Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 42, 2017, 551–562

DYADIC-BMO FUNCTIONS, THE DYADIC GUROV–RESHETNYAK CONDITION ON $[0, 1]^n$ AND EQUIMEASURABLE REARRANGEMENTS OF FUNCTIONS

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Abstract. We introduce the space of dyadic bounded mean oscillation functions f defined on $[0, 1]^n$ and study the behavior of the non increasing rearrangement of f, as an element of the space BMO $((0, 1])$. We also study the analogous class of functions that satisfy the dyadic Gurov-Reshetnyak condition and look upon their integrability properties.

1. Introduction

It is well known that the space of bounded mean oscillation plays a central role in harmonic analysis and especially in the theory of maximal operators and weights. It is defined by the following way. For an integrable function $f\colon Q_0 \equiv [0, 1]^n \to \mathbf{R}$ we define the mean oscillation of f on Q, where Q is a subcube of Q_0,

$$\Omega(f, Q) = \frac{1}{|Q|} \int_Q |f(x) - f_Q| \, \mathrm{d}x \tag{1.1}$$

where $f_Q = \frac{1}{|Q|} \int_Q f(y) \, \mathrm{d}y$, is the integral average of f on Q. We will say that f is of bounded mean oscillation on Q_0, if the following is satisfied

$$\|f\|_\star \equiv \sup \{\Omega(f, Q)\colon Q \text{ is a subcube of } Q_0\} < +\infty.$$

We will then write $f \in \mathrm{BMO}(Q_0)$. We are interested about the behavior of the nonincreasing rearrangement $f^\star$, as an element of $\mathrm{BMO}((0, 1])$, when $f \in \mathrm{BMO}(Q_0)$.

We denote by $f^\star$ the unique equimeasurable to $|f|$,[1] with domain $(0, 1]$, function which also satisfies the following requirements: it is nonincreasing and left continuous. A discussion about this definition can be seen in [4]. There is also an equivalent definition of $f^\star$, which is given by the following formula:

$$f^\star(t) = \sup_{\substack{E \subseteq [0,1]^n \\ |E| = t}} \left[\inf_{x \in E} |f(x)| \right], \quad \text{for} \quad t \in (0, 1]. \tag{1.2}$$

This can be seen in [8].

There is also an analogous function, corresponding to f, denoted by f_d which is now equimeasurable to f, left continuous and nonincreasing. This function now rearranges f, and not $|f|$, as $f^\star$ does, so that it is real valued. Also an analogous formula as (1.2) holds for f_d, if we replace the term $|f(x)|$ by $f(x)$. For a discussion on the topic of rearrangements of functions one can also see [2]. As it can be seen now in [1] or [3] the following is true.

https://doi.org/10.5186/aasfm.2017.4228
2010 Mathematics Subject Classification: Primary 42B35.
Key words: Dyadic-BMO functions, dyadic Gurov–Reshetnyak condition, equimeasurable rearrangements of functions.

[1] in the sense that $|\{f^\star > \lambda\}| = |\{|f| > \lambda\}|$, for any $\lambda > 0$.

Theorem A. *Let* $f \in \mathrm{BMO}([0,1]^n)$*. Then* $f^\star \in \mathrm{BMO}((0,1])$*. Moreover there exists a constant* c *depending only on the dimension of the space such that*

$$\|f^\star\|_\star \le c\|f\|_\star. \tag{1.3}$$

For instance a choice for such a constant is $c = 2^{n+5}$*.*

Until now it has not been found the best possible value of c in order that (1.3) holds for any $f \in \mathrm{BMO}([0,1]^n)$, for dimensions $n \ge 2$. As for the case $n = 1$ the following is true as can be seen in [6].

Theorem B. *Let* $f \in \mathrm{BMO}([0,1])$*. Then* $f^\star, f_d \in \mathrm{BMO}((0,1])$ *and the following inequalities hold:*

$$\|f^\star\|_\star \le \|f\|_\star, \tag{1.4}$$

$$\|f_d\|_\star \le \|f\|_\star. \tag{1.5}$$

Our aim in this paper is to find a better estimation for the constant c that appears in (1.3). For this reason we work on the respective dyadic analogue problem. We consider integrable functions defined on $[0,1]^n$ such that the following holds

$$\|f\|_{\star,\mathcal{D}} \equiv \sup\{\Omega(f,Q) \colon Q \in \mathcal{D}\} < +\infty. \tag{1.6}$$

Here by $\mathcal{D}$ we denote the tree of the dyadic subcubes of $Q_0 \equiv [0,1]^n$, that is the cubes that are produced if we bisect each side of Q_0 and continue this process to any resulting cube. Then if (1.6) holds for f, we will say that it belongs to the dyadic BMO space, denoted by $\mathrm{BMO}_{\mathcal{D}}([0,1]^n)$. Our first result is the following:

Theorem 1. *Let* $f \in \mathrm{BMO}_{\mathcal{D}}([0,1]^n)$*. Then* $f_d \in \mathrm{BMO}((0,1])$ *and*

$$\|f_d\|_\star \le 2^n\|f\|_{\star,\mathcal{D}}. \tag{1.7}$$

This of course gives us as a consequence that the constant c that appears in (1.3), can be replaced effectively by 2^n.

As in the usual case, Theorem 1 enables us to prove an inequality of the type of John–Nirenberg (see for example [5]), which is described by the following:

Theorem 2. *Let* $f \in \mathrm{BMO}_{\mathcal{D}}([0,1]^n)$*. Then the following inequality is true*

$$|\{x \in [0,1]^n : f(x) - f_Q > \lambda\}| \le B\exp\left(-\frac{b\lambda}{\|f\|_{\star,\mathcal{D}}}\right), \tag{1.8}$$

for any $\lambda > 0$*, where* b *depends only on the dimension of the space, while* B *is independent of* n*. For example* (1.8) *is satisfied for* $b = \frac{1}{2^{n-1}\mathrm{e}}$ *and* $B = \mathrm{e}$*.*

After proving the above Theorems we devote our study to the class of functions that satisfy the dyadic Gurov–Reshetnyak condition. More precisely we consider functions $f \colon Q_0 \equiv [0,1]^n \to \mathbf{R}^+$ which are integrable and satisfy

$$\Omega(f,Q) \le \varepsilon f_Q, \tag{1.9}$$

for any $Q \in \mathcal{D}$ and some $\varepsilon \in (0,2)$, independent of the cude Q. We say then that f satisfies the dyadic Gurov–Reshetnyak condition on $[0,1]^n$ with constant ε and write $f \in \mathrm{GR}_{\mathcal{D}}(Q_0,\varepsilon)$ (note that for any $f \in L^1(Q_0)$, (1.9) is satisfied for any cube Q, for the constant $\varepsilon = 2$). The study of such class of functions is of much importance in harmonic analysis and especially in the theory of weights. An extensive presentation of the study of such a class in the non-dyadic case can be seen in [8].

