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EXISTENCE AND UNIQUENESS RESULTS FOR LINEAR SECOND-ORDER EQUATIONS IN THE HEISENBERG GROUP

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Abstract. In this manuscript, we prove uniqueness and existence results of viscosity solutions for a class of linear second-order equations in the Heisenberg group. We state uniqueness by proving a comparison result to our class of equations, and existence via an application of Perron's method adapted to our framework. We also provide the explicit construction of the appropriate sub- and supersolutions employed by Perron's method for a variety of domains in the Heisenberg group.

1. Introduction and main results

Throughout the paper, we shall consider linear second-order differential equations in non-divergence form:

$$-\operatorname{tr}\left(A(p)\nabla^2_{\mathcal{H}_0}u(p)\right) + \langle b(p), \nabla_{\mathcal{H}}u(p)\rangle = f(p), \quad p \in \Omega, \tag{1.1}$$

where Ω is a bounded and smooth domain in the first-order Heisenberg group $\mathcal{H}$, $A\colon \overline{\Omega} \to \mathbf{R}^{2\times 2}$ and $b\colon \overline{\Omega} \to \mathbf{R}^3$. For definitions and notation, we refer the reader to Section 2. Observe that equation (1.1) does not depend on the variable u explicitly. This is because the u-dependence has been already treated in the literature [20, 22, 21].

The Heisenberg group is one of the simplest models which fit into the category of Carnot groups. These structures are connected, finite-dimensional, nilpotent, and simply connected Lie groups whose Lie algebras admit a stratification in terms of vector spaces. Moreover, one of these vector spaces generates, as a Lie algebra, the whole Lie algebra of the group and it is called the horizontal distribution. In this way, the relevant structures, such as the differential and the topological ones, are then defined in terms of the horizontal distribution. We refer the reader to [6] and [7] for further details.

The geometry of the Heisenberg group may be applied in different contexts. The differential operators defined in $\mathcal{H}$ can be used to model non-commutative quantum operators, as position and momentum, in the framework of Quantum Mechanics [6, 25]. Also, the analysis on the Heisenberg group may be used in mathematical finance, for instance to derive closed form of solutions for distributions of stock prices [26]. In these applications, the subelliptic Laplacian operator appears naturally. Observe that this operator arises in the right hand side of (1.1) when A equals the identity matrix field and b is identically zero.

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In this work we are interested in deriving general comparison and existence results for viscosity solutions of equations like (1.1). In spite of the fact that equation (1.1) is a relatively simple model, to the best of our knowledge, there are no comparison and uniqueness results that may be applied to the full form (1.1) (compare with the recent works [22] and [21]). Indeed, for the available comparison principles for second-order elliptic equations $F(x, u, \nabla_{\mathcal{H}} u, \nabla^2_{\mathcal{H}_0} u) = 0$, the following assumptions appear, in general, as essential tools:

- F is independent of the gradient of u
- the derivative $\partial F/\partial u$ is bounded away from zero, that is, there is a constant $\gamma > 0$ so that:
$$\frac{\partial F}{\partial u} > \gamma.$$
- the p-derivative of F grows at most quadratically in the gradients and linearly in the Hessians, that is, there are constants $R > 0$, $C_R > 0$ and $\tau \in [0,1]$ so that
$$|F(p, u, \eta, \mathcal{X}) - F(q, u, \eta, \mathcal{X})| \leq C_R + \omega_R(|p-q|)|p-q|^{\tau+2} + \mu_R(|p-q|\|\mathcal{X}\|),$$
for all $p, q \in \overline{\Omega}$, $|u| \leq R$, all vector $\eta \in \mathbf{R}^n$ and matrix $\mathcal{X}$. Moreover, ω_R and μ_R are modulus of continuity satisfying $\int_{0^+} (\mu_R(\sigma)/\sigma)\, d\sigma < \infty$.

For precise statements of the above assumptions and other sufficient conditions for comparison and maximum principles, see the references [20], [16] and [9]. Observe that the operator F driving equation (1.1) takes the form:
$$F(p, \eta, \mathcal{X}) = -\operatorname{tr}(A(p)\mathcal{X}) + \langle b(p), \eta \rangle - f(p).$$

Hence, in general, the above listed assumptions are not satisfied for (1.1). It is worth mentioning that most of the comparison results obtained for second-order and non-degenerate linear elliptic equations in the Heisenberg group are derived from the corresponding adaptation of the hypothesis and strategies presented mainly in [16]. We refer to [20] for an illustration of this observation and for an overview of maximum, comparison and uniqueness results in the Heisenberg scenario.

Based on the above comments, we firstly face this lack of general comparison principles for (1.1), by stating a uniqueness result for viscosity solutions. This will achieve by using the argument of perturbing a given viscosity (sub-) supersolution to obtain a strict viscosity (sub-) supersolution. Hence by an application of subelliptic principles and by the nature of (1.1), it is possible to derive the result. This strategy is also presented in [16] but it was used, in the Euclidean framework, in conjunction with additional hypothesis on the equation under study that we do not have in (1.1).

In addition to the application to uniqueness of viscosity solutions, comparison principles have been also used to study the connection between different notions of weak solutions. For instance, the relation between distributional and viscosity solutions for the p-Laplace equations has been obtained in [19] and [17] for the Euclidean case, and in [4] for the Heisenberg group. In these studies, comparison principles for, from one side, distributional solutions and, from the other one, for viscosity solutions were useful tools. Moreover, the knowledge of that relation is interesting since it allows for the formulation of Radó type theorem for p-harmonic functions (see [18]). Therefore, the comparison result developed in the present work is also potentially useful from that point of view.

On the other hand, existence of solutions in $\mathcal{H}$ has been recently treated in different works [1, 10, 24], etc, employing different procedures including approximation schemes, Perron's method, stochastic games, among others. In the present article, we study the problem of existence of viscosity solutions to (1.1) via the celebrated Perron's method adapted to the theory of viscosity solutions (see [14]) and to the Heisenberg group. As it is well known, Perron's method readily yields the existence of solutions with prescribed boundary conditions provided we are able to construct appropriate sub- and supersolutions having the right boundary values. Moreover, to conclude the existence of continuous solutions, the method requires a comparison result for semicontinuous solutions. This is another application of our contribution in getting comparison results for (1.1). Finally, we shall also provide the appropriate sub- and supersolutions for general continuous Dirichlet boundary values under different assumptions on the underlying domain Ω.

Throughout the article, we shall refer to the following set of assumptions on the data:

(H1) $A\colon \overline{\Omega} \to \mathbf{R}^{2\times 2}$, $b\colon \overline{\Omega} \to \mathbf{R}^3$ and $f\colon \overline{\Omega} \to \mathbf{R}$ are continuous in $\overline{\Omega}$, and A is symmetric and positive definite.

(H2) A is uniformly elliptic, that is, there exists a constant $\gamma > 0$ so that

$$\langle A(p)\xi, \xi\rangle \geq \gamma \|\xi\|^2, \text{ for all } p \in \overline{\Omega},\ \xi \in \mathbf{R}^n.$$

(H3) For each $p_0 \in \Omega$, there exist a neighbourhood U of p_0, a constant $C > 0$ and an even $m \geq 4$ such that:

$$\|b(p) - b(q)\| \leq C\phi(p\cdot q^{-1})^{1/m}, \text{ for all } p, q \in U,$$

where ϕ is given by

$$\phi(p\cdot q^{-1}) := (p_1 - q_1)^m + (p_2 - q_2)^m + (p_3 - q_3 + 2(p_1 q_2 - q_1 p_2))^m. \tag{1.2}$$

For notational purposes, we let

$$F(p, \eta, \mathcal{X}) := -\operatorname{tr}\big(A(p)\mathcal{X}\big) + \langle b(p), \eta\rangle - f(p),$$

for $(p, \eta, \mathcal{X}) \in \overline{\Omega} \times \mathbf{R}^3 \times S^2(\mathbf{R})$. The main contributions of our article are contained in the following

Theorem 1.1. *Suppose that (H1)-(H3) hold and let $u_0 \in \mathcal{C}(\overline{\Omega})$. Then there exists a unique solution to the following boundary value problem:*

$$\begin{cases} F(p, \nabla_{\mathcal{H}} u, \nabla^2_{\mathcal{H}_0} u) = 0 & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

The above result will be a direct consequence of the comparison principle stated in Section 3 and the Perron's method implemented in Section 4. Additional assumptions on the domain will be imposed in order to construct the appropriate sub- and supersolutions required by the method.

Finally, observe that hypothesis (H1) and (H2) are standard in the theory of elliptic differential equations. Moreover, (H3) imposes a Hölder-like regularity on the linear term.

