

PERFECT FRACTAL SETS WITH ZERO FOURIER DIMENSION AND ARBITRARY LONG ARITHMETIC PROGRESSION

Chun-Kit Lai

San Francisco State University, Department of Mathematic
1600 Holloway Avenue, San Francisco, CA 94132, U.S.A.; cklai@sfsu.edu

Abstract. By considering a Moran-type construction of fractals on $[0, 1]$, we show that for any $0 \leq s \leq 1$, there exists some Moran fractal sets, which is perfect, with Hausdorff dimension s whose Fourier dimension is zero and it contains arbitrarily long arithmetic progressions.

1. Introduction

1.1. Background and main results. Let E be a set inside a locally compact abelian group G (which may be \mathbf{R} , \mathbf{T} , $\mathbf{Z}/n\mathbf{Z}$). It has been widely believed that the decay of the Fourier transform of a measure supported on E and the length of arithmetic progressions (AP) that E can contain is closely related. For example, on \mathbf{R} , any sets of positive Lebesgue measure must contain arbitrarily long arithmetic progressions due to the Lebesgue density theorem and it supports measures decaying to zero due to the Riemann–Lebesgue lemma. On the other hand, the uniformity norm of a set $A \subset \mathbf{Z}/n\mathbf{Z}$ is defined as

$$\|A\|_u = \sup_{k \neq 0} |\widehat{\mathbf{1}_A}(k)|.$$

Small enough $\|A\|_u$ will produce 3-term non-trivial AP [18, Chapter 4 and 10]. Variants and generalization of such observation lead to many far reaching consequences, including the Green–Tao Theorem [6] on arbitrary long AP on primes.

Throughout the paper, we will focus on the group \mathbf{R} . The Fourier transform of a finite Borel measure μ on \mathbf{R} is defined to be

$$\widehat{\mu}(\xi) = \int e^{-2\pi i \xi x} d\mu(x).$$

Similar questions on AP and Fourier decay can be asked in the sparse set on \mathbf{R} . Fourier decay will be characterized by Fourier dimension. Let E be a Borel set in \mathbf{R} , the *Fourier dimension* of E is defined to be

$$\dim_F(E) = \sup\{\beta: \exists \mu \text{ such that } \text{spt} \mu = E \text{ and } |\widehat{\mu}(\xi)| \leq C|\xi|^{-\beta/2}\}.$$

Here, $\text{spt} \mu$ is the support of μ . It is well known Fourier dimension is always less than or equal to the Hausdorff dimension [4, Chapter 4]. A set E is called a *Salem set* if the Hausdorff dimension and the Fourier dimension of E are equal. It is difficult to construct Salem set deterministically, but it does exist in abundance under many probability models.

Laba and Pramanik [11] initiated the study about the connection between Fourier decay and the existence of AP. They showed, under some additional assumption, the existence of measures with Fourier decay implies that its support contain a 3-term non-trivial AP. Their results generated immense interest investigating how Fourier

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decay implies the existence of points of some prescribed configuration [1, 2, 16]. On the other hand, Shmerkin [16] recently showed however that there exist Salem sets without any 3-term AP. In this paper, we ask a converse question: *Suppose that a perfect set contains an arbitrarily long AP, can we say something about the Fourier dimension of the sets?*

Recall that a *perfect set* on \mathbf{R}^d is a compact set without isolated points. Note that the converse question will be trivial if we consider arbitrary sets. For example, we can take union of a Cantor set with the rational numbers. Then it must contain arbitrarily long AP from the rational numbers, but its Fourier dimension is completely determined by the Cantor sets we are taking. The following is our main conclusion:

Theorem 1.1. *For any $0 \leq s \leq 1$, there exists a perfect set E whose Hausdorff dimension equals s containing arbitrarily long arithmetic progression and its Fourier dimension equals zero.*

The set E is in fact a Cantor-type construction, which is automatically perfect. More specifically, it is called *Moran-type construction* of fractal sets. It means that the contraction ratio at the same level is the same but are allowed to vary at different levels (see Section 2). We show that if at each level, the fundamental intervals we choose are “biased” towards one end, then the Moran fractal will have Fourier dimension zero. This is in stark contrast to known results that random Moran construction generically are Salem sets [16] (see also [11, Section 6]). By doing so, we construct perfect sets with arbitrary Hausdorff dimensions that have zero Fourier dimension but it has arbitrary long AP.

