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VALUE DISTRIBUTION OF THE SEQUENCES OF THE DERIVATIVES OF ITERATED POLYNOMIALS

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Abstract. We establish the equidistribution of the sequence of the averaged pullbacks of a Dirac measure at any value in $\mathbf{C}\setminus\{0\}$ under the derivatives of the iterations of a polynomials $f\in\mathbf{C}[z]$ of degree more than one towards the f-equilibrium (or canonical) measure μ_f on $\mathbf{P}^1$. We also show that for every C^2 test function on $\mathbf{P}^1$, the convergence is exponentially fast up to a polar subset of exceptional values in $\mathbf{C}$. A parameter space analog of the latter quantitative result for the monic and centered unicritical polynomials family is also established.

1. Introduction

Let $f\in\mathbf{C}[z]$ be a polynomial of degree $d>1$. Let μ_f be the f-equilibrium (or canonical) measure on $\mathbf{P}^1$, which coincides with the harmonic measure $\mu_{K(f)}$ on the filled-in Julia set $K(f)$ of f with respect to ∞. The exceptional set $E(f):=\{a\in\mathbf{P}^1\colon \#\bigcup_{n\in\mathbf{N}}f^{-n}(a)<\infty\}$ of f contains ∞ and $\#E(f)\le 2$. Brolin [2, Theorem 16.1] studied the value distribution of the sequence $(f^n\colon\mathbf{P}^1\to\mathbf{P}^1)$ of the iterations of f, and established

$$\left\{a\in\mathbf{P}^1\colon \lim_{n\to\infty}\frac{(f^n)^*\delta_a}{d^n}=\mu_f \text{ weakly on } \mathbf{P}^1\right\}=\mathbf{P}^1\setminus E(f), \tag{1.1}$$

which is more precise than the classical inclusion $\partial K(f)\subset\overline{\bigcup_{n\in\mathbf{N}}f^{-n}(a)}$ for every $a\in\mathbf{P}^1\setminus E(f)$. Here for every $h\in\mathbf{C}(z)$ of degree >0 and every Radon measure ν on $\mathbf{P}^1$, the pullback $h^*\nu$ of ν under h is a Radon measure on $\mathbf{P}^1$ so that for every $a\in\mathbf{P}^1$, when $\nu=\delta_a$, $h^*\delta_a=\sum_{w\in h^{-1}(a)}(\deg_w h)\delta_a$ on $\mathbf{P}^1$. Pursuing the analogy between the roles played by $E(f)$ in (1.1) and by the set of *Valiron exceptional values* in $\mathbf{P}^1$ of a transcendental meromorphic function on $\mathbf{C}$, Sodin [20], Russakovskii–Sodin [19], and Russakovskii–Shiffman [18] (see also [7], [15]) studied the value distribution of a *sequence of* rational maps between projective spaces from the viewpoint of *Nevanlinna theory*, in a quantitative way (cf. [22, Chapter V, §2]). Gauthier and Vigny [10, 1. in Theorem A] studied the value distribution of the sequence $((f^n)'\colon\mathbf{P}^1\to\mathbf{P}^1)$ of the *derivatives* of iterations of a polynomial $f\in\mathbf{C}[z]$ of degree >1 (cf. [23]) possibly with a polar subset of exceptional values in $\mathbf{C}\setminus\{0\}$, in terms of dynamics of the *tangent map* $F(z,w):=(f(z),f'(z)w)$ on the tangent bundle $T\mathbf{C}$. The aim of this article is to improve their result in two ways.

The first improvement of [10, 1. in Theorem A] is qualitative, but with no exceptional values.

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Theorem 1. *Let $f \in \mathbf{C}[z]$ be of degree $d > 1$. Then for every $a \in \mathbf{C} \setminus \{0\}$,*

$$\lim_{n\to\infty} \frac{((f^n)')^*\delta_a}{d^n - 1} = \mu_f$$

weakly on $\mathbf{P}^1$.

In Theorem 1, the values $a = 0, \infty$ are excluded since it is clear that for every $n \in \mathbf{N}$, $((f^n)')^*\delta_\infty/(d^n - 1) = \delta_\infty (\neq \mu_f)$, and it immediately follows from (1.1) and the chain rule that $\lim_{n\to\infty}((f^n)'\delta_0)/(d^n - 1) = \mu_f$ weakly on $\mathbf{P}^1$ if and only if $E(f) = \{\infty\}$. In Gauthier–Vigny [10, 2. and 3. in Theorem A], they also established a result similar to Theorem 1 under the assumption that f has no Siegel disks (or the assumption that f is hyperbolic). Our proof of Theorem 1 is independent of their argument even in those cases.

The second improvement of [10, 1. in Theorem A] is quantitative, but with an at most polar subset of exceptional values in $\mathbf{C}$.

Theorem 2. *Let $f \in \mathbf{C}[z]$ be of degree $d > 1$, and suppose that $E(f) = \{\infty\}$. Then for every $\eta > \sup_{z\in\mathbf{C}:\text{ superattracting periodic point of } f} \limsup_{n\to\infty}(\deg_z(f^n))^{1/n}$, there is a polar subset $E = E_{f,\eta}$ in $\mathbf{C}$ such that for every $a \in \mathbf{C} \setminus E$ and every C^2-test function ϕ on $\mathbf{P}^1$,*

$$\int_{\mathbf{P}^1} \phi\, \mathrm{d}\left(\frac{((f^n)')^*\delta_a}{d^n - 1} - \mu_f\right) = o((\eta/d)^n)$$

as $n \to \infty$.

The proof of Theorem 2 is based on Russakovskii–Shiffman [18] mentioned above, and on an improvement of it for the *sequence of the iterations* of a rational function of degree > 1 by Drasin and the author [6] (see also [4] and [21] in higher dimensions).