For the study of the class $\mathrm{GR}_{\mathcal{D}}(Q_0,\varepsilon)$ we define for any f belonging to it, the following function

$$v(f;\sigma)=\sup\left\{\frac{\Omega(f,Q)}{f_Q}: Q\in\mathcal{D},\ \text{ with }\ \ell(Q)\le\sigma\right\}, \tag{1.10}$$

for ant $0\le\sigma\le\ell(Q_0)$, where by $\ell(Q)$ we denote the length of the side of the cube Q. We will prove the following independent result.

Theorem 3. *Let $f\in L^1(Q_0)$ with non-negative values. Then for any $t\in[0,1]$ the following inequality is true:*

$$\frac{1}{t}\int_0^t |f^\star(u)-f^{\star\star}(t)|\,\mathrm{d}u\le 2^n f^{\star\star}(t)v(f;\sigma_t), \tag{1.11}$$

where $\sigma_t=\min\left(2t^{\frac{1}{n}},1\right)$.

Here by $f^{\star\star}(t)$ we denote the Hardy function of $f^\star$ defined as

$$f^{\star\star}(t)=\frac{1}{t}\int_0^t f^\star(u)\mathrm{d}u,$$

for $t\in(0,1]$. Moreover we prove the following result by applying Theorem 3.

Theorem 4. *Let $f\in Q_0\to\mathbf{R}^+$, $f\in L_1(Q_0)$. Then there exist constants c_i for $i=1,2,3,4$ depending only on n such that the following hold: $c_4>1$ and*

$$f^{\star\star}(t)\le c_1 f_{Q_0}\exp\left(c_2\int_{c_3t^{\frac{1}{n}}}^1 v(f;\sigma)\frac{\mathrm{d}\sigma}{\sigma}\right), \tag{1.12}$$

for every $t\in\left(0,\frac{1}{c_4}\right]$.

The proof of Theorem 4 depends on Theorem 3, and can be effectively used for the proof of the following:

Theorem 5. *Let $f\in\mathrm{GR}_{\mathcal{D}}(Q_0,\varepsilon)$ for some $\varepsilon\in\left(0,\frac{1}{2^{n-1}}\right)$. Then for any $t\in(0,1]$, we have that $f^{\star\star}(t)\le\frac{p}{p-1}f_{Q_0}t^{-\frac{1}{p}}$, where $p>1$ is defined by the equation*

$$\frac{p^p}{(p-1)^{p-1}}=\frac{1}{2^{n-1}\varepsilon} \tag{1.13}$$

An immediate consequence is the following.

Corollary 1. *Let $f\in\mathrm{GR}_{\mathcal{D}}(Q_0,\varepsilon)$ for some $\varepsilon\in\left(0,\frac{1}{2^{n-1}}\right]$. Then $f\in L^q(Q_0)$ for any $q\in[1,p)$, where p is defined by* (1.13).

In this way we increase the integrability properties of f, if this function belongs to the space $\mathrm{GR}_{\mathcal{D}}(Q_0,\varepsilon)$, with ε restricted in the above range.

The paper is organized as follows: In Section 2 we give some preliminaries (Lemmas) needed in subsequent Sections. In Section 3 we prove Theorems 1 and 2 and in Section 4 we provide proofs of Theorems 3,4 and 5. We also mention that for the proofs of Theorems 1–5, we are inspired from [8], where the non-dyadic case has been studied. Here in this paper we study the dyadic one. At last we note that problems related to dyadic weights have been studied extensively in the past (see for example [10], [11] and [12]).

2. Preliminaries

In this section we state some Lemmas needed in subsequent sections. These can be found in [8]. The first one is the following.

Lemma 2.1. *Let $f \in L^1(Q_0)$. Then if we define $\Omega(f,Q)$ by (1.1), for a fixed cube $Q \subseteq Q_0$ we have the equalities*

$$\Omega(f,Q) = \frac{2}{|Q|} \int\limits_{\{x \in Q : f(x) > f_Q\}} (f(x) - f_Q)\,\mathrm{d}x = \frac{2}{|Q|} \int\limits_{\{x \in Q : f(x) < f_Q\}} (f_Q - f(x))\,\mathrm{d}x.$$

We will also need the following.

Lemma 2.2. *Let $f\colon I_1 \equiv [a_1, b_1] \to \mathbf{R}$ be monotone integrable on I_1. Suppose we are given $I = [a,b] \subseteq I_1$ such that $f_I = f_{I_1}$. Then the inequality $\Omega(f,I) \le \Omega(f,I_1)$, is true.*

Finally we will use:

Lemma 2.3. *Let f be non-increasing, summable on $(0,1]$ and let also $F(t) = \frac{1}{t}\int_0^t f(u)\mathrm{d}u$, for $t \in (0,1]$. Then for any constant $\gamma > 1$, the following inequality is true:*

$$F\left(\frac{t}{\gamma}\right) - F(t) \le \frac{\gamma}{2}\frac{1}{t}\int_0^t |f(u) - F(t)|\mathrm{d}u, \quad t \in (0,1].$$

3. f_d as an element of BMO((0, 1])

From now on we suppose that f is defined on $Q_0 \equiv [0,1]^n$, is real valued and integrable. We proceed to the presentation of the proof of Theorem 1 following [6].