1.2. Open problems. This result suggests us some broader open problems in connection to the current research between AP and Fourier decay. Suppose that $\mathcal{E} := \{E_j\}_{j=1}^\infty$ is a countable collection of finite sets on \mathbf{R}^d .

(Qu1): Does there exist any countable collection such that if a perfect set E contains some affine copies of E_j , for all j , then E supports a measure with decaying Fourier transform?

Our theorem showed that $E_j = \{0, 1, \dots, j - 1\}$ does not guarantee (Qu1) to hold. We may actually ask (Qu1) on \mathbf{R}^1 in the most general form:

(Qu2): Suppose that a perfect set E contains affine copies of any finite pattern on \mathbf{R}^1 , then E supports a measure μ with decaying Fourier transform? Can the support of μ be a Salem set?

In fact, perfect sets of zero Lebesgue measure containing affine copies of all finite patterns exist and it were first constructed by Erdős and Kakutani [3] in 1957. Recently, Molter and Yavicoli [14] constructed closed sets of arbitrary Hausdorff dimension on \mathbf{R}^1 containing all possible finite patterns. However, no Fourier analysis on such sets has been studied and (Qu1) and (Qu2) remains unknown.

On \mathbf{R}^2 , one may try

$$(1.1) \quad E_j = \{(\cos(2\pi k/j), \sin(2\pi k/j)) : k = 0, 1, \dots, j - 1\},$$

the set of points on \mathbf{R}^2 forming vertices of polygons with number of edges equal j .

(Qu3): Suppose that a perfect set E contains some affine copies of E_j in (1.1) for all j . Does E support a measure μ with decaying Fourier transform? Is the support of μ inside E a Salem set?

It is clear that the unit circle contains all E_j in (1.1), and yet unit circle is a Salem set and it does not give a counterexample to the question. Perfect fractal sets containing all such E_j is abundant in nature. One explicit example is called the *Apollonian gasket* (see Figure 1). For a more detailed exposition of Apollonian gasket, we refer the reader to [7, 10].

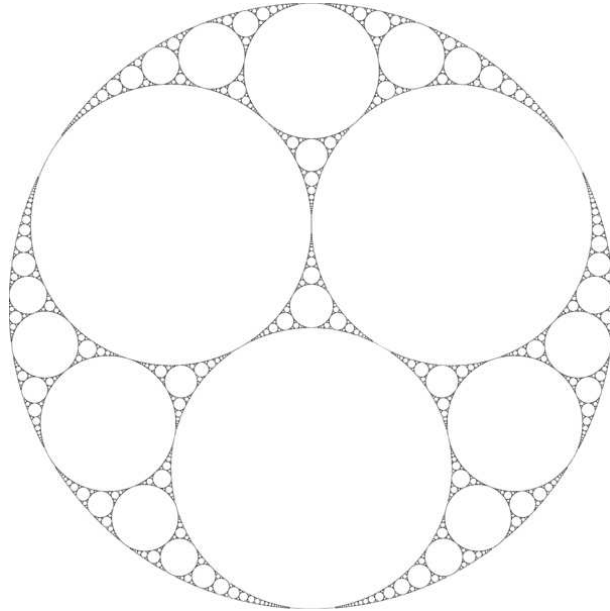


Figure 1. Apollonian gasket is formed by deleting the interiors of three circles tangential to the unit circle. Repeating the process by deleting the circle tangential and inside each curvilinear triangle left. The set that remains will be perfect and it contains all E_j in (1.1). However, its exact Fourier dimension is largely unknown.