We mention now that our results apply, among many other equations, to the subelliptic Laplace equation and to more general linear equations in divergence form as

$$-\operatorname{div}_{\mathcal{H}_0}\big(M(p)\nabla_{\mathcal{H}_0} u(p)\big) = f(p), \tag{1.4}$$

where $M\colon \overline{\Omega} \to \mathbf{R}^{2\times 2}$ is a $\mathcal{C}^1_{\mathcal{H}}$-symmetric and positive definite matrix field. Observe that solving equation (1.4) is equivalent to finding solutions of

$$-\operatorname{tr}\big(A(p)\nabla^2_{\mathcal{H}_0}u(p)\big) + \langle b(p), \nabla_{\mathcal{H}}u(p)\rangle = f(p),$$

where

$$A(p) = M(p), \quad b(p) = -\operatorname{div}_{\mathcal{H}_0} M(p).$$

Under assumptions (H1)-(H3) on A and b, which translate to additional conditions on the field M and its first-order derivatives, we may apply Theorem 1.1. Interesting cases appear when M is divergence-free or a constant matrix.

The paper is organized as follows. Section 2 is devoted to the preliminary definitions and results needed in the sequel. Precisely, we introduce the Heisenberg group, the notion of viscosity solutions and the subelliptic maximum principle. In Section 3, we provide the proof of the uniqueness part of Theorem 1.1, by means of a comparison principle. Finally, in Section 4, we show the existence of a viscosity solution to the boundary value problem (1.3) by means of Perron's method. We also provide specific sub- and supersolutions for different domains.

2. Preliminaries

Basic notation. We shall use the following standard notation in the work. The set of positive real numbers is denoted by $\mathbf{R}_+$. For real numbers a and b, we let:

$$a \wedge b := \min\{a, b\}, \quad a \vee b := \max\{a, b\}.$$

The Euclidean interior product is denoted by $\langle \cdot, \cdot \rangle$. If γ, $\beta \in \mathbf{R}^n$, the vector $\gamma \oplus \beta$ is defined as the vector in $\mathbf{R}^{2n}$ whose first n entries are those of γ, followed by the components of β. The canonical basis in $\mathbf{R}^n$ is $\{e_1, \ldots, e_n\}$. For matrices, we consider the following partial order

$$A \preceq B$$

if and only if $B - A$ is positive semi-definite. By $A \prec B$ we mean that $B - A$ is positive definite. Also, we denote by $S^n(\mathbf{R})$ the set of $n \times n$ symmetric and positive semi-definite matrices with real coefficients. The trace of a matrix A is denoted by $\operatorname{tr}(A)$. By Ω we shall always denote an open, bounded, connected and smooth domain in $\mathbf{R}^3$. We also introduce the following functional spaces

$$LSC(\Omega) = \{u\colon \Omega \to \mathbf{R}\colon u \text{ is lower semicontinuous in } \Omega\},$$
$$USC(\Omega) = \{u\colon \Omega \to \mathbf{R}\colon u \text{ is upper semicontinuous in } \Omega\}.$$

Finally, for $u\colon \Omega \to \mathbf{R} \cup \{-\infty, \infty\}$, we denote by u_* and u^* the lower- and upper semicontinuous envelopes of u, respectively. Defined as follows for $p \in \overline{\Omega}$:

$$u^*(p) = \lim_{r\to 0} \sup\left\{u(q)\colon |p-q| < r, q \in \Omega\right\},$$
$$u_*(p) = \lim_{r\to 0} \inf\left\{u(q)\colon |p-q| < r, q \in \Omega\right\}.$$

Recall that the envelopes are defined in $\overline{\Omega}$.

2.1. Heisenberg group. We denote by $\mathcal{H}$ the first-order Heisenberg group whose underlying manifold is $\mathbf{R}^3$, and whose group operation is given by

$$p \cdot q = (x_0 + x_1, y_0 + y_1, z_0 + z_1 + 2(x_1 y_0 - x_0 y_1)),$$

for all $p = (x_0, y_0, z_0)$, $q = (x_1, y_1, z_1) \in \mathbf{R}^3$. The group $\mathcal{H}$ is a Lie group with Lie algebra $\mathfrak{h}$ generated by the left-invariant basis

$$X_1 = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial z}, \quad X_2 = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial z}, \quad X_3 = \frac{\partial}{\partial z} = -\frac{1}{4}[X_1, X_2], \tag{2.1}$$

where $p = (x, y, z) \in \mathbf{R}^3$. We equip $\mathfrak{h}$ with an interior product (a Riemann structure) so that the frame (2.1) is orthonormal. We recall that the exponential mapping is a global diffeomorphism that takes the vector $xX_1 + yX_2 + zX_3$ in the Lie algebra $\mathfrak{h}$ to the point (x, y, z) in the Lie group $\mathcal{H}$. This allows us to identify vectors in $\mathfrak{h}$ with points in $\mathcal{H}$.

The two dimensional linear space generated by the vectors $X_1(p)$ and $X_2(p)$ is denoted by $\mathcal{H}_{0,p}$. The distribution $\mathcal{H}_0$ is called the horizontal distribution.

The metric structure on $\mathcal{H}$ is given by the Carnot–Carathéodory distance (CC distance in brief) which is defined as following: an absolutely continuous curve $\gamma \in W^{1,2}((0,1), \mathbf{R}^3)$ is said to be horizontal if there is a control $v \in L^2((0,1), \mathbf{R}^3)$ such that

$$\gamma'(t) = \sum_{i=1}^{2} v_i(t) X_i(\gamma(t))$$

for a.e. t in $(0,1)$. For any $p, q \in \mathcal{H}$, the CC distance between p and q is defined as

$$d_{CC}(p,q) := \inf\left\{\int_0^1 |v(t)|\,dt\right\}$$

where the infimum is taken over all horizontal curves γ, with associated control v, so that $\gamma(0) = p$ and $\gamma(1) = q$.

For computational purposes, we shall also use a smooth gauge out of the diagonal, called the Heisenberg gauge, which is equivalent to the CC distance, and it is defined as follows

$$|p|_{\mathcal{H}} := \left((x^2 + y^2)^2 + z^2\right)^{1/4}, \quad \text{for } p = (x, y, z) \in \mathcal{H}.$$

The corresponding distance is

$$d_{\mathcal{H}}(p,q) := |q^{-1} \cdot p|_{\mathcal{H}}, \quad \text{for all } p, q \in \mathcal{H}.$$

Also, for any $p \in \mathcal{H}$ and $\delta > 0$, we write

$$B_{\mathcal{H}}(p,\delta) := \{q \in \mathcal{H} \colon |q^{-1} \cdot p|_{\mathcal{H}} < \delta\},$$

to denote the ball in the Heisenberg group with center at p and radius δ.

Having established the basic structure on $\mathcal{H}$, our attention turns to differentiation and calculus. Given a smooth function u in $\mathbf{R}^3$, and a multi-index $I = (i_1, i_2, i_3)$, the derivative $X^I u$ is defined by

$$X^I u = X_1^{i_1} X_2^{i_2} X_3^{i_3} u.$$

The function u belongs to $\mathcal{C}^k_{\mathcal{H}}(\Omega)$ if $X^I u$ is continuous in Ω for all multi-indices I such that

$$d(I) := i_1 + i_2 + 2i_3 \le k.$$

Remark 2.1. Observe that, in general, the class of $\mathcal{C}^k_{\mathcal{H}}$ functions is larger than the class of Euclidean $\mathcal{C}^k(\Omega)$. Indeed, let $f(x,y,z) = x - g(y, 2xy + z)$, where

$$g(a,b) = \frac{|a|^\alpha b}{a^4 + b^2}, \quad \text{if } (a,b) \ne (0,0), \quad \text{and} \quad g(0,0) = 0.$$

Hence, $f \in \mathcal{C}^1_{\mathcal{H}}$ in the Heisenberg group for $\alpha \in (3,4)$ but it does not have continuous partial derivatives in the Euclidean sense (see [12, Remark 5.9]).

For an Euclidean smooth function $u\colon \mathbf{R}^3 \to \mathbf{R}$, Taylor expansion around 0 implies

$$u(p) = u(0) + \langle \nabla u(0), p\rangle + \frac{1}{2}\left\langle \nabla^2 u(0)p, p\right\rangle + o(|p|^2), \tag{2.2}$$

for $p \to 0$, and where ∇u and $\nabla^2 u$ stand for the Euclidean gradient and Hessian of u, respectively. Recalling that terms containing z^2, xz or yz are of order $o(|p|^2_{\mathcal{H}})$, we derive from (2.2) the horizontal Taylor expansion of u at 0

$$u(p) = u(0) + \langle \nabla_{\mathcal{H}} u(0), p\rangle + \frac{1}{2}\left\langle \nabla^2_{\mathcal{H}_0} u(0)p, p\right\rangle + o(|p|^2_{\mathcal{H}}),$$

where the gradient of u with respect to the frame (2.1) is

$$\nabla_{\mathcal{H}} u := (X_1 u)X_1 + (X_2 u)X_2 + (X_3 u)X_3,$$

and the symmetrized horizontal second derivative matrix, denoted by $\nabla^2_{\mathcal{H}_0} u$, is given by

$$\nabla^2_{\mathcal{H}_0} u := \begin{bmatrix} X_1^2 u & \frac{1}{2}(X_1X_2u + X_2X_1u) \\ \dfrac{1}{2}(X_1X_2u + X_2X_1u) & X_2^2 u \end{bmatrix}.$$

In addition, it will be useful to introduce the projection of $\nabla_{\mathcal{H}} u$ onto the horizontal distribution

$$\nabla_{\mathcal{H}_0} u := (X_1 u)X_1 + (X_2 u)X_2.$$

Finally, for a vector field $V = (V_1, V_2)$, its horizontal divergence operator is defined by

$$\operatorname{div}_{\mathcal{H}_0} V := X_1 V_1 + X_2 V_2,$$

while for a smooth matrix field

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix},$$

we let

$$\operatorname{div}_{\mathcal{H}_0} M := \begin{bmatrix} X_1 M_{11} + X_2 M_{12} \\ X_1 M_{21} + X_2 M_{22} \end{bmatrix}.$$

For a more complete discussion of the Heisenberg group and more general Carnot groups, we refer the reader to [7], [11] and [20].