Remark 1.1. Under the assumption $E(f) = \{\infty\}$ in Theorem 2, we have $\sup_{z\in\mathbf{C}:\text{ superattracting periodic point of } f} \limsup_{n\to\infty}(\deg_z(f^n))^{1/n} \in \{1, 2, \ldots, d-1\}$, and $= 1$ if and only if there is no superattracting cycles of f in $\mathbf{C}$. Here we adopt the convention $\sup_\emptyset = 1$. In the case that $E(f) \neq \{\infty\}$, we point out the following better estimate than that in Theorem 2

$$\int_{\mathbf{P}^1} \phi\, \mathrm{d}\left(\frac{((f^n)')^*\delta_a}{d^n - 1} - \mu_f\right) = O(nd^{-n}) \quad \text{as } n \to \infty$$

for every $a \in \mathbf{C} \setminus \{0\}$ and every C^2-test function ϕ on $\mathbf{P}^1$, with no exceptional values; indeed, we can assume that $f(z) = z^d$ without loss of generality (see Remark 3.1), and then $f^n(z) = z^{d^n}$ for every $n \in \mathbf{N}$ and μ_f is the normalized Lebesgue measure $m_{\partial\mathbf{D}}$ on the unit circle $\partial\mathbf{D} = \partial K(f)$. For every $a = re^{i\theta}$ ($r > 0, \theta \in \mathbf{R}$), every C^1-test function ϕ on $\mathbf{P}^1$, and every $n \in \mathbf{N}$, we have $\left|\int_{\mathbf{P}^1} \phi \mathrm{d}(((f^n)')^*\delta_a - \sum_{j=1}^{d^n-1} \delta_{e^{i(\theta+j\cdot 2\pi)/(d^n-1)}})/(d^n - 1)\right| \leq \|\phi\|_{C^1} \cdot \left|e^{(\log(rd^{-n}))/(d^n-1)} - 1\right| \leq \|\phi\|_{C^1} \cdot Cnd^{-n}$ for some $C > 0$ independent of both ϕ and n, and if ϕ is C^2, then by the *midpoint method* in numerically computing definite integrals, we also have $\left|\int_{\mathbf{P}^1} \phi\, \mathrm{d}\left(\sum_{j=1}^{d^n-1} \delta_{e^{i(\theta+j\cdot 2\pi)/(d^n-1)}}/(d^n - 1) - m_{\partial\mathbf{D}}\right)\right| \leq \|\phi\|_{C^2} \cdot C'd^{-n}$ for some $C' > 0$ independent of both ϕ and n.

Finally, let us focus on the (monic and centered) *unicritical polynomials family*

$$f\colon \mathbf{C} \times \mathbf{P}^1 \ni (\lambda, z) \mapsto z^d + \lambda =: f_\lambda(z) \in \mathbf{P}^1 \tag{1.2}$$

of degree $d > 1$. The parameter space analog of Theorem 1 for the sequence $((f_\lambda^n)'(\lambda))$ in $\mathbf{C}[\lambda]$ of the derivative of f_λ^n at its unique critical value $z = \lambda$ in $\mathbf{C}$ is also obtained by Gauthier–Vigny [10, Theorem 3.7]. We will also establish a parameter space analog of Theorem 2.

Theorem 3. *Let f be the monic and centered unicritical polynomials family of degree $d > 1$ defined as in* (1.2)*. Then for every $\eta > 1$, there is a polar subset $E = E_{f,\eta}$ in $\mathbf{C}$ such that for every $a \in \mathbf{C} \setminus E$ and every C^2-test function ϕ on $\mathbf{P}^1$,*

$$\int_{\mathbf{P}^1} \phi(\lambda)\,\mathrm{d}\left(\frac{((f_\lambda^n)'(\lambda))^*\delta_a}{d^n-1} - \mu_{C_d}\right)(\lambda) = O((\eta/d)^n)$$

as $n \to \infty$. Here C_d is the connectedness locus of the family f in the parameter space $\mathbf{C}$ and μ_{C_d} is the harmonic measure on C_d with pole ∞.

The proof of Theorem 3 is based on Russakovskii–Shiffman [18] mentioned above, and on a quantitative equidistribution of superattracting parameters by Gauthier–Vigny [9].

In Section 2, we recall a background from complex dynamics. In Sections 3, 4, and 5, we show Theorems 1, 2, and 3, respectively.

Notation 1.2. We adopt the convention $\mathbf{N} = \mathbf{Z}_{>0}$. For every $a \in \mathbf{C}$ and every $r > 0$, set $\mathbf{D}(a,r) := \{z \in \mathbf{C} \colon |z-a| < r\}$. Let δ_z be the Dirac measure on $\mathbf{P}^1$ at each $z \in \mathbf{P}^1$. Let $[z,w]$ be the chordal metric on $\mathbf{P}^1$ normalized as $[\cdot,\infty] = 1/\sqrt{1+|\cdot|^2}$ on $\mathbf{P}^1$ (following the notation in Nevanlinna's and Tsuji's books [14, 22]). Let ω be the Fubini–Study area element on $\mathbf{P}^1$ normalized as $\omega(\mathbf{P}^1) = 1$. The Laplacian dd^c on $\mathbf{P}^1$ is normalized as $\mathrm{dd}^c(-\log[\cdot,\infty]) = \omega - \delta_\infty$ on $\mathbf{P}^1$.

2. Background

2.1. Dynamics of rational functions. Let $f \in \mathbf{C}(z)$ be of degree $d > 1$. Let $C(f)$ be the critical set of f. The Julia and Fatou sets of f are defined by $J(f) := \{z \in \mathbf{P}^1 \colon \text{the family } (f^n \colon \mathbf{P}^1 \to \mathbf{P}^1)_{n\in\mathbf{N}} \text{ is not normal at } z\}$ and $F(f) := \mathbf{P}^1 \setminus J(f)$, respectively. A component of $F(f)$ is called a *Fatou component* of f. A Fatou component U of f is mapped by f properly onto a Fatou component of f. A Fatou component U of f is said to be *cyclic* if there is $n \in \mathbf{N}$ such that $f^n(U) = U$. For more details on complex dynamics, see e.g. Milnor's book [13].

The *f-equilibrium* (*or canonical*) *measure* μ_f on $\mathbf{P}^1$ is the unique probability Radon measure ν on $\mathbf{P}^1$ such that

$$f^*\nu = d \cdot \nu \quad \text{on } \mathbf{P}^1 \tag{2.1}$$

and that $\nu(\{a\}) = 0$ for every $a \in E(f)$; the *exceptional set* of f is $E(f) := \{a \in \mathbf{P}^1 \colon \#\bigcup_{n\in\mathbf{N}} f^{-n}(a) < \infty\} = \{a \in \mathbf{P}^1 \colon f^{-2}(a) = \{a\}\}$. Then in fact $\operatorname{supp}\mu_f = J(f)$, and for every $n \in \mathbf{N}$, $\mu_{f^n} = \mu_f$ on $\mathbf{P}^1$. For more details, see Brolin [2], Lyubich [12], Freire–Lopes–Mañé [8].