We also remark that special cases of the Moran measures were considered by the author in the study of fractal measures with exponential bases and frames [8, 12].

For the organization of the rest of the paper, we will present our setup and main result in Section 2 and prove our theorems in Section 3.

2. Setup and main results

Let $(K_j)_{j=1}^\infty$ and $(N_j)_{j=1}^\infty$ be sequences of positive integers such that $0 < K_j < N_j$ and $N_j \geq 2$. For each $j \geq 1$, let also B_j be subset of $\{0, 1, \dots, N_j - 1\}$ satisfying $\#B_j = K_j$. Moreover, we define

$$(2.1) \quad c := \sup_{j=1,2,\dots} \frac{\max B_j}{N_j}.$$

Define the following multi-index notation:

$$\Sigma_n = \{(b_1, \dots, b_n) : b_j \in B_j\}, \quad \Sigma^* = \bigcup_{n=1}^\infty \Sigma_n.$$

For each $\mathbf{j} := (b_1, \dots, b_n) \in \Sigma^*$, we associate an interval

$$I_{\mathbf{j}} = \left[\sum_{j=1}^n \frac{b_j}{N_1 \dots N_j}, \sum_{j=1}^n \frac{b_j}{N_1 \dots N_j} + \frac{1}{N_1 \dots N_n} \right].$$

These intervals form the fundamental intervals of a fractal. The *Moran set* associated with (N_j, B_j) is the unique compact set in $[0, 1]$ such that

$$(2.2) \quad E = E(N_j, B_j) = \bigcap_{n=1}^{\infty} \bigcup_{\mathbf{j} \in \Sigma_n} I_{\mathbf{j}}.$$

E also admits a representation in terms of digit expansion

$$(2.3) \quad E = \left\{ \sum_{j=1}^{\infty} \frac{b_j}{N_1 \dots N_j} : b_j \in B_j \ \forall j \geq 1 \right\}.$$

Shmerkin [16, Theorem 2.1] proved recently that the Moran set in (2.3) is almost surely a Salem set if we choose randomly the digit B_j from $\{0, 1, \dots, N_j - 1\}$ with $\#B_j = K_j$ under a natural assumption on N_j :

$$(2.4) \quad \lim_{j \rightarrow \infty} \frac{\log N_{j+1}}{\log(N_1 \dots N_j)} = 0,$$

In contrast to the theorem of Shmerkin, we prove if we pick digits biased towards one end, we don't have any decaying Fourier transform.

Theorem 2.1. *Suppose that $c < 1/2$ where c is defined in (2.1). Then any measures supported on E in (2.2) does not decay and thus*

$$\dim_F(E) = 0.$$

The above theorem leads to the following main theorem.

Theorem 2.2. *There exists fractal sets with zero Fourier dimension but it contains arbitrarily long arithmetic progression.*

The condition $c < 1/2$ is sharp in the sense that if all $N_j = 2$ and $B_j = \{0, 1\}$, then $c = 1/2$ and the measure is the Lebesgue measure on $[0, 1]$, whose Fourier transform will decay.

To avoid such triviality, we consider $N_j \geq 3$ and the consecutive digits set $B_j = \{0, 1, \dots, K_j - 1\}$ with $0 < K_j < N_j$, so that none of the E generated are intervals. We show that for a larger value of c , there exists non-decaying measures on E .

Theorem 2.3. *Let $N_j \geq 3$ and $B_j = \{0, 1, \dots, K_j - 1\}$ with $K_j < N_j$. Then*

- (1) if $c < 2/3$, $\dim_F(E) = 0$,
- (2) if

$$c < \frac{\sqrt{6}}{\pi} \simeq 0.7796968,$$

the Fourier transform of the standard measure supported on E which distributes the mass by $\mu(I_{\mathbf{j}}) = \frac{1}{K_1 \dots K_n}$ for all $\mathbf{j} \in \Sigma_n$ and $n \geq 1$ does not decay in the following sense:

$$\limsup_{n \rightarrow \infty} |\widehat{\mu}(n)| > 0.$$

We don't know whether $\dim_F(E) = 0$ for c in between $2/3$ and $\sqrt{6}/\pi$. From all the examples we have, to the best of our knowledge, if E supports a measure with non-decaying Fourier transform, then Fourier transform of all other measures supported on E does not decay.

