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COMPOSITION OPERATORS ON HARDY–ORLICZ SPACES ON PLANAR DOMAINS

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Abstract. We show isomorphic and isometric characterizations of Hardy–Orlicz spaces on multiply-connected domains whose boundary consists of finitely many disjoint analytic Jordan curves. We also study composition operators on these spaces. In particular we obtain characterization of compact composition operators on Hardy–Orlicz spaces in terms of Carleson's measures defined in the paper.

1. Introduction

Let $H(\Omega)$ denote the space of all holomorphic functions on Ω, where Ω is a domain on the Riemann sphere. For analytic map $\varphi \in H(\Omega)$, $\varphi\colon \Omega \to \Omega$, the composition operator is defined by

$$C_\varphi f := f \circ \varphi, \quad f \in H(\Omega).$$

This map may act on various quasi-Banach spaces X of analytic functions on Ω. Composition operators are fundamental objects of study in analysis that arise naturally in many situations. For example a classical result due to Forelli (see [4]) states that all surjective isometries of the Hardy space $H^p(\mathbf{D})$, $1 < p < \infty$, $p \neq 2$ are weighted composition operators. The problem of relating operator theoretic properties (e.g., boundedness, compactness, weak compactness, order boundedness, spectral properties) of composition operators C_φ to function theoretic properties of generating function (symbol of C_φ) has been a subject of great interest for quite some time. One of the famous problem was to characterize compact composition operators on Hardy spaces $H^p(\mathbf{D})$. In the eighties MacCluer (see [10] or [1]) proved that $C_\varphi\colon H^p(\mathbf{D}) \to H^p(\mathbf{D})$ is compact if and only if the pullback measure μ_φ defined by the formula

$$\mu_\varphi(B) := m(\varphi^{*-1}(B)),$$

(where m is normalized Lebesgue measure on $\partial\mathbf{D}$ and φ^* is radial limit of φ) is vanishing Carleson measure, i.e., $\mu_\varphi(W(a,h)) = o(h)$ as $h \to 0$ for any Carleson window $W(a,h) := \{z \in \mathbf{D}\colon 1-h < |z| < 1, |\arg(\overline{a}z)| < h\}$, $a \in \partial\mathbf{D}$. We note that this problem was also solved by Shapiro [18, 19]. He described compact composition operators in terms of Nevanlinna counting function. It should be emphasized that MacCluer's result (together with the famous Carleson Lemma) contributes to study another class of operators—inclusion operators $j_\mu\colon H^p(\mathbf{D}) \to L^p(\mathbf{D},\mu)$, where μ is finite Borel measure. This idea is very useful and often used in contemporary research—recently composition operators acting between Hardy-type spaces are

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studied thoroughly. The problems of characterizing compactness, weak compactness, absolute p-summability and other properties are under consideration on various variants of Hardy spaces. We refer to [6, 7, 8, 9] where the authors have extended the results of Shapiro and MacCluer to the case of composition operators on Hardy–Orlicz spaces.

In this paper we investigate composition operators on Hardy–Orlicz spaces on multiply-connected domains Ω whose boundary consists of finitely many disjoint analytic Jordan curves. These spaces are generalizations of classical Hardy spaces $H^p(\Omega)$ on multiply-connected domains Ω introduced by Rudin in [16]. Notice that in spite of many similarities there are significant differences between theory of Hardy spaces on unit disk and multiply connected domains (we refer to the paper of Sarason [17] and book of Fisher [2], where $H^p(\Omega)$ spaces are studied). Our goal is to show isomorphic and isometric characterizations of Hardy–Orlicz spaces on Ω as well as to provide a complete description of compact composition operators $C_\varphi\colon H^\Phi(\Omega) \to H^\Phi(\Omega)$ in terms of "Φ-Carleson measure on Ω", where Φ is an Orlicz function.

2. Preliminaries

In this section we set out some prerequisites which occur in this paper. In particular we recall some basic facts about Hardy–Orlicz spaces and harmonic measures.

Hardy–Orlicz spaces on disc. Let $\Phi\colon [0,\infty) \to [0,\infty)$ be an Orlicz function, i.e., continuous and nondecreasing function such that $\lim_{t\to\infty} \Phi(t) = \infty$ and $\Phi(t) = 0$ if and only if $t = 0$. The Orlicz function Φ satisfies the ∇_2-condition ($\Phi \in \nabla_2$) if for some constant $\beta > 1$ and for some $t_0 > 0$, one has $\Phi(\beta t) \geqslant 2\beta\Phi(t)$, for $t \geqslant t_0$.

Given a measure space (Ω, Σ, μ) the Orlicz space $L^\Phi(\Omega) := L^\Phi(\Omega, \Sigma, \mu)$ is the space of all (equivalence classes of) Σ-measurable functions $f\colon \Omega \to \mathbf{C}$ for which there is a constant $\lambda > 0$ such that

$$\int_\Omega \Phi(\lambda|f|)\, d\mu < \infty.$$

It is easy to check that if there exists $C > 0$, such that $\Phi(t/C) \leqslant \Phi(t)/2$ for all $t > 0$, then $L^\Phi(\Omega)$ is a quasi-Banach lattice equipped with the quasi-norm

$$\|f\|_\Phi := \inf\left\{\lambda > 0;\ \int_\Omega \Phi\Big(\frac{|f|}{\lambda}\Big)\, d\mu \leqslant 1\right\}.$$

It is well known that $\|\cdot\|_\Phi$ is a norm in the case when Φ is a convex function. For $\Phi(t) = t^p$, $p \in (0,\infty]$, we have $L^\Phi(\Omega) = L^p(\Omega)$ and the norms coincide. We refer the reader to [14] for more complete information about Orlicz spaces.

Let $\mathbf{D}$ be the unit disc of the complex plane. Throughout the paper we identify $\partial\mathbf{D}$ with $\mathbf{T} = [0, 2\pi)$. In the same way Hardy spaces $H^p(\mathbf{D})$ are defined out of the Lebesgue spaces $L^p(\mathbf{T})$, we define Hardy–Orlicz spaces $H^\Phi := H^\Phi(\mathbf{D})$ from the Orlicz spaces $L^\Phi(\mathbf{T})$. For $f \in H(\mathbf{D})$ and $r \in (0,1)$ denote by $f_r\colon \mathbf{T} \to \mathbf{C}$ the function given by $f_r(e^{it}) = f(re^{it})$. Following [11] H^Φ consists of analytic functions $f\colon \mathbf{D} \to \mathbf{C}$ such that

$$\|f\|_{H^\Phi} := \sup_{0\leqslant r<1} \|f_r\|_{L^\Phi(\mathbf{T})} < \infty. \tag{1}$$

The formula (1) defines a quasi-norm in H^Φ and it is a norm when Φ is a convex function. We note that for every $f \in H^\Phi$ the radial limit

$$f^*(t) := \lim_{r\to 1^-} f(re^{it}), \quad t \in \mathbf{T},$$

exists a.e. and $\|f\|_{H^\Phi} = \|f^*\|_{L^\Phi(\mathbf{T})}$. Recall that (see [8]) the inverse is also true: for given $f^* \in L^\Phi(\mathbf{T})$ such that its Fourier coefficients $\widehat{f^*}(n)$ vanish for $n < 0$, the analytic extension

$$f(z) = P[f^*](z) := \sum_{n=0}^{\infty} \widehat{f^*}(n) z^n, \quad z \in \mathbf{D},$$

belongs to H^Φ and $\|f\|_{H^\Phi} = \|f^*\|_{L^\Phi(\mathbf{T})}$.

We denote by HM^Φ the subspace of finite elements of H^Φ, i.e., the space of all $f \in H(\mathbf{D})$ such that for every $\lambda > 0$ we have

$$\sup_{0 \leqslant r < 1} \int_{\mathbf{T}} \Phi(\lambda |f(re^{it})|)\, dt < \infty.$$

Harmonic measures. Let Ω be a domain on the Riemann sphere and let $u\colon \Omega \to \mathbf{R}$ be a continuous function on $\Gamma = \partial\Omega$. The Dirichlet problem is to find (if there exists) a function $\tilde{u}\colon \bar{\Omega} \to \mathbf{R}$ which is continuous and satisfies two conditions:

1. $\tilde{u}$ is harmonic on Ω,
2. $\tilde{u} = u$ on Γ.

The Dirichlet problem can be solved for many domains. For our considerations the following result is sufficient (see [2, Corollary 1.4.5.]).

Proposition 2.1. *If each component of $\partial\Omega$ is nontrivial, then the Dirichlet problem is solvable in Ω.*

In particular if Ω is a multiply-connected domain whose boundary consists of finitely many disjoint analytic Jordan curves then the Dirichlet problem is solvable for Ω. Recall that an analytic arc is the image $\psi((-1,1))$ of the open interval $(-1,1)$ under a map which is one-to-one and analytic on a neighborhood in $\mathbf{C}$ of $(-1,1)$ to $\mathbf{C}$ and an analytic Jordan curve is a Jordan curve that is finite union of open analytic arcs.

