),$$

where

$$E_\ell = \bigcup_j \overline{B}_{\ell,j}.$$

It follows that

$$\sum_k 2^{kp} w \left(\bigcup_j B_{k,j} \right) \le C \sum_k 2^{kp} w \left(\bigcup_{\ell=k}^{N} E_\ell \right) \le C \sum_k 2^{kp} \left(\sum_{\ell=k}^{N} w(E_\ell) \right)$$

$$= C \sum_k \left(\sum_{\ell \le k} 2^{\ell p} \right) w(E_k) \le C \sum_k 2^{kp} w(E_k)$$

$$= C \sum_k 2^{kp} w \left(\bigcup_j \overline{B}_{k,j} \right),$$

which is exactly (2.3).

There holds by (2.2) and (2.3)

$$\int_{\bigcup_k K_k} M_{\mathfrak{B},w} f(x)^p w(x)\, dx \le C \sum_k 2^{kp} w(\bigcup_j B_{k,j}) \le C \sum_k 2^{kp} w(\bigcup_j \overline{B}_{k,j})$$

$$\le C \sum_{k,j} w(\overline{B}_{k,j}) \left(\frac{1}{w(\overline{B}_{k,j})} \int_{\overline{B}_{k,j}} |f(y)| w(y)\, dy \right)^p.$$

Recalling $t = p - (p-1)/s$,

$$\left(\frac{1}{w(\overline{B}_{k,j})} \int_{\overline{B}_{k,j}} |f(y)| w(y)\, dy \right)^p \le C \frac{1}{\overline{B}_{k,j}(w)} \left(\frac{1}{|\overline{B}_{k,j}|} \int_{\overline{B}_{k,j}} |f(y)|^{p/t} w(y)^{1/t}\, dy \right)^t.$$

Thus, by (2.4),

$$\int_{\bigcup_k K_k} M_{\mathfrak{B},w} f(x)^p w(x)\, dx \le C \sum_{k,j} |\overline{B}_{k,j}| \left(\frac{1}{|\overline{B}_{k,j}|} \int_{\overline{B}_{k,j}} |f(y)|^{p/t} w(y)^{1/t}\, dy \right)^t$$

$$\le C \sum_{k,j} |E(\overline{B}_{k,j})| \left(\frac{1}{|\overline{B}_{k,j}|} \int_{\overline{B}_{k,j}} |f(y)|^{p/t} w(y)^{1/t}\, dy \right)^t$$

$$\le C \sum_{k,j} \int_{E(\overline{B}_{k,j})} M_{\mathfrak{B}}[|f|^{p/t} w^{1/t}](x)^t\, dx$$

$$\le C \int_{\mathbf{R}^n} M_{\mathfrak{B}}[|f|^{p/t} w^{1/t}](x)^t\, dx \le C \int_{\mathbf{R}^n} |f(x)|^p w(x)\, dx,$$

where we have used the sets $E(\overline{B}_{k,j})$ are pairwise disjoint. This completes the proof. □

Remark 2.11. The weighted Tauberian condition for general maximal operators appears in [10, 13], for instance. It is shown in [8] that the condition (A) is equivalent to $M_{\mathfrak{B}} \colon L^p(w) \to L^p(w)$ for sufficient large p provided that $\mathfrak{B}$ is a homothecy invariant basis consisting of convex sets.

3. The class $\mathrm{RH}_{\infty,\mathfrak{B}}$

In this section, following [4], we show some properties of $\mathrm{RH}_{\infty,\mathfrak{B}}$ weights. Throughout this section we fix a general basis $\mathfrak{B}$. We need some lemmas.

Lemma 3.1. *Let* $1 < p, s < \infty$. *A weight* w *is in* $A_{p,\mathfrak{B}} \cap \mathrm{RH}_{s,\mathfrak{B}}$ *if and only if* w^s *is in* $A_{q,\mathfrak{B}}$ *where* $q = s(p-1) + 1$.

Proof. Suppose that $w \in A_{p,\mathfrak{B}} \cap \mathrm{RH}_{s,\mathfrak{B}}$. Then, noticing $q-1 = s(p-1)$, we have

$$
\begin{aligned}
B(w^s)B\left((w^s)^{-1/(q-1)}\right)^{q-1} &= B(w^s)B(w^{-1/(p-1)})^{s(p-1)} \\
&\leq C\{B(w)B(w^{-1/(p-1)})^{p-1}\}^s < \infty,
\end{aligned}
$$

where we have used $B(w^s) \leq CB(w)^s$.

Suppose, conversely, that $w^s \in A_{q,\mathfrak{B}}$ with $q = s(p-1) + 1$. We have then

$$
(3.1) \qquad\qquad B(w^s)B(w^{-1/(p-1)})^{s(p-1)} \leq C.
$$

Noticing $B(w)^s \leq B(w^s)$,

$$
B(w)B(w^{-1/(p-1)})^{p-1} \leq \{B(w^s)B(w^{-1/(p-1)})^{s(p-1)}\}^{1/s} \leq C^{1/s}.
$$

Thus, $w \in A_{p,\mathfrak{B}}$. It follows from (3.1) that

$$
B(w^s)^{1/s} \leq CB(w^{-1/(p-1)})^{1-p} \leq CB(w),
$$

where we have used $1 \leq B(w)B(w^{-1/(p-1)})^{p-1}$. □

Lemma 3.2. *Let* $p \geq 2$. *If* $u \in A_{p,\mathfrak{B}}$ *and* $v \in A_{p',\mathfrak{B}}$, *then there exists a constant* $C > 0$ *such that, for every* $B \in \mathfrak{B}$,

$$
(3.2) \qquad\qquad B(u)^{1/p}B(v)^{1/p'} \leq CB(u^{1/p}v^{1/p'}).
$$

Proof. Fix a $B \in \mathfrak{B}$. Then there is a constant C such that

$$
B(u)B(u^{1-p'})^{p-1} \leq C, \qquad B(v)B(v^{1-p})^{p'-1} \leq C.
$$

Raise the first equation to the power $1/p$ and the second to the power $1/p'$ and multiply them together. Then after re-arranging terms we get

$$
B(u)^{1/p}B(v)^{1/p'} \leq CB(u^{1-p'})^{-1/p'}B(v^{1-p})^{-1/p}.
$$

Since $p/p' = p-1$ and $p'/p = p'-1$, by Hölder's inequality

$$
B(u^{-1/p}v^{-1/p'}) \leq B(u^{1-p'})^{1/p'}B(v^{1-p})^{1/p}.
$$

Therefore, by combining these two inequalities we see that

$$
B(u)^{1/p}B(v)^{1/p'} \leq CB(u^{-1/p}v^{-1/p'})^{-1}.
$$

By the use of the inequality $1 \leq B(f)B(f^{1-p'})^{p-1}$,

$$
B(u^{-1/p}v^{-1/p'})^{-1} \leq B((u^{-1/p}v^{-1/p'})^{1-p'})^{p-1} = B((u^{1/p}v^{1/p'})^{p'-1})^{p-1}.
$$

Noticing $p \geq 2$, by applying Hölder's inequality the above expression can be bounded above by $B(u^{1/p}v^{1/p'})$. Therefore (3.2) holds. $\qquad \square$

Lemma 3.3. *Let $1 < p < \infty$. If $u \in A_{\infty,\mathfrak{B}} \cap \mathrm{RH}_{p,\mathfrak{B}}$ and $v \in A_{\infty,\mathfrak{B}} \cap \mathrm{RH}_{p',\mathfrak{B}}$, then there exists a constant C such that, for every $B \in \mathfrak{B}$,*

$$(3.3) \qquad B(u^p)^{1/p}B(v^{p'})^{1/p'} \leq CB(uv).$$

Proof. By the inclusion property of the class $A_{p,\mathfrak{B}}$ we may assume that

$$u \in A_{q_1,\mathfrak{B}}, \quad v \in A_{r_1,\mathfrak{B}} \quad \text{with} \quad p(r_1 - 1) = p'(q_1 - 1) = s > 1.$$