Finally, we study the Hausdorff dimension of the fractal sets in Theorem 2.1 and Theorem 2.3. Denote by \dim_H the Hausdorff dimension, Using result in [5], we have

Theorem 2.4. (1) For the fractal sets in Theorem 2.1,

$$\liminf_{j \rightarrow \infty} \frac{\log(K_1 \dots K_j)}{\log(N_1 \dots N_j) + \log(N_{j+1}/K_{j+1})} \leq \dim_H(E) \leq \liminf_{j \rightarrow \infty} \frac{\log(K_1 \dots K_j)}{\log(N_1 \dots N_j)}.$$

(2) For the fractal sets in Theorem 2.3,

$$\dim_H(E) = \liminf_{j \rightarrow \infty} \frac{\log(K_1 \dots K_j)}{\log(N_1 \dots N_j) + \log(N_{j+1}/K_{j+1})}.$$

In particular, for any $0 \leq s \leq 1$, there exists fractal set E with $\dim_H(E) = s$, $\dim_F(E) = 0$ and it contains arbitrarily long AP.

3. Proofs of the theorems

We first prove Theorem 2.1. This theorem follows from a classical result of Rajchman. We identify $[0, 1]$ as the circle $\mathbf{T} = \mathbf{R}/\mathbf{Z}$. Denote by $\mathcal{M}(\mathbf{T})$ the set of all finite Borel measures. Consider the following class of measures in $\mathcal{M}(\mathbf{T})$

$$R := \left\{ \mu \in \mathcal{M}(\mathbf{T}) : \lim_{|n| \rightarrow \infty} \widehat{\mu}(n) = 0 \right\}.$$

The measure μ in R are called *Rajchman measure*. For more detailed results about Rajchman measures, one can refer to [13]. Suppose that \mathcal{E} is a collection of subsets of \mathbf{T} , we define also

$$\mathcal{E}^\perp := \{ \mu \in \mathcal{M}(\mathbf{T}) : \forall E \in \mathcal{E}, \mu(E) = 0 \}.$$

For a set E and an integer n , we define the set

$$nE := \{ nx \pmod{1} : x \in E \}.$$

Let also \overline{E} be the closure of E under the standard Euclidean topology. The following theorem can be found in [9], [17] and [13]. We provide a proof based on the concept of set of uniqueness for completeness.

Theorem 3.1. Let

$$\mathcal{H} = \left\{ E \subset \mathbf{T} : \exists n_k \uparrow \infty \text{ such that } \overline{\bigcup_{k=1}^{\infty} n_k E} \neq \mathbf{T} \right\}.$$

Then $R \subset \mathcal{H}^\perp$. In particular, if $E \in \mathcal{H}$, then $\dim_F(E) = 0$.

Proof. We prove it using the idea of the set of uniqueness. A set $E \subset [0, 1]$ is called a *set of uniqueness* if

$$\sum_{n=0}^{\infty} (a_n \cos(2\pi nx) + b_n \sin(2\pi nx)) = 0$$

for all $x \in [0, 1] \setminus E$, then $a_n = b_n = 0$. We know that if $E \in \mathcal{H}$, then E is a set of uniqueness [15, p.50 Theorem II]. When E is a set of uniqueness, it does not support any measures whose Fourier transform decays at infinity [19, p.348 Theorem 6.13]. Therefore, for any $\mu \in R$, $\mu(E) = 0$ for all $E \in \mathcal{H}$. Thus, $R \subset \mathcal{H}^\perp$. Finally, this means also that E does not support any measure with decaying Fourier transform, which shows $\dim_F(E) = 0$. \square

As a simple example of the theorem, for the standard middle-third Cantor set K_3 , the set $3^n K_3 = K_3$. Hence, $\overline{\bigcup_{k=1}^{\infty} 3^k K_3} = K_3 \neq \mathbf{T}$. Thus, $\dim_F(K_3) = 0$.