Proof of Theorem 1. Suppose that $f \in \mathrm{BMO}_{\mathcal{D}}([0,1]^n)$. We shall prove that $f_d \in \mathrm{BMO}((0,1])$, and that

$$\|f_d\|_\star \le 2^n \|f\|_{\star,\mathcal{D}}. \tag{3.1}$$

For the proof of (3.1), we obviously need to prove the inequality

$$\frac{1}{|J|}\int_J |f_d(u) - (f_d)_J|\,\mathrm{d}u \le 2^n \|f\|_{\star,\mathcal{D}}, \tag{3.2}$$

for any J interval of $(0,1]$. Fix such a J. We set $\alpha = \frac{1}{|J|}\int_J f_d = (f_d)_J$.

i) We first consider the case where

$$\alpha \ge \int_{[0,1]^n} f(x)\mathrm{d}x \tag{3.3}$$

We consider now the family $(D_j)_j$ of those cubes $I \in \mathcal{D}$ maximal with respect to the relation $\subseteq$ under the condition $\frac{1}{|I|}\int_I f > \alpha$. Certainly, because of (3.3) we have that any such cube must be a strict subset of $[0,1]^n$. Additionally, because of the maximality of every D_j and the tree structure of $\mathcal{D}$ we have that $(D_j)_j$ is a pairwise disjoint subfamily of the tree $\mathcal{D}$. Certainly for any such cube D_j we have that $\frac{1}{|D_j|}\int_{D_j} f > \alpha$, so as a consequence $\frac{1}{|E|}\int_E f > \alpha$, where E denotes the union of the elements of the family $(D_j)_j$, that is $E = \bigcup_j D_j$. Now for any dyadic cube $I \neq [0,1]^n$ we denote as $I^\star$ the father of I in $\mathcal{D}$, that is the dyadic cube for which if we bisect it's sides we produce 2^n dyadic subcubes of $I^\star$, one of which is I. Now we consider for any D_j the respective element of $\mathcal{D}$, $D_j^\star$.

We look at the family $(D_j^\star)_j$. Certainly this is not necessarily pairwise disjoint. We consider now a maximal subfamily of $(D_j^\star)_j$, denoted as $(D_{j_k}^\star)_k$, under the relation of $\subseteq$. This is pairwise disjoint and $\bigcup_k D_{j_k}^\star = \bigcup_j D_j$. Moreover for any k, $D_{j_k}^\star \supsetneq D_{j_k}$. Additionally by the definition of E and the dyadic version of the Lebesque differentiation theorem, we have that $f(x) \le \alpha$, for almost every $x \in [0,1]^n \setminus E$, because of the construction of the family $(D_j)_j$. Moreover by the maximality of D_{j_k} we must have that $f_{D_{j_k}^\star} \le \alpha$, for any k. We now set $E^\star = \bigcup_k D_{j_k}^\star$. Then $E \subsetneq E^\star$ and certainly $|E| \ge \frac{|E^\star|}{2^n}$, by construction.

We look now to the function $f_d\colon (0,1] \to \mathbf{R}$. Since $\alpha = (f_d)_J > f_{[0,1]^n}$ it is easy to see (since f_d is non-increasing) that there exists $t \in (0,1]$, such that $J \subseteq [0,t]$ and $\frac{1}{t}\int_0^t f_d(u)\,\mathrm{d}u = \alpha$. That is $(f_d)_{[0,t]} = (f_d)_J$. We now take advantage of Lemma 2.2. We obtain immediately that

$$\Omega(f_d, J) = \frac{1}{|J|}\int_J |f_d(u) - (f_d)_J|\,\mathrm{d}u \le \Omega(f_d,[0,t]) = \frac{1}{t}\int_0^t |f_d(u)-\alpha|\,\mathrm{d}u. \tag{3.4}$$

Since now $\frac{1}{|E|}\int_E f > \alpha$, we obviously have because f_d is non-increasing, that

$$\frac{1}{|E|}\int_0^{|E|} f_d(u)\,\mathrm{d}u \ge \frac{1}{|E|}\int_E f(x)\,\mathrm{d}x > \alpha = \frac{1}{t}\int_0^t f_d(u)\,\mathrm{d}u,$$

and as a consequence the measure of E must satisfy $|E| \le t$. Thus by the comments mentioned above, we see that $|E^\star| \le 2^n|E| \le 2^n t$.

By (3.4), it is enough to prove that

$$\frac{1}{t}\int_0^t |f_d(u)-\alpha|\,\mathrm{d}u \le 2^n\|f\|_{\star,\mathcal{D}},$$

for the case i) to be completed. For this purpose we proceed as follows: By using Lemma 2.1 we have that

$$\int_0^t |f_d(u)-\alpha|\,\mathrm{d}u = 2\int_{\{u\in(0,t]:f_d(u)>\alpha\}} (f_d(u)-\alpha)\,\mathrm{d}u, \tag{3.5}$$

since $\alpha = (f_d)_{(0,t]}$.

The right side of (3.5) equals $2\int_{\{f>\alpha\}}(f(x)-\alpha)\,\mathrm{d}x$, because of the equimeasurability of f and f_d and the fact that $\alpha = \frac{1}{t}\int_0^t f_d(u)\,\mathrm{d}u \ge f_d(t)$. Thus, since $f(x) \le \alpha$, for almost every element of $[0,1]^n \setminus E^\star$, (3.5) and the remarks above give that

$$\begin{aligned}\int_0^t |f_d(u)-\alpha|\,\mathrm{d}u &= 2\int_{\{x\in E^\star \equiv \cup_k D_{j_k}^\star\,:\, f(x)>\alpha\}} (f(x)-\alpha)\,\mathrm{d}x \\ &= 2\int_{(\cup_k D_{j_k}^\star)\cap\{f>\alpha\}} (f(x)-\alpha)\,\mathrm{d}x = 2\sum_k \int_{D_{j_k}^\star\cap\{f>\alpha\}} (f(x)-\alpha)\,\mathrm{d}x.\end{aligned} \tag{3.6}$$

We prove now that for any k, the following inequality holds

$$\int_{D_{j_k}^\star\cap\{f>\alpha\}} (f(x)-\alpha)\,\mathrm{d}x \le \int_{D_{j_k}^\star\cap\{f>f_{D_{j_k}^\star}\}} \left(f(x)-f_{D_{j_k}^\star}\right)\mathrm{d}x. \tag{3.7}$$

Indeed, (3.7) is equivalent to

$$\ell_k \equiv \int\limits_{D_{j_k}^\star\cap\{f_{D_{j_k}^\star}<f\le\alpha\}} f(x)\,\mathrm{d}x \ge f_{D_{j_k}^\star}\left|D_{j_k}^\star \cap \left\{f > f_{D_{j_k}^\star}\right\}\right| - \alpha\left|D_{j_k}^\star\cap\{f>\alpha\}\right|, \tag{3.8}$$