Since $u \in \mathrm{RH}_{p,\mathfrak{B}}$ and $v \in \mathrm{RH}_{p',\mathfrak{B}}$, then, by Lemma 3.1,

$$u^p \in A_{q,\mathfrak{B}}, \quad v^{p'} \in A_{r,\mathfrak{B}} \quad \text{with} \quad q = p(q_1 - 1) + 1, \quad r = p'(r_1 - 1) + 1.$$

Since $q = s(p - 1)$ and $r = s(p' - 1)$, by Lemma 3.1 again, this is equivalent to

$$u^{p/s} \in A_{p,\mathfrak{B}} \cap \mathrm{RH}_{s,\mathfrak{B}}, \quad v^{p'/s} \in A_{p',\mathfrak{B}} \cap \mathrm{RH}_{s,\mathfrak{B}}.$$

This relation and Lemma 3.2 yield

$$B(u^p)^{1/sp}B(v^{p'})^{1/sp'} \leq CB(u^{p/s})^{1/p}B(v^{p'/s})^{1/p'} \leq CB((uv)^{1/s}) \leq CB(uv)^{1/s}.$$

If we raise both sides to the power s, we get inequality (3.3). $\qquad \square$

Theorem 3.4. *If the weight w is in $\mathrm{RH}_{\infty,\mathfrak{B}}$, then w^r is in $\mathrm{RH}_{\infty,\mathfrak{B}}$ for $r \geq 1$. If the weight w is in $A_{\infty,\mathfrak{B}} \cap \mathrm{RH}_{\infty,\mathfrak{B}}$, then w^r is in $\mathrm{RH}_{\infty,\mathfrak{B}}$ for $0 < r < 1$.*

Proof. Let $r \geq 1$. Hölder's inequality gives, for any $B \in \mathfrak{B}$ and any $x \in B$,

$$w(x)^r \leq CB(w)^r \leq CB(w^r).$$

Let $0 < r < 1$. Assuming $w \in A_{q,\mathfrak{B}}$ for some large $q > 1$ we have by Lemma 3.1

$$w^r \in A_{p,\mathfrak{B}} \cap \mathrm{RH}_{\mathfrak{B},1/r} \quad \text{with} \quad q = \frac{1}{r}(p - 1) + 1 \quad \text{and} \quad p > 1.$$

Thus,

$$w(x)^r \leq CB(w)^r \leq CB(w^r). \qquad \square$$

Theorem 3.5. *If u and v belong to $A_{\infty,\mathfrak{B}} \cap \mathrm{RH}_{\infty,\mathcal{B}}$, then uv belongs to $\mathrm{RH}_{\infty,\mathfrak{B}}$.*

Proof. Fix a $B \in \mathfrak{B}$ and an $x \in B$. Then

$$u(x)v(x) \leq CB(u)B(v) \leq CB(u^2)^{1/2}B(v^2)^{1/2}.$$

Since u and v belong to $A_{\infty,\mathfrak{B}} \cap \mathrm{RH}_{2,\mathfrak{B}}$, by Lemma 3.3, the above expression can be bounded above by $CB(uv)$. This means that $uv \in \mathrm{RH}_{\infty,\mathfrak{B}}$. $\qquad \square$

Theorem 3.6. *Fix $p > 1$. If w is in $A_{1,\mathfrak{B}}$, then w^{1-p} is in $A_{p,\mathfrak{B}} \cap \mathrm{RH}_{\infty,\mathfrak{B}}$. Conversely, if w is in $A_{p,\mathfrak{B}} \cap \mathrm{RH}_{\infty,\mathfrak{B}}$, then $w^{1-p'}$ is in $A_{1,\mathfrak{B}}$.*

Proof. That $w \in A_{1,\mathfrak{B}}$ implies $w \in A_{p',\mathfrak{B}}$. So, $w^{1-p} \in A_{p,\mathfrak{B}}$. It follows for $B \in \mathfrak{B}$ that

$$1 \leq B(w)B(w^{1-p})^{1/(p-1)} \leq B(w^{1-p})^{1/(p-1)} \operatorname*{ess\,inf}_{x \in B} w(x)$$

and hence

$$\operatorname*{ess\,sup}_{x \in B} w(x)^{1-p} \leq B(w^{1-p}).$$

This means that $w^{1-p} \in A_{p,\mathfrak{B}} \cap \mathrm{RH}_{\infty,\mathfrak{B}}$. Conversely, if $w \in A_{p,\mathfrak{B}} \cap \mathrm{RH}_{\infty,\mathfrak{B}}$, then, for $B \in \mathfrak{B}$,

$$\operatorname*{ess\,sup}_{x \in B} w(x)B(w^{1-p'})^{p-1} \leq CB(w)B(w^{1-p'})^{p-1} \leq C < \infty.$$

This implies

$$B(w^{1-p'}) \le C \operatorname*{ess\,inf}_{x \in B} w(x)^{1-p'},$$

which means that $w^{1-p'} \in A_{1,\mathfrak{B}}$.   □

Remark 3.7. Suppose that $M_{\mathfrak{B}} \colon L^p(\mathbf{R}^n) \to L^p(\mathbf{R}^n)$ for some $p > 1$. Then, using the well-known Rubio de Francia algorithm, one can produce many $A_{1,\mathfrak{B}}$-weights. Theorem 3.6 asserts that, if w is in $A_{1,\mathfrak{B}}$, then w^{-1} is in $A_{\infty,\mathfrak{B}} \cap \mathrm{RH}_{\infty,\mathfrak{B}}$. Hence, we have many weights belonging to $A_{\infty,\mathfrak{B}} \cap \mathrm{RH}_{\infty,\mathfrak{B}}$.

4. The classes $A_{\infty,\mathfrak{B}}$ and $\mathrm{RH}_{1,\mathfrak{B}}$

In this section we discuss the equivalence between the classes $A_{\infty,\mathfrak{B}}$ and $\mathrm{RH}_{1,\mathfrak{B}}$ when the basis $\mathfrak{B}$ has dyadic substructure with the Stein property. The following definition expresses the relevant property of a basis needed for our purposes. In the case when $\mathfrak{B} = \mathfrak{D}$, the usual collection of dyadic cubes in $\mathbf{R}^n$, it was obtained by Stein in [20].

Definition 4.1. We say that a basis $\mathfrak{B}$ satisfies the Stein property if and only if there exists a constant c such that, for any non-negative function $f \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$, $B \in \mathfrak{B}$ and $\lambda > B(f)$, we have

$$(4.1) \qquad \int_{\{x \in B \colon f(x) > \lambda\}} f(x)\,dx \le c\lambda |\{x \in B \colon M_{\mathfrak{B}}[f\mathbf{1}_B](x) > \lambda\}|.$$

Under the assumption of the Stein property, Lerner and Ombrosi established the self-improving property of general maximal operators on quasi-Banach function spaces [12].

Definition 4.2. We say that the basis $\mathfrak{B}$ has a dyadic substructure if and only if, for all $B \in \mathfrak{B}$, there exists a dyadic subset $\mathfrak{D}(B) \subset \mathfrak{B}$ that satisfies the following:
  (1) If $Q, R \in \mathfrak{D}(B)$ then $Q \cap R \in \{Q, R, \emptyset\}$;
  (2) $B \in \mathfrak{D}(B)$ and B is maximal in $\mathfrak{D}(B)$ with respect to inclusion.
The family $\mathfrak{D}(B)$ is referred to as a dyadic basis. We further assume that the basis $\mathfrak{D}(B)$ is unique and it is a density basis. That is, for all $B \in \mathfrak{B}$ and any $P \in \mathfrak{D}(B)$, we have $\mathfrak{D}(P) = \{P \cap Q \colon Q \in \mathfrak{D}(B)\}$. The basis $\mathfrak{D}(B)$ is called a density basis if it differentiates $L^\infty(B)$, namely, if for every $f \in L^\infty(B)$ and for almost every $x \in B$, there exists a sequence $\{Q_k\} \subset \mathfrak{D}(B)$ such that $Q_k \ni x$ and $\lim_{k \to \infty} Q_k(f) = f(x)$.