2.2. Viscosity solutions. The basic reference in what follows is [3]. Let $u \in USC(\Omega)$. The second-order superjet of u at p is defined as follows

$$\mathcal{J}^{2,+}(u,p) := \Big\{(\eta, \mathcal{X}) \in \mathbf{R}^3 \times S^2(\mathbf{R}) \text{ such that } u(q) \le u(p) + \left\langle \eta, p^{-1}\cdot q\right\rangle + \frac{1}{2}\left\langle \mathcal{X}(p^{-1}\cdot q)_0, (p^{-1}\cdot q)_0\right\rangle + o\big(d_{\mathcal{H}}(p,q)^2\big)\Big\},$$

as $q \to p$. Here, $(p^{-1}\cdot q)_0$ is the projection of $p^{-1}\cdot q$ onto the horizontal distribution $\mathcal{H}_0$. Similarly, if $v \in LSC(\Omega)$, we define the second-order subjet of v at the point p as

$$\mathcal{J}^{2,-}(v,p) := \Big\{(\eta, \mathcal{Y}) \in \mathbf{R}^3 \times S^2(\mathbf{R}) \text{ such that } v(q) \ge v(p) + \left\langle \eta, p^{-1}\cdot q\right\rangle + \frac{1}{2}\left\langle \mathcal{Y}(p^{-1}\cdot q)_0, (p^{-1}\cdot q)_0\right\rangle + o\big(d_{\mathcal{H}}(p,q)^2\big)\Big\},$$

as $q \to p$. It is well-known (see [3]) that subelliptic jets may be seen as appropriate derivatives of test functions touching the given function by above or below. More precisely, if u is upper semicontinuous, let us consider

$$\mathcal{K}^{+,2}(u,p) := \Big\{ \big(\nabla_{\mathcal{H}}\phi(p), \nabla^2_{\mathcal{H}_0}\phi(p)\big) \colon \text{so that } \phi \text{ is } \mathcal{C}^2_{\mathcal{H}} \text{ and} \\ (u-\phi)(q) \le (u-\phi)(p) \text{ for all } q \text{ close to } p \Big\}$$

and similarly define $\mathcal{K}^{-,2}(v,p)$ for test functions touching the lower semicontinuous function v from below around p. Hence, by the results in [3], it follows that

$$\mathcal{J}^{2,+}(u,p) = \mathcal{K}^{2,+}(u,p) \tag{2.3}$$

and

$$\mathcal{J}^{2,-}(v,p) = \mathcal{K}^{2,-}(v,p). \tag{2.4}$$

Finally, we shall also consider the theoretic closure of the sets defined above. We define $\overline{\mathcal{J}}^{2,+}(u,p)$ as the set of $(\eta,\mathcal{X})$ in $\mathbf{R}^3 \times S^2(\mathbf{R})$ so that there exists a sequence $(p_n, u(p_n), \eta_n, \mathcal{X}_n)$ converging to $(p, u(p), \eta, \mathcal{X})$ satisfying $(\eta_n, \mathcal{X}_n) \in \mathcal{J}^{2,+}(u,p_n)$ for all n. In a similar way, we define $\overline{\mathcal{J}}^{2,-}(v,p)$.

In the next definition, we introduce the notion of viscosity solutions of general second-order equations driven by an operator $\mathcal{F}$. Afterwards, we give the notion of solution of a boundary value problem.

Definition 2.2. Let $\mathcal{F}$ be a continuous function in $\Omega \times \mathbf{R} \times \mathbf{R}^3 \times S^2(\mathbf{R})$. An upper (resp. lower) semicontinuous function $u \colon \Omega \to \mathbf{R} \cup \{-\infty\}$ (resp. $\mathbf{R} \cup \{+\infty\}$) is a subsolution (resp. supersolution) of

$$\mathcal{F}(p, u, \nabla_{\mathcal{H}} u, \nabla^2_{\mathcal{H}_0} u) = 0,$$

in Ω if for any $p \in \Omega$:

(1) $u(p) < +\infty$ (resp. $u(p) > -\infty$);
(2) for every $(\eta, \mathcal{X}) \in \mathcal{J}^{2,+}(u,p)$ (resp. $(\eta, \mathcal{Y}) \in \mathcal{J}^{2,-}(v,p)$) there holds

$$\mathcal{F}(p, u, \eta, \mathcal{X}) \le 0, \quad (\text{resp. } \mathcal{F}(p, u, \eta, \mathcal{Y}) \ge 0).$$

Finally, a viscosity solution is both a viscosity subsolution and a viscosity supersolution.

Remark 2.3. In view of Remark 2.1 the set of $\mathcal{C}^2_{\mathcal{H}}$ test functions is larger than the set of Euclidean $\mathcal{C}^2$ test functions. Therefore, by (2.3) and (2.4), the set of viscosity solutions from Definition 2.2 is larger than the set of viscosity solution to the same equation, thought as depending on Euclidean gradient and hessian, obtained by testing with Euclidean test functions.

Definition 2.4. Let $u_0 \in \mathcal{C}(\overline{\Omega})$ and consider the Dirichlet boundary value problem

$$\begin{cases} \mathcal{F}(p, u, \nabla_{\mathcal{H}} u, \nabla^2_{\mathcal{H}_0} u) = 0 & \text{in } \Omega \quad (E) \\ u = u_0 & \text{on } \partial\Omega \quad (BD). \end{cases} \tag{2.5}$$

We say that $u \in USC(\overline{\Omega})$ is a viscosity subsolution to (2.5) if u is a subsolution to equation (E) in Ω and $u \le u_0$ on $\partial\Omega$. Similarly, we define the notion of viscosity supersolution. Finally, a viscosity solution to (2.5) is a viscosity sub- and supersolution.

Observe that the boundary condition in (2.5) is attained pointwise on $\partial\Omega$.

2.3. The subelliptic maximum principle in $\mathcal{H}$. The following version of the subelliptic maximum principle is a consequence of the Euclidean Crandall–Ishii Lemma (see [8] and also [9]) and the passage from Euclidean jets into subjets [3]. It is extracted from [5, Lemma 3.6].

Theorem 2.5. *Let $u \in USC(\Omega)$ and $v \in LSC(\Omega)$. Consider the function ϕ as defined in* (1.2)*, and assume that for each $\tau > 0$, p_τ and q_τ are points in $\Omega \times \Omega$ at which $u(p) - v(p) - \tau\phi(p \cdot q^{-1})$ has a local maximum. Moreover, suppose that $u - v$ has an interior maximum so that*

$$\sup_{\Omega}(u - v) > 0.$$

Then, there exist vectors $\eta_\tau = \nabla^p_{\mathcal{H}}\phi(p_\tau \cdot q_\tau^{-1})$ and matrices $\mathcal{X}_\tau$, $\mathcal{Y}_\tau$ in $S^2(\mathbf{R})$ so that

$$(\tau\eta_\tau, \mathcal{X}_\tau) \in \overline{\mathcal{J}}^{2,+}(u, p_\tau), \quad (\tau\eta_\tau, \mathcal{Y}_\tau) \in \overline{\mathcal{J}}^{2,-}(v, q_\tau)$$

and

$$\langle \mathcal{X}_\tau\xi, \xi\rangle - \langle \mathcal{Y}_\tau\xi, \xi\rangle \le \tau\|M_\tau\|^2\|\xi\|^2, \tag{2.6}$$

where M_τ is the following matrix of Euclidean derivatives

$$M_\tau := \begin{bmatrix} \nabla^2_{pp}\phi(p_\tau \cdot q_\tau^{-1}) & \nabla^2_{pq}\phi(p_\tau \cdot q_\tau^{-1}) \\ \nabla^2_{pq}\phi(p_\tau \cdot q_\tau^{-1}) & \nabla^2_{qq}\phi(p_\tau \cdot q_\tau^{-1}) \end{bmatrix}.$$

3. Proof of Theorem 1.1: uniqueness.

The uniqueness of solutions to (1.3) will be a corollary of the next comparison principle. Its proof is based on a perturbation argument to obtain strict supersolutions.