This is now easy to prove since

$$\begin{aligned}\ell_k &\geq f_{D^\star_{j_k}} \left|D^\star_{j_k} \cap \left\{f_{D^\star_{j_k}} < f \leq \alpha\right\}\right| \\ &= f_{D^\star_{j_k}} \left|D^\star_{j_k} \cap \left\{f > f_{D^\star_{j_k}}\right\}\right| - f_{D^\star_{j_k}} \left|D^\star_{j_k} \cap \{f > \alpha\}\right| \\ &\geq f_{D^\star_{j_k}} \left|D^\star_{j_k} \cap \left\{f > f_{D^\star_{j_k}}\right\}\right| - \alpha \left|D^\star_{j_k} \cap \{f > \alpha\}\right|\end{aligned}$$

where the last inequality is true because of the fact that $f_{D^\star_{j_k}} \leq \alpha$, for every k. But the last inequality is exactly (3.8), so by (3.6) and (3.7) we have that:

$$\begin{aligned}\int_0^t |f_d(u) - \alpha|\,\mathrm{d}u &\leq 2 \sum_k \int_{D^\star_{j_k} \cap \{f > f_{D^\star_{j_k}}\}} \left(f(x) - f_{D^\star_{j_k}}\right) \mathrm{d}x \\ &= \sum_k \int_{D^\star_{j_k}} \left|f(x) - f_{D^\star_{j_k}}\right| \mathrm{d}x = \sum_k \left|D^\star_{j_k}\right| \Omega\left(f, D^\star_{j_k}\right) \\ &\leq \left(\sum_k \left|D^\star_{j_k}\right|\right) \|f\|_{\star,\mathcal{D}} = |E^\star|\, \|f\|_{\star,\mathcal{D}} \leq 2^n t \|f\|_{\star,\mathcal{D}},\end{aligned} \tag{3.9}$$

where the first equality in (3.9) holds due to Lemma 2.1. Thus we have proved that

$$\frac{1}{t} \int_0^t |f_d(u) - \alpha|\,\mathrm{d}u \leq 2^n \|f\|_{\star,\mathcal{D}},$$

and the proof of case i) is complete.

We are now going to give a brief discussion for the second case, since this is analogous to the first one.

ii) We assume that J is a subinterval of $(0,1]$ and that

$$\alpha = \frac{1}{|J|} \int_J f_d(u)\,\mathrm{d}u < \int_{[0,1]^n} f(x)\,\mathrm{d}x. \tag{3.10}$$

We prove that $\Omega(f_d, J) \leq 2^n \|f\|_{\star,\mathcal{D}}$.

By (3.10), we choose $t \in [0,1)$ such that $\alpha = \frac{1}{t}\int_{1-t}^1 f_d(u)\,\mathrm{d}u$ and $J \subseteq [1-t, 1]$. We choose the maximal of $(D_j)_j$, $D_j \in \mathcal{D}$ for every j such that $\frac{1}{|D_j|}\int_{D_j} f \leq \alpha$. This is possible in view of (3.10), from which we also have that $D_j \neq X$ and because of their maximality, $(D_j)_j$ is pairwise disjoint. We pass as before to the pairwise disjoint family $(D^\star_{j_k})_k$, for which we have $E^\star = \bigcup_k D^\star_{j_k} = \bigcup_j D^\star_j \supseteq \bigcup D_j = E$, $|E^\star| \leq 2^n |E|$ and such that $f(x) \geq \alpha$, for almost every $x \in [0,1]^n \setminus E$. As before we have

$$\begin{aligned}\Omega(f_d, J) &= \frac{1}{|J|} \int_J |f_d(u) - (f_d)_J|\,\mathrm{d}u \leq \frac{1}{t} \int_{1-t}^1 |f_d(u) - \alpha|\,\mathrm{d}u \\ &= \frac{2}{t} \int_{\{u \in [1-t,1]:\, f_d(u) < \alpha\}} (\alpha - f_d(u))\,\mathrm{d}u \\ &= \frac{2}{t} \int_{\{x \in Q_0 \equiv [0,1]^n:\, f(x) < \alpha\}} (\alpha - f(x))\,\mathrm{d}x \\ &= \frac{2}{t} \int_{\{x \in E^\star:\, f(x) < \alpha\}} (\alpha - f(x))\,\mathrm{d}x = \frac{2}{t} \sum_k \int_{D^\star_{j_k} \cap \{f < \alpha\}} (\alpha - f(x))\,\mathrm{d}x.\end{aligned} \tag{3.11}$$

By the fact that $f_{D^\star_{j_k}} \geq \alpha$ and the same reasoning as before we conclude that for any k:

$$\int_{D^\star_{j_k}\cap\{f<\alpha\}} (\alpha - f(x))\,\mathrm{d}x \leq \int_{D^\star_{j_k}\cap\{f<f_{D^\star_{j_k}}\}} \left(f_{D^\star_{j_k}} - f(x)\right)\mathrm{d}x. \tag{3.12}$$

Thus by (3.11) and (3.12) we obtain:

$$\begin{aligned}\Omega(f_d, J) &\leq \frac{2}{t}\sum_k \int_{D^\star_{j_k}\cap\{f<f_{D^\star_{j_k}}\}} \left(f_{D^\star_{j_k}} - f(x)\right)\mathrm{d}x \\ &= \frac{1}{t}\sum_k |D^\star_{j_k}|\Omega\left(f, D^\star_{j_k}\right) \leq \frac{|E^\star|}{t}\|f\|_{\star,\mathcal{D}} \leq \frac{2^n|E|}{t}\|f\|_{\star,\mathcal{D}}.\end{aligned} \tag{3.13}$$

As before we can prove that $|E| \leq t$, so then (3.13) gives the desired result. Thus we proved that for any $J \subseteq [0,1]$ we have $\Omega(f_d, J) \leq 2^n\|f\|_{\star,\mathcal{D}}$, or that $\|f_d\|_\star \leq 2^n\|f\|_{\star,\mathcal{D}}$ and our proof is now complete. □

We are now able to prove the following

Theorem 3.1. *Let* $f\colon Q_0 \equiv [0,1]^n \to \mathrm{R}$ *be such that* $\int_{Q_0} f = 0$ *and that* $f \in \mathrm{BMO}_{\mathcal{D}}([0,1]^n)$. *Then*

$$f_d(t) \leq \frac{\|f\|_{\star,\mathcal{D}}}{b}\ln\left[\frac{B}{t}\right],$$

for some constants $b, B > 0$ *depending only in the dimension* n.