Definition 4.3. We say that the basis $\mathfrak{B}$ has dyadic substructure with the Stein property if and only if $\mathfrak{B}$ has dyadic substructure and each basis $\mathfrak{D}(B)$ satisfies the Stein property with uniform constant c for varying $B \in \mathfrak{B}$.

We first verify some lemmas.

Lemma 4.4. *Suppose that the basis $\mathfrak{B}$ has dyadic substructure with the Stein property. Let f be a non-negative, locally integrable function on $\mathbf{R}^n$ and $B \in \mathfrak{B}$. Let $P, Q \in \mathfrak{D}(B)$ such that $P \subset Q$. Assume that*

$$P(f) > \lambda \quad and \quad Q(f) \le \lambda.$$

Then $\lambda < P(f) \le c\lambda$.

Proof. We shall apply the Stein property to Q, $g \equiv P(f)\mathbf{1}_P$ and λ. This is justified by the fact that

$$Q(g) = Q(P(f)\mathbf{1}_P) \le Q(f) \le \lambda.$$

Thus,

$$\int_P g\,dx = \int_{\{x\in Q\,:\,g(x)>\lambda\}} g(x)\,dx \le c\lambda|\{x\in Q\colon M_{\mathfrak{D}(Q)}g(x)>\lambda\}| = c\lambda|P|,$$

where we have used the facts that $M_{\mathfrak{D}(Q)}g(x) = Q(f) \le \lambda$, when $x \in Q \setminus P$, and $M_{\mathfrak{D}(Q)}g(x) = P(f) > \lambda$, when $x \in P$. This proves $P(f) \le c\lambda$. $\qquad\square$

Lemma 4.5. *Suppose that the basis $\mathfrak{B}$ has dyadic substructure with the Stein property. Let $B \in \mathfrak{B}$.*

(1) *If the weight w is in $A_{p,\mathfrak{D}(B)}$, $1 < p < \infty$, then there exists a sequence of weights $\{w_k\} \in A_{p,\mathfrak{D}(B)} \cap L^\infty(B)$ such that $\lim_{k\to\infty} w_k = w$ a.e.*

(2) *If the weight w is in $\mathrm{RH}_{p,\mathfrak{D}(B)}$, $1 < p < \infty$, then there exists a sequence of weights $\{w_k\} \in \mathrm{RH}_{p,\mathfrak{D}(B)} \cap L^\infty(B)$ such that $\lim_{k\to\infty} w_k = w$ a.e.*

Proof. Fix a $w \in A_{p,\mathfrak{B}}$. We perform the Calderón–Zygmund decomposition of w at level k, $k \gg 1$, with respect to $\mathfrak{D}(B)$. Namely, let $\{Q_{k,j}\}_j \subset \mathfrak{D}(B)$ be the collection of dyadic bases in $\mathfrak{D}(B)$ which are maximal among the $Q \in \mathfrak{D}(B)$ that satisfy $Q(w) > k$. The maximality of $Q_{k,j}$ yields that the sets $Q_{k,j}$ are pairwise disjoint for varying j and, by Lemma 4.4, that $Q_{k,j}(w) \le ck$. We now let

$$w_k = w\mathbf{1}_{E_k} + \sum_j Q_{k,j}(w)\mathbf{1}_{Q_{k,j}},$$

where $E_k = B \setminus \bigcup_j Q_{k,j}$. Then w_k is in $L^\infty(B)$, because the basis $\mathfrak{D}(B)$ is a density basis, and w_k converges to w as $k \to \infty$, because

$$\sum_j |Q_{k,j}| \le \frac{w(B)}{k}.$$

To verify $w_k \in A_{p,\mathfrak{B}}$, we need only observe that

$$1 \le Q_{k,j}(w)Q_{k,j}(w^{-1/(p-1)})^{p-1}$$

and hence

$$Q_{k,j}(w)^{-1/(p-1)} \le Q_{k,j}(w^{-1/(p-1)}).$$

When $w \in \mathrm{RH}_{p,\mathfrak{B}}$, we need only also observe that

$$Q_{k,j}(w)^p \le Q_{k,j}(w^p). \qquad\square$$

A well known result in the theory of weighted norm inequalities due to Gehring (cf. [7]) states that if $w \in \mathrm{RH}_{p,\mathfrak{Q}}$, then there exists $\varepsilon > 0$ such that $w \in \mathrm{RH}_{p+\varepsilon,\mathfrak{Q}}$. Here, the symbol $\mathfrak{Q}$ denotes the collection of all cubes in $\mathbf{R}^n$ with sides parallel to the coordinate axes. We now extend this lemma to the general basis $\mathfrak{B}$.

Theorem 4.6. *Suppose that the basis $\mathfrak{B}$ has dyadic substructure with the Stein property. Then, if $w \in \mathrm{RH}_{p,\mathfrak{B}}$ for some $1 < p < \infty$, there exists $\varepsilon > 0$ such that $w \in \mathrm{RH}_{p+\varepsilon,\mathfrak{B}}$.*

Proof. Fix a $w \in \mathrm{RH}_{p,\mathfrak{B}}$. Given $B \in \mathfrak{B}$, we shall prove that the inequality

$$\left(\frac{1}{|B|}\int_B w(x)^{p(1+\delta)}\,dx\right)^{1/(p(1+\delta))} \le CB(w)$$

holds for some $C > 0$ and $\delta > 0$ independent of B. By a standard limiting argument based upon Lemma 4.5, we may assume without loss of generality that w is in $L^\infty(B)$.

We now let $a = B(w^p)$. It follows from Fubini's theorem that

$$\frac{1}{|B|} \int_B w(x)^{p(1+\delta)} \, dx = \frac{\delta}{|B|} \int_0^\infty w^p(\{x \in B : w(x)^p > \lambda\}) \lambda^{\delta-1} \, d\lambda$$

$$\leq a^{1+\delta} + \frac{\delta}{|B|} \int_a^\infty w^p(\{x \in B : w(x)^p > \lambda\}) \lambda^{\delta-1} \, d\lambda.$$

The Stein property yields

$$\frac{\delta}{|B|} \int_a^\infty w^p(\{x \in B : w(x)^p > \lambda\}) \lambda^{\delta-1} \, d\lambda$$

$$\leq c \frac{\delta}{|B|} \int_a^\infty |\{x \in B : M_{\mathfrak{D}(B)}[w^p \mathbf{1}_B](x) > \lambda\}| \lambda^\delta \, d\lambda$$

$$\leq c \frac{\delta}{(1+\delta)|B|} \int_B M_{\mathfrak{D}(B)}[w^p \mathbf{1}_B](x)^{1+\delta} \, dx.$$

We notice that

$$M_{\mathfrak{D}(B)}[w^p \mathbf{1}_B](x) \leq [w]_{\mathrm{RH}_{p,\mathfrak{B}}} M_{\mathfrak{D}(B)}[w \mathbf{1}_B](x)^p.$$

