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A SIMPLE PROOF FOR THE BOUNDARY SCHWARZ LEMMA FOR PLURIHARMONIC MAPPINGS

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Abstract. In this paper, we give a simple proof for the boundary Schwarz lemma for pluriharmonic mappings between Euclidean unit balls. We also give some generalization to C^1-mappings between domains with smooth boundaries.

1. Introduction

Let B^n be the Euclidean unit ball in $\mathbf{C}^n$ and let $\mathbf{B}^{2n}$ be the Euclidean unit ball in $\mathbf{R}^{2n}$. Each $z = x + iy \in \mathbf{C}^n$ corresponds to $z' = (x, y)^T \in \mathbf{R}^{2n}$, where T denotes the transpose of vectors and matrices. For $z \in \mathbf{C}^n$, $\|z\|$ denotes the Euclidean norm on $\mathbf{C}^n$. For $x \in \mathbf{R}^m$, $\|x\|$ denotes the Euclidean norm on $\mathbf{R}^m$. For each $z'_0 \in \partial\mathbf{B}^{2n}$, the tangent space $T_{z'_0}(\partial\mathbf{B}^{2n})$ is defined by

$$T_{z'_0}(\partial\mathbf{B}^{2n}) = \{\beta \in \mathbf{R}^{2n} \colon z_0'^T \beta = 0\}.$$

A C^2 mapping $f \colon B^n \to \mathbf{C}^m$ is said to be pluriharmonic if the restriction of each component f_j to every complex line is harmonic.

Let Ω be a domain in $\mathbf{R}^m$. For a C^1 mapping $f \colon \Omega \to \mathbf{R}^M$, let $J_f(x)$ denote the $M \times m$ Jacobian matrix of f at $x \in \Omega$.

In recent years, the Schwarz lemma at the boundary for holomorphic mappings has been studied by many authors [1, 2, 3, 5, 6]. More recently, the following boundary Schwarz lemma for pluriharmonic mappings between Euclidean unit balls was proved by Liu, Dai and Pan [4].

Theorem 1.1. *Let $f \colon B^n \to B^N$ be a pluriharmonic mapping for $n, N \geq 1$. If f is $C^{1+\alpha}$ at $z_0 \in \partial B^n$ for some $\alpha \in (0, 1)$ and $f(z_0) = w_0 \in \partial B^N$, then we have*

(I) *$J_f(z'_0)\beta \in T_{w'_0}(\partial\mathbf{B}^{2N})$ for any $\beta \in T_{z'_0}(\partial\mathbf{B}^{2n})$;*

(II) *There exists a positive $\lambda \in \mathbf{R}$ such that $J_f(z'_0)^T w'_0 = \lambda z'_0$,*

where z'_0 and w'_0 are real versions of z_0 and w_0 respectively, and

$$\lambda \geq \frac{1 - \|f(0)\|}{2^{2n-1}} > 0.$$

For the proof, they used the Schwarz lemma for pluriharmonic mappings [4, Theorem 1.1], a technical lemma [4, Lemma 2.1] and the Harnack inequality for nonnegative harmonic functions on the Euclidean unit ball in $\mathbf{R}^n$ [4, Theorem 3.1]. In this paper, we will prove the following theorem by using the Harnack inequality for nonnegative harmonic functions on the unit disc U in $\mathbf{C}$ and elementary arguments.

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We do not use the Schwarz lemma for pluriharmonic mappings [4, Theorem 1.1] and a technical lemma [4, Lemma 2.1].

The novelty of our theorem is as follows. We only need that f is C^1 at $z_0 \in \partial B^n$. Also, in (I), the assumption that f is pluriharmonic is not needed. In (II), we give an improvement of the lower estimate for λ.