Let Ω be a domain on the sphere for which the Dirichlet problem is solvable (we write $\Omega \in (SDP)$) and let $p \in \Omega$. If $u \in C(\partial\Omega)$ then the map $u \mapsto \tilde{u}(p)$ is linear and bounded by the Maximum Modulus Principle. The Riesz representation theorem implies that there is a unique real measure ω_p on $\Gamma = \partial\Omega$ such that

$$\tilde{u}(p) = \int_\Gamma u\, d\omega_p.$$

This measure is called harmonic measure on Γ for p. Note that ω_p is probability measure which has no atoms.

Now we recall important properties of harmonic measures. First notice that ω_p depends of the point $p \in \Omega$, but it can be shown that for p and $q \in \Omega$, ω_p and ω_q are boundedly mutually absolutely continuous. Further, if K is a compact subset of Ω (we write $K \subset\subset \Omega$), then there is a constant M such that $\omega_q(E) \leqslant M\omega_p(E)$ for all $q \in K$ and for all measurable set $E \subset \Omega$. Suppose now that Ω_1 and Ω_2 are two domains and f is holomorphic function which maps $\bar{\Omega}_1$ onto $\bar{\Omega}_2$ homeomorphically. If $\Omega_1 \in (SDP)$, then also $\Omega_2 \in (SDP)$. Let $p_1 \in \Omega_1$ and $p_2 = f(p_1)$. Denote by ω_1 the harmonic measure on $\partial\Omega_1$ for p and define measure μ on $\partial\Omega_2$ as follows:

$$\mu(E) = \omega_1(f^{-1}(E)), \quad E \subset \partial\Omega_2.$$

Then μ is harmonic measure on $\partial\Omega_2$ for p_2. Let $\Omega_1, \Omega_2 \in (SDP)$, $\Omega_1 \subset \Omega_2$. Let $p \in \Omega_1$ and let ω_1 and ω_2 be harmonic measure on $\partial\Omega_1$ and $\partial\Omega_2$ respectively, for p. Then for each compact set $E \subset \partial\Omega_1 \cap \partial\Omega_2$ we have $\omega_1(E) \leqslant \omega_2(E)$.

For next properties we need the Green's function of a domain. Assume that Ω is a domain on the Riemann sphere and $\Omega \in (SDP)$, $p \in \Omega$. A function $g(\cdot, p)$ is a Green's function for Ω with a pole (or singularity) at p ($p \neq \infty$), if

$$z \mapsto g(z;p) \text{ is harmonic on } \Omega \setminus \{p\},$$

$$g(z;p) + \log|z-p| \text{ is harmonic in a neighbourhood of } p,$$

$$\lim_{z\to\zeta} g(z;p) = 0 \text{ for all } \zeta \in \partial\Omega.$$

Assume now that $\partial\Omega$ consist of $m+1$ disjoint analytic Jordan curves. Let $p \in \Omega$ and let $g(z;p)$ be Green's function for Ω at p. Denote by $h(z) = h(z;p)$ the harmonic conjugate of $g(z;p)$ (note that h is multivalued). Then we have that locally $Q = g+ih$ is analytic and its derivative is single-valued on Ω. The following results (see [2]) show relationships between harmonic measure arc length and Green's function.

Theorem 2.2. [2, Theorem 1.6.4.] *Suppose Ω is bounded by a finite number of disjoint analytic Jordan curves. Then for each $z \in \Omega$ we have*

$$d\omega_z = -\frac{1}{2\pi}\frac{\partial}{\partial n} g(\cdot, z)\, ds,$$

where $g(\cdot, p)$ is the Green's function for Ω with pole at z, $\frac{\partial}{\partial n}$ is the derivative in the direction of outwards normal at $\partial\Omega$ and ds is arc length.

The function $P_z(\zeta) := \frac{d\omega_z}{ds}(\zeta) = -\frac{1}{2\pi}\frac{\partial}{\partial n} g(\zeta, z)$, where $\zeta \in \partial\Omega$ and $z \in \Omega$ is called Poisson kernel for Ω. It satisfies inequalities

$$c_1 \leqslant \frac{d\omega_z}{ds} \leqslant c_2$$

for positive constants c_1, c_2. For convienience we assume that s is normalized Lebesgue measure on $\partial\Omega$.

Theorem 2.3. [2, Proposition 1.6.5.] $d\omega_p(\zeta) = \frac{i}{2\pi} Q'(\zeta)\, d\zeta$.

3. Hardy–Orlicz spaces on planar domains

In this section we define Hardy–Orlicz spaces on planar domains and prove basic properties of these spaces. We start from the study of certain properties of domains and subharmonic functions.

Let Ω be a domain on the Riemann sphere. Recall that a regular exhaustion of Ω is a sequence $\{\Omega_n\}_{n=1}^{\infty}$ of subdomains of Ω which satisfy the following conditions:

(1) $\bar{\Omega}_n \subset \Omega_{n+1}$ for $n \in \mathbf{N}$,
(2) $\bigcup_{n=1}^{\infty} = \Omega$,
(3) every component of $\partial\Omega_n$, is nontrivial for each $n \in \mathbf{N}$.

It can be proved that each domain has a regular exhaustion (see [2, Proposition 1.5.3]). Recall that upper semicontinuous function $u\colon \Omega \to [-\infty, \infty)$ is called subharmonic (we write $u \in \mathrm{subh}(\Omega)$), if for every compact subset $K \subset \Omega$ and every continuous function h harmonic on interior of K with $h \geqslant u$ on ∂K, we have $h \geqslant u$ on K as well. In this case h is called harmonic majorant of u. It can be proved that if u has harmonic majorant, then it has the least harmonic majorant and it is unique. It is well known that subharmonic functions satisfy Maximum Modulus Principle. Using this fact and the Harnack Theorem we can obtain the following result.

Theorem 3.1. *Let $\Omega \in (SDP)$ and $u \in \mathrm{subh}(\Omega)$, $p \in \Omega$. Then u has harmonic majorant if and only if for each regular exhaustion $\{\Omega_n\}$ of Ω there exist a constant C such that*

$$\int_{\partial\Omega_n} u\, d\omega_{p,n} \leqslant C,$$

where $\omega_{p,n}$ is harmonic measure on Ω_n for p.

In fact infimum over those constants is equal to $v(p)$, where v is the least harmonic majorant of u.

Now we can define Hardy–Orlicz spaces on general domains. Let $\Phi\colon [0,\infty) \to [0,\infty)$ be convex Orlicz function. Suppose that $\Omega \in (SDP)$, $\{\Omega_n\}$ is regular exhaustion of Ω and $z_0 \in \Omega_1$. For $g\colon \Omega \to \mathbf{C}$ denote $g_n := g|_{\partial\Omega_n}$. We define Hardy–Orlicz space on Ω by the following condition:

$$H^\Phi(\Omega) := \{f \in H(\Omega)\colon \|f\|_{H^\Phi(\Omega)} < \infty\},$$

where

$$\|f\|_{H^\Phi(\Omega)} := \lim_{n\to\infty} \|f_n\|_\Phi = \lim_{n\to\infty} \inf\left\{\varepsilon > 0\colon \int_{\partial\Omega_n} \Phi\Big(\frac{|f_n|}{\varepsilon}\Big)\, d\omega_{n,z_0} \leqslant 1\right\}.$$

Notice that by Theorem 3.1 we can also describe $H^\Phi(\Omega)$ in terms of harmonic majorant; it is a set of all holomorphic functions f, for which there exists $\lambda > 0$, such that subharmonic function $\Phi(\lambda|f|)$ has harmonic majorant. Moreover

$$\|f\|_{H^\Phi(\Omega)} = \inf\{\varepsilon > 0\colon v_{f,\varepsilon}(z_0) \leqslant 1\},$$

where $v_{f,\varepsilon}$ is the least harmonic majorant of $\Phi\big(\frac{|f|}{\varepsilon}\big)$. It's clear that $H^\Phi(\Omega)$ is a Banach space. We denote by $HM^\Phi(\Omega)$ the subspace of finite elements of $H^\Phi(\Omega)$, i.e., closure of $H^\infty(\Omega)$ in $H^\Phi(\Omega)$.