Theorem 3.1. (Comparison principle) *Assume (H1)–(H3). Let $u \in USC(\overline{\Omega})$ be a subsolution of (1.1) and $v \in LSC(\overline{\Omega})$ be a supersolution of (1.1) so that $u \le v$ on $\partial\Omega$. Then*

$$u \le v \quad \text{in } \Omega.$$

Proof. Reasoning by contradiction, suppose that

$$\sup_{\Omega}(u - v) > 0.$$

Since $\Omega \subset \mathcal{H}$ is bounded, there exists $r_0 > 0$ such that $\overline{\Omega} \subset (-r_0, r_0)^3$. Define $\varphi\colon \Omega \to \mathbf{R}$ by

$$\varphi(p) := \varphi(x, y, z) = -\exp\left(2\frac{\kappa}{\gamma}x\right) - \exp\left(2\frac{\kappa}{\gamma}y\right) + 2\exp\left(2\frac{\kappa}{\gamma}r_0\right),$$

where $\kappa > 0$ so that $\|b(x)\| \le \kappa$ for all $x \in \overline{\Omega}$. We note that φ is a positive function in $\overline{\Omega}$. For latter reference, we compute

$$\nabla_{\mathcal{H}}\varphi(p) = -2\frac{\kappa}{\gamma}\left(\exp\left(2\frac{\kappa}{\gamma}x\right), \exp\left(2\frac{\kappa}{\gamma}y\right), 0\right),$$

and

$$\nabla^2_{\mathcal{H}_0}\varphi(p) = -4\left(\frac{\kappa}{\gamma}\right)^2 \begin{bmatrix} \exp\left(2\frac{\kappa}{\gamma}x\right) & 0 \\ 0 & \exp\left(2\frac{\kappa}{\gamma}y\right) \end{bmatrix} \prec 0,$$

in Ω. Let $v_\delta = v + \delta\varphi$. Note that for δ small enough, the function $u - v_\delta$ satisfies

$$\sup_{\Omega}(u - v_\delta) > 0. \tag{3.1}$$

We note that the function v_δ is a strict supersolution of the equation (1.1). Indeed, let $\psi \in \mathcal{C}^2_{\mathcal{H}}(\Omega)$ be a smooth test function touching v_δ from below at $p_0 \in \Omega$

$$\psi(p_0) = v_\delta(p_0), \quad \psi(p) < v_\delta(p), \quad p \neq p_0.$$

Then $\psi(p) - \delta\varphi(p)$ is a smooth test function that touches v from below at p_0. As v is a supersolution of (1.1), we derive

$$F\big(p_0, \nabla_{\mathcal{H}}\psi(p_0) - \delta\nabla_{\mathcal{H}}\varphi(p_0), \nabla^2_{\mathcal{H}_0}\psi(p_0) - \delta\nabla^2_{\mathcal{H}_0}\varphi(p_0)\big) \geq 0. \tag{3.2}$$

Moreover, observe that

$$\begin{aligned} -\delta\ \mathrm{tr}(A(p_0)\nabla^2_{\mathcal{H}_0}\varphi(p_0)) &\geq -\gamma\delta\ \mathrm{tr}(\nabla^2_{\mathcal{H}_0}\varphi(p_0)) \\ &= 4\frac{\delta\kappa^2}{\gamma}\left[\exp\left(2\frac{\kappa}{\gamma}x\right) + \exp\left(2\frac{\kappa}{\gamma}y\right)\right], \end{aligned}$$

and

$$\delta\,\langle b(p_0), \nabla_{\mathcal{H}}\varphi(p_0)\rangle \geq -\delta\kappa \parallel \nabla_{\mathcal{H}}\varphi(p_0) \parallel \geq -2\frac{\delta\kappa^2}{\gamma}\left[\exp\left(2\frac{\kappa}{\gamma}x\right) + \exp\left(2\frac{\kappa}{\gamma}y\right)\right].$$

Therefore, the difference

$$F(p_0, \nabla_{\mathcal{H}}\psi(p_0), \nabla^2_{\mathcal{H}_0}\psi(p_0)) - F(p_0, \nabla_{\mathcal{H}}\psi(p_0) - \delta\nabla_{\mathcal{H}}\varphi(p_0), \nabla^2_{\mathcal{H}_0}\psi(p_0) - \delta\nabla^2_{\mathcal{H}_0}\varphi(p_0))$$

may be estimated from below by the scalar

$$\alpha := 2\frac{\delta\kappa^2}{\gamma}\exp\bigg(-2\frac{\kappa}{\gamma}r_0\bigg).$$

The above, together with (3.2), imply

$$F\left(p_0, \nabla_{\mathcal{H}}\psi(p_0), \nabla^2_{\mathcal{H}_0}\psi(p_0)\right) \geq \alpha > 0.$$

Since ψ was arbitrary and F is continuous, we conclude that

$$F(p, \eta, \mathcal{X}) \geq \alpha > 0 \tag{3.3}$$

holds for all $(\eta, \mathcal{X}) \in \overline{\mathcal{J}}^{2,-}(v_\delta, p)$. Now we apply the subelliptic maximum principle to $u - v_\delta$. In order to do so, we shall check that if $p_\tau, q_\tau \in \overline{\Omega}$ are so that

$$M_\tau := \sup_{\overline{\Omega}\times\overline{\Omega}}\left\{u(p) - v_\delta(q) - \tau\phi(p\cdot q^{-1})\right\} = u(p_\tau) - v_\delta(q_\tau) - \tau\phi(p_\tau\cdot q_\tau^{-1}), \tag{3.4}$$

then we indeed have that p_τ and q_τ belong to Ω. The proof is standard, we quote it for completeness. To prove the statement, observe that in view of the compactness of $\overline{\Omega}$, there exist $p_0, q_0 \in \overline{\Omega}$ so that, up to a subsequence that we do not re-label

$$p_\tau \to p_0 \ \text{ and } \ q_\tau \to q_0 \ \text{ as } \ \tau \to \infty. \tag{3.5}$$

To reach a contradiction, assume that $p_0 \in \partial\Omega$. Observe that by (3.4), we have for any $p \in \overline{\Omega}$

$$u(p_\tau) - v_\delta(q_\tau) - \tau\phi(p_\tau\cdot q_\tau^{-1}) \geq u(p) - v_\delta(p), \tag{3.6}$$

which, together with the boundedness of u and $-v_\delta$ from above in $\overline{\Omega}$, imply that there is a constant $C > 0$ independent of τ so that

$$\tau\phi(p_\tau\cdot q_\tau^{-1}) \leq C, \quad \text{for all } \tau.$$

Hence, we must have

$$\lim_{\tau\to\infty} \phi(p_\tau \cdot q_\tau^{-1}) = 0.$$

Thus $p_0 = q_0$ in (3.5). Moreover, by (3.6), it follows

$$\tau\phi(p_\tau \cdot q_\tau^{-1}) \le u(p_\tau) - v_\delta(q_\tau) - u(p_0) + v_\delta(p_0).$$

Taking limsup and recalling (3.5), we certainly derive

$$\lim_{\tau\to\infty} \tau\phi(p_\tau \cdot q_\tau^{-1}) = 0. \tag{3.7}$$

By (3.1), there is $\tilde{p} \in \Omega$ so that $(u - v_\delta)(\tilde{p}) > 0$. Another application of (3.4), together with (3.7), give

$$\begin{aligned} u(p_0) - v(p_0) - \delta\varphi(p_0) &\ge \limsup_{\tau\to\infty} \left\{ u(p_\tau) - v(q_\tau) - \delta\varphi(q_\tau) - \tau\phi(p_\tau \cdot q_\tau^{-1}) \right\} \\ &\ge u(\tilde{p}) - v(\tilde{p}) - \delta\varphi(\tilde{p}) > 0, \end{aligned} \tag{3.8}$$

where the latter inequality follows by the choice of $\tilde{p}$. Hence we arrive at $u(p_0) - v(p_0) > 0$, which contradicts the boundary assumption. Then $p_0 \in \Omega$ and we derive $p_\tau, q_\tau \in \Omega$ for all sufficiently large τ.

By Theorem 2.5, for each $\tau > 0$, there exist $\eta_\tau \in \mathbf{R}^3$ and symmetric matrices $\mathcal{X}_\tau, \mathcal{Y}_\tau \in S^2(\mathbf{R})$ such that

$$(\tau\eta_\tau, \mathcal{X}_\tau) \in \overline{\mathcal{J}}^{2,+}(u, p_\tau), \quad (\tau\eta_\tau, \mathcal{Y}_\tau) \in \overline{\mathcal{J}}^{2,-}(v_\delta, q_\tau).$$

By (3.3), the following holds

$$0 < \alpha \le F(q_\tau, \tau\eta_\tau, \mathcal{Y}_\tau), \quad \text{for each } \tau. \tag{3.9}$$

Adding and subtracting $\mathrm{tr}(A(p_\tau)\mathcal{X}_\tau)$, $\langle b(p_\tau), \tau\eta_\tau\rangle$ and $f(p_\tau)$ give

$$F(q_\tau, \tau\eta_\tau, \mathcal{Y}_\tau) \le \mathrm{tr}(A(p_\tau)\mathcal{X}_\tau - A(q_\tau)\mathcal{Y}_\tau) + \langle b(q_\tau) - b(p_\tau), \tau\eta_\tau\rangle + f(p_\tau) - f(q_\tau), \tag{3.10}$$

where we have used the fact that $(\tau\eta_\tau, \mathcal{X}_\tau) \in \overline{\mathcal{J}}^{2,+}(u, p_\tau)$. Our goal is to arrive to a contradiction to (3.9) by showing that the right-hand side in (3.10) converges to 0. Hence, we proceed to estimate the right-hand side of (3.10). Firstly, we have

$$\langle b(p_\tau) - b(q_\tau), \tau\eta_\tau\rangle \le \tau\|b(p_\tau) - b(q_\tau)\|\|\eta_\tau\| = \tau\|b(p_\tau) - b(q_\tau)\|\|\nabla^p_{\mathcal{H}}\phi(p_\tau \cdot q_\tau^{-1})\|.$$

