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GROWTH PROPERTIES OF POTENTIALS IN CENTRAL MORREY–ORLICZ SPACES ON THE UNIT BALL

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Abstract. We introduce central Morrey–Orlicz spaces $M^{\Phi,\omega}(\mathbf{B})$ on the unit ball and study the existence of weighted spherical limits:

$$\liminf_{r\to 1-} (1-r)^{d_1}\omega(1-r)^{d_2}\left(\int_{S(0,r)} \Phi((1-r)^{d_3}|I_\alpha f(x)|)^q\, dS(x)\right)^{1/q}$$

for some $d_1, d_2, d_3 \in \mathbf{R}$, $1 \le q < \infty$, and all Riesz potentials $I_\alpha f$ with $f \in M^{\Phi,\omega}(\mathbf{B})$. We also deal with the existence of weighted spherical limits for Green potentials and monotone Sobolev functions.

1. Introduction

Let $\mathbf{R}^N$, $N \ge 2$, denote the N-dimensional Euclidean space. We use the notation $B(x,r)$ to denote the open ball centered at x with radius $r > 0$, whose boundary is denoted by $S(x,r)$. The L^q means over the spherical surface $S(0,r)$ for u is defined by

$$S_q(u,r) = \left(\frac{1}{|S(0,r)|}\int_{S(0,r)} |u(x)|^q\, dS(x)\right)^{1/q} = \left(\frac{1}{\omega_{N-1}}\int_{S(0,1)} |u(r\sigma)|^q\, dS(\sigma)\right)^{1/q}$$

when $1 \le q < \infty$, where $|S(0,r)| = \omega_{N-1} r^{N-1}$ with ω_{N-1} being the area of the unit sphere. Gardiner [4, Theorem 2] showed that

$$\liminf_{r\to 1-} (1-r)^{(N-1)(1-1/q)} S_q(u,r) = 0$$

when u is a Green potential in the unit ball $\mathbf{B} = B(0,1)$, $(N-3)/(N-1) < 1/q \le (N-2)/(N-1)$ and $q > 0$, as an extension of the result by Stoll [21] in the plane case. In [12], The first author gave versions of Gardiner's result in [4] to the half space. The first and third authors [17] studied the existence of boundary limits for BLD (Beppo Levi and Deny) functions u on the unit ball $\mathbf{B}$ of $\mathbf{R}^N$ satisfying

$$\int_{\mathbf{B}} |\nabla u(x)|^p (1-|x|)^\gamma\, dx < \infty, \tag{1.1}$$

where ∇ denotes the gradient, $1 < p < \infty$ and $-1 < \gamma < p-1$. In fact, we showed that

$$\liminf_{r\to 1-} (1-r)^{(N-p+\gamma)/p-(N-1)/q} S_q(u,r) = 0$$

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when $q > 0$ and $(N-p-1)/(p(N-1)) < 1/q < (N-p+\gamma)/(p(N-1))$, as a result corresponding to [16, Theorem 2.1] given in half spaces. In [17], we also studied the existence of boundary limits for monotone BLD functions u on the unit ball $\mathbf{B}$ of $\mathbf{R}^N$ satisfying (1.1).

We denote by $M^{\Phi,\omega}(\mathbf{B})$ the class of measurable functions f on the unit ball $\mathbf{B}$ satisfying

$$\|f\|_{M^{\Phi,\omega}(\mathbf{B})} = \sup_{0<r<1} \omega(1-r)\|f\|_{L^{\Phi}(\mathbf{B}\setminus B(0,r))} < \infty$$

with a convex function Φ and a doubling weight ω; the space $M^{\Phi,\omega}(\mathbf{B})$ is referred to as a central Morrey–Orlicz space (see Section 2 for the definitions of Φ and ω). For these spaces, see e.g. [1, 2, 3, 18]. When $\Phi(r) = r^p$ and $\gamma < 0$, one can find u such that $|\nabla u| \in M^{\Phi,\omega}(\mathbf{B})$ but u does not satisfy (1.1); see also Remark 3.5.

For $0 < \alpha < N$, we define the Riesz potential of order α for locally integrable function f on $\mathbf{B}$ by

$$I_\alpha f(x) = \int_{\mathbf{B}} |x-y|^{\alpha-N} f(y)\, dy.$$

Our main aim in this paper is to discuss the weighted limit

$$(1-r)^{d_1}\omega(1-r)^{d_2} S_q(\Phi((1-r)^{d_3} I_\alpha f), r)$$

as $r \to 1-0$ for $I_\alpha f$ with $f \in M^{\Phi,\omega}(\mathbf{B})$; d_1, d_2 and d_3 will be given later (see Theorem 3.1 below). The result is new even for $M^{p,\nu}(\mathbf{B})$, that is, for the case $\Phi(r) = r^p$ and $\omega(r) = r^{-\nu}$. The sharpness of the exponent of $1-r$ will be discussed later.

In Section 4, as an application of Theorem 3.1, we treat functions f on $\mathbf{B}$ satisfying the weighted condition $(1-|y|)^{\beta p_1} f(y)^{p_1} \in M^{\Phi,\omega}(\mathbf{B})$ for $1 < p_1 < \infty$ and $\beta > 0$ (see Theorem 4.2 below).

Let $G(x,y)$ be a Green kernel on $\mathbf{B}$. We define the Green potential for locally integrable function f on $\mathbf{B}$ by

$$Gf(x) = \int_{\mathbf{B}} G(x,y) f(y)\, dy.$$

In Section 5, we study the existence of weighted spherical limits for Green potentials Gf with $(1-|y|)f(y) \in M^{\Phi,\omega}(\mathbf{B})$ in our settings (see Theorem 5.3 below).

A continuous function u on an open set Ω is called monotone in the sense of Lebesgue [7] if for every relatively compact open set $G \subset \Omega$,

$$\max_{\overline{G}} u = \max_{\partial G} u \quad \text{and} \quad \min_{\overline{G}} u = \min_{\partial G} u.$$

Harmonic functions on Ω are monotone in Ω. More generally, solutions of elliptic partial differential equations of second order and weak solutions for variational problems may be monotone (see [5]). See also [6], [9], [10], [14], [15], [24], [25] and [26].

In the last section, we study the existence of weighted spherical limits for monotone functions u with $|\nabla u(y)|^{p_1} \in M^{\Phi,\omega}(\mathbf{B})$ with $p_1 > N-1$ in our settings (see Theorem 6.1 below). Essential tool in treating monotone functions is Lemma 6.2 below.

For related results on spherical means, see [11], [13], [15], [20], [22] and [23]. We also refer the reader to the papers [8] and [19] for weighted integral means over balls.

2. Preliminaries and lemmas

Throughout this paper, let C denote various positive constants independent of the variables in question. The symbol $g \sim h$ means that $C^{-1}h \le g \le Ch$ for some constant $C > 0$.

Let Φ be a convex function on $[0, \infty)$ such that

(Φ1) $\Phi(0) = 0$ and $\Phi(r) > 0$ for $r > 0$;

(Φ2) Φ is doubling, that is, there exists a constant $A_1 > 0$ such that

$$\Phi(2r) \le A_1 \Phi(r) \quad \text{for } r > 0;$$

(Φ3) for some $p \ge 1$, $r^{-p}\Phi(r)$ is almost increasing, that is, there exists a constant $A_2 > 0$ such that

$$\Phi(rt) \le A_2 r^p \Phi(t) \quad \text{when } 0 < r < 1 \text{ and } t > 0.$$

Further consider a weight ω such that

(ω1) $\omega(r) > 0$ for $r > 0$;

(ω2) ω is almost decreasing in $(0, \infty)$, that is, there is a constant $C > 0$ such that

$$\omega(t) \le C\omega(s) \quad \text{when } 0 < s < t < \infty;$$

(ω3) ω is doubling.

We see that $\omega(r) = r^{-\nu}(\log(e + r^{-1}))^{\tau}$ is almost decreasing when $\nu > 0$ and $\tau \in \mathbf{R}$.