For further work we need additional assumption on domain Ω. Let Ω be a bounded domain whose boundary consists of $m+1$ disjoint analytic Jordan curves, i.e.

$$\Gamma := \partial\Omega = \bigcup_{k=0}^{m} \Gamma_k,$$

where Γ_k is analytic Jordan curve and $\Gamma_k \cap \Gamma_j = \emptyset$ for $k \neq j$. Assume that Γ_0 is the boundary of the unbounded component of the complement of Ω. Denote by E_0 bounded component of $S^2 \setminus \Gamma_0$ and for $k \in \{1,2,\dots,m\}$, denote by E_k unbounded component of $S^2 \setminus \Gamma_k$, where S^2 is the Riemann sphere. From now Ω will be always a set of this type. We also define by $H_0^\Phi(E_k)$ the subspace of $H^\Phi(E_k)$ which consists those functions which vanish at ∞. The first result shows that $H^\Phi(\Omega)$ can be represented as a certain direct sum:

Theorem 3.2. *For each $f \in H^\Phi(\Omega)$ we have the following decomposition*

$$f(z) = f_0(z) + f_1(z) + \ldots + f_m(z), \quad z \in \Omega, \tag{2}$$

where $f_0 \in H^\Phi(E_0)$ and $f_k \in H_0^\Phi(E_k)$ for each $1 \leqslant k \leqslant m$. Moreover, the map $f \mapsto f_0$ is a bounded linear projection of $H^\Phi(\Omega)$ onto $H^\Phi(E_0)$ and $f \mapsto f_k$ is a bounded linear projection of $H^\Phi(\Omega)$ onto $H_0^\Phi(E_k)$.

Proof. Fix $f \in H^{\Phi}(\Omega)$, $z \in \Omega$ and let $C_0, \ldots, C_m$ be a smooth Jordan curves so close to $\Gamma_0, \ldots, \Gamma_m$ respectively, that z is exterior to $C_1, \ldots, C_m$ and interior to C_0. Now for each $0 \leqslant k \leqslant m$

$$f_k(z) := \frac{1}{2\pi i} \int_{C_k} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

It is clear that $f = f_0 + f_1 + \ldots + f_m$ and $f \mapsto f_k$ is a linear operator. Further $f_k \in H(E_k)$ and it is independent of the choice of C_k, $0 \leqslant k \leqslant m$. We have also $f_k(\infty) = 0$ for each $1 \leqslant k \leqslant m$. It's easily seen that if $f \in H_0^{\Phi}(E_k)$ for some k then $f_l = 0$ for $l \neq k$. Hence $f = f_k$. We need to show that $f_k \in H^{\Phi}(E_k)$. Note that for all $l \neq k$ the function f_l is bounded (pointwise) on some neighborhood U of Γ_k. From the formula (2) and the fact that $H^{\Phi}(\Omega)$ is a linear space we conclude that $f \in H^{\Phi}(U)$, and so this implies that $f \in H^{\Phi}(E_k)$. □

Note that interpolation theorem for general variants of Hardy spaces on planar domains (which are defined as a similar direct sums) has been recently proved in [13].

Recall that polynomials are dense in HM^{Φ}. Denote by $R(\Omega)$ the set of rational functions whose poles are off $\bar{\Omega}$. We formulate analogue of this result for $HM^{\Phi}(\Omega)$.

Proposition 3.3. *$R(\Omega)$ is dense in $HM^{\Phi}(\Omega)$.*

Proof. Fix $k \in \{0, \ldots, m\}$. Since $\Omega \subset E_k$ then $\omega_{z,\Omega}(A) \leqslant \omega_{z,E_k}(A)$ for every point $z \in \Omega$ and every subset $A \subset \Gamma_k$. Thus the $H^{\Phi}(E_k)$ norm is larger then $H^{\Phi}(\Omega)$. We need to show that f_k is a limit in $HM^{\Phi}(E_k)$ of a sequence of functions holomorphic in a neighborhood of $\bar{E}_k$. Let η_k be a Riemann mapping of $\mathbf{D}$ onto E_k. Since $\partial\mathbf{D}$ is analytic, this mapping can be extended to holomorphic and one-to-one in some neighborhood of $\bar{\mathbf{D}}$. The same is true for the inverse function η_k^{-1}. Put $g_k := f_k \circ \eta_k$, then $g_k \in HM^{\Phi}$ and therefore, by Runge's theorem, there is a function G analytic on a neighborhood of $\bar{\mathbf{D}}$ and such that $\|G - g_k\|_{H^{\Phi}} < \varepsilon$ and it is equivalent to

$$\|G \circ \eta_k^{-1} - f_k\|_{H^{\Phi}(E_k)} < \varepsilon,$$

and $G \circ \eta_k^{-1}$ is analytic in a neighborhood of $\bar{E}_k$. Applying again Runge's theorem we can approximate $G \circ \eta_k^{-1}$ uniformly on $\bar{E}_k$ by elements of $R(\Omega)$. □

Proposition 3.4. [2, Proposition 4.4.3.] *If $u \in L^1(\Gamma, ds)$ and*

$$0 = \int_{\Gamma} \frac{u(\zeta)}{\zeta - z}\, d\zeta, \quad z \notin \Gamma$$

then $u = 0$ a.e. on Γ.

Now we formulate the main result of this section. Let us remark that $\omega := \omega_z$ for a point $z \in \Omega$.

Theorem 3.5. *Every $f \in H^{\Phi}(\Omega)$ has boundary values f^* almost everywhere $(d\omega)$ on Γ and $f^* \in L^{\Phi}(\Gamma, \omega)$. Moreover*

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f^*(w)}{w - z}\, dw, \quad z \in \Omega, \tag{3}$$

$$0 = \int_{\Gamma} \frac{f^*(w)}{w - z}\, dw, \quad z \notin \bar{\Omega}, \tag{4}$$

$$f(z) = \int_{\Gamma} f^*(\zeta)\, d\omega_z(\zeta), \quad z \in \Omega. \tag{5}$$

Finally, the mapping $f \mapsto f^*$ is an isomorphism of $H^\Phi(\Omega)$ onto closed subspace of $L^\Phi(\Gamma,\omega)$ and it is an isometry of $HM^\Phi(\Omega)$ onto closed subspace of $L^\Phi(\Gamma,\omega)$.

Proof. By Theorem 3.2 it is enough to show that f_k has boundary values a.e. ds on Γ and that this boundary-value functions lies in $L^\Phi(\Gamma,\omega)$. Fix $k \in \{0,\dots,m\}$. For $l \neq k$ the function f_k is analytic on Γ_l. So (3), (4) and (5) hold immediately. Let η_k be the Riemann mapping of $\mathbf{D}$ onto E_k. Note that η_k extends to be analytic and conformal on a neighborhood of $\bar\Omega$ since Γ_k is analytic (the same is true for $\eta_k^{-1}\colon E_k \to \mathbf{D}$). Further $g_k = f_k \circ \eta_k \in H^\Phi$ so g_k^* exists a.e. on $\mathbf{T}$ and $g_k^* \in L^\Phi(\mathbf{T})$. Then $f_k = g_k \circ \eta_k^{-1}$ has boundary values a.e. ds on Γ_k and $f_k^* = g_k^* \circ \eta_k^{-1}$ a.e. so that $f_k^* \in L^\Phi(\Gamma_k, ds)$ and hence $f_k^* \in L^\Phi(\Gamma,\omega)$. Now if $z \in \Omega$ then

$$f_k(z) = g_k(\eta_k^{-1}(z)) = \frac{1}{2\pi i}\int_{|\xi|=1} \frac{g_k^*(\xi)}{\xi - \eta_k^{-1}(z)}\,d\xi.$$

Putting $\xi = \eta_k^{-1}(\zeta)$ we have

$$f_k(z) = \frac{1}{2\pi i}\int_{\Gamma_k} \frac{f_k^*(\zeta)}{\eta_k^{-1}(\zeta) - \eta_k^{-1}(z)}(\eta_k^{-1}(\zeta))'\,d\zeta.$$

Now we notice that

$$\frac{(\eta_k^{-1}(\zeta))'}{\eta_k^{-1}(\zeta) - \eta_k^{-1}(z)} = \frac{1}{\zeta - z} + S(z),$$

where S is analytic in a neighborhood of $\bar\Omega$, since the function in the left-hand side of the equality has a simple pole at z with residue equal to 1. So we have

$$\int_{\Gamma_k} S(\zeta) f_k^*(\zeta) = 0.$$

If $k \neq l$ then $\int_{\Gamma_l} \frac{f_k(\zeta)}{\zeta - z}\,d\zeta = 0$ and so (3) and (4) hold.To prove (5) recall that $d\omega_z(\zeta) = \frac{i}{2\pi}Q_z'(\zeta)d\zeta$, where $Q_z(\zeta) = g(\zeta;z) + ih(\zeta;z)$. Then $Q_z'(\zeta) = \frac{1}{z-\zeta} + R(\zeta)$, where R is holomorphic on $\bar\Omega$, hence $\int_\Gamma f_k^* R(\zeta)\,d\zeta = 0$. Consequently

$$\int_\Gamma f^*(\zeta)\,d\omega_z(\zeta) = -\frac{i}{2\pi}\int_\Gamma \frac{f^*(\zeta)}{\zeta - z}\,d\zeta + \frac{i}{2\pi}\int_\Gamma f^*(\zeta) R(\zeta)\,d\zeta = f(z).$$