Thus,

$$\frac{\delta}{|B|} \int_a^\infty w^p(\{x \in B : w(x)^p > \lambda\}) \lambda^{\delta-1} \, d\lambda$$

$$\leq \frac{C\delta}{(1+\delta)|B|} \int_B M_{\mathfrak{D}(B)}[w \mathbf{1}_B](x)^{p(1+\delta)} \, dx$$

$$\leq \frac{C\delta}{1+\delta} \| M_{\mathfrak{D}(B)} \|_{L^{p(1+\delta)}(\mathbf{R}^n)}^{p(1+\delta)} \frac{1}{|B|} \int_B w(x)^{p(1+\delta)} \, dx.$$

By using an elementary fact

$$\lim_{\delta \to +0} \| M_{\mathfrak{D}(B)} \|_{L^{p(1+\delta)}(\mathbf{R}^n)}^{p(1+\delta)} = (p')^p,$$

we can select $\delta > 0$ so that

$$\frac{C\delta}{1+\delta} \| M_{\mathfrak{D}(B)} \|_{L^{p(1+\delta)}(\mathbf{R}^n)}^{p(1+\delta)} = \frac{1}{2}.$$

Hence, notice that the integral is finite, since w is in $L^\infty(B)$ and B is a bounded set,

$$\frac{1}{|B|} \int_B w(x)^{p(1+\delta)} \, dx \leq 2a^{1+\delta} \leq 2(CB(w))^{p(1+\delta)},$$

which is our desired inequality. $\square$

The next theorem (Theorem 4.8) states that $A_{\infty,\mathfrak{B}} \subset \mathrm{RH}_{1,\mathfrak{B}}$. To prove the theorem, we need the following lemma, which is well-known, but, for the reader's convenience, we will quote the proof from the lecture notes of Hytönen "Weighted norm inequalities", since now we are not able to get this from his website.

Lemma 4.7. *Let $\mathfrak{D}$ be a countable collection of measurable subsets of $\mathbf{R}^n$ with the property that*

$$\forall Q, R \in \mathfrak{D} : Q \cap R \in \{Q, R, \emptyset\}.$$

Let μ be a positive measure on $\mathbf{R}^n$ and f be a locally μ-integrable function on $\mathbf{R}^n$. Define the maximal operator $M_{\mathfrak{D},\mu}^0$ by

$$M_{\mathfrak{D},\mu}^0 f(x) = \sup_{Q \in \mathfrak{D}} \mathbf{1}_Q(x) \exp\left(\frac{1}{\mu(Q)} \int_Q \log |f| \, d\mu \right).$$

Then, for all $p \in (0, \infty]$,

$$\|M_{\mathfrak{D},\mu}^0 f\|_{L^p(\mu)} \le e^{1/p} \|f\|_{L^p(\mu)}.$$

Proof. Define the maximal operator $M_{\mathfrak{D},\mu}$ by

$$M_{\mathfrak{D},\mu} f(x) = \sup_{Q \in \mathfrak{D}} \mathbf{1}_Q(x) \frac{1}{\mu(Q)} \int_Q |f| \, d\mu.$$

By Jensen's inequality and the basic properties of the logarithm, we have

$$M_{\mathfrak{D},\mu}^0 f \le M_{\mathfrak{D},\mu} f, \quad (M_{\mathfrak{D},\mu}^0 f)^p = (M_{\mathfrak{D},\mu}^0 [|f|^{p/q}])^q \le (M_{\mathfrak{D},\mu}[|f|^{p/q}])^q, \quad q \in (0, \infty).$$

By the $L^q(\mu)$ boundedness of $M_{\mathfrak{D},\mu}$ for $q > 1$, we have

$$\int_{\mathbf{R}^n} (M_{\mathfrak{D},\mu}^0 f)^p \, d\mu \le \int_{\mathbf{R}^n} (M_{\mathfrak{D},\mu}[|f|^{p/q}])^q \, d\mu \le (q')^q \int_{\mathbf{R}^n} (|f|^{p/q})^q \, d\mu = (q')^q \int_{\mathbf{R}^n} |f|^p \, d\mu.$$

As $q \to \infty$, we have $(q')^q \to e$, and hence

$$\|M_{\mathfrak{D},\mu}^0 f\|_{L^p(\mu)}^p \le e\|f\|_{L^p(\mu)}^p \quad \text{for} \quad p \in (0, \infty).$$

It is trivial for the case $p = \infty$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Theorem 4.8. *Suppose that the basis $\mathfrak{B}$ has a dyadic substructure with the Stein property. Then, if $w \in A_{\infty,\mathfrak{B}}$, there exists $\varepsilon > 0$ such that $w \in \mathrm{RH}_{1+\varepsilon,\mathfrak{B}}$.*

Proof. Fix a $w \in A_{\infty,\mathfrak{B}}$, that is, fix $w \in A_{p,\mathfrak{B}}$ for some large p. Given $B \in \mathfrak{B}$, we shall prove that the inequality

$$\left(\frac{1}{|B|} \int_B w(x)^{1+\delta} \, dx \right)^{1/(1+\delta)} \le C B(w)$$

holds for some $C > 0$ and $\delta > 0$ independent of B. In the same manner as above we may assume that w is in $L^\infty(B)$.

We now let $a = B(w)$. It follows from Fubini's theorem that

$$\frac{1}{|B|} \int_B w(x)^{1+\delta} \, dx = \frac{\delta}{|B|} \int_0^\infty w(\{x \in B : w(x) > \lambda\}) \lambda^{\delta-1} \, d\lambda$$

$$\le a^{1+\delta} + \frac{\delta}{|B|} \int_a^\infty w(\{x \in B : w(x) > \lambda\}) \lambda^{\delta-1} \, d\lambda.$$

By the Stein property

$$\frac{\delta}{|B|} \int_a^\infty w(\{x \in B : w(x) > \lambda\}) \lambda^{\delta-1} \, d\lambda$$

$$\le c \frac{\delta}{|B|} \int_a^\infty |\{x \in B : M_{\mathfrak{D}(B)}[w\mathbf{1}_B](x) > \lambda\}| \lambda^\delta \, d\lambda$$

$$\le c \frac{\delta}{(1+\delta)|B|} \int_B M_{\mathfrak{D}(B)}[w\mathbf{1}_B](x)^{1+\delta} \, dx.$$

We notice that,

$$M_{\mathfrak{D}(B)}[w\mathbf{1}_B](x) \le [w]_{A_{p,\mathfrak{B}}} M_{\mathfrak{D}(B),dx}^0[w\mathbf{1}_B](x).$$

Thus, by Lemma 4.7,

$$\frac{\delta}{|B|} \int_a^\infty w(\{x \in B : w(x) > \lambda\}) \lambda^{\delta-1} \, d\lambda$$

$$\le C \frac{\delta}{(1+\delta)|B|} \int_B M_{\mathfrak{D}(B),dx}^0[w\mathbf{1}_B](x)^{1+\delta} \, dx \le \frac{Ce\delta}{1+\delta} \frac{1}{|B|} \int_B w(x)^{1+\delta} \, dx.$$

We select $\delta > 0$ so that

$$\frac{Ce\delta}{1+\delta} = \frac{1}{2}.$$

Hence,

$$\frac{1}{|B|} \int_B w(x)^{1+\delta}\,dx \le 2a^{1+\delta} = 2B(w)^{1+\delta},$$

which is our desired inequality. $\qquad\square$

To verify the converse relation $\mathrm{RH}_{1,\mathfrak{B}} \subset A_{\infty,\mathfrak{B}}$, we must assume that w satisfies the following assumption.

Assumption 4.9. Let w be a weight. Suppose that $\mathfrak{B}$ has a dyadic substructure. There exists a uniform constant c for varying $B \in \mathfrak{B}$ such that

$$(4.2) \qquad \int_{\{x \in B\colon f(x) > \lambda\}} f(x) w(x)\,dx \le c\lambda w(\{x \in B\colon M_{\mathfrak{D}(B),w}[f\mathbf{1}_B](x) > \lambda\})$$

holds for any non-negative function $f \in L^1_{\mathrm{loc}}(wdx)$, $B \in \mathfrak{B}$ and $\lambda > w(B)^{-1} \int_B f\,wdx$.