Since Ω is bounded, we derive the bounds

$$\Big|\frac{\partial\phi}{\partial x_i}(p_\tau, q_\tau)\Big| \le C\phi(p_\tau, q_\tau)^{(m-1)/m}, \quad \text{for } i = 1, 2, 3.$$

Therefore using (H3) and recalling (3.7), we deduce

$$\langle b(p_\tau) - b(q_\tau), \tau\eta_\tau\rangle \le o(1).$$

We next estimate the second-order term. Observe that (2.6) means

$$\mathcal{X}_\tau - \mathcal{Y}_\tau \preceq \tau\|M_\tau\|^2 I.$$

Therefore

$$\mathrm{tr}(A(p_\tau)\mathcal{X}_\tau - A(q_\tau)\mathcal{Y}_\tau) = \mathrm{tr}(A_\tau\mathcal{Z}_\tau) \le C\tau\|M_\tau\|^2\|A_\tau\|,$$

where $A_\tau = \mathrm{diag}(A(p_\tau), A(q_\tau))$ and $\mathcal{Z}_\tau = \mathrm{diag}(\mathcal{X}_\tau, -\mathcal{Y}_\tau)$. To estimate the norm of M_τ, we appeal to the following second-derivative bounds

$$\Big|\frac{\partial^2\phi}{\partial x_i \partial x_j}(p_\tau, q_\tau)\Big| \le C\phi(p_\tau, q_\tau)^{(m-2)/m}, \quad \text{for } i, j = 1, 2, 3.$$

Therefore, using again (3.7), we obtain

$$\mathrm{tr}(A(p_\tau)\mathcal{X}_\tau - A(q_\tau)\mathcal{Y}_\tau) \le C\tau\phi(p_\tau, q_\tau)^{(2m-4)/m}$$

converges to 0 as $\tau \to \infty$ for $m \geq 4$. Letting $\tau \to \infty$ in (3.10) and recalling (3.9), we arrive at a contradiction. This ends the proof of the theorem. $\square$

Remark 3.2. Observe that the comparison principle implies the statement

$$M \geq \sup_{\partial\Omega} (u^* - v_*) \implies u - v \leq M \text{ in } \Omega,$$

for u and v bounded viscosity sub- and supersolutions to (1.1). This is a version of the so called maximum principle.

4. Proof of Theorem 1.1: Perron's method

The classical Perron's method adapted to viscosity solutions may be formulated as follows:

Theorem 4.1. *Suppose that there exist a viscosity subsolution $\underline{u} \in USC(\Omega)$ and a viscosity supersolution $\overline{u} \in LSC(\Omega)$ such that*

$$(\underline{u})_* = (\overline{u})^* = u_0 \tag{4.1}$$

on $\partial\Omega$. Then

$$u(p) := \sup\big\{v(p)\colon \underline{u} \leq v \leq \overline{u} \text{ in } \Omega \text{ and } v \text{ is a subsolution of (1.3)}\big\} \tag{4.2}$$

for $p \in \overline{\Omega}$, is a continuous viscosity solution of (1.3).

As it is well-known, the proof of the above theorem relies on two steps:

(I) Closedness under supremum: the function u^* is a viscosity subsolution in Ω of (1.1).

(II) Maximal solution: if u_* is not a supersolution at some point $p_0 \in \Omega$, then for all ϵ small, there exists a viscosity subsolution v_ϵ so that

$$v_\epsilon(x) \geq u(x), \quad \text{for all } x \in \Omega, \quad \sup_\Omega (v_\epsilon - u) > 0$$

and

$$v_\epsilon(x) = u(x) \text{ for all } x \in \Omega \setminus B_{\mathcal{H}}(p_0, \epsilon).$$

The proofs of these properties in the Euclidean case are standard (see [13, Chapter 2] or [9] for further details) and their adaptations to the Heisenberg group are straightforward. With the help of the comparison principle, the proof of Theorem 4.1 is simple. We give it here just for completeness.

Proof of Theorem 4.1. By **(I)**, the function u^* is a subsolution of equation (1.1) in Ω. By construction

$$(\underline{u})_* \leq u_* \leq u \leq u^* \leq (\overline{u})^*$$

in $\overline{\Omega}$. Hence $u^* = 0$ on $\partial\Omega$. By comparison principle, $u^* \leq \overline{u}$ in Ω which yields $u = u^*$ and then u is a subsolution of (1.3) (by definition, $u \leq 0$ on $\partial\Omega$). On the other hand, if u_* is not a supersolution at some $p_0 \in \Omega$, by **(II)** and for ϵ small enough, we can find a subsolution v_ϵ so that $v_\epsilon = 0$ on $\partial\Omega$ and $v_\epsilon(p) > u(p)$ for some p in Ω. Observe $v_\epsilon \geq u \geq \underline{u}$ in Ω. Moreover, by comparison again $v_\epsilon \leq \overline{u}$ in Ω. Therefore, v_ϵ belongs to the set defining u in (4.2), and we arrive at the contradiction $v_\epsilon \leq u$ in Ω. We conclude that u_* is a supersolution, and by comparison $u \leq u_*$ in $\overline{\Omega}$. This shows that u is a continuous viscosity solution of (1.3). $\square$

Perron's method ends the proof of Theorem 1.1. In the next section, we shall focus the question of how to find $\underline{u}$ and $\overline{u}$ satisfying the boundary values (4.1).

4.1. Construction of $\overline{u}$ and $\underline{u}$. The next proposition allows us to reduce the issue of constructing the appropriate sub- and supersolutions satisfying (4.1) for (1.3) to the framework of homogeneous boundary values. This is another application of the maximum principle.

Proposition 4.2. *The problem of solving* (1.3) *is equivalent to find the solutions of a sequence of appropriate homogeneous boundary value problems.*

Proof. Suppose first that $u_0 \in \mathcal{C}^2(\overline{\Omega})$. Observe that if u solves (1.3), then $v := u - u_0$ solves the homogeneous boundary value problem

$$\begin{cases} -\mathrm{tr}\big(A(p)\nabla^2_{\mathcal{H}_0} v(p)\big) - \langle b(p), \nabla_{\mathcal{H}} v(p)\rangle = \tilde{f}(p) & \text{if } p \in \Omega, \\ v(p) = 0 & \text{for } p \in \partial\Omega, \end{cases}$$

where

$$\tilde{f}(p) := f(p) + \mathrm{tr}\left(A(p)\nabla_{\mathcal{H}_0} u_0(p)\right) - \langle b(p); \nabla_{\mathcal{H}} u_0(p)\rangle ,$$

and conversely. The solution v, and consequently u, may be obtained by Perron's method with zero boundary values.

If u_0 is only continuous in $\overline{\Omega}$, then there exists a sequence $u_{0,\varepsilon} \in \mathcal{C}^2(\overline{\Omega})$ which converges, uniformly in $\overline{\Omega}$, to u_0. According to the above argument, there exists a unique solution u_ε to (1.1) with boundary values $u_{0,\varepsilon}$. In view of Remark 3.2, we have for all ϵ_1 and ϵ_2

$$\sup_{\overline{\Omega}} |u_{\epsilon_1} - u_{\epsilon_2}| \le \sup_{\partial\Omega} |u_{\epsilon_1} - u_{\epsilon_2}| \le \sup_{\partial\Omega} |u_{0,\epsilon_1} - u_{0,\epsilon_2}|.$$

Thus, u_ϵ converges uniformly to a function $u \in \mathcal{C}(\overline{\Omega})$. By standard stability results (see Proposition 2.1 in [15]), u is the solution to (1.3). $\square$

In what follows, we shall assume that $u_0 \equiv 0$ in $\overline{\Omega}$. In general terms, in order to find solutions satisfying

$$(\overline{u})_* = (\overline{u})^* = 0 \quad \text{on } \partial\Omega, \tag{4.3}$$

we shall start with a subsolution and a supersolution of (1.1) in Ω and then we shall modify them near the boundary $\partial\Omega$.

Throughout this section, we shall assume (H1) and (H2) from Section 1, and moreover

(D) There exist a function g and scalars $c \in \mathbf{R}$ and $\lambda_0 > 0$ so that $\partial\Omega = \{p \in \mathcal{H} \colon g(p) = c\}$, the set

$$\Omega_{\lambda_0} := \left\{ p \in \overline{\Omega} \colon c - g(p) < \frac{1}{\lambda_0} \right\}$$

is an open neighborhood of $\partial\Omega$ in $\overline{\Omega}$, and $g \in \mathcal{C}^2(\overline{\Omega}_{\lambda_0})$.