Note here that (Φ3) holds if and only if

(Φ4) $\Phi(rt) \ge {A_2}^{-1} r^p \Phi(t) \quad$ when $r \ge 1$ and $t > 0$.

Moreover, if Φ is of the form $r^{p_1}(\log(e + r))^{\theta}$, then ($\Phi$3) holds when $p_1 > p$ or when $p_1 = p$ and $\theta \ge 0$.

For an open set G in $\mathbf{R}^N$, we define the Luxemburg–Nakano–Orlicz norm for $f \in L^1_{\mathrm{loc}}(G)$ by

$$\|f\|_{L^\Phi(G)} = \inf \left\{ \lambda > 0 \colon \int_G \Phi(|f(y)|/\lambda)\, dy \le 1 \right\};$$

we set $f = 0$ outside G for the sake of convenience.

We consider the family $M^{\Phi,\omega}(\mathbf{B})$ of all measurable functions f on $\mathbf{B}$ satisfying

$$\|f\|_{M^{\Phi,\omega}(\mathbf{B})} = \sup_{0<r<1} \omega(1-r)\|f\|_{L^\Phi(\mathbf{B}\setminus B(0,r))} < \infty.$$

When $\Phi(r) = r^p$ and $\omega(r) = r^{-\nu}$, $M^{\Phi,\omega}(\mathbf{B})$ will be written as $M^{p,\nu}(\mathbf{B})$. It is easy to see that

$$\sup_{0<r<1} \omega(1-r)\|f\|_{L^\Phi(\mathbf{B}\setminus B(0,r))} < \infty \tag{2.1}$$

if and only if

$$\sup_{0<r<1} \int_{\mathbf{B}\setminus B(0,r)} \Phi(\omega(1-r)|f(y)|)\, dy < \infty. \tag{2.2}$$

Moreover it is useful to note the following result.

Lemma 2.1. *Let $\Phi(r) = r^p(\log(c+r))^{\theta}$ and $\omega(r) = r^{-\nu}(\log(c+r))^{\tau}$ for $p > 1$ and real numbers θ, ν, τ, where $c > 1$ is chosen so large that Φ is convex. If $0 \le \nu < 1/p$, then the following are equivalent:*

(1) *there exists a constant $C_1 > 0$ such that*

$$\sup_{0<r<1} \omega(1-r)\|f\|_{L^\Phi(\mathbf{B}\setminus B(0,r))} \le C_1;$$

(2) *there exists a constant $C_2 > 0$ such that*

$$\sup_{0<r<1} \omega(1-r)^p \int_{\mathbf{B}\setminus B(0,r)} \Phi(|f(y)|)\, dy \le C_2;$$

here $C_1 \sim C_2$.

Proof. We treat only the case when $\theta \geq 0$, since the case $\theta < 0$ is similarly treated. Let $0 < r < 1$ and $t > 0$. By (Φ4) and (ω2), we have

$$\omega(1-r)^p\Phi(t) \leq C\Phi\left(\omega(1-r)t\right), \tag{2.3}$$

so that (1) implies (2). Next, if $\omega(1-r)t \leq t^{1+A}$ for $A > 0$, then

$$\begin{aligned}\Phi(\omega(1-r)t) &= (\omega(1-r)t)^p\left(\log(c+\omega(1-r)t)\right)^\theta \\ &\leq (\omega(1-r)t)^p\left(\log(c+t^{1+A})\right)^\theta \\ &\leq C\omega(1-r)^p t^p(\log(c+t))^\theta = C\omega(1-r)^p\Phi\left(t\right)\end{aligned}$$

and if $\omega(1-r)t > t^{1+A}$, then $t \leq \omega(1-r)^{1/A}$, so that

$$\Phi(\omega(1-r)t) \leq \Phi(\omega(1-r)^{1+1/A}).$$

Hence

$$\Phi(\omega(1-r)t) \leq C\left\{\omega(1-r)^p\Phi\left(t\right) + \Phi(\omega(1-r)^{1+1/A})\right\}. \tag{2.4}$$

If $\nu p < 1$, then we note that

$$\int_{\mathbf{B}\setminus B(0,r)} \Phi(\omega(1-r)^{1+1/A})\,dy = C(1-r)\Phi(\omega(1-r)^{1+1/A}) < \infty,$$

when A is so large that $\nu p(1+1/A) < 1$. Now the equivalence of assertions (1) and (2) is obtained. □

Here we give an estimate for spherical means for Riesz kernels.

Lemma 2.2. *Let $0 < a < N$ and c_1, c_2 be positive constants. If $c_1|y| < t < c_2|y|$ and $1/2 < |y| < 1$, then there exists a constant $C > 0$ such that*

$$\int_{S(0,1)\cap B(y,1-t)} |t\sigma - y|^{a-N}\,dS(\sigma) \leq C\begin{cases}|t-|y||^{a-1} & \text{when } a < 1;\\ (1-t)^{a-1} & \text{when } a > 1.\end{cases}$$

Proof. By an application of polar coordinates, we note that

$$\begin{aligned}&\int_{S(0,1)\cap B(y,1-t)} |t\sigma - y|^{a-N}\,dS(\sigma) \\ &\leq C\int_0^{\sin^{-1}2(1-t)} ||y| + t^2 - 2|y|t\cos\theta|^{(a-N)/2}\sin^{N-2}\theta\,d\theta \\ &\leq C\int_0^{c(1-t)} (||y|-t|^2 + t^2\theta^2)^{(a-N)/2}\theta^{N-2}\,d\theta \\ &= Ct^{1-N}||y|-t|^{a-1}\int_0^{c(1-t)t/||y|-t|} (1+s^2)^{(a-N)/2}s^{N-2}\,ds \\ &\leq Ct^{1-N}||y|-t|^{a-1}\times\begin{cases}1 & \text{when } a < 1,\\ ((1-t)t/||y|-t|)^{a-1} & \text{when } a > 1.\end{cases}\end{aligned}$$

Thus the present lemma is obtained. □

Lemma 2.3. *Let $0 < a < N$ and c_1, c_2 be positive constants. If $c_1|y| < t < c_2|y|$ and $1/2 < |y| < 1$, then there exists a constant $C > 0$ such that*

$$\int_{S(0,1)} |t\sigma - y|^{a-N}\,dS(\sigma) \geq C\begin{cases}|t-|y||^{a-1} & \text{when } a < 1;\\ 1 & \text{when } a > 1.\end{cases}$$

Proof. By an application of polar coordinates, we have

$$\begin{aligned}\int_{S(0,1)} |t\sigma - y|^{a-N}\, dS(\sigma) &= C\int_0^{\pi} ||y| + t^2 - 2|y|t\cos\theta|^{(a-N)/2}\sin^{N-2}\theta\, d\theta \\ &\geq C\int_0^{\pi/2} (||y|-t|^2 + t^2\theta^2)^{(a-N)/2}\theta^{N-2}\, d\theta \\ &= Ct^{1-N}||y|-t|^{a-1}\int_0^{\pi t/(2||y|-t|)} (1+s^2)^{(a-N)/2}s^{N-2}\, ds \\ &\geq Ct^{1-N}||y|-t|^{a-1} \times \begin{cases} 1 & \text{when } a<1, \\ (\pi t/||y|-t|)^{a-1} & \text{when } a>1 \end{cases}\end{aligned}$$

since

$$\pi t/(2||y|-t|) \geq \pi t/(2(|y|+t)) \geq \pi c_1/(2(1+c_1)) > 0$$

when $c_1|y| < t < c_2|y|$. Thus the present lemma is obtained. □

For a nonnegative function $f \in L^1_{\mathrm{loc}}(\mathbf{B})$ and $x \in \mathbf{B}$, write

$$\begin{aligned} I_\alpha f(x) &= \int_{B(x,(1-|x|)/2)} |x-y|^{\alpha-N} f(y)\, dy \\ &\quad + \int_{\{y\in\mathbf{B}\setminus B(x,(1-|x|)/2):1-|y|\leq 1-|x|\}} |x-y|^{\alpha-N} f(y)\, dy \\ &\quad + \int_{\{y\in\mathbf{B}\setminus B(x,(1-|x|)/2):1-|y|> 1-|x|\}} |x-y|^{\alpha-N} f(y)\, dy \\ &= I_1(x) + I_2(x) + I_3(x). \end{aligned}$$