Theorem 4.10. *Suppose that w satisfies Assumption 4.9. Then, if $w \in \mathrm{RH}_{s,\mathfrak{B}}$ for some $1 < s < \infty$, there exists $p > 1$ such that $w \in A_{p,\mathfrak{B}}$.*

Proof. If $w \in \mathrm{RH}_{s,\mathfrak{B}}$, then for any $B \in \mathfrak{B}$

$$B(w^s)^{1/s} \le [w]_{\mathrm{RH}_{s,\mathfrak{B}}} B(w).$$

This implies that

$$\left(\frac{1}{w(B)} \int_B w(x)^{-1} w(x)\,dx\right) \left(\frac{1}{w(B)} \int_B (w(x)^{-1})^{1-s} w(x)\,dx\right)^{s'-1} \le [w]^{s'}_{\mathrm{RH}_{s,\mathfrak{B}}},$$

which means that

$$w^{-1} \in A^w_{s',\mathfrak{B}},$$

where the class $A^w_{s',\mathfrak{B}}$ is defined by the Muckenhoupt class replacing the Lebesgue measure dx by wdx. Thanks to Assumption 4.9 and Lemma 4.7, the same manipulation as in the proof of Theorem 4.8 gives us that the inequality

$$\left(\frac{1}{w(B)} \int_B (w(x)^{-1})^{1+\varepsilon} w(x)\,dx\right)^{1/(1+\varepsilon)} \le C\frac{|B|}{w(B)}$$

holds for some $C > 0$ and $\varepsilon > 0$ independent of B. A calculation shows that

$$\left(\frac{w(B)}{|B|}\right)^{\varepsilon/(1+\varepsilon)} \left(\frac{1}{|B|} \int_B w(x)^{-\varepsilon}\,dx\right)^{1/(1+\varepsilon)} \le C$$

and that

$$B(w)B(w^{-\varepsilon})^{1/\varepsilon} \le C^{(1+\varepsilon)/\varepsilon},$$

which means $w \in A_{p,\mathfrak{B}}$ with $1/(p-1) = \varepsilon$. $\qquad\square$

In the remainder of this section we discuss an equivalent way of defining the reverse Hölder class $\mathrm{RH}_{1,\mathfrak{B}}$ (Theorem 4.12) following the papers [9] and [15].

Definition 4.11. A function $\Phi\colon [0,\infty) \to [0,\infty]$ is said to be a Young function if it is left-continuous, convex and increasing, and if $\Phi(0) = 0$ and $\Phi(t) \to \infty$ as $t \to \infty$. Given a Young function Φ, define the mean Luxemburg norm of f on a finite set $B \subset \mathbf{R}^n$ by

$$\|f\|_{\Phi,B} = \inf\left\{\lambda > 0\colon \frac{1}{|B|} \int_B \Phi\left(\frac{|f(x)|}{\lambda}\right)\,dx \le 1\right\}.$$

If $\Phi(t) = t \log(e + t)$, we write the mean Luxemburg norm as $\|f\|_{L\log L, B}$.

Theorem 4.12. *Suppose that the basis $\mathfrak{B}$ has a dyadic substructure with the Stein property. Then w is in $\mathrm{RH}_{1,\mathfrak{B}}$ if and only if*

$$(4.3) \qquad [w]_{\mathrm{RH}_{1,\mathfrak{B}}} = \sup_{B \in \mathfrak{B}} \frac{\|w\|_{L\log L, B}}{B(w)} < \infty.$$

Before proving Theorem 4.12, we show two equivalent quantities to the quantity $[w]_{1,\mathfrak{B}}$.

Proposition 4.13. *The following two quantities are equivalent to the quantity $[w]_{1,\mathfrak{B}}$:*

$$(4.4) \qquad [w]'_{\mathrm{RH}_{1,\mathfrak{B}}} = \sup_{B \in \mathfrak{B}} \frac{1}{w(B)} \int_B w(x) \log\left(e + \frac{w(x)}{B(w)}\right) dx$$

and

$$(4.5) \qquad [w]''_{\mathrm{RH}_{1,\mathfrak{B}}} = \sup_{B \in \mathfrak{B}} \frac{1}{w(B)} \int_B M_{\mathfrak{D}(B)} w(x) \, dx.$$

The quantity $[w]''_{\mathrm{RH}_{1,\mathfrak{B}}}$ is known as Fujii–Wilson constant; see [9].

Proof of Proposition 4.13. If $[w]'_{\mathrm{RH}_{1,\mathfrak{B}}} = C_2 < \infty$, then we have

$$\frac{1}{|B|} \int_B \frac{w(x)}{B(w)} \log\left(e + \frac{w(x)}{B(w)}\right) dx = \frac{1}{w(B)} \int_B w(x) \log\left(e + \frac{w(x)}{B(w)}\right) dx < C_2.$$

We notice that, for any Young function Φ, $\Phi(\theta t) \le \theta \Phi(t)$ holds for all $0 < \theta < 1$.
This fact and $C_2 > 1$ imply

$$\frac{1}{|B|} \int_B \frac{w(x)}{C_2 B(w)} \log\left(e + \frac{w(x)}{C_2 B(w)}\right) dx \le 1$$

and, by the definition of the mean Luxemburg norm,

$$\|w\|_{L\log L, B} \le C_2 B(w),$$

which means that $[w]_{\mathrm{RH}_{1,\mathfrak{B}}} \le C_2$.
If $[w]_{\mathrm{RH}_{1,\mathfrak{B}}} = C_1 < \infty$, then we have

$$\|w\|_{L\log L, B} \le C_1 B(w).$$

By the definition of the mean Luxemburg norm, we see that

$$\frac{1}{|B|} \int_B \frac{w(x)}{C_1 B(w)} \log\left(e + \frac{w(x)}{C_1 B(w)}\right) dx \le 1.$$

Noticing that $C_1 > 1$ and

$$\frac{1}{C_1}\left(e + \frac{w(x)}{B(w)}\right) \le \left(e + \frac{w(x)}{C_1 B(w)}\right),$$

we have

$$1 \ge \frac{1}{wB)} \int_B \frac{w(x)}{C_1} \log\left(e + \frac{w(x)}{B(w)}\right) dx - \frac{\log C_1}{C_1}$$

and

$$\frac{1}{wB)} \int_B w(x) \log\left(e + \frac{w(x)}{B(w)}\right) dx \le 2C_1,$$

which means that $[w]'_{\mathrm{RH}_{1,\mathfrak{B}}} \le 2C_1$.

The equivalence of the second and third quantities holds from the following lemma.