We also introduce the characteristic set

$$C_g := \{p \in \partial\Omega \colon \nabla_{\mathcal{H}_0} g(p) = 0\} .$$

In the sequel, we divide the exposition into two main cases according to the scenarios $C_g = \emptyset$ or $C_g \neq \emptyset$.

Case I. Suppose:

(DI) the characteristic set C_g is empty, that is, the surface $\partial\Omega$ is non-characteristic.

Then we have the following

Proposition 4.3. *Assume (D) and (DI). Then there exist a subsolution* $\underline{u} \in \mathcal{C}(\overline{\Omega})$ *and a supersolution* $\overline{u} \in \mathcal{C}(\overline{\Omega})$ *to* (1.1) *in* Ω *so that* (4.3) *holds.*

Proof. As Ω is a bounded domain, there exists $r > 0$ such that $\Omega \subset [-r, r]^3$. Furthermore, let $C > 0$ so that

$$\|b\|_{L^\infty(\overline{\Omega})}, \|f\|_{L^\infty(\overline{\Omega})} \leq C.$$

We start with the following function

$$u_1(p) = -\exp[\alpha(p)] + \exp(N),$$

where $p = (x, y, z) \in \overline{\Omega}$, $\alpha(p) := -N\left(\frac{x-r}{2r}\right)$ and $N > 0$ to be determined. We shall prove that u_1 is a supersolution of (1.1) in Ω. Firstly, notice that

$$\nabla_{\mathcal{H}} u_1(p) = \frac{N}{2r}\big(e^{\alpha(p)}, 0, 0\big), \quad \nabla^2_{\mathcal{H}_0} u_1(p) = -\frac{N^2}{4r^2}\begin{bmatrix} e^{\alpha(p)} & 0 \\ 0 & 0 \end{bmatrix}.$$

A straightforward calculation and the uniform ellipticity of A show that

$$F(p, \nabla_{\mathcal{H}} u_1(p), \nabla^2_{\mathcal{H}_0} u_1(p)) \geq \frac{N}{2r}\left(e^N N \frac{\gamma}{2r} - C\right) - C \geq 0$$

if N is sufficiently large. Hence, u_1 is a supersolution of $F = 0$ in Ω. In order to produce (4.3), we modify u_1 near the boundary. Choose $\lambda \geq \lambda_0$ so that $\nabla_{\mathcal{H}_0} g \neq 0$ in $\overline{\Omega}_\lambda$. Define

$$v_1(p) = M(1 - e^{-\lambda d(p)}), \quad p \in \overline{\Omega}_\lambda, \tag{4.4}$$

where $d(p) := c - g(p)$ and $M > 0$ is chosen so that

$$u_1(p) \leq M(1 - e^{-1/2}) \quad \text{for all } p \in \overline{\Omega}. \tag{4.5}$$

Observe that $v_1 = 0$ on $\partial\Omega$. Moreover, for any $p \in \Omega_\lambda$ we have

$$\begin{aligned} &F(p, \nabla_{\mathcal{H}} v_1(p), \nabla^2_{\mathcal{H}_0} v_1(p)) \\ &= \lambda M e^{-\lambda d(p)}\big[\lambda \langle A(p)\nabla_{\mathcal{H}_0} d(p), \nabla_{\mathcal{H}_0} d(p)\rangle - \operatorname{tr}\big(A(p)\nabla^2_{\mathcal{H}_0} d(p)\big) + \langle b(p), \nabla_{\mathcal{H}} d(p)\rangle\big] - f(p) \\ &= \lambda M e^{-\lambda d(p)}\big[\lambda \langle A(p)\nabla_{\mathcal{H}_0} g(p), \nabla_{\mathcal{H}_0} g(p)\rangle + \operatorname{tr}\big(A(p)\nabla^2_{\mathcal{H}_0} g(p)\big) - \langle b(p), \nabla_{\mathcal{H}} g(p)\rangle\big] - f(p). \end{aligned}$$

Since $\nabla_{\mathcal{H}_0} g \neq 0$ in $\overline{\Omega}_\lambda$, we may choose λ large so that v_1 is a supersolution. Define

$$\overline{u}(p) = \begin{cases} v_1(p) \wedge u_1(p) & \text{in } \overline{\Omega}_\lambda, \\ u_1(p) & \text{in } \Omega \setminus \overline{\Omega}_\lambda. \end{cases} \tag{4.6}$$

By (4.5), the localization property of supersolutions and the property of closedness under minimum of supersolutions, $\overline{u}$ is a viscosity supersolution of (1.1) in Ω. Finally, since $u_1 \geq 0$ in $\overline{\Omega}$, there holds $\overline{u} = v_1 = 0$ on $\partial\Omega$.

On the other hand, to construct the subsolution $\underline{u}$ satisfying (4.3), we start with

$$u_2(p) = \exp\left[N\left(\frac{x + r}{2r}\right)\right] - \exp(N),$$

and we show as before that u_2 is a subsolution of F. Pick $M > 0$ so that:

$$u_2(p) \geq -M(1 - e^{-1/2}) \quad \text{for all } p \in \overline{\Omega}. \tag{4.7}$$

Introducing the function

$$v_2(p) = -M(1 - e^{-\lambda d(p)}),$$

for $p \in \overline{\Omega}_\lambda$, $\lambda \geq \lambda_0$, we obtain

$$F(p, \nabla_{\mathcal{H}} v_2(p), \nabla^2_{\mathcal{H}_0} v_2(p)) = -\lambda M e^{-\lambda d(p)} \big[\lambda \langle A(p) \nabla_{\mathcal{H}_0} g(p), \nabla_{\mathcal{H}_0} g(p) \rangle + \operatorname{tr}(A(p) \nabla^2_{\mathcal{H}_0} g(p)) - \langle b(p), \nabla_{\mathcal{H}} g(p) \rangle \big] - f(p).$$

By proceeding as before, we deduce that v_2 is a subsolution in Ω_λ. Define:

$$\underline{u}(p) = \begin{cases} v_2(p) \vee u_2(p) & \text{in } \overline{\Omega}_\lambda, \\ u_2(p) & \text{in } \Omega \setminus \overline{\Omega}_\lambda. \end{cases}$$

Hence, $\underline{u}$ is a viscosity subsolution of (1.1) in Ω, and, since $u_2 \leq 0$ in $\overline{\Omega}$, we have $\underline{u} = v_2 = 0$ on $\partial\Omega$. □

Remark 4.4. For non-characteristic boundary surfaces, we may choose for $c - g$ the Heisenberg distance function $d_{\partial\Omega}$ to the boundary defined as

$$d_{\partial\Omega}(p) := \inf \{ d_{CC}(p, q) \colon q \in \partial\Omega \}, \quad p \in \overline{\Omega}.$$

Indeed, in view of [2, Theorem 1.1], for Euclidean $\mathcal{C}^2$ boundaries $\partial\Omega$, the horizontal gradient of the distance function and the distance function itself are of class $\mathcal{C}^1$ in a neighbourhood of $\partial\Omega$. Moreover, by the results obtained in [23], $d_{\partial\Omega}$ satisfies the Eikonal equation on $\partial\Omega$, hence we may perform the arguments in the proof of Proposition 4.3 to obtain $\underline{u}$ and $\overline{u}$.

Next, we shall give examples of domains whose boundaries have no characteristic points.

Torus. Let $\Omega := \{(x, y, z) \in \mathcal{H} \colon g(x, y, z) < r\}$, where $g(x, y, z) = z^2 + (\sqrt{x^2 + y^2} - R)^2$ and $R > r > 0$. Observe

$$\nabla_{\mathcal{H}_0} g(p) = \frac{2}{\sqrt{x^2 + y^2}} \begin{bmatrix} x(\sqrt{x^2 + y^2} - R) + 2yz\sqrt{x^2 + y^2} \\ y(\sqrt{x^2 + y^2} - R) - 2xz\sqrt{x^2 + y^2} \end{bmatrix}.$$

By straightforward calculations, $\nabla_{\mathcal{H}_0} g(p) \neq 0$ for all $p \in \partial\Omega$. Indeed, suppose that for some boundary point p, we have $\nabla_{\mathcal{H}_0} g(p) = 0$. Then we may distinguish the following scenarios:

(1) If $x = 0$, then

$$\begin{cases} 2yz\sqrt{y^2} = 0, \\ y(\sqrt{y^2} - R) = 0, \end{cases}$$

which imply $y = 0$ or $z = 0$, and $y = 0$ or $|y| = R$. Since $y \neq 0$ because $(0, 0, z) \notin \partial\Omega$, we derive that $|y| = R$ and $z = 0$ is the only plausible choice. However, $z = 0$ implies $|y| = R - r$ or $|y| = R + r$, which is a contradiction. Hence we conclude $x \neq 0$. In a similar way, $y \neq 0$.