Proof. We define $F(t) = \frac{1}{t}\int_0^t f_d(u)\,\mathrm{d}u$. Then by Lemma 2.3, $F\left(\frac{t}{\alpha}\right) - F(t) \leq \frac{\alpha}{2}\frac{1}{t}\int_0^t |f_d(u) - F(t)|\,\mathrm{d}u$, for any $t \in (0,1]$ and $\alpha > 1$. Thus

$$F\left(\frac{t}{\alpha}\right) - F(t) \leq \frac{\alpha}{2}\Omega\left(f_d, [0,t]\right) \leq \frac{\alpha}{2}\|f_d\|_\star \leq 2^{n-1}\alpha\|f\|_{\star,\mathcal{D}}, \tag{3.14}$$

by using Theorem 1. By (3.14) now we have for any $\alpha > 1$ the inequalities

$$F\left(\frac{1}{\alpha^i}\right) - F\left(\frac{1}{\alpha^{i-1}}\right) \leq 2^{n-1}\alpha\|f\|_{\star,\mathcal{D}}, \tag{3.15}$$

for any $i = 1, 2, \ldots, k, k+1$ and any fixed $k \in \mathbf{N}$. Summing inequalities (3.15) we obtain as a consequence that

$$\begin{aligned}&F\left(\frac{1}{\alpha^{k+1}}\right) - F(1) \leq (k+1)2^{n-1}\alpha\|f\|_{\star,\mathcal{D}} \implies (\text{since } \int_{Q_0} f = 0) \\ &F\left(\frac{1}{\alpha^{k+1}}\right) \leq \left((k+1)2^{n-1}\alpha\right)\|f\|_{\star,\mathcal{D}}.\end{aligned} \tag{3.16}$$

Fix now $t \in (0,1]$ and $\alpha > 1$. Then for a unique $k \in \mathbf{N}$ we have that

$$\begin{aligned}&\frac{1}{\alpha^{k+1}} < t \leq \frac{1}{\alpha^k} \implies k \leq \frac{1}{\ln(\alpha)}\ln\left(\frac{1}{t}\right) \overset{(3.16)}{\implies} \\ &f_d(t) \leq \frac{1}{t}\int_0^t f_d(u)\,\mathrm{d}u = F(t) \leq F\left(\frac{1}{\alpha^{k+1}}\right) \leq \left((k+1)2^{n-1}\alpha\right)\|f\|_{\star,\mathcal{D}} \\ &\leq \left(\left[\frac{1}{\ln(\alpha)}\ln\left(\frac{1}{t}\right) + 1\right]2^{n-1}\alpha\right)\|f\|_{\star,\mathcal{D}}.\end{aligned} \tag{3.17}$$

Now the function h defined for any $\alpha > 1$, by $h(\alpha) = \frac{\alpha}{\ln(\alpha)}$ attains its minimum value at $\alpha = \mathrm{e}$. Thus for this value of α, we obtain by (3.17) that

$$f_d(t) \le \left[\ln\left(\frac{1}{t}\right) 2^{n-1}\mathrm{e} + 2^{n-1}\mathrm{e}\right] \|f\|_{\star,\mathcal{D}} = \frac{\|f\|_{\star,\mathcal{D}}}{b}\left[\ln\left(\frac{B}{t}\right)\right], \quad \text{for any } t \in (0,1]$$

where $b = \frac{1}{2^{n-1}\mathrm{e}}$, $B = \mathrm{e}$. □

We now proceed to the

Proof of Theorem 2. For any $f \in \mathrm{BMO}_{\mathcal{D}}([0,1]^n)$ we prove that

$$|\{x \in Q_0 \colon (f(x) - f_{Q_0}) > \lambda\}| \le B \exp\left(-\frac{b\lambda}{\|f\|_{\star,\mathcal{D}}}\right), \tag{3.18}$$

for every $\lambda > 0$ and the above values of b, B.

We fix a $\lambda > 0$ and suppose without loss of generality that $f_{Q_0} = 0$. We set $A_\lambda = \{x \in Q_0 \colon f(x) > \lambda\}$. In order to prove (3.18), we just need to prove that $|A_\lambda| \le B\exp\left(-\frac{b\lambda}{\|f\|_{\star,\mathcal{D}}}\right)$, for this value of $\lambda > 0$. We have $|A_\lambda| = |\{f > \lambda\}| = |\{f_d > \lambda\}| \le \left|\left\{t \in (0,1] \colon \frac{\|f\|_{\star,\mathcal{D}}}{b}\ln\left(\frac{B}{t}\right) > \lambda\right\}\right|$, since by Theorem 3.1 $f_d(t) \le \frac{\|f\|_{\star,\mathcal{D}}}{b}\ln\left(\frac{B}{t}\right)$, for every $t \in (0,1]$. Thus we have that

$$|A_\lambda| \le \left|\left\{t \in (0,1] \colon t < \exp\left(-\frac{b\lambda}{\|f\|_{\star,\mathcal{D}}}\right)\right\}\right| \le B\exp\left(-\frac{b\lambda}{\|f\|_{\star,\mathcal{D}}}\right). \quad □$$

Remark 3.1. By considering the results of this section it is worth mentioning the following. Suppose that $f \colon [0,1]^n \to \mathrm{R}^+$ be such that $\|f\|_{\star,\mathcal{D}} < +\infty$. Because f is non-negative we must have that $f_d = |f|_d = f^\star$, on $(0,1]$. Thus we have that for any such f we must have that $\|f^\star\|_\star \le 2^n \|f\|_{\star,\mathcal{D}}$ and the inequality $|\{x \in Q_0 \colon |f(x) - f_{Q_0}| > \lambda\}| \le B\exp\left(-\frac{b\lambda}{\|f\|_{\star,\mathcal{D}}}\right)$, for every $\lambda > 0$ and the above mentioned values of b and B.