Theorem 1.2. *Let $f\colon B^n \to B^N$ be a C^1 mapping for $n, N \geq 1$. Assume that f is C^1 at $z_0 \in \partial B^n$ and $f(z_0) = w_0 \in \partial B^N$.*

(I) *Then we have $J_f(z_0')\beta \in T_{w_0'}(\partial \mathbf{B}^{2N})$ for any $\beta \in T_{z_0'}(\partial \mathbf{B}^{2n})$ and there exists a nonnegative $\lambda \in \mathbf{R}$ such that $J_f(z_0')^T w_0' = \lambda z_0'$;*

(II) *Moreover, if f is pluriharmnic on B^n, then*

$$\lambda \geq \frac{1-(f(0)')^T w_0'}{2} \geq \frac{1-\|f(0)\|}{2} > 0.$$

We give an example of a real anaytic mapping f such that $\lambda = 0$ in (I) of the above theorem. So, we cannot conclude that $\lambda > 0$ in (I) of the above theorem.

Example 1.3. Let $f(z) = (e^{-(x_1-1)^2}, 0, \ldots, 0)^T$, $f\colon B^n \to B^N$, where $z = (x_1, \ldots, x_n)^T + i(y_1, \ldots, y_n)^T$. Let $z_0 = (1, \ldots, 0)^T \in \partial B^n$. Then we have $w_0 = f(z_0) = (1, \ldots, 0)^T \in \partial B^N$ and $J_f(z_0') = O$. Therefore, $\lambda = 0$.

We give a generalization of Theorem 1.2 (I) to C^1-mappings between domains with smooth boundaries. Let Ω be a domain in $\mathbf{R}^n$. Ω is said to have C^r boundary ($r \geq 1$), if there exist a neighbourhood U of $\partial\Omega$ and a real valued C^r function ρ on U such that $\Omega \cap U = \{x \in U\colon \rho(x) < 0\}$, $\nabla \rho \neq 0$ on $\partial\Omega$, where

$$\nabla\rho(x) = \left(\frac{\partial\rho}{\partial x_1}(x), \ldots, \frac{\partial\rho}{\partial x_n}(x)\right)^T, \quad \text{for } x = (x_1, \ldots, x_n)^T \in U.$$

ρ is called the defining function for Ω. For each $x_0 \in \partial\Omega$, the tangent space to $\partial\Omega$ at x_0 is defined as follows:

$$T_{x_0}(\partial\Omega) = \{\beta \in \mathbf{R}^n\colon \nabla\rho(x_0)^T\beta = 0\}.$$

Proposition 1.4. *Let $\Omega_1 \subset \mathbf{R}^m$ be a domain with C^2-boundary and $\Omega_2 \subset \mathbf{R}^M$ be a domain with C^1-boundary for $m, M \geq 1$. Let ρ_j be the defining function for Ω_j for $j = 1, 2$, respectively. Let $f\colon \Omega_1 \to \Omega_2$ be a C^1 mapping. Assume that f is C^1 at $x_0 \in \partial\Omega_1$ and $f(x_0) = y_0 \in \partial\Omega_2$. Then we have $J_f(x_0)\beta \in T_{y_0}(\partial\Omega_2)$ for any $\beta \in T_{x_0}(\partial\Omega_1)$ and there exists a nonnegative $\lambda \in \mathbf{R}$ such that $J_f(x_0)^T\nabla\rho_2(y_0) = \lambda\nabla\rho_1(x_0)$.*

2. Proof of Theorem 1.2

Proof of (I). Let $\beta \in T_{z_0'}(\partial\mathbf{B}^{2n})$ be fixed. We may assume that $\|\beta\| = 1$. Let $\gamma(t) = (1-t^2)z_0' + t\beta$. Then $\gamma(t) \in \mathbf{B}^{2n}$ for $t \in (-1,1) \setminus \{0\}$, $\gamma(0) = z_0'$ and $\frac{d}{dt}\gamma(t)|_{t=0} = \beta$. Therefore the real valued function $(f(\gamma(t))')^T w_0'$ attain its local maximum at $t = 0$. Since this function is C^1 on $(-1,1)$, we have

$$\frac{d}{dt}(f(\gamma(t))')^T w_0'|_{t=0} = (J_f(z_0')\beta)^T w_0' = 0.$$

This implies that $J_f(z_0')\beta \in T_{w_0'}(\partial\mathbf{B}^{2N})$. Next, assume that $J_f(z_0')^T w_0' = \lambda z_0' + \beta$ for some $\lambda \in \mathbf{R}$ and $\beta \in T_{z_0'}(\partial\mathbf{B}^{2n})$. Then