To show that $f \mapsto f^*$ is isomorphism from $H^\Phi(\Omega)$ into $L^\Phi(\Gamma,\omega)$ it is enough to use Theorem 3.2 and the facts that $f_k \mapsto f_k^*$ is isomorphism from $H^\Phi(E_k)$ into $L^\Phi(\Gamma_k,\omega)$ and for $l \neq k$ the function f_k is analytic on Γ_l. Now we show that $f \mapsto f^*$ is isometry from $HM^\Phi(\Omega)$ into $L^\Phi(\Gamma,\omega)$. Let $q \in R(\Omega)$ and let u be the harmonic function on Ω given by

$$u(z) = \int_\Gamma \Phi\Big(\frac{|q(\zeta)|}{\|q\|_{H^\Phi(\Omega)}}\Big)\,d\omega_z, \quad z \in \Omega. \tag{6}$$

Then $\Phi\Big(\frac{|q(z)|}{\|q\|_{H^\Phi(\Omega)}}\Big) \leqslant u(z)$, and if v is any harmonic majorant of $\Phi\Big(\frac{|q(z)|}{\|q\|_{H^\Phi(\Omega)}}\Big)$, then

$$u(x) = \Phi\Big(\frac{|q(x)|}{\|q\|_{H^\Phi(\Omega)}}\Big) \leqslant \liminf\{v(z) : z \to x\}.$$

It implies that the harmonic function $v-u$ is non-negative at Γ and hence for all $z \in \Omega$. Thus, the function u given by (6) is the least harmonic majorant of $\Phi\Big(\frac{|q|}{\|q\|_{H^\Phi(\Omega)}}\Big)$ and we conclude that $\|q\|_{H^\Phi(\Omega)} = \|q\|_{L^\Phi(\Gamma,\omega)}$ for $q \in R(\Omega)$. Now take $f \in HM^\Phi(\Omega)$. By

Proposition 3.3 there exist $\{q_n\}$ in $R(\Omega)$ such that $q_n \to f$ in $H^\Phi(\Omega)$. Then $q_n \to f$ uniformly on compact subsets of Ω and it's clear that

$$\|q_n - q_m\|_{H^\Phi(\Omega)} = \|q_n - q_m\|_{L^\Phi(\Gamma)}.$$

Hence $\{q_n\}$ is a Cauchy sequence in $L^\Phi(\Gamma)$. So if $q_n \to g$ in $L^\Phi(\Gamma, \omega)$, then

$$f(z) = \int_\Gamma g(\zeta)\, d\omega_z(\zeta), \quad z \in \Omega,$$

since all the harmonic measures are boundedly mutually absolutely continuous. Moreover we have

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{g(\zeta)}{\zeta - z}\, d\zeta, \quad z \in \Omega,$$

and

$$0 = \int_\Gamma \frac{g(\zeta)}{\zeta - z}\, d\zeta = \int_\Gamma \frac{f^*(\zeta)}{\zeta - z}\, d\zeta, \quad z \notin \bar{\Omega}.$$

Now by Proposition 3.4 we have $g = f^*$ a.e. $d\omega$. Finally $\|q_n - f^*\|_{L^\Phi(\Gamma,\omega)} \to 0$ and

$$\|f^*\|_{L^\Phi(\Gamma,\omega)} = \lim \|q_n\|_{L^\Phi(\Gamma,\omega)} = \lim \|q_n\|_{H^\Phi(\Omega)} = \lim \|f\|_{H^\Phi(\Omega)}. \qquad \square$$

Notice that in the case when $\Phi \in \Delta_2$, i.e., $\Phi(2x) \leqslant K\Phi(x)$ for some constants K, x_0 and $x \geqslant x_0$, then $H^\Phi(\Omega) = HM^\Phi(\Omega)$. So from Theorem 3.2 it follows the analogous result proved by Rudin in [16, Theorem 3.2.] for Hardy space $H^p(\Omega)$, $1 \leqslant p < \infty$, since every power function $\Phi(t) = t^p$ satisfies Δ_2 condition.

Recall that a Banach space X of holomorphic functions on an open subset Ω of the complex plane has the Fatou property if X is continuously embedded in $H(\Omega)$, the space of holomorphic functions on Ω, equipped with its natural topology of compact convergence, and if it has the following property: for every bounded sequence $\{f_n\}$ in X which converges uniformly on compact subsets of Ω to a function f, one has $f \in X$. Note that $H^\Phi(\Omega)$ has the Fatou property.

We have seen that $H^\Phi(\Omega)$ is isomorphic to $H^\Phi(E_0) \oplus H_0^\Phi(E_1) \oplus \cdots \oplus H_0^\Phi(E_m)$. Since conformal maps are isometries in the class of Hardy–Orlicz spaces we may assume that $E_0 = \mathbf{D}$, $E_i = a_i + r_i\mathbf{D}$ for $i = 1, \ldots, m$, where $r_i \in (0,1)$, $a_i \in \mathbf{D}$, $a_i \neq a_j$ if $i \neq j$ and the circles $\partial\mathbf{D}, a_i + r_i\partial\mathbf{D}$ are pairwise disjoint and $E_i \cap E_j = \emptyset$. Spaces of those type (on such Ω, which is called *circular domain*) are isomorphic to a general (described before) so from now Ω denotes this special case. Let us note that conformal maps (from E_i onto unit disc and inverse) which we used before for this particular Ω are of the form:

$$\eta_i(z) = \begin{cases} \frac{r_i}{z} + a_i & \text{dla } z \in \mathbf{D} \setminus \{0\}, \\ \infty & \text{dla } z = 0, \end{cases}$$

$$\eta_i^{-1}(z) = \begin{cases} \frac{r_i}{z - a_i} & \text{dla } z \in E_i \setminus \{\infty\}, \\ 0 & \text{dla } z = \infty. \end{cases}$$

for $1 \leqslant i \leqslant m$ and we put $\eta_0 = id_{\mathbf{D}}$. It was proved in [8] that the norm of the evaluation functional $\delta_z \colon H^\Phi(\mathbf{D}) \mapsto \mathbf{C}$ at $z \in \mathbf{D}$ satisfies condition $\|\delta_z\| \approx \Phi^{-1}(\frac{1}{1-|z|})$. Using this fact, Theorem 3.2 and conformal maps it is easy to prove, that for $z \in \Omega$, the norm of $\delta_z \colon H^\Phi(\Omega) \mapsto \mathbf{C}$ satisfies $\|\delta_z\| \leqslant C\Phi^{-1}(\frac{1}{\mathrm{dist}(z,\Gamma)})$, where the constant C depends only on Ω and z.

At the end of this section we introduce useful family of functions. Recall that (see [8]) for $a \in \partial\mathbf{D}$ and $r \in (0,1)$

$$u_{a,r}(z) = \Big(\frac{1-r}{1-\bar{a}rz}\Big)^2,$$

is holomorphic function on $\mathbf{D}$ with $\|u_{a,r}\|_{H^1} \leqslant 1-r$, $\|u_{a,r}\|_{H^\infty} = 1$ and $\|u_{a,r}\|_{H^\Phi} \approx \frac{1}{\Phi^{-1}(\frac{1}{1-r})}$. For $1 \leqslant i \leqslant m$ and a and r as above we define

$$u^i_{a,r}(z) = (u_{a,r} \circ \eta_i^{-1})(z) = \Big(\frac{1-r}{1-\frac{\bar{a}rr_i}{z-a_i}}\Big)^2, \quad z \in \Omega,$$

and $u^0_{a,r} = u_{a,r}|_\Omega$. Note that for each $0 \leqslant i \leqslant m$ function $u^i_{a,r}$ extends to holomorphic function on E_i. It is also clear that we have analogous norm estimation: $\|u^i_{a,r}\|_{H^1(E_i)} \leqslant 1-r$, $\|u^i_{a,r}\|_{H^\infty(E_i)} = 1$, $\|u^i_{a,r}\|_{H^\Phi(E_i)} \approx \frac{1}{\Phi^{-1}(\frac{1}{1-r})}$, and $\|u^i_{a,r}\|_{H^1(\Omega)} \leqslant C(1-r)$, $\|u^i_{a,r}\|_{H^\infty(\Omega)} \approx 1$, $\|u^i_{a,r}\|_{H^\Phi(\Omega)} \approx \frac{1}{\Phi^{-1}(\frac{1}{1-r})}$, for some constant C.