4. The dyadic Gurov–Reshetnyak condition

We again consider functions $f \colon Q_0 \equiv [0,1]^n \to \mathbf{R}^+$ such that $f \in L^1(Q_0)$ and for which the following condition is satisfied

$$\Omega(f,Q) \equiv \frac{1}{|Q|}\int_Q |f(x) - f_Q|\,\mathrm{d}x \le \varepsilon f_Q, \quad \forall Q \in \mathcal{D},$$

for some $\varepsilon \in (0,2)$, independent of the cube Q. As we noted in Section 1 we say then that $f \in \mathrm{GR}_{\mathcal{D}}(Q_0,\varepsilon)$. Define the function $v(f;\cdot)$ by (1.10). We provide now the

Proof of Theorem 3:. We define $\sigma_t = \min\left(2t^{\frac{1}{n}}, 1\right)$, for every $t \in (0,1]$ and $B_t = v(f;\sigma_t)$. We shall prove that for every $t \in (0,1]$, we have that

$$\frac{1}{t}\int_0^t |f^\star(u) - f^{\star\star}(t)|\,\mathrm{d}u \le 2^n B_t f^{\star\star}(t).$$

For this proof we work as in Theorem 1. Fix $t \in (0,1]$ and set $\alpha = f^{\star\star}(t)$. Then $\alpha > f_{Q_0} = \int_{[0,1]^n} f(x)\,\mathrm{d}x = f^{\star\star}(1)$, since $f^\star$ is non-increasing. We define the following maximal operator

$$M_d\varphi(x) = \sup\left\{\frac{1}{|Q|}\int_Q |\varphi(y)|\,\mathrm{d}y \colon x \in Q \in \mathcal{D}\right\}$$

for every $\varphi \in L^1(Q_0)$, where $\mathcal{D}$ is as usual the class of all dyadic subcubes of Q_0. This is called the dyadic maximal operator with respect to the tree $\mathcal{D}$.

We consider the set $E = \{M_d f > \alpha\}$. It is easy to see that E can be written as $E = \bigcup_j D_j$, where $(D_j)_j$ is a pairwise disjoint family of cubes in $\mathcal{D}$, maximal under the condition $\frac{1}{|D_j|}\int_{D_j} f > \alpha$, with respect to the relation $\subseteq$. Since $\alpha > f_{Q_0} = \int_{[0,1]^n} f$ for any such cube we have that $D_j \neq [0,1]^n$. Let also $D_j^\star$ be the father of D_j in $\mathcal{D}$, for every j. By the maximality of D_j we have that $\frac{1}{|D_j^\star|}\int_{D_j^\star} f \le \alpha$, or that $f_{D_j^\star} \le \alpha$. We set now $E^\star = \bigcup_j D_j^\star$. Then $E^\star$ can be written as $E^\star = \bigcup_k D_{j_k}^\star$, where the family $\left(D_{j_k}^\star\right)_k$ is a maximal subfamily of $(D_j)_j$ under the relation $\subseteq$. Because of its maximality, this must be disjoint. Of course by the dyadic form of the Lebesque differentiation theorem we have that for almost every $x \notin E$, $x \in [0,1]^n$, the following is satisfied:

$$f(x) \le M_d f(x) \le \alpha = f^{\star\star}(t). \tag{4.1}$$

We consider now the following quantity: $L_t \equiv \int_0^t |f^\star(u) - f^{\star\star}(t)|\,\mathrm{d}u$, which in view of lemma 2.1 can be written as

$$L_t = 2\int_{\{u\in(0,t]:f^\star(u)\ge\alpha\}} (f^\star(u)-\alpha)\,\mathrm{d}u. \tag{4.2}$$

By (4.2) we have that

$$L_t = 2\int_{\{x\in[0,1]^n:f(x)>\alpha\}} (f(x)-\alpha)\,\mathrm{d}x, \tag{4.3}$$

because of the equimeasurability of f and $f^\star$ and the fact that $\alpha = f^{\star\star}(t) = \frac{1}{t}\int_0^t f^\star(u)\,\mathrm{d}u \ge \int_0^1 f^\star$.

Then since $E \subseteq E^\star$, and because of (4.1), we have because of (4.3) that

$$\begin{aligned} L_t &= 2\int_{E^\star\cap\{x\in Q_0:\, f(x)>\alpha\}} (f(x)-\alpha)\,\mathrm{d}x = 2\int_{(\cup D_{j_k}^\star)\cap\{f>\alpha\}} (f(x)-\alpha)\,\mathrm{d}x \\ &= \sum_k \int_{D_{j_k}^\star\cap\{f>\alpha\}} (f(x)-\alpha)\,\mathrm{d}x. \end{aligned} \tag{4.4}$$

Is is now easy to show, is done in Section 3, that the following inequality is true

$$\int_{D_{j_k}^\star\cap\{f>\alpha\}} (f(x)-\alpha)\,\mathrm{d}x \le \int_{D_{j_k}^\star\cap\left\{f>f_{D_{j_k}^\star}\right\}} \left(f(x)-f_{D_{j_k}^\star}\right)\mathrm{d}x, \text{ for any } k. \tag{4.5}$$

Now (4.4), in view of (4.5) becomes:

$$\begin{aligned} L_t &\le \sum_k 2\int_{\left\{x\in D_{j_k}^\star:\, f(x)>f_{D_{j_k}^\star}\right\}} \left(f(x)-f_{D_{j_k}^\star}\right)\mathrm{d}x \\ &= \sum_k \int_{D_{j_k}^\star} \left|f(x)-f_{D_{j_k}^\star}\right|\mathrm{d}x = \sum \left|D_{j_k}^\star\right| \Omega\left(f, D_{j_k}^\star\right), \end{aligned}$$

where the first equality holds in view of Lemma 2.1. Now by the definition of E and α we immediately have that

$$\frac{1}{|E|}\int_0^{|E|} f^\star(u)\,\mathrm{d}u \ge \frac{1}{|E|}\int_E f(x)\,\mathrm{d}x > \alpha = \frac{1}{t}\int_0^t f^\star(u)\,\mathrm{d}u \implies |E| \le t,$$

since $f^\star$ is non-increasing. Thus by the construction of $E^\star$ we have that

$$|E^\star| = \sum_k |D^\star_{j_k}| \le \sum_k 2^n \left|D^\star_{j_k} \cap E\right| = 2^n|E| \le 2^n t.$$