Lemma 4.14. *Suppose that the basis $\mathfrak{B}$ has a dyadic substructure with the Stein property. Then, for any $B \in \mathfrak{B}$ and any measurable function w, we have*

(4.6)
$$\frac{1}{1+c} \int_B w(x) \log\left(e + \frac{w(x)}{B(w)}\right) dx \leq \int_B M_{\mathfrak{D}(B)}[w\mathbf{1}_B](x)\, dx$$
$$\leq 10 \int_B w(x) \log\left(e + \frac{w(x)}{B(w)}\right) dx.$$

Proof. Fix a $B \in \mathfrak{B}$. By homogeneity, we may assume that $B(w) = 1$. It follows from Fubini's theorem that

$$\int_B w \log(e + w)\, dx = \int_0^\infty w(\{x \in B \colon w(x) > \lambda\}) \frac{d\lambda}{e + \lambda} = \int_0^1 \cdots + \int_1^\infty \cdots = I + II.$$

There holds

$$I \leq w(B) \leq \int_B M_{\mathfrak{D}(B)}[w\mathbf{1}_B]\, dx,$$

because $\mathfrak{D}(B)$ is a density basis, and by the Stein property

$$II = \int_1^\infty w(\{x \in B \colon w(x) > \lambda\}) \frac{d\lambda}{e + \lambda}$$
$$\leq c \int_1^\infty |\{x \in B \colon M_{\mathfrak{D}(B)}[w\mathbf{1}_B](x) > \lambda\}| \frac{\lambda}{e + \lambda}\, d\lambda$$
$$\leq c \int_0^\infty |\{x \in B \colon M_{\mathfrak{D}(B)}[w\mathbf{1}_B](x) > \lambda\}|\, d\lambda = c \int_B M_{\mathfrak{D}(B)}[w\mathbf{1}_B](x)\, dx.$$

This gives the first inequality in (4.6). For the second inequality, we invoke the elementary inequality

(4.7) $|\{x \in B \colon M_{\mathfrak{D}(B)}[w\mathbf{1}_B](x) > \lambda\}| \leq \frac{2}{\lambda} w(\{x \in B \colon w(x) > \lambda/2\}), \quad \lambda > 0.$

This implies

$$\int_2^\infty |\{x \in B \colon M_{\mathfrak{D}(B)}[w\mathbf{1}_B](x) > \lambda\}|\, d\lambda \leq 2 \int_2^\infty w(\{x \in B \colon w(x) > \lambda/2\}) \frac{d\lambda}{\lambda}$$
$$= 2 \int_1^\infty w(\{x \in B \colon w(x) > \lambda\}) \frac{d\lambda}{\lambda}$$
$$\leq 8 \int_1^\infty w(\{x \in B \colon w(x) > \lambda\}) \frac{d\lambda}{e + \lambda}$$
$$\leq 8 \int_B w \log(e + w)\, dx.$$

Noticing

$$\int_0^2 |\{x \in B \colon M_{\mathfrak{D}(B)}[w\mathbf{1}_B](x) > \lambda\}|\, d\lambda \leq 2|B| = 2w(B) \leq 2 \int_B w \log(e + w)\, dx,$$

we obtain the second inequality in (4.6).   □

To prove Theorem 4.12, we need one more lemma.

Lemma 4.15. *Let* $[w]'_{\mathrm{RH}_{1,\mathfrak{B}}} = C_2 < \infty$ *and* $B \in \mathfrak{B}$. *Then for any measurable subset* $S \subset B$ *and* $\lambda > 0$, *we have*

(4.8) $$\frac{|S|}{|B|} < e^{-\lambda} \quad \text{implies} \quad \frac{w(S)}{w(B)} \leq \frac{2C_2}{\lambda} + e^{-\lambda/2}.$$

Proof. Let

$$E_\lambda = \{x \in B \colon w(x) > e^\lambda B(w)\}.$$

Then

$$w(E_\lambda) \leq \frac{1}{\lambda} \int_B w(x) \log\left(e + \frac{w(x)}{B(w)}\right) dx \leq \frac{C_2}{\lambda} w(B)$$

by (4.4). Therefore,

$$w(S) = w(S \cap E_{\lambda/2}) + w(S \setminus E_{\lambda/2}) \leq \frac{2C_2}{\lambda} w(B) + e^{\lambda/2} B(w)|S|$$

$$\leq \frac{2C_2}{\lambda} w(B) + e^{\lambda/2} e^{-\lambda} w(B) \qquad \text{by the hypothesis in (4.8)}$$

$$= \frac{2C_2}{\lambda} w(B) + e^{-\lambda/2} w(B).$$

This proves (4.8). $\qquad\square$

Remark 4.16. If λ is sufficiently large, then we obtain $\alpha = e^{-\lambda} < 1$ and $\beta = \frac{2C_2}{\lambda} + e^{-\lambda/2} < 1$. Thus, by (4.8)

(4.9) $$\frac{|S|}{|B|} < \alpha \quad \text{implies} \quad \frac{w(S)}{w(B)} \leq \beta.$$

Proof of Theorem 4.12. Fix a $w \in \mathrm{RH}_{p,\mathfrak{B}}$ for some $1 < p < \infty$. The fact that $t \log(e + t) \leq (1 + e)t^p$, $t > 1$, implies that

$$[w]_{\mathrm{RH}_{1,\mathfrak{B}}} \leq (1 + e)[w]_{\mathrm{RH}_{p,\mathfrak{B}}} < \infty.$$

We shall prove the converse. We follow the interesting argument in [15, p.18 proof of Theorem 5.4]. Let $[w]'_{\mathrm{RH}_{1,\mathfrak{B}}} = C_2 < \infty$. We have to prove that, given $B \in \mathfrak{B}$, the inequality

$$\left(\frac{1}{|B|} \int_B w(x)^{1+\delta} dx\right)^{1/(1+\delta)} \leq CB(w)$$

holds for some $C > 0$ and $\delta > 0$ independent of B. Observe that by homogeneity we can assume that $B(w) = 1$. In the same manner as in the proof of Lemma 4.5 based upon (4.5), we may assume without loss of generality that w is in $L^\infty(B)$.

Set $a > 1$ so that $\frac{c}{a} = \alpha$, where c is the constant appearing in the definition of the Stein property and α is the constant in Remark 4.16. It follows that

$$\int_B w(x)^{1+\delta} dx \leq \int_B M_{\mathfrak{D}(B)}[w\mathbf{1}_B](x)^\delta w(x) dx$$

$$= \delta \int_0^\infty w(\{x \in B \colon M_{\mathfrak{D}(B)}[w\mathbf{1}_B](x) > \lambda\})\lambda^{\delta-1} d\lambda$$

$$\leq w(B) + \delta \sum_{k=0}^\infty \int_{c^k}^{c^{k+1}} w(\{x \in B \colon M_{\mathfrak{D}(B)}[w\mathbf{1}_B](x) > \lambda\})\lambda^{\delta-1} d\lambda$$

$$\leq |B| + \delta c^\delta \sum_{k=0}^\infty \int_{a^k}^{a^{k+1}} w(\{x \in B \colon M_{\mathfrak{D}(B)}[w\mathbf{1}_B](x) > a^k\})a^{k\lambda}\,\frac{d\lambda}{\lambda}$$

$$= |B| + \delta a^\delta \log a \sum_{k=0}^\infty a^{k\lambda} w(\Omega_k),$$

where

$$\Omega_k = \{x \in B \colon M_{\mathfrak{D}(B)}[w\mathbf{1}_B](x) > a^k\}.$$

Noticing $a^k > 1 = B(w)$, we can apply the Calderón–Zygmund decomposition of w at level a^k to obtain a family of maximal non-overlapping dyadic bases $\{Q_{k,j}\}_j \subset \mathfrak{D}(B)$ for which $\Omega_k = \bigcup_j Q_{k,j}$ and

$$(4.10) \qquad\qquad a^k < Q_{k,j}(w) \leq ca^k,$$

where we have used Lemma 4.4. Now,

$$\sum_{k=0}^\infty a^{k\delta} w(\Omega_k) = \sum_{k,j} a^{k\delta} w(Q_{k,j}) \leq \sum_{k,j} Q_{k,j}(w)^\delta w(Q_{k,j}).$$

Let $E_{k,j} = Q_{k,j} \backslash \Omega_{k+1}$ and $F_{k,j} = Q_{k,j} \cap \Omega_{k+1}$. Of course, we have $|Q_{k,j}| = |E_{k,j}| + |F_{k,j}|$. There holds by (4.10) that

$$|F_{k,j}| = \sum_{Q_{k+1,i} \subset Q_{k,j}} |Q_{k+1,i}| \leq \frac{1}{a^{k+1}} \sum_{Q_{k+1,i} \subset Q_{k,j}} w(Q_{k+1,i})$$

$$\leq \frac{1}{a^{k+1}} w(Q_{k,j}) \leq \frac{ca^k}{a^{k+1}}|Q_{k,j}| = \frac{c}{a}|Q_{k,j}| = \alpha|Q_{k,j}|.$$