2.2. Dynamics of polynomials. Let $f \in \mathbf{C}[z]$ be of degree $d > 1$. We note that $\infty \in E(f)$, $\#(C(f) \cap \mathbf{C}) \le d-1$, and $C(f) \cap \mathbf{C} = (\operatorname{supp} \mathrm{dd}^c \log|f'|) \cap \mathbf{C}$.

The filled-in Julia set $K(f)$ of f is defined by

$$K(f) := \{z \in \mathbf{C} \colon \limsup_{n\to\infty} |f^n(z)| < \infty\},$$

whose complement in $\mathbf{P}^1$ coincides with the immediate superattractive basin

$$I_\infty(f) := \{z \in \mathbf{P}^1 \colon \lim_{n\to\infty} f^n(z) = \infty\}$$

of the superattracting fixed point ∞ of f; in particular, $\lim_{n\to\infty} f^n = \infty$ locally uniformly on $I_\infty(f)$, and $K(f)$ is a compact subset in $\mathbf{C}$. We note that $F(f) = I_\infty(f) \cup \operatorname{int} K(f)$ and that $J(f) = \partial K(f)$.

By a standard telescope argument, there exists the locally uniform limit

$$g_f := \lim_{n\to\infty} \frac{-\log[f^n(\cdot),\infty]}{d^n}$$

on $\mathbf{C}$. Setting $g_f(\infty) := +\infty$, we have $g_f \circ f = d \cdot g_f$ on $\mathbf{P}^1$, and for every $n \in \mathbf{N}$, we also have $g_{f^n} = g_f$ on $\mathbf{P}^1$. The restriction of g_f to $I_\infty(f)$ coincides with the Green function on $I_\infty(f)$ with pole ∞, and the measure

$$\mu_{K(f)} := \mathrm{dd}^c g_f + \delta_\infty \quad \text{on } \mathbf{P}^1$$

coincides with the harmonic measure on $K(f)$ with pole ∞. In particular, $\operatorname{supp}\mu_{K(f)} \subset \partial K(f)$, and in fact $\mu_{K(f)} = \mu_f$ on $\mathbf{P}^1$. The function $z \mapsto g_f(z) - \log|z|$ extends harmonically to an open neighborhood of ∞ in $I_\infty(f)$ so the function $z \mapsto -\log[z,\infty] - g_f(z)$ extends continuously to $\mathbf{P}^1$.

The following is substantially shown in Buff [3, the proof of Theorem 4].

Theorem 2.1. (Buff) *Let $f \in \mathbf{C}[z]$ be of degree $d > 1$, and let $z_0 \in \mathbf{C}$. If $g_f(z_0) \geq \max_{c\in C(f)\cap\mathbf{C}} g_f(c)$, then $|f'(z_0)| \leq d^2 \cdot e^{(d-1)g_f(z_0)}$, and the equality never holds if $(C(f)\cap\mathbf{C})\cap I_\infty(f) \neq \emptyset$.*

For more details on polynomial dynamics and potential theory, see Brolin [2, Chapter III], and also Ransford's book [17].

3. Proof of Theorem 1

Let $f \in \mathbf{C}[z]$ be of degree $d > 1$. For every $a \in \mathbf{C}$ and every $n \in \mathbf{N}$, the functions $(\log|(f^n)' - a|)/(d^n - 1) - g_f$ and $(\log\max\{1, |(f^n)'|\})/(d^n - 1) - g_f$ extend continuously to $\mathbf{P}^1$. Set $a_d = a_d(f) := \lim_{n\to\infty} f(z)/z^d \in \mathbf{C}\setminus\{0\}$.

Remark 3.1. Since the question is affine invariant, we could assume $|a_d| = 1$ without loss of generality, by replacing f with $c^{-1}\circ f\circ c$ for such $c \in \mathbf{C}\setminus\{0\}$ that $c^{d-1} = a_d^{-1}$ if necessary (for every $c \in \mathbf{C}\setminus\{0\}$, $z \mapsto c\cdot z$ is also denoted by c). In this article, we would not normalize f as $|a_d| = 1$ in order to make it explicit which computations would be independent of such a normalization.

Lemma 3.2. *On $I_\infty(f)\setminus\bigcup_{n\in\mathbf{N}\cup\{0\}} f^{-n}(C(f)\cap\mathbf{C})$,*

$$\lim_{n\to\infty}\left(\frac{\log|(f^n)'|}{d^n-1} - g_f\right) = 0$$

locally uniformly.

Proof. For every $n \in \mathbf{N}$ and every $z \in \mathbf{C}$, by a direct calculation, we have

$$\begin{aligned}
&\frac{\log|(f^n)'(z)|}{d^n-1} - \frac{\log\left|d^n\cdot a_d^{(d^n-1)/(d-1)}\right|}{d^n-1} = \frac{1}{d^n-1}\int_{\mathbf{C}} \log|z-u|(\mathrm{dd}^c\log|(f^n)'|)(u)\\
&= \frac{1}{d^n-1}\int_{\mathbf{C}}\sum_{j=0}^{n-1}\left(\int_{\mathbf{C}}\log|z-\cdot|\mathrm{d}((f^j)^*\delta_w)\right)(\mathrm{dd}^c\log|f'|)(w)\\
&= \frac{1}{d^n-1}\int_{\mathbf{C}}\sum_{j=0}^{n-1}\left(\log|f^j(z)-w| - \log|a_d|^{(d^j-1)/(d-1)}\right)(\mathrm{dd}^c\log|f'|)(w) \qquad (3.1)\\
&= \frac{1}{d^n-1}\int_{\mathbf{C}}\sum_{j=0}^{n-1}(\log[f^j(z),w] - \log[f^j(z),\infty] - \log[w,\infty])(\mathrm{dd}^c\log|f'|)(w)\\
&\quad - \log|a_d|^{\frac{1}{d-1}-\frac{n}{d^n-1}}.
\end{aligned}$$