Additionally, for any k we have that $|D^\star_{j_k}| \le |E^\star| \le 2^n t$, thus $\ell(D^\star_{j_k}) \le 2t^{\frac{1}{n}}$, for every k. Thus we immediately have by the definition of the function $v(f;\cdot)$, that $\Omega(f, D^\star_{j_k}) \le v(f;\sigma_t) f_{D^\star_{j_k}} \le v(f;\sigma_t)\alpha = v(f;\sigma_t) f^{\star\star}(t)$, where $\sigma_t = \min\left\{2t^{\frac{1}{n}}, 1\right\}$, for any $t \in (0,1]$. So as a consequence from (4.5) and the above comments, we obtain $L_t \le |E^\star| v(f;\sigma_t) f^{\star\star}(t) \le 2^n t f^{\star\star}(t) B_t \implies \frac{1}{t}\int_0^t |f^\star(u) - f^{\star\star}(t)|\,\mathrm{d}u \le 2^n B_t f^{\star\star}(t)$. Thus the proof of our Theorem is complete. □

We proceed now to the

Proof of Theorem 4. We suppose that we are given $f\colon Q_0 \equiv [0,1]^n \to \mathbf{R}^+$ such that $f \in L^1(Q_0)$. By Lemma 2.3 we have that

$$f^{\star\star}\left(\frac{t}{\gamma}\right) - f^\star(t) \le \frac{\gamma}{2}\frac{1}{t}\int_0^t |f^\star(u) - f^{\star\star}(t)|\,\mathrm{d}u, \tag{4.6}$$

for $t \in (0,1]$ and any $\gamma > 1$. If $t \in (0,1]$, because of Theorem 3 and (4.6) we have that

$$f^{\star\star}\left(\frac{t}{\gamma}\right) - f^{\star\star}(t) \le 2^{n-1}\gamma B_t f^{\star\star}(t) \implies f^{\star\star}\left(\frac{t}{\gamma}\right) \le \left(1 + 2^{n-1}\gamma B_t\right) f^{\star\star}(t), \tag{4.7}$$

We consider now those t for which $t \in \left(0, \frac{1}{2^n\gamma}\right]$. The choice of γ will be made later.

We set $s = \left[\frac{\ln\left(\frac{1}{2^n t}\right)}{\ln(\gamma)}\right] \in \mathbf{N}^\star$. Then we have that $\gamma^s \le \frac{1}{2^n t} < \gamma^{s+1} \implies \gamma^s t > \frac{1}{2^n\gamma}$. As a consequence we produce

$$f^{\star\star}(\gamma^s t) \le f^{\star\star}\left(\frac{1}{2^n\gamma}\right) = 2^n\gamma\int_0^{\frac{1}{2^n\gamma}} f^\star \le 2^n\gamma\int_0^1 f^\star = 2^n\gamma f_{Q_0}. \tag{4.8}$$

Now in view of (4.7) we must have that

$$\begin{aligned} f^{\star\star}\left(\frac{t}{\gamma}\right) &\le \left(1 + 2^{n-1}\gamma B_t\right) f^{\star\star}(t) \le \left(1 + 2^{n-1}\gamma B_t\right)\left(1 + 2^{n-1}\gamma B_{\gamma t}\right) f^\star(2t) \\ &\le \ldots \le \prod_{i=0}^{s}\left(1 + 2^{n-1}\gamma B_{\gamma^i t}\right) f^{\star\star}(\gamma^s t), \end{aligned} \tag{4.9}$$

where s is as above. As a consequence (4.8) and (4.9) give

$$f^{\star\star}\left(\frac{t}{\gamma}\right) \le 2^n\gamma f_{Q_0}\exp\left(2^{n-1}\gamma\sum_{i=0}^{s} B_{\gamma^i t}\right), \tag{4.10}$$

in view of the inequality $1 + x \le \mathrm{e}^x$, which holds for every $x > 0$. By the choice of s we have that $(\gamma^i t)2^n \le 1$, for every $i \in \{0,1,2,\ldots,s\}$. Thus by the definition of the function $t \longmapsto B_t$ we have

$$B_{\gamma^i t} = v\left(f; 2(\gamma^i t)^{\frac{1}{n}}\right), \quad \text{for every } \; i = 0,1,2,\ldots,s. \tag{4.11}$$

Thus

$$\ell_{k,n} = \int_{2(\gamma^k t)^{\frac{1}{n}}}^{2(\gamma^{k+1} t)^{\frac{1}{n}}} v(f;\sigma)\,\frac{\mathrm{d}\sigma}{\sigma} \ge v\Big(f; 2(\gamma^k t)^{\frac{1}{n}}\Big)\Big\{\ln\Big[2(\gamma^{k+1}t)^{\frac{1}{n}}\Big] - \ln\Big[2(\gamma^k t)^{\frac{1}{n}}\Big]\Big\}, \tag{4.12}$$

for every $k \in \{0,1,2,\dots,s\}$, in view of the fact that the function $\sigma \longmapsto v(f;\sigma)$ is non-decreasing. We immediately get from (4.12) that

$$\int_{2(\gamma^k t)^{\frac{1}{n}}}^{2(\gamma^{k+1}t)^{\frac{1}{n}}} v(f;\sigma)\,\frac{\mathrm{d}\sigma}{\sigma} \geq v\left(f;2(\gamma^k t)^{\frac{1}{n}}\right)\ln\left(\gamma^{\frac{1}{n}}\right), \tag{4.13}$$

From (4.10), (4.11) and (4.13) we see that

$$f^{\star\star}\left(\frac{t}{\gamma}\right) \leq 2^n\gamma f_{Q_0}\exp\left[2^{n-1}\gamma\sum_{k=0}^{s-1}\frac{n}{\ln(\gamma)}\int_{2(\gamma^k t)^{\frac{1}{n}}}^{2(\gamma^{k+1}t)^{\frac{1}{n}}} v(f;\sigma)\,\frac{\mathrm{d}\sigma}{\sigma}+2^{n-1}\gamma B_{\gamma^s t}\right]. \tag{4.14}$$