This implies by (4.9)

$$\frac{w(F_{k,j})}{w(Q_{k,j})} \leq \beta$$

and hence

$$(1 - \beta)w(Q_{k,j}) \leq w(E_{k,j}).$$

We also notice that the sets $E_{k,j}$ are pairwise disjoint. Thus,

$$\sum_{k,j} Q_{k,j}(w)^\delta w(Q_{k,j}) \leq \frac{1}{1-\beta} \sum_{k,j} Q_{k,j}(w)^\delta w(E_{k,j})$$

$$\leq \frac{1}{1-\beta} \sum_{k,j} \int_{E_{k,j}} M_{\mathfrak{D}(B)}[w\mathbf{1}_B](x)^\delta w(x)\,dx$$

$$= \frac{1}{1-\beta} \int_B M_{\mathfrak{D}(B)}[w\mathbf{1}_B](x)^\delta w(x)\,dx.$$

Choosing δ so that

$$\frac{\delta a^\delta \log a}{1-\beta} = \frac{1}{2},$$

we obtain

$$\int_B w(x)^{1+\delta}\,dx \leq 2|B|,$$

which is our desired inequality. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

5. Volume formula

In this section we extend the nice formula (1.2) to higher dimensions. Throughout this section we always assume that the weight w is a radial weight: $w(x) = w_0(|x|)$, $x \in \mathbf{R}^n$, for some non-negative function w_0 on the half interval $[0, \infty)$. We suppose further that w_0 satisfies the supremum condition (1.1). When w_0 satisfies (1.1), it is not difficult to deduce that w_0 has the doubling condition:

$$(5.1) \qquad \int_{r_1}^{r_2} w_0(r)\, dr \le C \int_{r_1'}^{r_2'} w_0(r)\, dr$$

for all $0 \le r_1 \le r_1' \le r_2' \le r_2 < \infty$ with $r_2 - r_1 = 2(r_2' - r_1')$. For an $A \subset \mathbf{R}^n$ we set

$$r_1(A) = \inf_{x \in A} |x|, \quad r_2(A) = \sup_{x \in A} |x| \quad \text{and} \quad \text{rad}\,(A) = r_2(A) - r_1(A).$$

Let $\mathfrak{R}$ be the family of all rectangles in $\mathbf{R}^n$, $n \ge 2$, with sides parallel to the coordinate axes.

Lemma 5.1. *Let $R \in \mathfrak{R}$ be any rectangle. Then*

$$\frac{w(R)}{|R|} \approx \frac{1}{\text{rad}\,(R)} \int_{r_1(R)}^{r_2(R)} w_0(r)\, dr.$$

Proof. Passing to polar coordinates, we observe that

$$w(R) = \int_{r_1(R)}^{r_2(R)} \sigma(S_r \cap R) w_0(r)\, dr,$$

where S_r is sphere of radius r and centered at the origin and σ is its surface measure. Since w_0 satisfies the supremum condition (1.1), we get

$$w(R) \le \sup_{r_1(R) < r < r_2(R)} w_0(r) \int_{r_1(R)}^{r_2(R)} \sigma(R \cap S_r)\, dr \le C \frac{1}{\text{rad}\,(R)} \int_{r_1(R)}^{r_2(R)} w_0(r)\, dr \cdot |R|.$$

Thus, we shall prove the converse:

$$w(R) \ge C \frac{1}{\text{rad}\,(R)} \int_{r_1(R)}^{r_2(R)} w_0(r)\, dr \cdot |R|.$$

Because of the rotation invariance and the symmetry of the problem, we may assume that the rectangle R is of the form

$$R = \prod_{i=1}^{n} (a_i, b_i), \quad 0 < a_i < b_i < b_1 < \infty.$$

For $j = 2, \dots, n$, set $R_j = (a_1, b_1) \times (a_j, b_j)$. We now observe that

$$\begin{aligned}
\text{rad}\,(R) &= \sqrt{b_1^2 + b_2^2 + \cdots + b_n^2} - \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} \\
&= \frac{(b_1^2 + b_2^2 + \cdots + b_n^2) - (a_1^2 + a_2^2 + \cdots + a_n^2)}{\sqrt{b_1^2 + b_2^2 + \cdots + b_n^2} + \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}} \\
&\le \sum_{j=2}^{n} \frac{(b_1^2 + b_j^2) - (a_1^2 + a_j^2)}{\sqrt{b_1^2 + b_j^2} + \sqrt{a_1^2 + a_j^2}} \\
&= \sum_{j=2}^{n} \left(\sqrt{b_1^2 + b_j^2} - \sqrt{a_1^2 + a_j^2} \right) = \sum_{j=2}^{n} \text{rad}\,(R_j).
\end{aligned}$$

By this observation, we may further assume that

$$(5.2) \qquad\qquad \operatorname{rad}(R) \leq (n-1)\operatorname{rad}(R_2).$$

We now verify that

$$(5.3) \qquad\qquad \operatorname{rad}(R_2) \leq 2\sqrt{n}\operatorname{rad}(R).$$

Indeed,

$$\begin{aligned}
\operatorname{rad}(R) &= \sqrt{b_1^2 + b_2^2 + \cdots + b_n^2} - \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} \\
&\geq \sqrt{(b_1^2 + b_2^2) + (b_3^2 + \cdots + b_n^2)} - \sqrt{(a_1^2 + a_2^2) + (b_3^2 + \cdots + b_n^2)} \\
&= \frac{(b_1^2 + b_2^2) - (a_1^2 + a_2^2)}{\sqrt{(b_1^2 + b_2^2) + (b_3^2 + \cdots + b_n^2)} + \sqrt{(a_1^2 + a_2^2) + (b_3^2 + \cdots + b_n^2)}} \\
&\geq \frac{(b_1^2 + b_2^2) - (a_1^2 + a_2^2)}{2\sqrt{n}b_1} \geq \frac{1}{2\sqrt{n}} \frac{(b_1^2 + b_2^2) - (a_1^2 + a_2^2)}{\sqrt{b_1^2 + b_2^2} + \sqrt{a_1^2 + a_2^2}} \\
&= \frac{1}{2\sqrt{n}}\left(\sqrt{b_1^2 + b_2^2} - \sqrt{a_1^2 + a_2^2}\right) = \frac{\operatorname{rad}(R_2)}{2\sqrt{n}}.
\end{aligned}$$

Let $\overline{R} = \prod_{i=3}^{n}(a_i, b_i)$. It follows by Fubini's theorem that

$$w(R) = \int_{\overline{R}}\left(\int_{R_2} w_0((x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2)^{1/2})\,dx_1 dx_2\right)dx_3\cdots dx_n.$$

Fix $\overline{x} \in \overline{R}$ and let $h = |\overline{x}|$. We shall estimate

$$I \equiv \int_{R_2} w_0((x_1^2 + x_2^2 + h^2)^{1/2})\,dx_1 dx_2.$$

In [18], it is shown that there exists a set $A \subset R_2$ such that

$$(5.4) \qquad\qquad \operatorname{rad}(R_2) \leq 8\operatorname{rad}(A)$$

and

$$(5.5) \qquad\qquad \operatorname{rad}(A)\inf_{r_1(A)<r<r_2(A)}\operatorname{arc}(A \cap \mathcal{C}_r) \geq \frac{|R_2|}{32},$$

where $\mathcal{C}_r$ is the circle of radius r and centered at the origin and $\operatorname{arc}(A \cap \mathcal{C}_r)$ is the arc length of the arc $A \cap \mathcal{C}_r$. Moreover, it is also shown that

$$(5.6) \qquad\qquad r_2(A) = \max\left(\sqrt{b_1^2 + a_2^2}, \sqrt{a_1^2 + b_2^2}\right).$$