Then noting that $g_f \circ f = d \cdot g_f$ on $\mathbf{P}^1$, for every $n \in \mathbf{N}$ and every $z \in \mathbf{P}^1$, we have

$$\begin{aligned}\frac{\log|(f^n)'(z)|}{d^n-1} - g_f(z) &= \frac{1}{d^n-1}\int_{\mathbf{C}}\left(\sum_{j=0}^{n-1}\log[f^j(z),w]\right)(\mathrm{dd}^c\log|f'|)(w)\\ &+ \frac{d-1}{d^n-1}\sum_{j=0}^{n-1}\left(-\log[f^j(z),\infty] - g_f(f^j(z))\right)\\ &+\left(-\int_{\mathbf{C}}\log[w,\infty](\mathrm{dd}^c\log|f'|)(w) + \log d + \log|a_d|\right)\frac{n}{d^n-1},\end{aligned} \tag{3.2}$$

which with $\sup_{z\in\mathbf{P}^1}|-\log[z,\infty] - g_f(z)| < \infty$ completes the proof. □

Lemma 3.3. *There is $C = C_f > 0$ such that for every $n \in \mathbf{N}$ and every $z \in \mathbf{P}^1$,*

$$\frac{\log\max\{1,|(f^n)'(z)|\}}{d^n-1} - g_f(z) \le \frac{Cn}{d^n-1}. \tag{3.3}$$

Proof. Set

$$\begin{aligned}C = C_f :=& (d-1)\cdot\sup_{z\in\mathbf{P}^1}|-\log[z,\infty] - g_f(z)|\\ &+ (d-1)\cdot\sup_{w\in C(f)\cap\mathbf{C}}|\log[w,\infty]| + \log d + |\log|a_d|| \in \mathbf{R}_{>0}.\end{aligned} \tag{3.4}$$

Then for every $n \in \mathbf{N}$ and every $z \in \mathbf{C}$, from (3.2), we have $|(f^n)'(z)| \le e^{Cn}\cdot e^{(d^n-1)g_f(z)}$, which with $g_f \ge 0$ on $\mathbf{P}^1$ completes the proof. □

We note that $\max_{c\in\bigcup_{n\in\mathbf{N}\cup\{0\}} f^{-n}(C(f)\cap\mathbf{C})} g_f(c) = \max_{c\in C(f)\cap\mathbf{C}} g_f(c) < \infty$ by $g_f\circ f = d\cdot g_f$ on $\mathbf{P}^1$.

Lemma 3.4. *For every $a \in \mathbf{C}\setminus\{0\}$,*

$$\lim_{n\to\infty}\int_{\mathbf{P}^1}\left|\frac{\log|(f^n)'-a|}{d^n-1} - g_f\right|\mathrm{d}\omega = 0.$$

Proof. Fix $a \in \mathbf{C}\setminus\{0\}$. The sequence $((\log|(f^n)'-a|)/(d^n-1))$ of subharmonic functions on $\mathbf{C}$ is locally uniformly bounded from above on $\mathbf{C}$; indeed, by the chain rule and $\liminf_{z\to\infty}|f'(z)| = +\infty$, for every $R > 0$ so large that $\{|z| = R\} \subset I_\infty(f)\setminus\bigcup_{n\in\mathbf{N}\cup\{0\}} f^{-n}(C(f)\cap\mathbf{C})$, we have $\liminf_{n\to\infty}\inf_{|z|=R}|(f^n)'(z)| = +\infty$, which with the maximum modulus principle yields $\sup_{|z|\le R}|(f^n)'(z) - a| \le \sup_{|z|=R} 2|(f^n)'(z)|$ for every $n \in \mathbf{N}$ large enough. Then by Lemma 3.3, we have $\limsup_{n\to\infty}\sup_{|z|\le R}(\log|(f^n)'-a|)/(d^n-1) \le \sup_{|z|=R} g_f(z) < \infty$. By Lemma 3.2 and $g_f > 0$ on $I_\infty(f)$, for every compact subset C in $I_\infty(f)\setminus\bigcup_{n\in\mathbf{N}\cup\{0\}} f^{-n}(C(f)\cap\mathbf{C})$, we also have $1/2 \le |((f^n)'-a)/(f^n)'| \le 2$ on C for every $n \in \mathbf{N}$ large enough, so in particular

$$\lim_{n\to\infty}\left(\frac{\log|(f^n)'-a|}{d^n-1} - g_f\right) = \lim_{n\to\infty}\left(\frac{\log|(f^n)'|}{d^n-1} - g_f\right) = 0 \tag{3.5}$$

locally uniformly on $I_\infty(f)\setminus\bigcup_{n\in\mathbf{N}\cup\{0\}} f^{-n}(C(f)\cap\mathbf{C})$.

Let m_2 be the Lebesgue measure on $\mathbf{C}$. By a *compactness principle* for a locally uniformly upper bounded sequence of subharmonic functions on a domain in $\mathbf{R}^m$ which is not locally uniformly convergent to $-\infty$ (see Azarin [1, Theorem 1.1.1], Hörmander's book [11, Theorem 4.1.9(a)]), we can choose a sequence (n_j) in $\mathbf{N}$ tending to $+\infty$ as $j\to\infty$ such that the $L^1_{\mathrm{loc}}(\mathbf{C}, m_2)$-limit $\phi := \lim_{j\to\infty}(\log|(f^{n_j})'-a|)/(d^{n_j}-1)$ exists and is subharmonic on $\mathbf{C}$. Choosing a subsequence of (n_j) if necessary, we have $\phi = \lim_{j\to\infty}(\log|(f^{n_j})' - a|)/(d^{n_j}-1)$ Lebesgue a.e. on $\mathbf{C}$. Then by (3.5), we have $\phi \equiv g_f$ Lebesgue a.e. on $\mathbf{C}\setminus(K(f)\cup\bigcup_{n\in\mathbf{N}\cup\{0\}} f^{-n}(C(f)\cap\mathbf{C}))$, and in turn on $\mathbf{C}\setminus K(f)$ by the subharmonicity of ϕ and the harmonicity of g_f there. Let us show that $\phi = g_f$ Lebesgue a.e. on the whole $\mathbf{C}$, and then $\lim_{n\to\infty}(\log|(f^n)'-a|)/(d^n-1) = g_f$ in $L^1_{\mathrm{loc}}(\mathbf{C}, m_2)$, which with the locally uniform convergence (3.5) will complete the proof since $\max_{c\in\bigcup_{n\in\mathbf{N}\cup\{0\}} f^{-n}(C(f)\cap\mathbf{C})} g_f(c) < \infty$ and the Radon–Nikodym derivative $\mathrm{d}\omega/\mathrm{d}m_2$ is continuous so locally bounded on $\mathbf{C}$.