From (4.14) we have as a consequence that

$$f^{\star\star}\left(\frac{t}{\gamma}\right) \leq 2^n\gamma f_{Q_0}\exp\left[2^{n-1}n\frac{\gamma}{\ln(\gamma)}\int_{2t^{\frac{1}{n}}}^{1} v(f;\sigma)\,\frac{\mathrm{d}\sigma}{\sigma}+2^{n-1}\gamma v\left(f;\frac{1}{2^n}\right)\right], \tag{4.15}$$

and this holds for every $t \in \left(0,\frac{1}{2^n\gamma}\right]$ and any $\gamma > 1$. We choose now in (4.15) $\gamma = \mathrm{e}$, so that the function $\gamma \longmapsto \frac{\gamma}{\ln(\gamma)}$, is minimized on $(1,+\infty)$. Then (4.15) implies

$$f^{\star\star}\left(\frac{t}{\mathrm{e}}\right) \leq 2^n\mathrm{e} f_{Q_0}\exp\left[2^{n-1}\mathrm{e}n\int_{2t^{\frac{1}{n}}}^{1} v(f;\sigma)\,\frac{\mathrm{d}\sigma}{\sigma}\right]\exp\left[2^{n-1}\mathrm{e}v\left(f;\frac{1}{2^n}\right)\right], \tag{4.16}$$

for every $t \in \left(0,\frac{1}{2^n\mathrm{e}}\right]$. Certainly $v\left(f;\frac{1}{2^n}\right) \leq 2$. Thus (4.16) gives

$$f^{\star\star}\left(\frac{t}{\mathrm{e}}\right) \leq C_1 f_{Q_0}\exp\left(C_2\int_{C_3' t^{\frac{1}{n}}}^{1} v(f;\sigma)\,\frac{\mathrm{d}\sigma}{\sigma}\right), \tag{4.17}$$

for every $t \in \left(0,\frac{1}{2^n\gamma\mathrm{e}}\right]$, for certain constants C_1, C_2, C_3'. By setting $y = \frac{t}{\mathrm{e}}$ in (4.17), we conclude that for every $y \in \left(0,\frac{1}{2^n\mathrm{e}^2}\right]$ the following inequality holds:

$$f^{\star\star}(y) \leq c_1 f_{Q_0}\exp\left(c_2\int_{c_3 y^{\frac{1}{n}}}^{1} v(f;\sigma)\,\frac{\mathrm{d}\sigma}{\sigma}\right), \tag{4.18}$$

where

$$c_1 = 2^n\mathrm{e}\exp[2^n\mathrm{e}] = 2^n\exp[2^n\mathrm{e}+1], \quad c_2 = 2^{n-1}\mathrm{e}n, \quad \text{and} \quad c_3 = C_3'\mathrm{e}^{\frac{1}{n}}.$$

So by setting $c_4 = \frac{1}{2^n\mathrm{e}^2}$, we derive the proof of our Theorem. □

We are now ready to give the

Proof of Theorem 5. We are given a function $f \in \mathrm{GR}_{\mathcal{D}}(Q_0,\varepsilon)$ for some $\varepsilon\colon 0 < \varepsilon < \frac{1}{2^{n-1}}$, and suppose that $t \in (0,1]$ is fixed. By using Theorem 3 we obtain:

$$\frac{1}{t}\int_0^t |f^\star(u) - f^{\star\star}(t)|\,\mathrm{d}u \leq 2^n\varepsilon f^{\star\star}(t). \tag{4.19}$$

Then by Lemma 2.3 we have in view of (4.13) that

$$f^{\star\star}\left(\frac{t}{\gamma}\right) \leq \left(2^{n-1}\gamma\varepsilon+1\right)f^{\star\star}(t), \quad \text{for any } \gamma > 1.$$

Let now p_0 be the unique $p > 1$ such that $\frac{p^p}{(p-1)^{p-1}} = \frac{1}{2^{n-1}\varepsilon}$. We set $\gamma = \left(\frac{p_0}{p_0-1}\right)^{p_0} \Longrightarrow \gamma^{\frac{1}{p_0}} = \frac{p_0}{p_0-1}$. Then

$$\left(2^{n-1}\gamma\varepsilon+1\right)^{p_0}=\left(2^{n-1}\varepsilon\left(\frac{p_0}{p_0-1}\right)^{p_0}+1\right)^{p_0}=\left(2^{n-1}\varepsilon\frac{1}{p_0-1}\frac{1}{2^{n-1}\varepsilon}+1\right)^{p_0}$$
$$=\left(1+\frac{1}{p_0-1}\right)^{p_0}=\left(\frac{p_0}{p_0-1}\right)^{p_0}=\gamma$$

implies $(2^{n-1}\gamma\varepsilon+1)=\gamma^{\frac{1}{p_0}}$, for the $\gamma>1$ given above. Thus

$$f^{\star\star}\left(\frac{t}{\gamma}\right)\le\gamma^{\frac{1}{p_0}}f^{\star\star}(t),\quad\forall t\in(0,1]. \tag{4.20}$$

Let now $j\in\mathbf{N}$ be such that

$$\gamma^{-j}<t\le\gamma^{-j+1}. \tag{4.21}$$

Then by (4.20) we inductively see that $f^{\star\star}(\gamma^{-k})\le\gamma^{\frac{k}{p_0}}f^{\star\star}(1)$, for any $k\in\mathbf{N}$, so by using (4.21) for our t we conclude that

$$f^{\star\star}(t)\le f^{\star\star}(\gamma^{-j})\le\gamma^{\frac{j}{p_0}}f^{\star\star}(1). \tag{4.22}$$

By (4.21) now $\gamma^{\frac{j}{p_0}}\le\left(\frac{\gamma}{t}\right)^{\frac{1}{p_0}}$. Thus from this last inequality and (4.22) we have that

$$f^{\star\star}(t)\le\frac{\gamma^{\frac{1}{p_0}}}{t^{\frac{1}{p_0}}}f^{\star\star}(1)=\left(\frac{p_0}{p_0-1}\right)f_{Q_0}t^{-\frac{1}{p_0}},$$

and this holds for any $t\in(0,1]$. The proof of Theorem 5 is now complete. □

At last we mention that the proof of Corollary 1, is immediate by the statement of Theorem 5.

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Received 13 June 2016 • Accepted 23 September 2016