Proof of Theorem 2.1. We show that the Moran set K in (2.3) belongs to \mathcal{H} . In fact, we will show that for $n_k = N_1 \dots N_k$,

$$(3.1) \quad \overline{\bigcup_{k=1}^{\infty} n_k E} \subset [0, 2c].$$

Since $c < 1/2$, this union cannot be the whole circle and hence, this will complete the proof by Theorem 3.1.

To show that (3.1) holds, we note from (2.3) that for $n_k = N_1 \dots N_k$

$$n_k E = \left\{ \sum_{j=1}^{\infty} \frac{b_{k+j}}{N_{k+1} \dots N_{k+j}} : b_{k+j} \in B_{k+j} \right\}.$$

Using the definition of c and $N_k \geq 2$, we have

$$(3.2) \quad \max(n_k E) = \sum_{j=1}^{\infty} \frac{\max B_{k+j}}{N_{k+1} \dots N_{k+j}} \leq \sum_{j=1}^{\infty} \frac{c}{N_{k+1} \dots N_{k+j-1}} \leq \sum_{j=1}^{\infty} \frac{c}{2^{j-1}} = 2c.$$

Hence, $n_k E \subset [0, 2c]$ for all k . Thus, this shows (3.1) holds and completes the proof. □

Proof of Theorem 2.2. We consider $B_j = \{0, 1, \dots, K_j - 1\}$, where K_j satisfies

- (1) $\lim_{j \rightarrow \infty} K_j = \infty$,
- (2) $\frac{K_j}{N_j} < \frac{1}{2}$ for all j .

Then the second condition and Theorem 2.1 shows that $\dim_F(E(N_j, B_j)) = 0$. On the other hand, from (2.3), we know that the arithmetic progression

$$\left\{ 0, \frac{1}{N_1 \dots N_j}, \dots, \frac{K_j - 1}{N_1 \dots N_j} \right\}$$

are inside $E(N_j, B_j)$. Condition (1) on K_j implies that we have arbitrarily long arithmetic progression. □

We now turn to prove Theorem 2.3. The first statement follows from the same proof as Theorem 2.1, except in the last step (3.2), we use $N_j \geq 3$ to obtain $3/2c$ in the last step and hence $c < 2/3$ will guarantee $\overline{\bigcup_{k=1}^{\infty} n_k E}$ does not cover \mathbf{T} .

To prove the second statement, we note that the measure assigning $\mu(I_j) = \frac{1}{K_1 \dots K_n}$ is actually an infinite convolution of discrete measures

$$\mu = \nu_1 * \nu_2 * \dots,$$

where

$$\nu_n = \frac{1}{K_n} \sum_{j=0}^{K_n-1} \delta_{j/N_1 \dots N_n}$$

with δ_a denotes the Dirac measure at a . Suppose that we write

$$(3.3) \quad \mu_n = \nu_1 * \nu_2 \dots * \nu_n, \quad \mu_{>n} = \nu_{n+1} * \nu_{n+2} * \dots$$

so that $\mu = \mu_n * \mu_{>n}$. Note that

$$(3.4) \quad \widehat{\mu}(N_1 \dots N_n) = \widehat{\mu}_n(N_1 \dots N_n) \widehat{\mu}_{>n}(N_1 \dots N_n) = \widehat{\mu}_{>n}(N_1 \dots N_n),$$

because of the integral periodicity that $\widehat{\mu}_n(N_1 \dots N_n) = 1$.