(2) Suppose further that $x \neq 0$ and $y \neq 0$. Then

$$\begin{cases} x(\sqrt{x^2 + y^2} - R) + 2yz\sqrt{x^2 + y^2} = 0, \\ y(\sqrt{x^2 + y^2} - R) - 2xz\sqrt{x^2 + y^2} = 0 \end{cases}$$

$$\iff \begin{cases} xy(\sqrt{x^2 + y^2} - R) + 2y^2 z\sqrt{x^2 + y^2} = 0, \\ xy(\sqrt{x^2 + y^2} - R) - 2x^2 z\sqrt{x^2 + y^2} = 0 \end{cases}$$

$$\implies 2y^2 z\sqrt{x^2 + y^2} = -2x^2 z\sqrt{x^2 + y^2}$$

$$\implies y^2 z = -x^2 z \implies (x^2 + y^2) z = 0.$$

Hence $z = 0$, which in turn implies

$$\begin{cases} x(\sqrt{x^2+y^2} - R) = 0, \\ y(\sqrt{x^2+y^2} - R) = 0, \end{cases}$$

since $x \neq 0$ and $y \neq 0$, we obtain $\sqrt{x^2+y^2} = R$. Thus $|z| = r$ which contradicts that $z = 0$.

This shows that the torus is non-characteristic.

Solid of revolutions around the vertical axis. This is a generalization of the above example. Let $\Omega := \{(x,y,z) \in \mathcal{H} \colon g(x,y,z) < 1\}$ where $g(x,y,z) = h(\pm\sqrt{x^2+y^2}, z)$, h smooth so that $\gamma\colon h(x,z) = 1$ is a $\mathcal{C}^2$ closed curve satisfying that there is $\epsilon > 0$ such that

$$h(x,z) > 1 \quad \text{for all } (x,z) \in (-\varepsilon, \varepsilon) \times \mathbf{R}; \tag{4.8}$$

$$\frac{\partial h}{\partial z} \neq 0 \quad \text{and} \quad \frac{\partial h}{\partial x} \neq 0 \;\text{ in } \partial\Omega. \tag{4.9}$$

Observe that (4.8) says that the curve γ does not intersect the z axis in the xz-plane. Moreover,

$$\nabla_{\mathcal{H}_0} g(x,y,z) = \frac{1}{\sqrt{x^2+y^2}} \begin{bmatrix} \dfrac{\partial h}{\partial x} x + 2y\sqrt{x^2+y^2}\dfrac{\partial h}{\partial z} \\[2ex] \dfrac{\partial h}{\partial x} y - 2x\sqrt{x^2+y^2}\dfrac{\partial h}{\partial z} \end{bmatrix},$$

where $\frac{\partial h}{\partial x}$ and $\frac{\partial h}{\partial z}$ are evaluated at $(\sqrt{x^2+y^2}, z)$. Note that the horizontal gradient is null on $\partial\Omega$ if and only if

$$\begin{cases} \dfrac{\partial g}{\partial u} x + 2y\sqrt{x^2+y^2}\dfrac{\partial g}{\partial z} = 0, \\ \dfrac{\partial g}{\partial u} y - 2x\sqrt{x^2+y^2}\dfrac{\partial g}{\partial z} = 0. \end{cases}$$

Reasoning as in the previous example we arrive at a contradiction with $x^2 + y^2 \neq 0$ on $\partial\Omega$ in view of (4.8) and (4.9).

Case II. In this framework, we shall make use of the following set of assumptions:

(DII) The characteristic set C_g is not empty, that is, the surface $\partial\Omega$ has characteristic points.

$(\mathrm{DII})_A$ $\operatorname{tr}\big(A(p)\nabla^2_{\mathcal{H}_0} g(p)\big) > 0$ in C_g, and either $b_3 = 0$ or $b_3 X_3 g < 0$ in the closure of a neighbourhood of C_g.

$(\mathrm{DII})_B$ $\nabla^2_{\mathcal{H}_0} g(p) = 0$ in C_g, and $b_3 X_3 g < 0$ in the closure of a neighbourhood of C_g.

Firstly, we have the following

Proposition 4.5. *Under hypothesis (D), (DII) and $(\mathrm{DII})_A$, there exist a subsolution $\underline{u} \in \mathcal{C}(\overline{\Omega})$ and a supersolution $\overline{u} \in \mathcal{C}(\overline{\Omega})$ to (1.1) in Ω so that (4.3) holds.*

Proof. As in the proof of Proposition 4.5, take the functions u_1 and u_2, with the same choices for M. We derive first preliminaries assertions, which are consequences of $(\mathrm{DII})_A$. Since the set C_g is compact, for the function

$$p \to \operatorname{tr}(A(p)\nabla^2_{\mathcal{H}_0} g(p)),$$

which is continuous and positive in C_g, there is $c > 0$ so that

$$\operatorname{tr}\big(A(p)\nabla^2_{\mathcal{H}_0} g(p)\big) > c,$$

for all $p \in C_g$. Hence, there is an open neighbourhood W of C_g such that

$$\mathrm{tr}\left(A(p)\nabla^2_{\mathcal{H}_0} g(p)\right) \geq c, \tag{4.10}$$

for all $p \in W$. Moreover, there exists a smaller open neighbourhood $U \subset W$ of C_g so that

$$\|b\|_{L^\infty(\overline{\Omega})}\|\nabla_{\mathcal{H}_0} g\|_{L^\infty(\overline{U})} < c/2. \tag{4.11}$$

On the other hand, if $p \in \overline{\Omega}_{\lambda_0} \setminus U$, then:

$$\langle A(p)\nabla_{\mathcal{H}_0} g(p), \nabla_{\mathcal{H}_0} g(p)\rangle > 0.$$

Therefore, by the compactness of $\overline{\Omega}_{\lambda_0} \setminus U$ and by continuity, there is a constant $c^* > 0$ so that

$$\langle A(p)\nabla_{\mathcal{H}_0} g(p), \nabla_{\mathcal{H}_0} g(p)\rangle > c^*, \tag{4.12}$$

for all $p \in \overline{\Omega}_{\lambda_0} \setminus U$. As in the previous case, introduce

$$v_1(p) = M(1 - e^{-\lambda d(p)}), \quad p \in \overline{\Omega}_\lambda, \tag{4.13}$$

with $d(p) := c - g(p)$, M as in (4.5) and $\lambda \geq \lambda_0$, to be specified below. Also

$$\Omega_\lambda \subset \left(\overline{\Omega}_{\lambda_0} \setminus U\right) \cup U.$$

Firstly, if $p \in U$, then by $(\mathrm{DII})_A$, (4.10) and (4.11), we derive

$$F(p, \nabla_{\mathcal{H}} v_1(p), \nabla^2_{\mathcal{H}_0} v_1(p)) \geq \lambda M e^{-\lambda d(p)}\left(c - \langle b(p), \nabla_{\mathcal{H}} g(p)\rangle\right) - f(p) \geq \frac{c\lambda M e^{-1}}{2} - f(p).$$

Making λ large so that

$$\frac{c\lambda M e^{-1}}{2} - \|f\|_{L^\infty(\overline{\Omega})} > 0,$$

we have

$$F(p, \nabla_{\mathcal{H}} v_1(p), \nabla^2_{\mathcal{H}_0} v_1(p)) \geq 0.$$

Finally, if p belongs to the compact set $\overline{\Omega}_{\lambda_0} \setminus U$, then we have to choose λ large enough so that

$$\lambda M e^{-1}\left(\lambda c^* + \mathrm{tr}\left(A(p)\nabla^2_{\mathcal{H}_0} g(p)\right) - \langle b(p), \nabla_{\mathcal{H}} g(p)\rangle\right) - \|f\|_{L^\infty(\overline{\Omega})} > 0. \tag{4.14}$$

In both cases, λ may be chosen independent of p, and we take the larger one to define v_1. In that way, we conclude that v_1 is a viscosity supersolution in Ω_λ. We conclude by defining $\overline{u}$ as in (4.6). Reasoning as before, it is also possible to get a subsolution $\underline{u}$ satisfying (4.1). □

Next, we provide examples of domains where the hypothesis in Proposition 4.5 are satisfied.

Ellipsoids. Let $p_0 = (x_0, y_0, z_0)$ and consider a, b and c as positive constants. Define:

$$\Omega := \left\{(x, y, z) \colon \frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} + \frac{(z - z_0)^2}{c^2} = 1\right\}.$$

Letting:

$$g(x, y, z) := \frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} + \frac{(z - z_0)^2}{c^2},$$

we have

$$\nabla^2_{\mathcal{H}_0} g(x, y, z) = 2\begin{bmatrix} \frac{1}{a^2} + \frac{4}{c^2}y^2 & -\frac{4}{c^2}xy \\ -\frac{4}{c^2}xy & \frac{1}{b^2} + \frac{4}{c^2}x^2 \end{bmatrix},$$

which is a positive definite matrix. As A is uniformly elliptic, assumption $(\mathrm{DII})_A$ is satisfied for appropriate b_3. In particular, $(\mathrm{DII})_A$ holds true in the Euclidean ball.

On the other hand, calling now hypothesis $(\mathrm{DII})_B$, we derive the following result.