4. Composition operators on Hardy–Orlicz spaces of planar domains

In this section we study properties of composition operators $C_\varphi\colon H^\Phi(\Omega) \to H^\Phi(\Omega)$ and canonical inclusion maps $j_\mu\colon H^\Phi(\Omega) \hookrightarrow L^\Phi(\Omega,\mu)$, where μ is a Borel measure μ on Ω. Recall that if φ is analytic map $\varphi\colon \Omega \to \Omega$ (in this case we write $\varphi \in \Upsilon := \Upsilon_\Omega$), then for $f \in H(\Omega)$ we define C_φ as follow:

$$(C_\varphi f)(z) := (f \circ \varphi)(z), \quad z \in \Omega.$$

First let's see that C_φ defines (bounded) operator. Linearity is clear. To show that $C_\varphi\colon H^\Phi(\Omega) \to H^\Phi(\Omega)$ is bounded take $f \in H^\Phi(\Omega)$ with $\|f\|_{H^\Phi(\Omega)} = \lambda$, where $\lambda = \inf\{\varepsilon > 0\colon v_{f,\varepsilon}(z_0) \leqslant 1\}$ and $v_{f,\varepsilon}$ being the least harmonic majorant of $\Phi\Big(\frac{|f|}{\varepsilon}\Big)$. We have

$$\Phi\Big(\frac{|f\circ\varphi|}{\varepsilon}\Big) \leqslant v_{f,\varepsilon} \circ \varphi,$$

and $v_{f,\varepsilon} \circ \varphi$ is harmonic function. Further $\Phi\Big(\frac{|f(\varphi(z_0))|}{\varepsilon}\Big) \leqslant v_{f,\varepsilon}(\varphi(z_0))$. Put $w_0 = \varphi(z_0)$, then from Harnack's inequality there exists constant $c > 0$ such that $v_{f,\varepsilon}(w_0) \leqslant c v_{f,\varepsilon}(z_0)$, now taking $C = \max(c,1)$ it is clear that $\|C_\varphi f\|_{H^\Phi(\Omega)} \leqslant C\|f\|_{H^\Phi(\Omega)}$.

In what follows we investigate compactness of composition operators on Hardy–Orlicz spaces of planar domains. We will use the following result (see [8, Proposition 3.8]).

Proposition 4.1. *Let X, Y be two Banach spaces of analytic functions on an open set $\Omega \subset \mathbf{C}$, X having the Fatou property. Let $\varphi \in \Upsilon$ be such that $C_\varphi \in Y$ whenever $f \in X$. Then $C_\varphi\colon X \to Y$ is compact if and only if for every bounded sequence $\{f_n\}$ in X which converges to 0 uniformly on compact subsets of Ω, one has $\|C_\varphi f_n\|_Y \to 0$.*

We can apply Proposition 4.1 in the case when $X = Y = H^\Phi(\Omega)$ getting necessary condition for compactness of C_φ.

Proposition 4.2. *If $C_\varphi\colon H^\Phi(\Omega) \to H^\Phi(\Omega)$ is compact operator, then for each $0 \leqslant i \leqslant m$ we have*

$$\lim_{s\to 1^-} \sup_{p\in\partial\mathbf{D}} \Phi^{-1}\Big(\frac{1}{1-s}\Big)\|C_\varphi u^i_{p,s}\|_{H^\Phi(\Omega)} = 0. \tag{7}$$

Proof. Let $\{p_n\}$ be any sequence in $\partial\mathbf{D}$, and $\{s_n\} \subset (0,1)$, with $s_n \to 1$. By the fact that $\|u^i_{p,s}\|_{H^\Phi} \approx \frac{1}{\Phi^{-1}(\frac{1}{1-s})}$, for every $s \in (0,1)$ we conclude that the sequence $\{f^i_n\}$ defined by

$$f^i_n(z) = \Phi^{-1}\Big(\frac{1}{1-s_n}\Big) u^i_{p_n,s_n}(z), \quad z \in \Omega,$$

is bounded in $H^\Phi(\Omega)$ for all $0 \leqslant i \leqslant m$. Moreover Φ^{-1} is concave and $\Phi^{-1}(0) = 0$, so there exists a constant $C > 0$ such that $\Phi^{-1}(x) \leqslant Cx$ for $x \geqslant 1$. But it implies that for $1 \leqslant i \leqslant m$

$$|f^i_n(z)| \leqslant \frac{C}{1-s_n}\Big|\frac{(1-s_n)^2}{(1-\frac{p_n s_n r_i}{z-a_i})^2}\Big|,$$

and

$$|f^0_n(z)| \leqslant \frac{C}{1-s_n}\Big|\frac{(1-s_n)^2}{(1-|z|)^2}\Big|,$$

We conclude that for $0 \leqslant i \leqslant m$

$$f^i_n \to 0,$$

on compact subsets of Ω and by Proposition 4.1 we have that $\|C_\varphi f_n\|_{H^\Phi(\Omega)} \to 0$. Therefore (7) clearly holds. $\square$

Carleson measures. Let $\varphi \in \Upsilon$. Since φ is bounded by Theorem 3.5, the boundary function φ^* exists and it is in $L^\Phi(\Gamma, \omega)$ or (what is equivalent) $\varphi^* \in L^\Phi(\Gamma, ds)$. For given $\varphi \in \Upsilon$ and any Borel subset of $\bar{\Omega}$ we define pullback measure as follows:

$$\mu_\varphi(B) := s((\varphi^*)^{-1}(B)).$$

Further we have

$$\begin{aligned} \|C_\varphi f\|_{H^\Phi(\Omega)} \approx \|(f\circ\varphi)^*\|_{L^\Phi(\Gamma,\omega)} &= \inf\Big\{\varepsilon > 0\colon \int_\Gamma \Phi\Big(\frac{|f\circ\varphi^*|}{\varepsilon}\Big)\, d\omega \leqslant 1\Big\} \\ &\approx \inf\Big\{\varepsilon > 0\colon \int_\Gamma \Phi\Big(\frac{|f\circ\varphi^*|}{\varepsilon}\Big)\, ds \leqslant 1\Big\} \\ &= \inf\Big\{\varepsilon > 0\colon \int_{\bar\Omega} \Phi\Big(\frac{|f|}{\varepsilon}\Big)\, d\mu_\varphi \leqslant 1\Big\} \\ &= \|f\|_{L^\Phi(\bar\Omega,\mu_\varphi)}. \end{aligned}$$

It shows that some properties of a composition operator $C_\varphi\colon H^\Phi(\Omega) \to H^\Phi(\Omega)$ can be expressed in terms of an inclusion operator $j_{\mu_\varphi}\colon H^\Phi(\Omega) \hookrightarrow L^\Phi(\bar\Omega, \mu_\varphi)$. We will consider inclusion operator in general (not only for a pullback measures).

Lemma 4.3. *Suppose that $j_\mu\colon H^\Phi(\Omega) \hookrightarrow L^\Phi(\bar\Omega, \mu_\varphi)$ is a compact operator. Then $\mu(\Gamma) = 0$.*

Proof. First note that lemma is true when $\Omega = \mathbf{D}$ (see [8, Lemma 4.8]). To have general case consider operators $T_i\colon H^\Phi \to H^\Phi(\Omega)$, defined as follow

$$T_i f = f \circ \eta_i^{-1}|_\Omega,$$

where $0 \leqslant i \leqslant m$. It is clear that T_i is a bounded operator for each $0 \leqslant i \leqslant m$. Since $\{e_n\}$ given by $e_n(z) := z^n$ is weakly null in HM^Φ then $\{T_i e_n\}$ is weakly null in

$HM^{\Phi}(\Omega)$ for $0 \leqslant i \leqslant m$. By compactness of j_μ we have that

$$\int_{\bar{\Omega}} \Phi\Big(\frac{r_i^n}{|z-a_i|^n}\Big)\, d\mu < \varepsilon,$$

where $1 \leqslant i \leqslant m$, and

$$\int_{\bar{\Omega}} \Phi(|z^n|)\, d\mu < \varepsilon$$

for every $\varepsilon \in (0,1)$ and sufficiently large n. Hence $\Phi(1)\mu(\Gamma_i) < \varepsilon$ for each $0 \leqslant i \leqslant m$ and it implies that $\mu(\Gamma) = 0$. □

Let $\{\Omega_n\}$ be (arbitrary) regular exhaustion of Ω and let μ be Borel measure on $\bar{\Omega}$. We define operator $I_n\colon H^{\Phi}(\Omega) \to L^{\Phi}(\bar{\Omega},\mu)$ as follow

$$I_n(f) := f\chi_{\bar{\Omega}\setminus\Omega_n}.$$

Theorem 4.4. *Let μ be a finite measure on $\bar{\Omega}$, and Φ an Orlicz function. The following assertions are equivalent*:

(1) *The inclusion operator $j_\mu\colon H^{\Phi}(\Omega) \to L^{\Phi}(\bar{\Omega},\mu)$ is well defined and compact.*
(2) *For every bounded sequence $\{f_n\}$ in $H^{\Phi}(\Omega)$ converging to 0 uniformly on compact sets, we have $\|f_n\|_{L^{\Phi}(\bar{\Omega},\mu)} \to 0$.*
(3) *$H^{\Phi}(\Omega)$ is included in $L^{\Phi}(\bar{\Omega})$ and*

$$\lim_{n\to\infty} \|I_n\|_{H^{\Phi}(\Omega)\to L^{\Phi}(\bar{\Omega},\mu)} = 0.$$

for any regular exhaustion $\{\Omega_n\}$ of Ω.