Proof of Theorem 2.3 (2). To prove the required statement, we will show that

$$\lim_{n \rightarrow \infty} |\widehat{\mu}(N_1 \dots N_n)| > 0.$$

By (3.4), it suffices to prove that

$$\inf |\widehat{\mu}(N_1 \dots N_n)| = \inf |\widehat{\mu}_{>n}(N_1 \dots N_n)| > 0.$$

We now compute $|\widehat{\mu}_{>n}(\xi)|$ using (3.3):

$$|\widehat{\mu}_{>n}(\xi)| = \prod_{j=1}^{\infty} |\widehat{\nu}_{n+j}(\xi)| = \prod_{j=1}^{\infty} \left| \frac{1}{K_{n+j}} \sum_{j=0}^{K_{n+j}-1} e^{-2\pi i j \xi / N_1 \dots N_{n+j}} \right|.$$

Hence,

$$\begin{aligned} |\widehat{\mu}_{>n}(N_1 \dots N_n)| &= \prod_{j=1}^{\infty} \left| \frac{1}{K_{n+j}} \sum_{j=0}^{K_{n+j}-1} e^{-2\pi i j / N_{n+1} \dots N_{n+j}} \right| \\ &= \prod_{j=1}^{\infty} \left| \frac{e^{2\pi i K_{n+j} / N_{n+1} \dots N_{n+j}} - 1}{K_{n+j} (e^{2\pi i / N_{n+1} \dots N_{n+j}} - 1)} \right| \\ &= \prod_{j=1}^{\infty} \left| \frac{\sin(\pi K_{n+j} / N_{n+1} \dots N_{n+j})}{K_{n+j} \sin(\pi / N_{n+1} \dots N_{n+j})} \right| \\ &\geq \prod_{j=1}^{\infty} \left| \frac{\sin(\pi K_{n+j} / N_{n+1} \dots N_{n+j})}{\pi K_{n+j} / N_{n+1} \dots N_{n+j}} \right| \quad (\text{by } \sin x \leq x) \\ &\geq \prod_{j=1}^{\infty} \left| 1 - \frac{1}{6} \cdot \left(\frac{\pi K_{n+j}}{N_{n+1} \dots N_{n+j}} \right)^2 \right| \quad \left(\text{by } \sin x \geq x - \frac{x^3}{6} \right) \end{aligned}$$

where the last inequality holds provided that the terms inside are positive, which we are going to check. Note that for $j > 1$,

$$\begin{aligned} &\prod_{j=2}^{\infty} \left| 1 - \frac{1}{6} \cdot \left(\frac{\pi K_{n+j}}{N_{n+1} \dots N_{n+j}} \right)^2 \right| \\ &\geq \prod_{j=2}^{\infty} \left| 1 - \frac{1}{6} \cdot \left(\frac{\pi}{N_{n+1} \dots N_{n+j-1}} \right)^2 \right| \quad (\text{by } K_{n+1} < N_{n+1}) \\ &\geq \prod_{j=2}^{\infty} \left| 1 - \frac{1}{6} \cdot \left(\frac{\pi}{3^{j-1}} \right)^2 \right| \quad (\text{by } N_j \geq 3) \\ &= \prod_{j=1}^{\infty} \left(1 - \frac{\pi^2}{6 \cdot 3^{2j}} \right) := c_0 \end{aligned}$$

and $c_0 > 0$ because $\sum \frac{\pi^2}{6 \cdot 3^{2j}} < \infty$. Hence,

$$|\widehat{\mu}_{>n}(N_1 \dots N_n)| \geq \left| 1 - \frac{1}{6} \cdot \left(\frac{\pi K_{n+1}}{N_{n+1}} \right)^2 \right| \cdot c_0 \geq \left(1 - \frac{c^2 \pi^2}{6} \right) \cdot c_0$$

using the fact that $c = \sup \frac{K_{n+1}}{N_{n+1}}$. Hence, if $c < \sqrt{6}/\pi$, the term above will be positive. This shows that

$$\inf |\widehat{\mu}_{>n}(N_1 \dots N_n)| > 0$$

completing the proof. □

We now prove Theorem 2.4 about the Hausdorff dimension. Under our setting in Section 2, we say that E is a *homogeneous Moran set* if for any $\mathbf{j} \in \Sigma_n$, if we