Set

$$A(0,r) = B(0, r+(1-r)/2) \setminus B(0, r-(1-r)/2).$$

Lemma 2.4. *Let $1 \leq q < \infty$.*

(1) *Suppose $0 < \varepsilon < \alpha$ and*

$$(N-1)/q < N - \alpha p + \varepsilon(p-1).$$

Then there exists a constant $C > 0$ such that

$$\begin{aligned} &S_q(\Phi((1-r)^{-\varepsilon}\omega(1-r)I_1), r) \\ &\leq C(1-r)^{-\varepsilon}\int_{A(0,r)} |r-|y||^{(\alpha-\varepsilon)p+\varepsilon-N+(N-1)/q}\Phi(\omega(1-r)f(y))\, dy \end{aligned}$$

for all $1/2 < r < 1$ and nonnegative measurable functions $f \in L^1_{\mathrm{loc}}(\mathbf{B})$.

(2) *Suppose $0 < \varepsilon < \alpha$ and*

$$(N-1)/q > N - \alpha p + \varepsilon(p-1) > 0.$$

Then there exists a constant $C > 0$ such that

$$S_q(\Phi((1-r)^{-\varepsilon}\omega(1-r)I_1), r) \leq C(1-r)^{(\alpha-\varepsilon)p-N+(N-1)/q}$$

for all $1/2 < r < 1$ and nonnegative measurable functions f on $\mathbf{B}$ with $\|f\|_{M^{\Phi,\omega}(\mathbf{B})} \leq 1$.

(3) *Suppose $0 < \varepsilon < \alpha$ and $(\alpha-\varepsilon)p + \varepsilon - N > 0$. Then there exists a constant $C > 0$ such that*

$$\Phi((1-r)^{-\varepsilon}\omega(1-r)I_1) \leq C(1-r)^{(\alpha-\varepsilon)p-N}$$

for all $1/2 < r < 1$ and nonnegative measurable functions f on $\mathbf{B}$ with $\|f\|_{M^{\Phi,\omega}(\mathbf{B})} \le 1$.

Proof. Let $0 < \varepsilon < \alpha$ and

$$(\alpha - \varepsilon)p + \varepsilon - N + (N-1)/q < 0.$$

For $1/2 < r = |x| < 1$, we have

$$\begin{aligned} I_1(x) &= \int_{B(x,(1-|x|)/2)} |x-y|^{\alpha-N} f(y)\,dy \\ &\le C \int_0^{1-r} \left(\frac{1}{|B(x,t)|} \int_{B(x,t)\cap A(0,r)} f(y)\,dy \right) t^{\alpha-1}\,dt \\ &\le C \int_0^{1-r} \left(\frac{1}{|B(x,t)|} \int_{B(x,t)\cap A(0,r)} t^{\alpha-\varepsilon} f(y)\,dy \right) t^{\varepsilon-1}\,dt \end{aligned}$$

since $B(x,(1-|x|)/2) \subset A(0,r)$, where $|B|$ denotes the volume of balls B.

We have by Jensen's inequality and ($\Phi 3$)

$$\begin{aligned} &\Phi((1-r)^{-\varepsilon}\omega(1-r)I_1(x)) \\ &\le C\Phi\left((1-r)^{-\varepsilon}\omega(1-r) \int_0^{1-r} \left(\frac{1}{|B(x,t)|} \int_{B(x,t)\cap A(0,r)} t^{\alpha-\varepsilon} f(y)\,dy \right) t^{\varepsilon-1}\,dt \right) \\ &\le C(1-r)^{-\varepsilon} \int_0^{1-r} \left(\frac{1}{|B(x,t)|} \int_{B(x,t)\cap A(0,r)} \Phi(t^{\alpha-\varepsilon}\omega(1-r)f(y))\,dy \right) t^{\varepsilon-1}\,dt \\ &\le C(1-r)^{-\varepsilon} \int_0^{1-r} t^{(\alpha-\varepsilon)p-N} \left(\int_{B(x,t)\cap A(0,r)} \Phi(\omega(1-r)f(y))\,dy \right) t^{\varepsilon-1}\,dt \\ &\le C(1-r)^{-\varepsilon} \int_{A(0,r)} |x-y|^{(\alpha-\varepsilon)p+\varepsilon-N} \Phi(\omega(1-r)f(y))\,dy \end{aligned}$$

since $(\alpha-\varepsilon)p + \varepsilon - N < 0$.

Hence in this case Minkowski's inequality and Lemma 2.2 yield

$$\begin{aligned} &S_q(\Phi((1-r)^{-\varepsilon}\omega(1-r)I_1), r) \\ &\le C(1-r)^{-\varepsilon} \int_{A(0,r)} S_q(|\cdot - y|^{(\alpha-\varepsilon)p+\varepsilon-N}, r)\Phi(\omega(1-r)f(y))\,dy \\ &\le C(1-r)^{-\varepsilon} \int_{A(0,r)} |r-|y||^{(\alpha-\varepsilon)p+\varepsilon-N+(N-1)/q} \Phi(\omega(1-r)f(y))\,dy \end{aligned}$$

since $(\alpha-\varepsilon)p + \varepsilon - N + (N-1)/q < 0$, $1/2 < r < 1$ and $r \sim |y|$ on $A(0,r)$, which gives assertion (1).

Next we shall show assertion (2). Similarly, under our assumptions, we obtain as above

$$\begin{aligned} &S_q(\Phi((1-r)^{-\varepsilon}\omega(1-r)I_1), r) \\ &\le C(1-r)^{-\varepsilon} \int_{A(0,r)} S_q(|\cdot - y|^{(\alpha-\varepsilon)p+\varepsilon-N}\chi_{B(y,(1-r)/2)}, r)\Phi(\omega(1-r)f(y))\,dy \\ &\le C(1-r)^{-\varepsilon}(1-r)^{(\alpha-\varepsilon)p+\varepsilon-N+(N-1)/q} \int_{A(0,r)} \Phi(\omega(1-r)f(y))\,dy \\ &\le C(1-r)^{(\alpha-\varepsilon)p-N+(N-1)/q} \end{aligned}$$

for $1/2 < r < 1$, since

$$\int_{A(0,r)} \Phi(\omega(1-r)f(y))\,dy \le \int_{\mathbf{B}\setminus B(0,s)} \Phi(\omega(2(1-s)/3)f(y))\,dy \le C\int_{\mathbf{B}\setminus B(0,s)} \Phi(\omega(1-s)f(y))\,dy \le C,$$

where $s = r-(1-r)/2$. Thus assertion (2) is proved.