Proof. To show that (2) implies (1) is a standard and it can be found in [15]. We show that (1) implies (2). From lemma we know that $\mu(\partial\Omega) = 0$; take $\{f_n\} \subset B_{H^{\Phi}(\Omega)}$ with $f_n \to 0$ uniformly on compact subsets of Ω. In particular $f_n(z) \to 0$ for all $z \in \Omega$. Since $\mu(\partial\Omega) = 0$ we have that $f_n \to 0$ μ-a.e. on $\bar{\Omega}$. Suppose that $\|f_n\|_{L^{\Phi}(\bar{\Omega},\mu)} \to 0$ is not true. From the compactness of inclusion operator we have that there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to g \neq 0$ in $L^{\Phi}(\bar{\Omega})$. But $f_{n_k} \to g$ μ-a.e. and this gives a contradiction.

Suppose now that (3) holds. We have that

$$\|j_\mu\| = \lim_{n\to\infty} \|j_\mu - I_n\|$$

Notice that $j_\mu - I_n$ is inclusion operator of $H^{\Phi}(\Omega)$ into $L^{\Phi}(\bar{\Omega},\nu)$ for a finite measure ν supported by $\bar{\Omega}_n$. Applying equivalence of (1) and (2) to $I_\nu^n := j_\mu - I_n$ we get that I_ν^n is compact for every $n \in \mathbf{N}$ and also j_μ is compact operator as the limit of compact operators.

Assume that (2) holds. Notice that the sequence $\{\|I_n\|\}$ is decreasing so if (3) were not true, there would exist a constant $\varepsilon_0 > 0$, a sequence $\{f_n\}$ in the unit ball of $H^{\Phi}(\Omega)$ and regular exhaustion $\{\Omega_n\}$ of Ω such that $\|I_n f_n\|_{L^{\Phi}(\bar{\Omega},\mu)} > \varepsilon_0$, for $n \in \mathbf{N}$. The sequence $\{f_n\}$ is uniformly bounded on compact subets of Ω. By Theorem 3.2 we can decompose each f_n as follow:

$$f_n(z) = f_{0,n}(z) + f_{1,n}(z) + \cdots + f_{m,n}(z), \quad z \in \Omega,$$

with $f_{i,n} \in H_0^{\Phi}(E_i)$. For each $1 \leqslant i \leqslant n$ define by $g_{i,n}(z) := \left(\frac{r_i}{z-a_i}\right)^n f_{i,n}(z)$ and $g_{0,n}(z) := z^n f_{0,n}(z)$. It is clear that $g_{i,n}$ is bounded in $H_0^{\Phi}(E_i)$ for $0 \leqslant i \leqslant m$ and $g_{i,n} \to 0$ uniformly on compact subets of E_i, so $g_n = g_{0,n} + \cdots + g_{m,n}$ is bounded in

$H^\Phi(\Omega)$ and $g_n \to 0$ uniformly on compact subets of Ω. By (2) we should have that if $\|g_n\|_{L^\Phi(\bar\Omega,\mu)} \to 0$. But if $\operatorname{dist}(\Gamma_i, z) \leqslant \frac{1}{n}$, or in other words $z \in A_i^n$, where

$$A_i^n = \left\{z \in E_i \colon z = a_i + r_i(1+\varepsilon)e^{i\theta},\ 0 \leqslant \varepsilon \leqslant \tfrac{1}{n},\ \theta \in [0, 2\pi]\right\},$$
$$A_0^n = \mathbf{D} \setminus \Big(1 - \frac{1}{n}\Big)\mathbf{D},$$

then

$$|g_{i,n}(z)| = \left|\frac{r_i}{z - a_i}\right|^n |f_{i,n}(z)| \geqslant \left(1 + \tfrac{1}{n}\right)^{-n} |f_{i,n}(z)| \geqslant \tfrac{1}{4}|f_{i,n}(z)|$$

for $1 \leqslant i \leqslant m$ and in a similar manner we obtain $|g_{0,n}(z)| \geqslant \frac{1}{4}|f_{0,n}(z)|$ for $z \in A_0^n$. Now if we renumber $\{\Omega_n\}$ to have $\Omega \setminus \Omega_n \subset A_n = \bigcup_{i=0}^m A_i^n$ and in the same way we renumber $\{f_n\}$, then

$$4\|g_n\|_{L^\Phi(\bar\Omega,\mu)} \geqslant \|I_n f_n\|_{L^\Phi(\bar\Omega,\mu)} \geqslant \varepsilon_0 > 0,$$

for n big enough and this is a contradiction to the fact $\|g_n\|_{L^\Phi(\bar\Omega,\mu)} \to 0$. □

Carleson window on $\Gamma_i \subset \Gamma = \partial\Omega$ of center $\xi \in \Gamma_i$ and radius $0 < h < \min_{i\neq j}\operatorname{dist}(\Gamma_i, \Gamma_j)$ is the set

$$W_0(\xi, h) = \{z \in \overline{\Omega} \colon 1 - h < |z|,\ |\arg(\overline{\xi} z)| < h\}, \quad i = 0$$
$$W_i(\xi, h) = \{z \in \overline{\Omega} \colon |z - a_i| < \frac{r_i}{1-h},\ |\arg(\overline{\xi} z)| < h\}, \quad 1 \leqslant i \leqslant m.$$

It is trivial that $z \in W_0(\xi, h)$ if and only if $\eta_i(z) \in W_i(\eta_i(\xi), h)$ when $i \geqslant 1$. Note also that area of Carleson window satisfies estimation $A(W_i(\xi, h)) \approx 2r_i^2 h^2$, and the constant $2r_i^2$ is optimal.

Let μ be a finite measure on $\bar\Omega$, $0 < h < \min_{i\neq j}\operatorname{dist}(\Gamma_i, \Gamma_j)$, $A > 0$, and Φ an Orlicz function, we denote:

- $\rho_\mu(h) := \max_{0\leqslant i\leqslant m} \sup_{\xi\in\Gamma_i} \mu(W(\xi, h))$,
- $K_\mu(h) := \sup_{0<t\leqslant h} \frac{\rho_\mu(t)}{t}$,
- $\gamma_A(h) := \frac{1}{\Phi\left(A\Phi^{-1}\left(\frac{1}{h}\right)\right)}$.

Following [8] we define the following conditions:

(R_0) $\lim_{h\to 0^+} \frac{\rho_\mu(h)}{\gamma_A(h)} = 0$ for every $A > 0$.

(K_0) $\lim_{h\to 0^+} \frac{K_\mu(h)h}{\gamma_A(h)} = 0$ for every $A > 0$.

(C_0) Inclusion operator $j_\mu \colon H^\Phi(\Omega) \hookrightarrow L^\Phi(\bar\Omega, \mu)$ is compact.

Our purpose is to show that we have the following implications

$$(K_0) \implies (C_0) \implies (R_0).$$

We will use the following proposition (see [15, Lemma 3.3.])

Proposition 4.5. *Condition* (R_0) *is equivalent to*:

$$\lim_{h\to 0^+} \frac{\Phi^{-1}\left(\frac{1}{h}\right)}{\Phi^{-1}\left(\frac{1}{\rho_\mu(h)}\right)} = 0.$$

We note that in [8] (or [15]) it was proved that (C_0) implies (R_0) in the case of $\Omega = \mathbf{D}$. Based on our Theorem 4.4 we prove below a more general result.

Theorem 4.6. *Condition* (C_0) *implies* (R_0).

Proof. Using Theorem 4.4, in the same way that Proposition 4.1 was used to prove Proposition 4.2, we deduce that for each $0 \leqslant i \leqslant m$ we have

$$\lim_{r\to 1^-} \sup_{p\in\partial\mathbf{D}} \Phi^{-1}\Big(\frac{1}{1-r}\Big)||u^i_{p,r}||_{L^\Phi(\bar\Omega,\mu)} = 0. \tag{8}$$

It is easy to check that $|u^0_{p,r}(z)| \geqslant \frac{1}{9}$, when $z \in W_0(p,h)$, and $r = 1-h$, thus

$$\frac{1}{\Phi\big(\frac{1}{\mu(W_0(p,h))}\big)} \leqslant 9||u^0_{p,r}||_{L^\Phi(\bar\Omega,\mu)}.$$

We obtain analogous inequality

$$\frac{1}{\Phi\big(\frac{1}{\mu(W_i(\eta_i(p),h))}\big)} \leqslant 9||u^i_{p,r}||_{L^\Phi(\bar\Omega,\mu)},$$

if $i \geqslant 1$ using the fact that $z \in W_0(p,h)$ if and only if $\eta_i(z) \in W_i(\eta_i(p),h)$. Further taking for each $i \geqslant 0$ the supremum over $\xi_i = \eta_i(p)$ (in fact we take $(m+1)$-times supremum over $p \in \partial\mathbf{D}$ and use the fact that η_i is bijection) and then maximum over $i = 0,\dots,m$ we obtain from (8) that

$$\lim_{h\to 0^+} \frac{\Phi^{-1}\big(\frac{1}{h}\big)}{\Phi^{-1}\big(\frac{1}{\rho_\mu(h)}\big)} = 0. \qquad \square$$

For $\xi \in \Gamma$, and $\alpha > 1$ consider the following family of sets

$$G^\alpha_\xi := \{z \in \Omega\colon |z-\xi| < \alpha \operatorname{dist}(z,\Gamma)\}.$$

Put $G_\xi := G^3_\xi$. The maximal (non-tangential) function M_f will be defined by:

$$M_f(\xi) := \sup\{|f(z)|\colon z \in G_\xi\}.$$

Using the fact that the maximal (non-tangential) function M_f is weak type $(1,1)$ and strong type (∞,∞) (see [5, p. 47–49]) we can extend (using the same methods) important result which was proved in [8, Proposition 3.5].