$$\|\beta\|^2 = (\lambda z_0' + \beta)^T\beta = (J_f(z_0')^T w_0')^T\beta = w_0'^T J_f(z_0')\beta = 0$$

by the above argument. Therefore, we have $J_f(z_0')^T w_0' = \lambda z_0'$ for some $\lambda \in \mathbf{R}$. Let

$$u(\zeta) = 1 - (f(\zeta z_0')')^T w_0', \quad \zeta \in U. \tag{2.1}$$

Then $u(r) \geq u(1)$ for $r \in (0,1)$. Therefore, we have

$$\lambda = (J_f(z_0')z_0')^T w_0' = \lim_{r \to 1-0} \frac{u(r) - u(1)}{1-r} \geq 0.$$

Proof of (II). Assume that f is pluriharmonic on B^n. Then the function u defined in (2.1) is nonnegative and harmonic on the unit disc U in $\mathbf{C}$. By Harnack's inequality on the unit disc, we have

$$\frac{1-r}{1+r} u(0) \leq u(\zeta) \leq \frac{1+r}{1-r} u(0), \quad \text{for } r = |\zeta| < 1.$$

Therefore,

$$\frac{1}{1+r} u(0) \leq \frac{u(r) - u(1)}{1-r}, \quad \text{for } 0 < r < 1.$$

Letting $r \to 1 - 0$, we have

$$\frac{1 - (f(0)')^T w_0'}{2} \leq (J_f(z_0')z_0')^T w_0' = \lambda.$$

This completes the proof. □

3. Proof of Proposition 1.4

Let $\beta \in T_{x_0}(\partial\Omega_1)$ be fixed. Let $\gamma(t) = x_0 + \varepsilon t\beta - t^2 \nabla\rho_1(x_0)$. Then $\gamma(0) = x_0$ and $\frac{d}{dt}\gamma(t)|_{t=0} = \varepsilon\beta$. Since Ω_1 has C^2-boundary, there exist $\varepsilon > 0$ and $t_0 > 0$ such that $\gamma(t) \in \Omega_1$ for $t \in (-t_0, t_0) \setminus \{0\}$. Therefore the real valued function $\rho_2(f(\gamma(t)))$ attain its local maximum at $t = 0$. Since this function is C^1 near $t = 0$, we have

$$\frac{d}{dt}\rho_2(f(\gamma(t)))|_{t=0} = (\nabla\rho_2(y_0))^T J_f(x_0)\varepsilon\beta = 0.$$

This implies that $J_f(x_0)\beta \in T_{y_0}(\partial\Omega_2)$. Next, assume that $J_f(x_0)^T \nabla\rho_2(y_0) = \lambda\nabla\rho_1(x_0) + \beta$ for some $\lambda \in \mathbf{R}$ and $\beta \in T_{x_0}(\partial\Omega_1)$. Then

$$\|\beta\|^2 = (\lambda\nabla\rho_1(x_0) + \beta)^T\beta = (J_f(x_0)^T\nabla\rho_2(y_0))^T\beta = \nabla\rho_2(y_0)^T J_f(x_0)\beta = 0$$

by the above argument. Therefore, we have $J_f(x_0)^T\nabla\rho_2(y_0) = \lambda\nabla\rho_1(x_0)$ for some $\lambda \in \mathbf{R}$. Since $x_0 - t\nabla\rho_1(x_0) \in \Omega_1$ for sufficiently small $t > 0$, we have

$$\begin{aligned}
\lambda\|\nabla\rho_1(x_0)\|^2 &= (\nabla\rho_2(y_0))^T J_f(x_0)\nabla\rho_1(x_0) \\
&= -\frac{d}{dt}\rho_2(f(x_0 - t\nabla\rho_1(x_0)))|_{t=0} \\
&= \lim_{t \to +0} \frac{\rho_2(f(x_0)) - \rho_2(f(x_0 - t\nabla\rho_1(x_0)))}{t} \\
&\geq 0.
\end{aligned}$$

Thus, $\lambda \geq 0$. This completes the proof. □

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