Finally we shall show assertion (3). When $0<\varepsilon<\alpha$ and $(\alpha-\varepsilon)p+\varepsilon-N>0$, we have

$$\begin{aligned}
&\Phi((1-r)^{-\varepsilon}\omega(1-r)I_1(x)) \\
&\le C(1-r)^{-\varepsilon}\int_0^{1-r} t^{(\alpha-\varepsilon)p-N}\left(\int_{B(x,t)\cap A(0,r)} \Phi(\omega(1-r)f(y))\,dy\right) t^{\varepsilon-1}\,dt \\
&\le C(1-r)^{-\varepsilon}(1-r)^{(\alpha-\varepsilon)p+\varepsilon-N}\int_{A(0,r)} \Phi(\omega(1-r)f(y))\,dy \le C(1-r)^{(\alpha-\varepsilon)p-N},
\end{aligned}$$

which proves assertion (3). □

Lemma 2.5. *Let $0<d<1$ and $M>0$. Set*

$$G(t) = (1-t)^d \int_{A(0,t)} |t-|y||^{-d} g(y)\,dy$$

for a nonnegative measurable function g such that $\sup_{0<t<1}\int_{A(0,t)} g(y)\,dy \le M$. Then there exists a constant $c>0$ such that

$$\inf_{1-2^{-j+1}<t<1-2^{-j}} G(t) < cM \quad \text{for each positive integer } j.$$

Proof. For each positive integer j, we have

$$\begin{aligned}
&\int_{1-2^{-j+1}}^{1-2^{-j}} G(t)\,\frac{dt}{1-t} \\
&\le C\int_{A(0,1-2^{-j})\cup A(0,1-2^{-j+1})} \left(2^{-j(d-1)}\int_{1-2^{-j+1}}^{1-2^{-j}} |t-|y||^{-d}\,dt\right) g(y)\,dy \\
&\le C\int_{A(0,1-2^{-j})\cup A(0,1-2^{-j+1})} g(y)\,dy \le CM.
\end{aligned}$$

Hence

$$\inf_{1-2^{-j+1}<t<1-2^{-j}} G(t) \le CM/(\log 2),$$

as required. □

Lemma 2.6. *Let $1 \le q < \infty$.*

(1) *Suppose $\varepsilon>0$ and*

$$(N-1)/q < N-\alpha p-\varepsilon(p-1).$$

Then there exists a constant $C>0$ such that

$$S_q(\Phi(\omega(1-r)(1-r)^{\varepsilon}I_2), r) \le C(1-r)^{(\alpha+\varepsilon)p-N+(N-1)/q}$$

for all $1/2<r<1$ and nonnegative measurable functions f on $\mathbf{B}$ with $\|f\|_{M^{\Phi,\omega}(\mathbf{B})} \le 1$.

(2) *Suppose* $\varepsilon > 0$ *and*

$$(N-1)/q > N - \alpha p - \varepsilon(p-1).$$

Then there exists a constant $C > 0$ *such that*

$$S_q(\Phi(\omega(1-r)(1-r)^{\varepsilon} I_2), r) \le C(1-r)^{\varepsilon}$$

for all $1/2 < r < 1$ *and nonnegative measurable functions* f *on* $\mathbf{B}$ *with* $\|f\|_{M^{\Phi,\omega}(\mathbf{B})} \le 1$.

Proof. Let $\varepsilon > 0$ such that

$$(N-1)/q < N - (\alpha+\varepsilon)p + \varepsilon.$$

For $1/2 < r = |x| < 1$, we have

$$\begin{aligned} I_2(x) &= \int_{\mathbf{B}} |x-y|^{\alpha-N} f_{2,x}(y)\,dy \\ &\le C \int_{(1-|x|)/2}^{2} \left(\frac{1}{|B(x,t)|} \int_{B(x,t)} f_{2,x}(y)\,dy \right) t^{\alpha-1}\,dt \\ &\le C \int_{(1-|x|)/2}^{2} \left(\frac{1}{|B(x,t)|} \int_{B(x,t)} t^{\alpha+\varepsilon} f_{2,x}(y)\,dy \right) t^{-\varepsilon-1}\,dt, \end{aligned}$$

where $f_{2,x}(y) = f(y)\chi_{E_{2,x}}(y)$ with $E_{2,x} = \{y \in \mathbf{B} \setminus B(x,(1-|x|)/2) : 1-|y| \le 1-|x|\}$.

We have by Jensen's inequality and (Φ3)

$$\begin{aligned} &\Phi(\omega(1-|x|)(1-|x|)^{\varepsilon} I_2(x)) \\ &\le C\Phi\left(\omega(1-|x|)(1-|x|)^{\varepsilon} \int_{(1-|x|)/2}^{2} \left(\frac{1}{|B(x,t)|} \int_{B(x,t)} t^{\alpha+\varepsilon} f_{2,x}(y)\,dy \right) t^{-\varepsilon-1}\,dt \right) \\ &\le C(1-|x|)^{\varepsilon} \int_{(1-|x|)/2}^{2} \left(\frac{1}{|B(x,t)|} \int_{B(x,t)} \Phi(t^{\alpha+\varepsilon}\omega(1-|x|) f_{2,x}(y))\,dy \right) t^{-\varepsilon-1}\,dt \\ &\le C(1-|x|)^{\varepsilon} \int_{(1-|x|)/2}^{2} t^{(\alpha+\varepsilon)p} \left(\frac{1}{|B(x,t)|} \int_{B(x,t)} \Phi(\omega(1-|x|) f_{2,x}(y))\,dy \right) t^{-\varepsilon-1}\,dt \\ &\le C(1-|x|)^{\varepsilon} \int_{\mathbf{B}} |x-y|^{(\alpha+\varepsilon)p-\varepsilon-N} \Phi(\omega(1-|x|) f_{2,x}(y))\,dy \end{aligned}$$

since $(\alpha+\varepsilon)p - \varepsilon - N < 0$.

By Lemma 2.2, we see that

$$\begin{aligned} &\int_{\{\sigma \in S(0,1):\, |t\sigma - y| > (1-t)/2\}} |t\sigma - y|^{a-N}\,dS(\sigma) \\ &\le \int_{\{\sigma \in S(0,1):\, |t\sigma - y| > (1-t)/2\}} (C|(1+(1-t))\sigma - y|)^{a-N}\,dS(\sigma) \\ &\le C|(1+(1-t)) - |y||^{a-1} \le C|1-t|^{a-1} \end{aligned}$$

for $1/2 < t < 1$ and $y \in \mathbf{B}$, when $a < 1$. Hence Minkowski's inequality yields

$$
\begin{aligned}
&S_q(\Phi(\omega(1-r)(1-r)^{\varepsilon} I_2), r) \\
&\le C(1-r)^{\varepsilon} \int_{\mathbf{B}} S_q(|\cdot - y|^{(\alpha+\varepsilon)p-\varepsilon-N} \chi_{E_{2,x}}(y), r) \Phi(\omega(1-|x|) f(y))\, dy \\
&\le C(1-r)^{\varepsilon} (1-r)^{(\alpha+\varepsilon)p-\varepsilon-N+(N-1)/q} \int_{\mathbf{B}\setminus B(0,r)} \Phi(\omega(1-r) f(y))\, dy \\
&\le C(1-r)^{(\alpha+\varepsilon)p-N+(N-1)/q}
\end{aligned}
$$

since $(\alpha+\varepsilon)p - \varepsilon - N + (N-1)/q < 0$, which gives assertion (1).

Next we shall show assertion (2). Suppose $\varepsilon > 0$ such that

$$(N-1)/q > N - (\alpha+\varepsilon)p + \varepsilon > 0.$$

Then we have by Jensen's inequality and ($\Phi3$)

$$
\begin{aligned}
&\Phi(\omega(1-|x|)(1-|x|)^{\varepsilon} I_2(x)) \\
&\le C\Phi\left(\omega(1-|x|)(1-|x|)^{\varepsilon} \int_{(1-|x|)/2}^{2} \left(\frac{1}{|B(x,t)|} \int_{B(x,t)} t^{\alpha+\varepsilon} f_{2,x}(y)\, dy \right) t^{-\varepsilon-1}\, dt \right) \\
&\le C(1-|x|)^{\varepsilon} \int_{(1-|x|)/2}^{2} \left(\frac{1}{|B(x,t)|} \int_{B(x,t)} \Phi(t^{\alpha+\varepsilon} \omega(1-|x|) f_{2,x}(y))\, dy \right) t^{-\varepsilon-1}\, dt \\
&\le C(1-|x|)^{\varepsilon} \int_{(1-|x|)/2}^{2} t^{(\alpha+\varepsilon)p} \left(\frac{1}{|B(x,t)|} \int_{B(x,t)} \Phi(\omega(1-|x|) f_{2,x}(y))\, dy \right) t^{-\varepsilon-1}\, dt \\
&\le C(1-|x|)^{\varepsilon} \int_{\mathbf{B}} |x-y|^{(\alpha+\varepsilon)p-\varepsilon-N} \Phi(\omega(1-|x|) f_{2,x}(y))\, dy.
\end{aligned}
$$

By Lemma 2.2 and Minkowski's inequality, we find

$$
\begin{aligned}
&S_q(\Phi(\omega(1-r)(1-r)^{\varepsilon} I_2), r) \\
&\le C(1-r)^{\varepsilon} \int_{\mathbf{B}} S_q(|\cdot - y|^{(\alpha+\varepsilon)p-\varepsilon-N} \chi_{E_{2,x}}(y), r) \Phi(\omega(1-r) f(y))\, dy \\
&\le C(1-r)^{\varepsilon} \int_{\mathbf{B}\setminus B(0,r)} \Phi(\omega(1-r) f(y))\, dy \le C(1-r)^{\varepsilon}
\end{aligned}
$$

since $(\alpha+\varepsilon)p - \varepsilon - N + (N-1)/q > 0$.