let $I_{j_{b_1}}, \dots, I_{j_{b_{n+1}}}$ be the intervals inside I_j enumerated from left to right, then the leftmost endpoint of $I_{j_{b_1}}$ is the same as I_j , the rightmost endpoint of $I_{j_{b_{n+1}}}$ is the same as I_j and the gap between each consecutive intervals are the same. E is a *partial homogeneous Moran set* if for any $\mathbf{j} \in \Sigma_n$, then the leftmost endpoint of $I_{j_{b_1}}$ is the same as I_j , and the gap between each consecutive intervals are zero (i.e. intervals are packed towards one end).

Feng, Wen and Wu proved that

Theorem 3.2. [5, Theorem 2.1 and Lemma 2.2] *Given two sequences of numbers, N_j and K_j , $j = 1, 2, \dots$. The Hausdorff dimension of the homogenous Moran set and the partial homogeneous Moran set are s_1 and s_2 respectively, where*

$$s_1 = \liminf_{j \rightarrow \infty} \frac{\log(K_1 \dots K_j)}{\log(N_1 \dots N_j)}, \quad s_2 = \liminf_{j \rightarrow \infty} \frac{\log(K_1 \dots K_j)}{\log(N_1 \dots N_j) + \log(N_{j+1}/K_{j+1})}.$$

Moreover, any other Moran sets E with $\#B_j = K_j$, its Hausdorff dimension must satisfy

$$s_2 \leq \dim_H(E) \leq s_1.$$

Note that if we assume that N_j satisfies

$$(3.5) \quad \lim_{j \rightarrow \infty} \frac{\log N_{j+1}}{\log(N_1 \dots N_j)} = 0,$$

which is the same as the assumption in (2.4), then we see that $s_1 = s_2$. Assumption (3.5) holds if N_j does not grow too fast. For example, $N_j = j$ will do.

Proof of Theorem 2.4. The first statement follows directly from Theorem 3.2. If we pick $B_j = \{0, 1, \dots, K_j - 1\}$, then the Moran set E is partial homogeneous, so the second statement also follows from Theorem 3.2.

We now choose N_j and K_j appropriately so that the last statement holds. Consider first $0 < s < 1$. Take $K_j = \lfloor N_j^s \rfloor$, where $\lfloor x \rfloor$ is the largest integers smaller than or equal to x . Take also $B_j = \{0, 1, \dots, K_j - 1\}$. Then $\max B_j/N_j \leq N_j^s/N_j = \frac{1}{N_j^{1-s}} < 1/2$ as we choose N_j to be strictly increasing. Hence, Theorem 2.3 holds. We now compute the Hausdorff dimension. Assuming (3.5) holds,

$$\dim_H(E) = \liminf_{j \rightarrow \infty} \frac{\log(N_1 \dots N_j)^s}{\log(N_1 \dots N_j)} = s.$$

When $s = 1$, we take $K_j = \lfloor N_j/3 \rfloor$. Then $\max B_j/N_j \leq 1/3$, the same argument works.

Suppose that $s = 0$. We choose a sequence $s_j > 0$ such that

$$\lim_{j \rightarrow \infty} s_j = 0 \quad \text{and} \quad \lim_{j \rightarrow \infty} N_j^{s_j} = \infty$$

(For example, take $s_j = (\ln N_j)^{-1}$). Let $K_j = \lfloor N_j^{s_j} \rfloor$. Then

$$\dim_H(E) \leq \liminf_{j \rightarrow \infty} \frac{\log(N_1 \dots N_j)^{s_j}}{\log(N_1 \dots N_j)} = \liminf_{j \rightarrow \infty} s_j = 0.$$

Moreover, K_j still tends to infinity with $\max B_j/N_j < 1/2$ clearly holds. Hence, Theorem 2.3 gives us the Moran set has zero Fourier dimension and arbitrarily long AP, completing the proof. \square

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