By $\log(1/[w,\infty]) - \log\max\{1,|w|\}) \le \log\sqrt{2}$ on $\mathbf{C}$ and Lemma 3.3, for every $n\in\mathbf{N}$, we have

$$\frac{\log|(f^n)'-a|}{d^n-1} - g_f = \frac{\log[(f^n)',a]}{d^n-1} + \left(\frac{\log(1/[(f^n)',\infty])}{d^n-1} - g_f\right) + \frac{\log(1/[a,\infty])}{d^n-1}$$
$$\le \frac{C_f\cdot n}{d^n-1} + \frac{\log\sqrt{2}+\log(1/[a,\infty])}{d^n-1}$$

on $\mathbf{C}$, so $\phi \le g_f$ Lebesgue a.e. on $\mathbf{C}$ and in turn on $\mathbf{C}$ by the subharmonicity of ϕ and the continuity of g_f on $\mathbf{C}$. Hence $\phi - g_f$ is ≤ 0 and is upper semicontinuous on $\mathbf{C}$.

Now suppose to the contrary that the open subset $\{z\in\mathbf{C}\colon \phi(z) < g_f(z)\}$ in $\mathbf{C}$ is non-empty. Then by $\phi\equiv g_f$ on $\mathbf{C}\setminus K(f)$, there is a bounded Fatou component U of f containing a component W of $\{z\in\mathbf{C}\colon \phi(z) < g_f(z)\}$. Since $\phi \le g_f = 0$ on $U\subset K(f)$, by the maximum principle for subharmonic functions, we in fact have $U = W$.

Taking a subsequence of (n_j) if necessary, we can assume that $(f^{n_j}|U)$ is locally uniformly convergent to a holomorphic function g on U as $j\to\infty$ without loss of generality. We claim that $g'\equiv a$ on U, so we can say $g\in\mathbf{C}[z]$; indeed, fixing a domain $D\Subset U = W$, by a version of Hartogs's lemma on subharmonic functions (see Hörmander's book [11, Theorem 4.1.9(b)]) and the upper semicontinuity of ϕ, we have $\limsup_{n\to\infty}\sup_{\overline{D}}(\log|(f^{n_j})'-a|)/(d^{n_j}-1) \le \sup_{\overline{D}}\phi < 0$. Hence $g' = (\lim_{j\to\infty} f^{n_j})' = \lim_{j\to\infty}(f^{n_j})' \equiv a$ on D, so $g'\equiv a$ on U by the identity theorem for holomorphic functions.

Hence, under the assumption that $a\neq 0$, the locally uniform limit g on U is non-constant. So by Hurwitz's theorem and the classification of cyclic Fatou components, there is $N\in\mathbf{N}$ such that $V := f^{n_N}(U) = g(U)(\supset g(\overline{D}))$ is a Siegel disk of f and, setting $p := \min\{n\in\mathbf{N}\colon f^n(V) = V\}$, that $p|(n_j - n_N)$ for every $j\ge N$. We can fix a holomorphic injection $h\colon V\to\mathbf{C}$ such that for some $\alpha\in\mathbf{R}\setminus\mathbf{Q}$, setting $\lambda := e^{2i\pi\alpha}$, we have $h\circ f^p = \lambda\cdot h$ on V, so for every $j\ge N$, $h\circ f^{n_j} = \lambda^{(n_j-n_N)/p}\cdot(h\circ f^{n_N})$ on U. Then taking a subsequence of (n_j) if necessary, there also exists the limit

$$\lambda_0 := \lim_{j\to\infty}\lambda^{(n_j-n_N)/p}$$

in $\partial\mathbf{D}$, so that $h\circ g = \lim_{j\to\infty} h\circ f^{n_j} = \lambda_0\cdot(h\circ f^{n_N})$ on U. In particular,

$$h\circ f^{n_j} - h\circ g = (\lambda^{(n_j-n_N)/p} - \lambda_0)\cdot(h\circ f^{n_N}) \tag{3.6}$$

on U. Set $w_0 := h^{-1}(0)\in V$, so that $f^p(w_0) = w_0$, and fix $z_0\in f^{-n_N}(w_0)\cap U$, so that $f^{n_j}(z_0) = w_0$ for every $j\ge N$ and $g(z_0) = \lim_{j\to\infty} f^{n_j}(z_0) = w_0$.

We claim that

$$\frac{\log|(f^{n_j})'(z_0)-a|}{d^{n_j}-1} = \frac{\log|\lambda^{(n_j-n_N)/p}-\lambda_0|}{d^{n_j}-1} + O(d^{-n_j}) \tag{3.7}$$

as $j \to \infty$; for, by the chain rule applied to both sides in (3.6) and $h'(w_0) \neq 0$ (and $g'(z_0) = a$), we have

$$(3.6') \qquad (f^{n_j})'(z_0) - a = (\lambda^{(n_j - n_N)/p} - \lambda_0) \cdot (f^{n_N})'(z_0),$$

which also yields $(f^{n_N})'(z_0) \neq 0$ by $(f^{n_j})'(z_0) = (f^{n_j - n_N})'(w_0) \cdot (f^{n_N})'(z_0)$ and the assumption $a \neq 0$. We also claim that

$$(3.8) \qquad \liminf_{j \to \infty} \frac{1}{d^{n_j}} \log |\lambda^{(n_j - n_N)/p} - \lambda_0| \geq 0$$

(cf. [16, Proof of Theorem 3]); indeed, for every domain $D \Subset U \backslash f^{-n_N}(w_0)$, since h^{-1} is Lipschitz continuous on $h(\bigcup_{n \in \mathbf{N}} (f^p)^n (f^{n_N}(D))) \cup g(D)) \Subset h(V)$ and $\sup_D |h \circ f^{n_N}| > 0$, from (3.6), we observe that

$$(*) \qquad \frac{1}{d^{n_j}} \sup_D \log |f^{n_j} - g| \leq \frac{1}{d^{n_j}} \log |\lambda^{(n_j - n_N)/p} - \lambda_0| + O(d^{-n_j})$$

as $j \to \infty$. On the other hand, for every domain $\tilde{D}$ intersecting ∂U in $\mathbf{C}$, fixing $\tilde{z} \in \tilde{D} \cap I_\infty(f) \neq \emptyset$, we observe that

$$(**) \qquad \liminf_{j \to \infty} \frac{1}{d^{n_j}} \sup_{\tilde{D}} \log |f^{n_j} - g| \geq g_f(\tilde{z}) > 0.$$