When $\varepsilon > 0$ and $(\alpha+\varepsilon)p - \varepsilon - N \ge 0$, taking $0 < \delta < (N-1)/q$, we have

$$
\begin{aligned}
&\Phi(\omega(1-|x|)(1-|x|)^{\varepsilon} I_2(x)) \\
&\le C(1-|x|)^{\varepsilon} \int_{(1-|x|)/2}^{2} t^{(\alpha+\varepsilon)p} \left(\frac{1}{|B(x,t)|} \int_{B(x,t)} \Phi(\omega(1-|x|) f_{2,x}(y))\, dy \right) t^{-\varepsilon-1}\, dt \\
&\le C(1-|x|)^{\varepsilon} \int_{\mathbf{B}} |x-y|^{-\delta} \Phi(\omega(1-|x|) f_{2,x}(y))\, dy
\end{aligned}
$$

and

$$
\begin{aligned}
S_q(\Phi(\omega(1-r)(1-r)^{\varepsilon} I_2), r) &\le C(1-r)^{\varepsilon} \int_{\mathbf{B}} S_q(|\cdot - y|^{-\delta} \chi_{E_{2,x}}(y), r) \Phi(\omega(1-r) f(y))\, dy \\
&\le C(1-r)^{\varepsilon} \int_{\mathbf{B}\setminus B(0,r)} \Phi(\omega(1-r) f(y))\, dy \le C(1-r)^{\varepsilon},
\end{aligned}
$$

which completes the proof of assertion (2). □

Lemma 2.7. *Let $1 \le q < \infty$.*

(1) *Suppose*

(ω4) $t^{\alpha p+\varepsilon_0-N+(N-1)/q}\omega(t)^{-p}$ *is almost decreasing on* $(0,1]$ *for some* $\varepsilon_0 > 0$.

Let $0 < \varepsilon < \varepsilon_0/(p-1)$. *Then there exists a constant* $C > 0$ *such that*

$$S_q(\Phi((1-r)^\varepsilon I_3), r) \le C(1-r)^{(\alpha+\varepsilon)p-N+(N-1)/q}\omega(1-r)^{-p}$$

for all $1/2 < r < 1$ *and nonnegative measurable functions* f *on* $\mathbf{B}$ *with* $\|f\|_{M^{\Phi,\omega}(\mathbf{B})} \le 1$.

(2) *Suppose* $\varepsilon > 0$ *and*

$$(N-1)/q > N - \alpha p - \varepsilon(p-1).$$

Then there exists a constant $C > 0$ *such that*

$$S_q(\Phi((1-r)^\varepsilon I_3), r) \le C(1-r)^\varepsilon$$

for all $1/2 < r < 1$ *and nonnegative measurable functions* f *on* $\mathbf{B}$ *with* $\|f\|_{M^{\Phi,\omega}(\mathbf{B})} \le 1$.

Proof. Let $1/2 < r = |x| < 1$. First note from (ω4) and $0 < \varepsilon < \varepsilon_0/(p-1)$ that $t^{(\alpha+\varepsilon)p-\varepsilon-N+(N-1)/q}\omega(t)^{-p}$ is almost decreasing on $(0,1]$ and

$$(\alpha+\varepsilon)p - \varepsilon - N + (N-1)/q < 0.$$

Further, note that

$$\int_{B(0,1/4)} |x-y|^{\alpha-N} f(y)\,dy \le C\int_{B(0,1/4)} f(y)\,dy \le C.$$

As in the proof of Lemma 2.6, we have

$$\begin{aligned} I_3(x) &= \int_{\mathbf{B}} |x-y|^{\alpha-N} f_{3,x}(y)\,dy \\ &\le C\int_{(1-|x|)/2}^{2} \left(\frac{1}{|B(x,t)|}\int_{B(x,t)} f_{3,x}(y)\,dy\right) t^{\alpha-1}\,dt \\ &\le C\int_{(1-|x|)/2}^{2} \left(\frac{1}{|B(x,t)|}\int_{B(x,t)} t^{\alpha+\varepsilon} f_{3,x}(y)\,dy\right) t^{-\varepsilon-1}\,dt, \end{aligned}$$

where $f_{3,x}(y) = f(y)\chi_{E_{3,x}}(y)$ with $E_{3,x} = \{y \in \mathbf{B}\setminus B(x,(1-|x|)/2) : 1-|y| > 1-|x|\}$. Since $r \sim |y|$ for $y \in \mathbf{B}\setminus B(0,1/4)$, in the same way as in the proof of Lemma 2.6, we see from Lemma 2.2 that

$$\begin{aligned} S_q(\Phi((1-r)^\varepsilon I_3), r) &\le C(1-r)^\varepsilon \left(\int_{\mathbf{B}} S_q(|\cdot - y|^{(\alpha+\varepsilon)p-\varepsilon-N}\chi_{E_{3,x}}(y), r)\Phi(f(y))\,dy + 1\right) \\ &\le C(1-r)^\varepsilon \left(\int_{B(0,r)} (1-|y|)^{(\alpha+\varepsilon)p-\varepsilon-N+(N-1)/q}\Phi(f(y))\,dy + 1\right). \end{aligned}$$

Let j_0 be the smallest integer such that $r \le 1 - 2^{-j_0-1}$. Note here that

$$\begin{aligned}
&\int_{B(0,r)} (1-|y|)^{(\alpha+\varepsilon)p-\varepsilon-N+(N-1)/q}\Phi(f(y))\,dy \\
&\le \sum_{j=0}^{j_0} \int_{A(0,1-2^{-j})} (1-|y|)^{(\alpha+\varepsilon)p-\varepsilon-N+(N-1)/q}\Phi(f(y))\,dy \\
&\le C\sum_{j=0}^{j_0} 2^{-j((\alpha+\varepsilon)p-\varepsilon-N+(N-1)/q)} \int_{A(0,1-2^{-j})} \Phi(f(y))\,dy \\
&\le C\sum_{j=0}^{j_0} 2^{-j((\alpha+\varepsilon)p-\varepsilon-N+(N-1)/q)}\omega(2^{-j})^{-p} \\
&\le C(1-r)^{(\alpha+\varepsilon)p-\varepsilon-N+(N-1)/q}\omega(1-r)^{-p}
\end{aligned}$$

by (ω4), which gives assertion (1).