Proposition 4.7. *Assume that the Orlicz function $\Phi \in \nabla_2$. Then every linear, or sublinear, operator which is of weak-type $(1,1)$ and (strong) type (∞,∞) is bounded from L^Φ into itself. In particular, for every $f \in L^\Phi(\Gamma)$, the maximal non-tangential function $M_f \in L^\Phi(\Gamma)$.*

Theorem 4.8. *Let μ be finite borel measure on Ω, $f\colon \Omega \to \mathbf{C}$ continuous function. Then for each $t > 0$ and for each $0 < h < \frac{1}{4}\min_{i\neq j}\operatorname{dist}(\Gamma_i,\Gamma_j)$ we have*

$$\mu(\{z \in \Omega\colon\ \operatorname{dist}(z,\Gamma) \leqslant h,\ |f(z)| > t\}) \leqslant C' K_\mu(h) s(\{\xi \in \Gamma\colon M_f(\xi) > t\}). \tag{9}$$

It was proved in [8] that for $\bar{\mathbf{D}}$ the inequality holds with $C' = 1$. Using exactly the same methods it is easy to obtain (9) for a circular domain.

Recall that the norm of evaluation functional at point $z \in \Omega$ on $H^\Phi(\Omega)$ is not larger than $C\Phi^{-1}\Big(\frac{1}{\operatorname{dist}(z,\Gamma)}\Big)$ for a constant C. Put $c_0 = \max(1,C)$ (to have $c_0 > 1$) and denote by $c_1 = \max(1,C')$, where C' is the constant from Theorem 4.8.

Lemma 4.9. *Let μ be a finite Borel measure on $\bar\Omega$. Suppose that $\Phi \in \nabla_2$ and there exists $A > 0$ and $0 < h_A < \frac{1}{4}\min_{i\neq j}\operatorname{dist}(\Gamma_i,\Gamma_j)$ such that for every $h \in (0,h_A)$ we have*

$$K_\mu(h) \leqslant \tfrac{1}{h}\gamma_A(h),$$

then for every $f \in B_{H^{\Phi}(\Omega)}$ and for every Borel subset $E \subset \bar{\Omega}$ we have

$$\int_E \Phi\Big(\frac{A}{2c_0}|f|\Big)\, d\mu \leqslant \mu(E)\Phi(x_A) + \frac{c_1}{c_0}\int_\Gamma \Phi(M_f)\, ds,$$

where $x_A = \frac{A}{2}\Phi^{-1}\big(\frac{1}{h_A}\big)$.

Proof. For every $w > 0$, the inequality $|f(z)| > w$ implies that

$$w < c_0\Phi^{-1}\Big(\frac{1}{\operatorname{dist}(z,\Gamma)}\Big),$$

hence $\operatorname{dist}(z,\Gamma) < \frac{1}{\Phi(w/c_0)}$. It follows by Theorem 4.8 that

$$\mu(\{z \in \Omega\colon |f(z)| > w\}) \leqslant c_1 K_\mu\Big(\frac{1}{\Phi(s/c_0)}\Big) s(\{\xi \in \Gamma\colon M_f(\xi) > w\}),$$

if $\Phi(w/c_0) \geqslant \frac{1}{4}\min_{i\neq j}\operatorname{dist}(\Gamma_i,\Gamma_j)$. Further

$$\int_E \Phi\Big(\frac{A}{2c_0}|f|\Big)\, d\mu = \int_0^\infty \Phi'(t)\mu\big(\{|f| > 2c_0 t/A\} \cap E\big)\, dt.$$

By assumption for $\Phi(w/c_0) > 1/h_A$ we have:

$$K_\mu\Big(\frac{1}{\Phi(w/c_0)}\Big) \leqslant \frac{\Phi(w/c_0)}{\Phi(Aw/c_0)}.$$

Then we have

$$\mu\bigg(\bigg\{z \in \Omega |f(z)| > \frac{2c_0 t}{A}\bigg\}\bigg) \leqslant c_1\frac{\Phi(2t/A)}{\Phi(2t)} s\bigg(\bigg\{\xi \in \Gamma\colon M_f(\xi) > \frac{2c_0 t}{A}\bigg\}\bigg).$$

and

$$\begin{aligned}\int_E \Phi\Big(\frac{A}{2c_0}|f|\Big)\, d\mu &= \int_0^{x_A} \Phi'(t)\mu(E)dt + \int_{x_A}^\infty \Phi'(t) s\bigg(\bigg\{M_f(\xi) > \frac{2c_0 t}{A}\bigg\}\bigg)dt \\ &\leqslant \Phi(x_A)\mu(E) + c_1\int_{x_A}^\infty \frac{\Phi(2t/A)}{\Phi(2t)}\Phi'(t) s\bigg(\bigg\{M_f(\xi) > \frac{2c_0 t}{A}\bigg\}\bigg)dt.\end{aligned}$$

Note that one has $\Phi(t) \leqslant t\Phi'(t) \leqslant \Phi(2t)$, for any Orlicz function Φ. Using these inequalities we obtain

$$\begin{aligned}\int_{x_A}^\infty \frac{\Phi(2t/A)}{\Phi(2t)}\Phi'(t)\, s\bigg(\bigg\{M_f(\xi) > \frac{2c_0 t}{A}\bigg\}\bigg)\, dt \\ \leqslant \int_0^\infty \frac{\Phi(2t/A)}{t} s\bigg(\bigg\{M_f(\xi) > \frac{2c_0 t}{A}\bigg\}\bigg)\, dt \\ \leqslant \frac{2}{A}\int_0^\infty \Phi'(2t/A) s\bigg(\bigg\{M_f(\xi) > \frac{2c_0 t}{A}\bigg\}\bigg)\, dt \\ \leqslant \frac{2}{A}\int_0^\infty \Phi'(x) s\Big(\Big\{M_f > c_0 x\Big\}\Big)\, dx \\ = \int_\Gamma \Phi\Big(\frac{1}{c_0}M_f\Big)\, ds \leqslant \frac{1}{c_0}\int_\Gamma \Phi\Big(M_f\Big)\, ds\end{aligned}$$

which implies desired inequality. ☐

Theorem 4.10. *If $\Phi \in \nabla_2$, then condition (K_0) implies (C_0).*

Proof. Take $\varepsilon > 0$. Denote by C the norm of maximal operator M_f. Set $A = \frac{4Cc_0c_1}{\varepsilon}$, where c_0 and c_1 are constants from the previous Lemma. Condition (K_0) implies existence of $0 < h_A < \frac{1}{4}\min_{i\neq j}\operatorname{dist}(\Gamma_i,\Gamma_j)$ such that

$$K_\mu(h) \leqslant \frac{1}{2h}\gamma_A(h),$$

when $h \leqslant h_A$. If $f \in B_{H^\Phi(\Omega)}$ then for each $n \in \mathbf{N}$ by Lemma 4.9 we have:

$$\begin{aligned}\int_{\bar\Omega\setminus\Omega_n}\Phi\Big(\frac{|f|}{\varepsilon}\Big)\,d\mu &= \int_{\bar\Omega\setminus\Omega_n}\Phi\Big(\frac{A|f|}{4Cc_0c_1}\Big)\,d\mu \leqslant \frac{1}{2}\int_{\bar\Omega\setminus\Omega_n}\Phi\Big(\frac{A|f|}{2Cc_0c_1}\Big)\,d\mu\\ &\leqslant \frac{1}{2}\Big(\mu(\bar\Omega\setminus\Omega_n)\Phi(x_A) + \frac{c_1}{c_0}\int_\Gamma \Phi\Big(\frac{M_f}{Cc_1}\Big)\,ds\Big)\\ &\leqslant \frac{1}{2}\Big(\mu(\bar\Omega\setminus\Omega_n)\Phi(x_A) + \frac{1}{c_0}\Big),\end{aligned}$$

for any regular exhaustion $\{\Omega_n\}$ of Ω. Since $c_0 > 1$ and $\mu(\Gamma) = 0$ we have $\frac{1}{2}(\mu(\bar\Omega\setminus\Omega_n)\Phi(x_A) + \frac{1}{c_0}) \leqslant 1$ for a sufficiently large n. This implies by Theorem 4.4 that j_μ is compact, i.e., (C_0) holds. □

We are able to show that in the case of $\mu = \mu_\varphi$ that conditions (C_0), (R_0) and (K_0) are equivalent. To prove that we need the following result:

Theorem 4.11. *There exists a constant $k > 0$ such that for every $\varphi \in \Upsilon$, each $0 \leqslant i \leqslant m$ and $\xi_i \in \Gamma_i$ we have*

$$\mu_\varphi(W_i(\xi_i, \varepsilon h)) \leqslant k\varepsilon\mu_\varphi(W_i(\xi_i, h)), \tag{10}$$

where $\varepsilon \in (0,1)$ and h is small enough.