Proposition 4.6. *Under the assumptions (D), (DII) and $(DII)_B$, there exist a subsolution $\underline{u} \in \mathcal{C}(\overline{\Omega})$ and a supersolution $\overline{u} \in \mathcal{C}(\overline{\Omega})$ to* (1.1) *in Ω so that* (4.3) *holds.*

Proof. As in the proof of Proposition 4.5, take the functions u_1 and u_2. We now modify them near $\partial\Omega$. By assumption $(\mathrm{DII})_B$, there exists an open neighbourhood V of C_g so that

$$\|b\|_{L^\infty(\overline{\Omega})}\|\nabla_{\mathcal{H}_0} g\|_{L^\infty(\overline{V})} + \|A\|_{L^\infty(\overline{\Omega})}\|\nabla^2_{\mathcal{H}_0} g\|_{L^\infty(\overline{V})} < \frac{c}{2}, \tag{4.15}$$

where $c > 0$ satisfies

$$-b_3(p)X_3(p) \geq c, \tag{4.16}$$

for all p in $\overline{\Omega}_{\lambda_0}$. Next, define v_1 as in (4.13), for $\lambda \geq \lambda_0$ to be determined a posteriori. Then, if $p \in V$, we derive from (4.15) and (4.16), the following

$$\begin{aligned} &F(p, \nabla_{\mathcal{H}} v_1(p), \nabla^2_{\mathcal{H}_0} v_1(p)) \\ &= \lambda M e^{-\lambda d(p)}\left[\lambda \langle A(p)\nabla_{\mathcal{H}_0} g(p), \nabla_{\mathcal{H}_0} g(p)\rangle + \operatorname{tr}\left(A(p)\nabla^2_{\mathcal{H}_0} g(p)\right) - \langle b(p), \nabla_{\mathcal{H}} g(p)\rangle\right] - f(p) \\ &\geq \frac{c\lambda M e^{-1}}{2} - \|f\|_{L^\infty(\overline{\Omega})}. \end{aligned}$$

Choosing λ large enough, and independent of $p \in V$, we get

$$F(p, \nabla_{\mathcal{H}} v_1(p), \nabla^2_{\mathcal{H}_0} v_1(p)) \geq 0.$$

Moreover, when $p \in \overline{\Omega}_{\lambda_0} \setminus U$, we choose λ so that (4.14) holds, where $c^* > 0$ is chosen now so that (4.12) is valid in $\overline{\Omega}_{\lambda_0} \setminus V$. We conclude as in the previous cases. □

As an example we quote the following.

Heisenberg ball. Let:

$$\Omega := B_{\mathcal{H}}(0,1) = \left\{(x,y,z) \colon (x^2+y^2)^2 + z^2 < 1\right\}.$$

Taking $g(x,y,z) := (x^2+y^2)^2 + z^2$, we obtain

$$\begin{aligned} \nabla_{\mathcal{H}_0} g(x,y,z) &= 4\begin{bmatrix} x(x^2+y^2) + yz \\ y(x^2+y^2) - xz \end{bmatrix}, \\ \nabla^2_{\mathcal{H}_0} g(x,y,z) &= 12\begin{bmatrix} x^2+y^2 & 0 \\ 0 & x^2+y^2 \end{bmatrix}. \end{aligned}$$

Observe that $\nabla_{\mathcal{H}_0} g(x,y,z) = 0$ implies $x = 0$ and $y = 0$. Hence $\nabla^2_{\mathcal{H}_0} g(x,y,z) = 0$. Moreover, $X_3 g(x,y,z) = 2z \neq 0$ in a neighbourhood of C_g. Hence, for appropriate b, assumption $(\mathrm{DII})_B$ holds true.

Remark 4.7. In the case where there is no linear term in (1.1), that is $b \equiv 0$ in $\overline{\Omega}$, we may apply the above reasoning assuming that, in a neighbourhood of C_g, we have $f \equiv 0$ and $\nabla^2_{\mathcal{H}_0} g \succeq 0$. For instance, this is the case of the subelliptic Laplace equation in $B_{\mathcal{H}}$ with arbitrary continuous boundary values and with a source term f compactly supported in $B_{\mathcal{H}}$.

Remark 4.8. Observe that in view of the left invariant property of the vector fields X_1, X_2 and X_3, to solve the problem (1.3) in a domain Ω it is equivalent to solve

$$\begin{cases} -\mathrm{tr}\big(A_0(p)\nabla^2_{\mathcal{H}_0} v(p)\big) - \langle b_0(p), \nabla_{\mathcal{H}} v(p)\rangle = f_0(p) & \text{if } p \in \Omega_0, \\ v(p) = v_0(p) & \text{for } p \in \partial\Omega_0, \end{cases}$$

where

$$\Omega_0 := \{p_0 \cdot p \colon p \in \Omega\},$$

and A_0, b_0, f_0 and v_0 are the corresponding left translation of A, b, f and u_0, respectively. As a corollary, we derive that if Proposition 4.3, Proposition 4.5 or Proposition 4.6 hold in a domain Ω, then the same conclusion is valid in the translated set Ω_0.

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References

[1] Achdou, Y., and N. Tchou: A finite difference scheme on a non commutative group. - Numer. Math. 89, 2001, 401–424.

[2] Arcozzi, N., and F. Ferrari: Metric normal and distance function in the Heisenberg group. - Math. Z. 256:3, 2007, 661–684.

[3] Bieske, T.: On ∞-harmonic functions on the Heisenberg group. - Comm. Partial Differential Equations 27:3, 2002, 727–761.

[4] Bieske, T.: Equivalence of weak and viscosity for the p-Laplace equation in the Heisenberg group. - Ann. Acad. Sci. Fenn. Math. 31, 2006, 363–379.

[5] Bieske, T.: A sub-Riemannian maximum principle and its application to the p-Laplacian in Carnot groups. - Ann. Acad. Sci. Fenn. Math. 37, 2012, 119–134.

[6] Calin, O., D. Chang, and P. Greiner: Geometric analysis on the Heisenberg group and its generalizations. - Stud. Adv. Math. 40, Amer. Math. Soc., 2007.

[7] Capogna, L., D. Danielli, S. Pauls, and J. Tyson: An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem. - Progr. Math. 259, Birkhauser-Verlag, Basel, 2007.

[8] Crandall, I., and H. Ishii: The maximum principle for semicontinuous functions. - Differential Integral Equations 3, 1001–1014.

[9] Crandall, M., H. Ishii, and P. L. Lions: User's guide to viscosity solutions of second order partial differential equations. - Bull. Amer. Math. Soc. 27:1, 1992, 1–67.

[10] Ferrari, F., Q. Liu, and J. Manfredi: On the horizontal mean curvature flow for axisymmetric surfaces in the Heisenberg group. - Commun. Contemp. Math. 16:3, 1350027, 2014, 1–41.

[11] Folland, G., and E. Stein: Hardy-spaces on homogeneous groups. - Princeton Univ. Press, 1982.

[12] Franchi, B., R. Serapioni, and F. Serra-Cassano: Rectifiability and perimeter in the Heisenberg group. - Math. Ann. 321:3, 2001, 479–531.

[13] Giga, Y.: Surface evolution equations: a level set method. - Monographs in Mathematics 99, Birkhauser Verlag, Basel, 2006.

[14] Ishii, H.: Perron's method for Hamilton–Jacobi equations. - Duke Math. J. 55:2, 1987, 369–384.

[15] Ishii, H.: On uniqueness and existence of viscosity solutions of fully non-linear second-order elliptic PDEs. - Comm. Pure Appl. Math. 42, 1989, 15–45.

[16] Ishii, H., and P. L. Lions: Viscosity solutions of fully nonlinear second-order elliptic partial equations. - J. Differential Equations 83, 1990, 26–78.

[17] Julin, V., and P. Juutinen: A new proof for the equivalence of weak and viscosity solutions for the p-Laplace equation. - Comm. Partial Differential Equations 37:5, 2012, 934–946.

[18] Juutinen, P., and P. Lindqvist: A theorem of Radó's type for the solutions of a quasi-linear equation. - Math. Res. Lett. 11, 2004, 31–34.

[19] Juutinen, P., P. Lindqvist, and J. Manfredi: On the equivalence of viscosity solutions and weak solutions for a quasilinear equation. - SIAM J. Math. Anal. 33:3, 2001, 699–717.

[20] Manfredi, J.: Non-linear subelliptic equations on Carnot groups: Analysis and geometry in metric spaces. - Notes of a course given at the Third School on Analysis and Geometry in Metric Spaces, Trento, 2003.

[21] Martino, V., and A. Montanari: Nonsmooth solutions for a class of fully nonlinear PDE's on Lie groups. - Nonlinear Anal. 126, 2015, 115–130.

[22] Martino, V. and A. Montanari: Lipschitz continuous viscosity solutions for a class of fully nonlinear equations on Lie groups. - J. Geom. Anal. 24:1, 2014, 169–189.

[23] Monti, R., and F. Serra Cassano: Surface measures in Carnot-Carathéodory spaces. - Calc. Var. Partial Differential Equations, 13:3, 2001, 339–376.

[24] Ochoa, P.: Approximation schemes for fully second-order equations in the Heisenberg group. - Comm. Pure Appl. Anal. 14:5, 2015, 1841–1863.

[25] Stein, E.: Harmonic analysis: real variable methods, orthogonality and oscillatory integrals. - Princeton Univ. Press, 1993.

[26] Stein, E., and J. Stein: Stock price distributions with stochastic volatility: an analytic approach. - Review Financial Study 4:4, 1991, 727–752.

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