For assertion (2), suppose $\varepsilon > 0$ such that

$$(N-1)/q > N - (\alpha+\varepsilon)p + \varepsilon > 0.$$

Then, in the same way as in the proof of Lemma 2.6, we see from Lemma 2.2 that

$$\begin{aligned}
&S_q(\Phi((1-r)^\varepsilon I_3), r) \\
&\le C(1-r)^\varepsilon \left(\int_{\mathbf{B}} S_q(|\cdot - y|^{(\alpha+\varepsilon)p-\varepsilon-N}\chi_{E_{3,x}}(y), r)\Phi(f(y))\,dy + 1 \right) \\
&\le C(1-r)^\varepsilon \left(\int_{B(0,r)} \Phi(f(y))\,dy + 1 \right) \le C(1-r)^\varepsilon.
\end{aligned}$$

When $\varepsilon > 0$ and $(\alpha+\varepsilon)p - \varepsilon - N \ge 0$, we see that

$$S_q(\Phi((1-r)^\varepsilon I_3), r) \le C(1-r)^\varepsilon \left(\int_{B(0,r)} \Phi(f(y))\,dy + 1 \right) \le C(1-r)^\varepsilon,$$

as in the proof of Lemma 2.6. Thus the present lemma is proved. □

Remark 2.8. If $\omega(r) = r^{-\nu}$, then (ω4) holds in case $(N-1)/q < N - \alpha p - \nu p$.

3. Spherical limits for Riesz potentials

We are now ready to show our main result.

Theorem 3.1. *Let $1 \le q < \infty$.*

(1) *Suppose (ω4) holds for some $\varepsilon_0 > 0$. If $0 < \varepsilon < \min\{\alpha, \varepsilon_0/(p-1)\}$ and*

$$N - \alpha p + \varepsilon(p-1) - 1 < (N-1)/q < N - \alpha p - \varepsilon(p-1),$$

then there exists a constant $C > 0$ such that

$$\liminf_{r\to 1-} (1-r)^{N-(\alpha+\varepsilon)p-(N-1)/q}\omega(1-r)^p S_q(\Phi((1-r)^\varepsilon I_\alpha f), r) \le C$$

for all nonnegative measurable functions f with $\|f\|_{M^{\Phi,\omega}(\mathbf{B})} \le 1$.

(2) *If $0 < \varepsilon < \alpha$ and*

$$\begin{aligned}
\max\{N - \alpha p - \varepsilon(p-1), N - \alpha p + \varepsilon(p-1) - 1\} &< (N-1)/q \\
&< N - \alpha p + \varepsilon(p-1),
\end{aligned}$$

then there exists a constant $C>0$ *such that*

$$\liminf_{r\to 1-} \min\{(1-r)^{N-(\alpha+\varepsilon)p-(N-1)/q}\omega(1-r)^p, (1-r)^{-\varepsilon}\} S_q(\Phi((1-r)^\varepsilon I_\alpha f), r) \le C$$

for all nonnegative measurable functions f *with* $\|f\|_{M^{\Phi,\omega}(\mathbf{B})} \le 1$.

(3) *If* $0<\varepsilon<\alpha$ *and* $(N-1)/q > N-\alpha p+\varepsilon(p-1) > 0$, *then there exists a constant* $C>0$ *such that*

$$\min\{(1-r)^{N-(\alpha+\varepsilon)p-(N-1)/q}\omega(1-r)^p, (1-r)^{-\varepsilon}\} S_q(\Phi((1-r)^\varepsilon I_\alpha f), r) \le C$$

for all $1/2<r<1$ *and all nonnegative measurable functions* f *with* $\|f\|_{M^{\Phi,\omega}(\mathbf{B})} \le 1$.

(4) *If* $0<\varepsilon<\alpha$ *and* $(\alpha-\varepsilon)p+\varepsilon-N>0$, *then there exists a constant* $C>0$ *such that*

$$\min\{(1-r)^{N-(\alpha+\varepsilon)p}\omega(1-r)^p, (1-r)^{-\varepsilon}\} S_q(\Phi((1-r)^\varepsilon I_\alpha f), r) \le C$$

for all $1/2<r<1$ *and all nonnegative measurable functions* f *with* $\|f\|_{M^{\Phi,\omega}(\mathbf{B})} \le 1$.

Proof. We shall show assertion (1). Let f be a nonnegative measurable function in $M^{\Phi,\omega}(\mathbf{B})$. For $x\in\mathbf{B}$, write

$$I_\alpha f(x) = I_1(x)+I_2(x)+I_3(x)$$

as before. Let $0<\varepsilon<\min\{\alpha, \varepsilon_0/(p-1)\}$ such that

$$-1 < (\alpha-\varepsilon)p+\varepsilon-N+(N-1)/q < (\alpha+\varepsilon)p-\varepsilon-N+(N-1)/q < 0.$$

Set

$$d = -(\alpha-\varepsilon)p-\varepsilon+N-(N-1)/q.$$

Then $0<d<1$. First note by Lemma 2.6 (1) that

$$(1-r)^{N-(\alpha+\varepsilon)p-(N-1)/q} S_q(\Phi(\omega(1-r)(1-r)^\varepsilon I_2), r) \le C,$$

so that by (Φ4)

$$(1-r)^{N-(\alpha+\varepsilon)p-(N-1)/q}\omega(1-r)^p S_q(\Phi((1-r)^\varepsilon I_2), r) \le C.$$

By Lemma 2.7 (1), we have

$$(1-r)^{N-(\alpha+\varepsilon)p-(N-1)/q}\omega(1-r)^p S_q(\Phi((1-r)^\varepsilon I_3), r) \le C.$$

Finally, we obtain by Lemma 2.4 (1)

$$S_q(\Phi((1-r)^{-\varepsilon}\omega(1-r)I_1), r) \le C(1-r)^{-\varepsilon}\int_{A(0,r)} |r-|y||^{-d} g(y)\,dy,$$

where $g(y)=\Phi(\omega(1-|y|)f(y))$. Therefore ($\Phi$4) gives

$$\begin{aligned}
&(1-r)^{N-(\alpha+\varepsilon)p-(N-1)/q}\omega(1-r)^p S_q(\Phi((1-r)^\varepsilon I_1), r)\\
&\le C(1-r)^{N-(\alpha+\varepsilon)p-(N-1)/q}(1-r)^{2\varepsilon p} S_q(\Phi((1-r)^{-\varepsilon}\omega(1-r)I_1), r)\\
&\le C(1-r)^d \int_{A(0,r)} |r-|y||^{-d} g(y)\,dy.
\end{aligned}$$

In view of Lemma 2.5, we can find a sequence $\{r_j\}$ of positive numbers such that $1-2^{-j+1}<r_j<1-2^{-j}$ and

$$\sup_j\, (1-r_j)^{N-(\alpha+\varepsilon)p-(N-1)/q}\omega(1-r_j)^p S_q(\Phi((1-r_j)^\varepsilon I_1), r_j) \le C.$$

Thus assertion (1) is obtained.

Next we shall show assertion (2). Suppose $0 < \varepsilon < \alpha$ such that

$$N - \alpha p - \varepsilon(p-1) < (N-1)/q < N - \alpha p + \varepsilon(p-1).$$

By Lemmas 2.6 (2) and 2.7 (2), we obtain

$$S_q(\Phi((1-r)^{\varepsilon} I_2), r) \le C S_q(\Phi(\omega(1-r)(1-r)^{\varepsilon} I_2), r) \le C(1-r)^{\varepsilon}$$

and

$$S_q(\Phi((1-r)^{\varepsilon} I_3), r) \le C(1-r)^{\varepsilon}$$

for all $1/2 < r < 1$. In view of (Φ4), we obtain

$$\begin{aligned}
&(1-r)^{N-(\alpha+\varepsilon)p-(N-1)/q}\omega(1-r)^p S_q(\Phi((1-r)^{\varepsilon} I_1), r) \\
&\le C(1-r)^{N-(\alpha+\varepsilon)p-(N-1)/q}(1-r)^{2p\varepsilon} S_q(\Phi(\omega(1-r)(1-r)^{-\varepsilon} I_1), r) \\
&\le C(1-r)^{N-(\alpha-\varepsilon)p-(N-1)/q} S_q(\Phi(\omega(1-r)(1-r)^{-\varepsilon} I_1), r)
\end{aligned}$$

for all $1/2 < r < 1$. Thus, by Lemma 2.4 (1), assertion (2) is proved.

For a proof of assertion (3), it suffices to apply Lemma 2.4 (2) in the proof of assertion (2).