Now fix $z_1 \in U$ and $z' \in \partial U$ such that $\mathbf{D}(z_1, |z' - z_1|) \subset U \setminus f^{-n_N}(w_0)$. Then for every $\epsilon \in (0, |z' - z_1|)$, using Cauchy's estimate applied to $f^{n_j} - g \in \mathbf{C}[z]$ around z_1, we have

$$\begin{aligned} |f^{n_j} - g| &\leq \sum_{k=0}^{d^{n_j}} \frac{\sup_{\partial \mathbf{D}(z_1, |z' - z_1| - \epsilon)} |f^{n_j} - g|}{(|z' - z_1| - \epsilon)^k} | \cdot - z_1|^k \\ &\leq \left(\sup_{\mathbf{D}(z_1, |z' - z_1| - \epsilon)} |f^{n_j} - g| \right) \cdot \sum_{k=0}^{d^{n_j}} \left(\frac{|z' - z_1| + \epsilon}{|z' - z_1| - \epsilon} \right)^k \end{aligned}$$

on $\mathbf{D}(z', \epsilon)$, so since $z' \in \mathbf{D}(z', \epsilon) \cap \partial U$ and $\mathbf{D}(z_1, |z' - z_1| - \epsilon) \Subset U \setminus f^{-n_N}(w_0)$, by (**) and (*), we have

$$\begin{aligned} 0 < \Bigg(&\liminf_{j \to \infty} \frac{1}{d^{n_j}} \log \sup_{\mathbf{D}(z', \epsilon)} |f^{n_j} - g| \\ &\leq \liminf_{j \to \infty} \frac{1}{d^{n_j}} \log \sup_{\mathbf{D}(z_1, |z' - z_1| - \epsilon)} |f^{n_j} - g| + \log \frac{|z' - z_1| + \epsilon}{|z' - z_1| - \epsilon} \\ \leq \Bigg) &\liminf_{j \to \infty} \frac{1}{d^{n_j}} \log |\lambda^{(n_j - n_N)/p} - \lambda_0| + \log \frac{|z' - z_1| + \epsilon}{|z' - z_1| - \epsilon}. \end{aligned}$$

This yields (3.8) as $\epsilon \to 0$.

Once (3.7) and (3.8) are at our disposal, using a version of Hartogs's lemma on subharmonic functions again, we have

$$\phi(z_0) \geq \limsup_{j \to \infty} \frac{\log |(f^{n_j})'(z_0) - a|}{d^{n_j} - 1} \geq \liminf_{j \to \infty} \frac{\log |\lambda^{(n_j - n_N)/p} - \lambda_0|}{d^{n_j} - 1} \geq 0,$$

which contradicts $\phi < g_f = 0$ on $U = W$. $\square$

For every $a \in \mathbf{C} \setminus \{0\}$ and every C^2-test function ϕ on $\mathbf{P}^1$, by Lemma 3.4, we have

$$\left|\int_{\mathbf{P}^1} \phi\, \mathrm{d}\left(\frac{((f^n)')^*\delta_a}{d^n-1} - \mu_f\right)\right| = \left|\int_{\mathbf{P}^1} \phi\, \mathrm{dd}^c\left(\frac{\log|(f^n)'(\cdot)-a|}{d^n-1} - g_f\right)\right|$$
$$\le \left(\sup_{\mathbf{P}^1}\left|\frac{\mathrm{dd}^c\phi}{\mathrm{d}\omega}\right|\right) \cdot \int_{\mathbf{P}^1}\left|\frac{\log|(f^n)'(z)-a|}{d^n-1} - g_f\right| \mathrm{d}\omega(z) \to 0 \quad \text{as } n \to \infty,$$

where the Radon–Nikodym derivative $(\mathrm{dd}^c\phi)/\mathrm{d}\omega$ on $\mathbf{P}^1$ is bounded on $\mathbf{P}^1$. □

4. Proof of Theorem 2

Let $f \in \mathbf{C}[z]$ be of degree $d > 1$, and suppose that $E(f) = \{\infty\}$. Then

$$\sup_{z\in\mathbf{C}\colon \text{superattracting periodic point of } f} \limsup_{n\to\infty}(\deg_z(f^n))^{1/n}$$
$$= \sup_{c\in C(f)\cap\mathbf{C}\colon \text{periodic under } f} \limsup_{n\to\infty}(\deg_c(f^n))^{1/n} \in \{1, 2, \ldots, d-1\}$$

(recall the convention $\sup_\emptyset = 1$). Set $a_d := a_d(f) = \lim_{n\to\infty} f(z)/z^d \in \mathbf{C}\setminus\{0\}$. For every $n \in \mathbf{N}$, the functions $(\log(1/[(f^n)',\infty])/(d^n-1) - g_f$ and $(\log\max\{1, |(f^n)'|\})/(d^n-1) - g_f$ extend continuously to $\mathbf{P}^1$.

Lemma 4.1. *For every* $\eta > \sup_{c\in C(f)\cap\mathbf{C}\colon \text{periodic under } f} \limsup_{n\to\infty}(\deg_c(f^n))^{1/n}$,

$$\int_{\mathbf{P}^1}\left|\frac{\log(1/[(f^n)',\infty])}{d^n-1} - g_f\right| \mathrm{d}\omega = o((\eta/d)^n)$$

as $n \to \infty$.