The inequality (5.5) yields

$$\begin{aligned}
I &= \int_{r_1(R_2)}^{r_2(R_2)} w_0((r^2 + h^2)^{1/2})\operatorname{arc}(R_2 \cap \mathcal{C}_r)\,dr \\
&\geq \int_{r_1(A)}^{r_2(A)} w_0((r^2 + h^2)^{1/2})\operatorname{arc}(A \cap \mathcal{C}_r)\,dr \\
&\geq \operatorname{rad}(A)\inf_{r_1(A)<r<r_2(A)}\operatorname{arc}(A \cap \mathcal{C}_r)\cdot\frac{1}{\operatorname{rad}(A)}\int_{r_1(A)}^{r_2(A)} w_0((r^2 + h^2)^{1/2})\,dr \\
&\geq \frac{1}{\operatorname{rad}(A)}\int_{r_1(A)}^{r_2(A)} w_0((r^2 + h^2)^{1/2})\,dr\cdot\frac{|R_2|}{32}.
\end{aligned}$$

There holds by (5.3)

$$(5.7) \qquad \operatorname{rad}(A) \leq \operatorname{rad}(R_2) \leq 2\sqrt{n} \operatorname{rad}(R).$$

Using the change of variables $t = (r^2 + h^2)^{1/2}$, we have

$$\int_{r_1(A)}^{r_2(A)} w_0((r^2 + h^2)^{1/2}) \, dr = \int_{(r_1(A)^2+h^2)^{1/2}}^{(r_2(A)^2+h^2)^{1/2}} w_0(t) \frac{t}{\sqrt{t^2 - h^2}} \, dt$$
$$\geq \int_{(r_1(A)^2+h^2)^{1/2}}^{(r_2(A)^2+h^2)^{1/2}} w_0(t) \, dt.$$

It follows that

$$(5.8) \qquad \operatorname{rad}(A) \leq 2\sqrt{n} \left(\sqrt{r_2(A)^2 + h^2} - \sqrt{r_1(A)^2 + h^2} \right).$$

Indeed,

$$\sqrt{r_2(A)^2 + h^2} - \sqrt{r_1(A)^2 + h^2} = \frac{r_2(A)^2 - r_1(A)^2}{\sqrt{r_2(A)^2 + h^2} + \sqrt{r_1(A)^2 + h^2}}$$
$$\geq \frac{r_2(A)^2 - r_1(A)^2}{2\sqrt{n} b_1} \geq \frac{r_2(A)^2 - r_1(A)^2}{2\sqrt{n} \sqrt{b_1^2 + a_2^2}}$$
$$\geq \frac{1}{2\sqrt{n}} \frac{r_2(A)^2 - r_1(A)^2}{r_2(A) + r_1(A)} = \frac{r_2(A) - r_1(A)}{2\sqrt{n}} = \frac{\operatorname{rad}(A)}{2\sqrt{n}} \qquad \text{by (5.6)},$$

where we have used (5.6). By the inequalities (5.2), (5.4) and (5.8), we see that there exists a open interval $(a, b) \subset ((r_1(A)^2 + h^2)^{1/2}, (r_2(A)^2 + h^2)^{1/2}) \subset (r_1(R), r_2(R))$ such that

$$\operatorname{rad}(R) = 16(n - 1)\sqrt{n}(b - a).$$

This yields, thanks to the doubling condition 5.1,

$$(5.9) \qquad \int_{(r_1(A)^2+h^2)^{1/2}}^{(r_2(A)^2+h^2)^{1/2}} w_0(r) \, dr \geq \int_a^b w_0(r) \, dr \geq C \int_{r_1(R)}^{r_2(R)} w_0(r) \, dr.$$

Thus, by (5.7) and (5.9),

$$I \geq C \frac{1}{\operatorname{rad}(R)} \int_{r_1(R)}^{r_2(R)} w_0(r) \, dr \cdot |R_2|.$$

Integrating both sides of this inequality over $\overline{R}$, we obtain

$$w(R) \geq C \frac{1}{\operatorname{rad}(R)} \int_{r_1(R)}^{r_2(R)} w_0(r) \, dr \cdot |R|,$$

which completes the proof. $\qquad\qquad\square$

Fix $N \gg 1$. Let $\mathfrak{K}_N$ be the family of all tubes in $\mathbf{R}^n$, $n \geq 2$, with eccentricity N (the ratio of the length of long-sides and short-sides is equal to N). Lemma 5.1 yields the following.

Theorem 5.2. *Suppose that $w_0 \colon [0, \infty) \to [0, \infty)$ satisfies (1.1). Let $w(x) = w_0(|x|)$, $x \in \mathbf{R}^n$, be a radial weight. Then the weight w belongs to the reverse Hölder class* $\mathrm{RH}_{\infty, \mathfrak{R}}$ *and the reverse Hölder class* $\mathrm{RH}_{\infty, \mathfrak{K}_N}$.

Proof. Lemma 5.1 and (1.1) imply, for any $R \in \mathfrak{R}$,

$$R(w) = \frac{w(R)}{|R|} \geq C \frac{1}{\text{rad}\,(R)} \int_{r_1(R)}^{r_2(R)} w_0(r)\, dr \geq C \sup_{r_1(R) < r < r_2(R)} w_0(r) = C \operatorname*{ess\,sup}_{x \in R} w(x),$$

which means that w belongs to $\text{RH}_{\infty, \mathfrak{R}}$. For the case $T \in \mathfrak{K}_N$, let L be the axis of the tube T and let Π be the plane which contains the origin and L. Rotating Π to the plane (x_1, x_2) and applying another rotation in the plane (x_1, x_2), we may assume that T is of the form

$$T = (a_1, b_1) \times (a_2, b_2) \times (-c/2, c/2)^{n-2},$$

where $0 < a_1 < b_1$, $0 < a_2 < b_2$, $b_1 - a_1 = Nc$, $b_2 - a_2 = c$ and $c > 0$. This observation, Lemma 5.1 and (1.1) yield that w belongs to $\text{RH}_{\infty, \mathfrak{K}_N}$. $\square$

It is well-known that the strong maximal operator $M_{\mathfrak{R}}$ is bounded on $L^p(\mathbf{R}^n)$ for $p > 1$; see for example [6, p. 452]. It is also known that the Kakeya maximal operator $M_{\mathfrak{K}_N}$ satisfies for $1 < p \leq (n+2)/2$ (see [22, Theorem 1] and, for the log-factors, [21, Theorem 1]. The result of [21] is based on the argument in [19].)

$$\|M_{\mathfrak{K}_N}\|_{L^p(\mathbf{R}^n)} \leq C N^{n/p-1} (\log N)^{\alpha_n}.$$

These facts, Corollary 2.7 and Theorem 5.2 yield the following.

Proposition 5.3. *Suppose that $w_0 \colon [0, \infty) \to [0, \infty)$ satisfies (1.1). Let $w(x) = w_0(|x|)$, $x \in \mathbf{R}^n$, be a radial weight. Then,*
  (a) *for $1 < p \leq \infty$,*
$$\|M_{\mathfrak{R}, w} f\|_{L^p(w)} \leq C \|f\|_{L^p(w)}.$$
  (b) *for $1 < p \leq (n+2)/2$,*
$$\|M_{\mathfrak{K}_N, w} f\|_{L^p(\mathbf{R}^n, w)} \leq C N^{n/p-1} (\log N)^{\alpha_n} \|f\|_{L^p(\mathbf{R}^n, w)},$$

  where the constants C and α_n are independent of N.

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