For a proof of assertion (4), note that by our assumption

$$(N-1)/q > 0 > N - \alpha p + \varepsilon(p-1) > N - \alpha p - \varepsilon(p-1)$$

and by (Φ4)

$$\begin{aligned}
&(1-r)^{N-(\alpha+\varepsilon)p}\omega(1-r)^p S_q(\Phi((1-r)^{\varepsilon} I_1), r) \\
&\le C(1-r)^{N-(\alpha-\varepsilon)p} S_q(\Phi(\omega(1-r)(1-r)^{-\varepsilon} I_1), r)
\end{aligned}$$

for all $1/2 < r < 1$. As in the proof of assertion (2), it suffices to apply Lemma 2.4 (3). □

Remark 3.2. The first and third authors [17, Theorem 1] treated the existence of boundary limits for BLD functions u on the unit ball $\mathbf{B}$ of $\mathbf{R}^N$ satisfying

$$\int_{\mathbf{B}} |\nabla u(x)|^p (1-|x|)^{\gamma}\, dx < \infty,$$

where ∇ denotes the gradient, $1 < p < \infty$ and $-1 < \gamma < p-1$. In fact, we showed that

$$\liminf_{r\to 1-} (1-r)^{(N-p+\gamma)/p-(N-1)/q} S_q(u, r) = 0$$

when $q > 0$ and $(N-p-1)/(p(N-1)) < 1/q < (N-p+\gamma)/(p(N-1))$. If u is in addition monotone in $\mathbf{B}$ in the sense of Lebesgue, then u is shown to have weighted boundary limit zero (see [17, Theorem 2]).

When $\Phi(r) = r^p$ and $\omega(r) = r^{-\nu}$, we obtain the following corollary.

Corollary 3.3. *Suppose* $1 \le q < \infty$, $\nu \ge 0$ *and*

$$\frac{N-\alpha p - 1}{N-1} < \frac{1}{q} < \frac{N - \alpha p - \nu p}{N-1}.$$

Then

$$\liminf_{r\to 1-} (1-r)^{(N-\alpha p-\nu p)/p-(N-1)/(pq)} S_{pq}(I_\alpha f, r) < \infty$$

for all nonnegative measurable functions $f \in M^{p,\nu}(\mathbf{B})$.

Proof. Let f be a nonnegative measurable functions $f \in M^{p,\nu}(\mathbf{B})$. First note that $(\omega 4)$ holds for some $\varepsilon_0 > 0$. Take $0 < \varepsilon < \min\{\alpha, \varepsilon_0/(p-1)\}$ such that

$$N - \alpha p + \varepsilon(p-1) - 1 < (N-1)/q.$$

Then Theorem 3.1 (1) gives

$$\liminf_{r \to 1-} (1-r)^{(N-\alpha p - \nu p)/p - (N-1)/(pq)} S_{pq}(I_\alpha f, r) < \infty,$$

as required. □

Corollary 3.4. *Suppose* $1 \le p \le q < \infty$ *and*

$$\frac{N - \alpha p - 1}{p(N-1)} < \frac{1}{q} < \frac{N - \alpha p - \nu p}{p(N-1)}.$$

Then

$$\liminf_{r \to 1-} (1-r)^{(N-\alpha p - \nu p)/p - (N-1)/q} S_q(I_\alpha f, r) < \infty$$

for all nonnegative measurable functions $f \in M^{p,\nu}(\mathbf{B})$.

Remark 3.5. We show that the exponent in Corollary 3.4 is the best possible. For this, let $1 \le p \le q < \infty$ and $\alpha + \nu - N/p + (N-1)/q < 0$. Consider the function

$$f(y) = |e - y|^{\nu - N/p}$$

for $0 < \nu < 1/p$, where $e = (0, \dots, 0, 1) \in \partial \mathbf{B}$. Then, by the proof of Lemma 2.2 and $0 < \nu < 1/p$, we see that

$$(1-r)^{-\nu p} \int_{\mathbf{B} \setminus B(0,r)} |f(y)|^p \, dy \le C(1-r)^{-\nu p} \int_r^1 (1-t)^{\nu p - 1} \, dt \le C$$

for $1/2 < r < 1$. Moreover,

$$\begin{aligned} I_\alpha f(x) &\ge \int_{\mathbf{B} \cap B(x, |e-x|/2)} |x-y|^{\alpha - N} |e-y|^{\nu - N/p} \, dy \\ &\ge C|e-x|^{\nu - N/p} \int_{\mathbf{B} \cap B(x, |e-x|/2)} |x-y|^{\alpha - N} \, dy \\ &\ge C|e-x|^{\alpha + \nu - N/p} \end{aligned}$$

for $x \in \mathbf{B}$. Lemma 2.3 gives

$$S_q(I_\alpha f, r) \ge C(1-r)^{\alpha + \nu - N/p + (N-1)/q}$$

for $1/2 < r < 1$, since $\alpha + \nu - N/p + (N-1)/q < 0$. Hence

$$\liminf_{r \to 1-} (1-r)^{(N-\alpha p - \nu p)/p - (N-1)/q} S_q(I_\alpha f, r) \ge C > 0.$$

Remark 3.6. Consider

$$f(y) = \sum_{j=1}^{\infty} |y - \mathbf{e}_j|^{\nu - N/p} \chi_{B(0,(1-2^{-j})+2^{-j-2}) \setminus B(0,(1-2^{-j})-2^{-j-2})}(y)$$

for $0 < \nu < 1/p$, where $\mathbf{e} = (0, \dots, 0, 1) \in \partial \mathbf{B}$ and $\mathbf{e}_j = (1 - 2^{-j})\mathbf{e}$. Let j_0 be the largest integer such that $1 - 2^{-j_0} - 2^{-j_0 - 2} < r \le 1 - 2^{-j_0 - 1} - 2^{-j_0 - 3}$. Then note from

Lemma 2.2 that

$$\int_{\mathbf{B}\setminus B(0,r)} f(y)^p\,dy \le \sum_{j=j_0}^{\infty} \int_{B(0,(1-2^{-j})+2^{-j-2})\setminus B(0,(1-2^{-j})-2^{-j-2})} |y-\mathbf{e}_j|^{\nu p-N}\,dy$$
$$\le C\sum_{j=j_0}^{\infty} \int_{(1-2^{-j})-2^{-j-2}}^{(1-2^{-j})+2^{-j-2}} |t-(1-2^{-j})|^{\nu p-1}\,dt$$
$$\le C\sum_{j=j_0}^{\infty} 2^{-j\nu p} \le C(1-r)^{\nu p}$$

since $0<\nu<1/p$. Further,

$$I_\alpha f(x) \ge C|x-\mathbf{e}_j|^{\alpha-N} \int_{B(\mathbf{e}_j,|x-e_j|/2)} |y-\mathbf{e}_j|^{\nu-N/p}\,dy \ge C|x-\mathbf{e}_j|^{\alpha+\nu-N/p}$$

for $x\in B(\mathbf{e}_j,2^{-j-2}/2)$, since $\nu-N/p+N>0$. We see that

$$S_q(I_\alpha f,1-2^{-j}) \ge C\left(\int_{S(0,1-2^{-j})\cap B(\mathbf{e}_j,2^{-j-2}/2)} |x-\mathbf{e}_j|^{(\alpha+\nu-N/p)q}\,dS(x)\right)^{1/q} = \infty,$$

when $(\alpha+\nu-N/p)q+N-1\le 0$. This implies the necessity of the lower limit in Theorem 3.1 when $\nu>0$ and

$$\frac{N-1}{q} \le \frac{N-\alpha p-\nu p}{p}.$$

Let $M_0^{\Phi,\omega}(\mathbf{B})$ denote the family of all measurable functions f on $\mathbf{B}$ such that

$$\lim_{r\to 1-}\int_{A(0,r)} \Phi(\omega(1-r)|f(y)|)\,dy = 0.$$

With a slight modification of the proof of Theorem 3.1, we can prove the following result.