Proof. For every $n \in \mathbf{N}$, from (3.2), we have

$$\begin{aligned}&\int_{\mathbf{P}^1}\left|\frac{\log|(f^n)'(z)|}{d^n-1} - g_f(z)\right| \mathrm{d}\omega(z)\\ &\le \frac{1}{d^n-1}\int_{\mathbf{C}}\left(\sum_{j=0}^{n-1}\int_{\mathbf{P}^1}\log\frac{1}{[f^j(z),w]}\,\mathrm{d}\omega(z)\right)(\mathrm{dd}^c\log|f'|)(w) + \frac{C_f\cdot n}{d^n-1},\end{aligned} \tag{4.1}$$

where $C_f > 0$ is defined in (3.4). By [6, Theorem 2], for every $\eta > \sup_{c\in C(f)\cap\mathbf{C}\colon \text{periodic under } f} \limsup_{n\to\infty}(\deg_c(f^n))^{1/n}$ and every $w \in \mathbf{C}(= \mathbf{P}^1 \setminus E(f)$ under the assumption $E(f) = \{\infty\})$, we have

$$\int_{\mathbf{P}^1}\log\frac{1}{[f^n(z),w]}\,\mathrm{d}\omega(z) = o(\eta^n)$$

as $n \to \infty$, which with Lemma 3.3 and $0 \le \log(1/[w,\infty]) - \log\max\{1,|w|\} \le \log\sqrt{2}$ on $\mathbf{C}$ completes the proof. □

Lemma 4.2. *For every* $\eta > 1$*, the Valiron exceptional set*

$$E_V(((f^n)'),(\eta^n)) := \left\{a \in \mathbf{P}^1\colon \limsup_{n\to\infty}\frac{1}{\eta^n}\int_{\mathbf{P}^1}\log\frac{1}{[(f^n)'(z),a]}\,\mathrm{d}\omega(z) > 0\right\}$$

of the sequence $((f^n)')$ *of the derivatives of the iterations of* f *with respect to the sequence* (η^n) *in* $\mathbf{R}_{>0}$ *is a polar subset in* $\mathbf{P}^1$.

Proof. This is an application of Russakovskii–Shiffman [18, Proposition 6.2] to the sequence $((f^n)')$ in $\mathbf{C}[z]$ since $\sum_{n\in\mathbf{N}} 1/\eta^n < \infty$ for every $\eta > 1$. □

For every $\eta > \sup_{c\in C(f)\cap\mathbf{C}\colon \text{periodic under } f} \limsup_{n\to\infty}(\deg_c(f^n))^{1/n}$, every $a \in \mathbf{C} \setminus E_V(((f^n)'),(\eta^n))$, and every C^2-test function ϕ on $\mathbf{P}^1$, by Lemmas 4.1 and 4.2, we have

$$\begin{aligned}\left|\int_{\mathbf{P}^1}\phi\,\mathrm{d}\left(\frac{((f^n)')^*\delta_a}{d^n-1}-\mu_f\right)\right| &= \left|\int_{\mathbf{P}^1}\phi\,\mathrm{dd^c}\left(\frac{\log[(f^n)',a]}{d^n-1}+\frac{\log(1/[(f^n)',\infty])}{d^n-1}-g_f\right)\right|\\ &\le \left(\sup_{\mathbf{P}^1}\left|\frac{\mathrm{dd^c}\phi}{\mathrm{d}\omega}\right|\right)\left(\frac{1}{d^n-1}\int_{\mathbf{P}^1}\log\frac{1}{[(f^n)'(z),a]}\,\mathrm{d}\omega(z)\right.\\ &\quad\left.+\int_{\mathbf{P}^1}\left|\frac{\log(1/[(f^n)'(z),\infty])}{d^n-1}-g_f\right|\mathrm{d}\omega(z)\right)\\ &= o((\eta/d)^n)\quad\text{as } n\to\infty,\end{aligned}$$

where the Radon–Nikodym derivative $(\mathrm{dd^c}\phi)/\mathrm{d}\omega$ on $\mathbf{P}^1$ is bounded on $\mathbf{P}^1$. □

5. Proof of Theorem 3

Let $f\colon \mathbf{C}\times\mathbf{P}^1 \ni (\lambda,z) \mapsto z^d+\lambda =: f_\lambda(z) \in \mathbf{P}^1$ be the monic and centered unicritical polynomials family of degree $d>1$. For every $n\in\mathbf{N}$, $f_\lambda^n(\lambda), (f_\lambda^n)'(\lambda) \in \mathbf{C}[\lambda]$ are of degree d^n, d^n-1, respectively.

5.1. Background on the family *f*. Recall the definitions in Subsection 2.2. The following constructions are due to Douady–Hubbard [5] and Sibony.

For every $\lambda\in\mathbf{C}$, $f_\lambda'(z) = d\cdot z^{d-1}$, so $C(f_\lambda)\cap\mathbf{C} = \{0\}$ and $f_\lambda(0)=\lambda$. The connectedness locus $C_d := \{\lambda\in\mathbf{C}\colon \lambda\in K(f_\lambda)\}$ of the family f is a compact subset in $\mathbf{C}$, and $H_\infty = H_{d,\infty} := \mathbf{P}^1\setminus C_d$ is a simply connected domain containing ∞ in $\mathbf{P}^1$. Moreover, the locally uniform limit

$$g_{H_\infty}(\lambda) := g_{f_\lambda}(\lambda) = d\cdot g_{f_\lambda}(0) = \lim_{n\to\infty}\frac{-\log[f_\lambda^n(\lambda),\infty]}{d^n}$$

exists on $\mathbf{C}$. Setting $g_{H_\infty}(\infty) := +\infty$, the restriction of g_{H_∞} to H_∞ coincides with the Green function on H_∞ with pole ∞, and the measure

$$\mu_{C_d} := \mathrm{dd^c} g_{H_\infty} + \delta_\infty \quad \text{on } \mathbf{P}^1$$

coincides with the harmonic measure on C_d with pole ∞. In particular, $z\mapsto g_{H_\infty}(z) - \log|z|$ extends harmonically to an open neighborhood of ∞ in H_∞, and $\operatorname{supp}\mu_{C_d}\subset \partial C_d$ (in fact, the equality holds).

5.2. Proof of Theorem 3. For every $n\in\mathbf{N}$, $\lambda\mapsto(\log|(f_\lambda^n)'(\lambda)|)/(d^n-1) - g_{H_\infty}(\lambda)$ and $\lambda\mapsto(\log\max\{1,|(f_\lambda^n)'(\lambda)|\})/(d^n-1)-g_{H_\infty}(\lambda)$ on $\mathbf{C}$ extend continuously to $\mathbf{P}^1$.