Proof. First notice that Theorem is true when $\Omega = \mathbf{D}$, $\varphi \in \Upsilon_{\mathbf{D}}$ (see [8], in this case $h < 1 - |\varphi(0)|$). To prove it in general case fix $0 \leqslant i \leqslant m$. Let $U_1^i,\dots,U_{n_i}^i \subset \Omega$ be finite family of domains which satisfies conditions:

(i) for every $1 \leqslant j \leqslant n_i$ the set U_j^i is simply connected,
(ii) for every $1 \leqslant j \leqslant n_i$ the boundary ∂U_j^i of U_j^i is formed by analytic Jordan curve,
(iii) $\Gamma_i \subset \bigcup_{j=1}^{n_i}\partial U_j^i$ and every arc contained in $\Gamma_i \cap \partial U_j^i$ is free analytic boundary arc for every $1 \leqslant j \leqslant n_i$,
(iv) for every $0 \leqslant l \leqslant m$ and $l \neq i$ we have $\Gamma_l \cap \bigcup_{j=1}^{n_i}\partial U_j^i = \emptyset$.

Let $\tau_{i,j}\colon U_j^i \mapsto \mathbf{D}$ be conformal map from U_j^i onto disc $\mathbf{D}$, then $\tau_{i,j}$ extends for each $1 \leqslant j \leqslant n_i$ to diffeomorphism $\tau_{i,j}\colon \bar U_j^i \mapsto \bar{\mathbf{D}}$. The same is true for the inverse map $\tau_{i,j}^{-1}$. Note that there exist finite constants $C_{i,j}, c_{i,j} > 0$ such that $c_{i,j} \leqslant |(\tau_{i,j}^{-1})'(z)| \leqslant C_{i,j}$. It implies that for every $z_1, z_2 \in \partial\mathbf{D}$ there exist finite constants $C'_{i,j}, c'_{i,j} > 0$ such that

$$c'_{i,j}|z_1 - z_2| \leqslant |\tau_{i,j}^{-1}(z_1) - \tau_{i,j}^{-1}(z_2)| \leqslant C'_{i,j}|z_1 - z_2|. \tag{11}$$

Put $C = \max_{0\leqslant i\leqslant m}\max_{1\leqslant j\leqslant n_i} C'_{i,j}$ and $c = \min_{0\leqslant i\leqslant m}\min_{1\leqslant j\leqslant n_i} c'_{i,j}$. Fix $\xi \in \Gamma_k \subset \partial\Omega$ and let h be as small that the preimage $\varphi^{*-1}(W_k(\xi,h))$ of Carleson window $W_k(\xi,h)$ is contained in a certain arc of ∂U_j^i for $0 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n_i$.

For $0 \leqslant k \leqslant m$ define function $\psi_{i,j,k}\colon \mathbf{D} \mapsto \mathbf{D}$ as follow:

$$\psi_{i,j,k}(z) = (\eta_k^{-1} \circ \varphi \circ \tau_{i,j}^{-1})(z), \quad z \in \mathbf{D}.$$

Since $\psi_{i,j,k} \in \Upsilon_{\mathbf{D}}$ then the measure $\mu_{\psi_{i,j,k}}$ satisfies (10). Thus for sufficiently small $h > 0$ we have

$$\begin{aligned}\mu_{\psi_{i,j,k}}\big(W_0(\eta^{-1}(\xi),h)\big) &= s\big(\{z \in \partial\mathbf{D}\colon\ \psi^*_{i,j,k}(z) \in W_0(\eta^{-1}(\xi),h)\}\big)\\ &= s\big(\{z \in \partial\mathbf{D}\colon\ \varphi^* \circ \tau^{-1}_{i,j}(z) \in W_k(\xi,h)\}\big).\end{aligned}$$

Using (11) with universal constants C, c we easily see that the last amount is comparable to

$$s\big(\{z \in \Gamma_i\colon \varphi^*(z) \in W_k(\xi,h)\}\big) = \mu_\varphi\big(W_k(\xi,h)\big),$$

for every $0 \leqslant k \leqslant m$ and every $\xi \in \Gamma_k$ and sufficiently small $h > 0$. Thus for sufficiently small Carleson windows we have

$$\mu_{\psi_{i,j,k}}(W_0(\eta^{-1}(\xi),h)) \approx \mu_\varphi(W_k(\xi,h))$$

for every $0 \leqslant k \leqslant m$. Then for $0 \leqslant k \leqslant m$ we have

$$\begin{aligned}\mu_\varphi(W_k(\xi,\varepsilon h)) &\leqslant K_1\mu_{\psi_{i,j,k}}\big(W_0(\eta_k^{-1}(\xi),\varepsilon h)\big)\\ &\leqslant CK_1\varepsilon\mu_{\psi_{i,j,k}}\big(W_0(\eta_k^{-1}(\xi),h)\big) \leqslant CK_1K_2\varepsilon\mu_\varphi\big(W_k(\xi,h)\big).\end{aligned}$$ □

Using this Theorem we obtain characterization of compact composition operators.

Theorem 4.12. *Let $\varphi \in \Upsilon$ and let $\Phi \in \nabla_2$ be the Orlicz function. Composition operator $C_\varphi\colon H^\Phi(\Omega) \to H^\Phi(\Omega)$ is compact if and only if for every $A > 0$*

$$\lim_{h\to 0^+} \frac{\rho_\varphi(h)}{\gamma_A(h)} = 0, \tag{12}$$

where $\rho_\varphi := \rho_{\mu_\varphi}$. Equivalently C_φ is compact if and only if

$$\lim_{h\to 0^+} \frac{\Phi^{-1}\big(\frac{1}{h}\big)}{\Phi^{-1}\big(\frac{1}{\rho_\varphi(h)}\big)} = 0. \tag{13}$$

Proof. Equivalence of (12) and (13) was proved in 4.5. For h small enough and $0 < t < h$ using (10) with $\varepsilon = \frac{t}{h}$ we obtain

$$\mu_\varphi(W_i(\xi_i,t)) \leqslant k\tfrac{t}{h}\mu_\varphi(W_i(\xi_i,h))$$

for each $\xi_i \in \Gamma$, $i = 0,\dots,m$. Taking supremum over $\xi \in \Gamma$ we get $\rho_\varphi(t) \leqslant k\frac{t}{h}\rho_\varphi(h)$. Hence for $\mu = \mu_\varphi$

$$K_\mu(h) = \sup_{0<t\leqslant h}\frac{\rho_\mu(t)}{t} \approx \frac{\rho_\mu(h)}{h},$$

so from Theorem 4.6 and Theorem 4.10 conditions (R_0), (K_0) and (C_0) are equivalent. □

Now we prove that condition (7) in Proposition 4.2 is also sufficient to the compactness of composition operator.

Corollary 4.13. *Let $\varphi \in \Upsilon$ and let $\Phi \in \nabla_2$ be the Orlicz function. Composition operator $C_\varphi\colon H^\Phi(\Omega) \to H^\Phi(\Omega)$ is a compact operator if and only if for each $0 \leqslant i \leqslant m$ we have*

$$\lim_{s\to 1^-}\sup_{p\in\partial\mathbf{D}} \Phi^{-1}\Big(\frac{1}{1-s}\Big)\|C_\varphi u^i_{p,s}\|_{H^\Phi(\Omega)} = 0. \tag{14}$$

Proof. We prove in Proposition 4.2 that condition (14) is necessary to compactness of C_φ. In Theorem 4.6 we showed that (13) follows from

$$\lim_{r\to 1^-}\sup_{p\in\partial\mathbf{D}} \Phi^{-1}\left(\frac{1}{1-r}\right)||u^i_{p,r}||_{L^\Phi(\bar\Omega,\mu)} = 0,$$

but this condition is equivalent to (14) for $\mu = \mu_\varphi$. By Theorem 4.12 we obtain the thesis. □

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