Lemma 5.1. *For every $n\in\mathbf{N}$ and every $\lambda\in\mathbf{C}$,*

$$\frac{\log\max\{1,|(f_\lambda^n)'(\lambda)|\}}{d^n-1} - g_{H_\infty}(\lambda) \le \frac{n\log(d^2)}{d^n-1}. \tag{3.3$'$}$$

Proof. For every $n\in\mathbf{N}$ and every $\lambda\in\mathbf{C}$, by $g_{f_\lambda^n} = g_{f_\lambda}$ on $\mathbf{P}^1$ and $g_{f_\lambda}\circ f_\lambda = d\cdot g_{f_\lambda}$ on $\mathbf{P}^1$, we have $g_{f_\lambda^n}(\lambda) = g_{f_\lambda}(\lambda) = d\cdot g_{f_\lambda}(0) \ge g_{f_\lambda}(0) = \max_{c\in C(f_\lambda)\cap\mathbf{C}} g_{f_\lambda}(c) = \max_{c\in C(f_\lambda^n)\cap\mathbf{C}} g_{f_\lambda^n}(c)$, so by Theorem 2.1, we have $|(f_\lambda^n)'(\lambda)| \le (d^n)^2 e^{(d^n-1)g_{f_\lambda^n}(\lambda)} = (d^n)^2 e^{(d^n-1)g_{f_\lambda}(\lambda)} = (d^n)^2 e^{(d^n-1)g_{H_\infty}(\lambda)}$. This with $g_{H_\infty}(\lambda)\ge 0$ completes the proof. □

Lemma 5.2.

$$\int_{\mathbf{P}^1}\left|\frac{\log(1/[(f_\lambda^n)'(\lambda),\infty])}{d^n-1}-g_{H_\infty}(\lambda)\right|\mathrm{d}\omega(\lambda)=O(n^2d^{-n})$$

as $n\to\infty$.

Proof. For every $n\in\mathbf{N}$, by the third equality in (3.1) for f_λ evaluated at $z=\lambda$, we have

$$\frac{\log|(f_\lambda^n)'(\lambda)|}{d^n-1}-\frac{n\log d}{d^n-1}=\frac{d-1}{d^n-1}\sum_{j=0}^{n-1}\log|f_\lambda^j(\lambda)|=\frac{d-1}{d^n-1}\sum_{j=0}^{n-1}\log|f_\lambda^{j+1}(0)|,$$

so that

$$\begin{aligned}&\int_{\mathbf{P}^1}\left|\frac{\log|(f_\lambda^n)'(\lambda)|}{d^n-1}-g_{H_\infty}(\lambda)\right|\mathrm{d}\omega(\lambda)\\ \text{(4.1}'\text{)}\qquad&\le\frac{d-1}{d^n-1}\sum_{j=0}^{n-1}\int_{\mathbf{P}^1}\left|\log|f_\lambda^{j+1}(0)|-d^j\cdot g_{H_\infty}(\lambda)\right|\mathrm{d}\omega(\lambda)+\frac{n\log d}{d^n-1}\\ &=O(n^2d^{-n})\quad\text{as } n\to\infty\end{aligned}$$

since by Gauthier–Vigny [9, §4.3, Proof of Theorem A], we have

$$\int_{\mathbf{P}^1}\left|\log|f_\lambda^{n+1}(0)|-d^n\cdot g_{H_\infty}(\lambda)\right|\mathrm{d}\omega(\lambda)=O(n)$$

as $n\to\infty$. This with Lemma 5.1 and $0\le\log(1/[w,\infty])-\log\max\{1,|w|\}\le\log\sqrt{2}$ on $\mathbf{C}$ completes the proof. □

Lemma 5.3. *For every* $\eta>1$, *the Valiron exceptional set*

$$E_V(((f_\lambda^n)'(\lambda)),(\eta^n)):=\left\{a\in\mathbf{P}^1:\ \limsup_{n\to\infty}\frac{1}{\eta^n}\int_{\mathbf{P}^1}\log\frac{1}{[(f_\lambda^n)'(\lambda),a]}\mathrm{d}\omega(\lambda)>0\right\}$$

of the sequence $((f_\lambda^n)'(\lambda))$ *in* $\mathbf{C}[\lambda]$ *with respect to the sequence* (η^n) *in* $\mathbf{R}_{>0}$ *is a polar subset in* $\mathbf{P}^1$.

Proof. This is an application of Russakovskii–Shiffman [18, Proposition 6.2] to the sequence $((f_\lambda^n)'(\lambda))$ in $\mathbf{C}[\lambda]$ since $\sum_{n\in\mathbf{N}}1/\eta^n<\infty$ for every $\eta>1$. □

For every $\eta>1$, every $a\in\mathbf{C}\setminus E_V(((f_\lambda^n)'(\lambda)),(\eta^n))$, and every C^2-test function ϕ on $\mathbf{P}^1$, by Lemmas 5.2 and 5.3, we have

$$\begin{aligned}&\left|\int_{\mathbf{P}^1}\phi(\lambda)\,\mathrm{d}\left(\frac{((f_\lambda^n)'(\lambda))^*\delta_a}{d^n-1}-\mu_{C_d}\right)(\lambda)\right|\\ &=\left|\int_{\mathbf{P}^1}\phi(\lambda)\,\mathrm{dd}^c\left(\frac{\log[(f_\lambda^n)'(\lambda),a]}{d^n-1}+\frac{\log(1/[(f_\lambda^n)'(\lambda),\infty])}{d^n-1}-g_{H_\infty}(\lambda)\right)\right|\\ &\le\left(\sup_{\mathbf{P}^1}\left|\frac{\mathrm{dd}^c\phi}{\mathrm{d}\omega}\right|\right)\left(\frac{1}{d^n-1}\int_{\mathbf{P}^1}\log\frac{1}{[(f_\lambda^n)'(\lambda),a]}\mathrm{d}\omega(\lambda)\right.\\ &\qquad\left.+\int_{\mathbf{P}^1}\left|\frac{\log(1/[(f_\lambda^n)(\lambda),\infty])}{d^n-1}-g_{H_\infty}(\lambda)\right|\mathrm{d}\omega(\lambda)\right)\\ &=o((\eta/d)^n)\quad\text{as } n\to\infty,\end{aligned}$$

where the Radon–Nikodym derivative $(\mathrm{dd}^c\phi)/\mathrm{d}\omega$ on $\mathbf{P}^1$ is bounded on $\mathbf{P}^1$. □

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