Corollary 3.7. *Let* $1\le q<\infty$. *Suppose* $(\omega4)$ *holds for some* $\varepsilon_0>0$. *If* $0<\varepsilon<\min\{\alpha,\varepsilon_0/(p-1)\}$ *and*

$$N-\alpha p+\varepsilon(p-1)-1 < (N-1)/q < N-\alpha p-\varepsilon(p-1),$$

then

$$\liminf_{r\to 1-} (1-r)^{N-(\alpha+\varepsilon)p-(N-1)/q}\omega(1-r)^p S_q(\Phi((1-r)^\varepsilon I_\alpha f),r) = 0$$

for all nonnegative measurable functions $f\in M_0^{\Phi,\omega}(\mathbf{B})$.

Proof. Let f be a nonnegative measurable function in $M_0^{\Phi,\omega}(\mathbf{B})$. For $x\in\mathbf{B}$, write

$$I_\alpha f(x) = I_1(x)+I_2(x)+I_3(x)$$

as before. Let $0<\varepsilon<\min\{\alpha,\varepsilon_0/(p-1)\}$ such that

$$-1 < (\alpha-\varepsilon)p+\varepsilon-N+(N-1)/q < (\alpha+\varepsilon)p-\varepsilon-N+(N-1)/q < 0.$$

Set

$$d = -(\alpha-\varepsilon)p-\varepsilon+N-(N-1)/q.$$

Then $0<d<1$. First note by the proof of Lemma 2.6 (1) that

$$(1-r)^{N-(\alpha+\varepsilon)p-(N-1)/q} S_q(\Phi(\omega(1-r)(1-r)^\varepsilon I_2),r) \le C\int_{\mathbf{B}\setminus B(0,r)} \Phi(\omega(1-r)f(y))\,dy,$$

so that by (Φ4)

$$(1-r)^{N-(\alpha+\varepsilon)p-(N-1)/q}\omega(1-r)^p S_q(\Phi((1-r)^\varepsilon I_2), r) \le C \int_{\mathbf{B}\setminus B(0,r)} \Phi(\omega(1-r)f(y))\,dy.$$

Take $0 < r_0 < 1$. We write

$$\begin{aligned} I_3(x) &= \int_{\{y\in\mathbf{B}\setminus B(x,(1-|x|)/2):\ 1-|y|>1-|x|,|y|\le r_0\}} |x-y|^{\alpha-N} f(y)\,dy \\ &\quad + \int_{\{y\in\mathbf{B}\setminus B(x,(1-|x|)/2):\ 1-|y|>1-|x|,|y|>r_0\}} |x-y|^{\alpha-N} f(y)\,dy \\ &= I_{3,1}(x) + I_{3,2}(x). \end{aligned}$$

By (Φ3) and the fact that $I_{3,1}(x) \le C$, we have

$$\begin{aligned} &\liminf_{r\to 1-} (1-r)^{N-(\alpha+\varepsilon)p-(N-1)/q}\omega(1-r)^p S_q(\Phi((1-r)^\varepsilon I_{3,1}), r) \\ &\le \liminf_{r\to 1-} (1-r)^{N-(\alpha+\varepsilon)p-(N-1)/q}\omega(1-r)^p S_q(\Phi((1-r)^\varepsilon C), r) = 0 \end{aligned}$$

and by the proof of Lemma 2.7 (1)

$$\begin{aligned} &(1-r)^{N-(\alpha+\varepsilon)p-(N-1)/q}\omega(1-r)^p S_q(\Phi((1-r)^\varepsilon I_{3,2}), r) \\ &\le C \sup_{\{j\in\mathbf{N}:\ 1-2^{-j-1}>r_0\}} \omega(2^{-j})^p \int_{A(0,1-2^{-j})} \Phi(f(y))\,dy \end{aligned}$$

for $r_0 < r < 1$, so that

$$\begin{aligned} &\liminf_{r\to 1-} (1-r)^{N-(\alpha+\varepsilon)p-(N-1)/q}\omega(1-r)^p S_q(\Phi((1-r)^\varepsilon I_3), r) \\ &\le C \sup_{\{j\in\mathbf{N}:\ 1-2^{-j-1}>r_0\}} \omega(2^{-j})^p \int_{A(0,1-2^{-j})} \Phi(f(y))\,dy. \end{aligned}$$

Letting $r_0 \to 1$, we infer that

$$\liminf_{r\to 1-} (1-r)^{N-(\alpha+\varepsilon)p-(N-1)/q}\omega(1-r)^p S_q(\Phi((1-r)^\varepsilon I_3), r) = 0.$$

Finally, we obtain by Lemma 2.4

$$S_q(\Phi((1-r)^{-\varepsilon}\omega(1-r)I_1), r) \le C(1-r)^{-\varepsilon} \int_{A(0,r)} |r-|y||^{-d} g(y)\,dy,$$

where $g(y) = \Phi(\omega(1-|y|)f(y))$. Therefore ($\Phi$4) gives

$$\begin{aligned} &(1-r)^{N-(\alpha+\varepsilon)p-(N-1)/q}\omega(1-r)^p S_q(\Phi((1-r)^\varepsilon I_1), r) \\ &\le C(1-r)^{N-(\alpha+\varepsilon)p-(N-1)/q}(1-r)^{2\varepsilon p} S_q(\Phi((1-r)^{-\varepsilon}\omega(1-r)I_1), r) \\ &\le C(1-r)^d \int_{A(0,r)} |r-|y||^{-d} g(y)\,dy. \end{aligned}$$

In view of Lemma 2.5, we can find a sequence $\{r_j\}$ of positive numbers such that $r_0 < 1-2^{-j+1} < r_j < 1-2^{-j}$ and

$$\begin{aligned} &\sup_j (1-r_j)^{N-(\alpha+\varepsilon)p-(N-1)/q}\omega(1-r_j)^p S_q(\Phi((1-r_j)^\varepsilon I_1), r_j) \\ &\le C \sup_{r_0<r<1} \int_{A(0,r)} \Phi(\omega(1-r)f(y))\,dy. \end{aligned}$$

Thus, letting $r_0 \to 1$, we obtain the required result. □

4. Spherical limits for Riesz potentials II

In this section we treat functions f on $\mathbf{B}$ satisfying the weighted condition $(1-|y|)^{\beta p_1} f(y)^{p_1} \in M^{\Phi,\omega}(\mathbf{B})$ for $1 < p_1 < \infty$ and $\beta > 0$.

Lemma 4.1. *Let $1 < p_1 < \infty$ and $\beta > 0$. Suppose $0 < \alpha p_1 - \alpha_1 < \beta p_1 < p_1 - 1$. Then there exists a constant $C > 0$ such that*

$$\left((1-|x|)^{-(\alpha-\alpha_1/p_1)+\beta} I_\alpha f(x)\right)^{p_1} \le C I_{\alpha_1} g(x)$$

for all $x \in \mathbf{B} \setminus B(0,1/2)$ and nonnegative measurable functions $f \in L^1_{loc}(\mathbf{B})$, where $g(y) = (1-|y|)^{\beta p_1} f(y)^{p_1}$.

Proof. Let f be a nonnegative measurable function $f \in L^1_{loc}(\mathbf{B})$. By Hölder's inequality and Lemma 2.2, we have

$$\begin{aligned}
\int_{\mathbf{B}} |x-y|^{\alpha-N} f(y)dy &\le \left(\int_{\mathbf{B}} |x-y|^{(\alpha-\alpha_1/p_1)p_1'-N}(1-|y|)^{-\beta p_1'}\, dy\right)^{1/p_1'} \\
&\quad \times \left(\int_{\mathbf{B}} |x-y|^{\alpha_1-N}(1-|y|)^{\beta p_1} f(y)^{p_1}\, dy\right)^{1/p_1} \\
&\le \left(\int_0^1 \left(\int_{S(0,r)} |x-y|^{(\alpha-\alpha_1/p_1)p_1'-N}\, dS(y)\right)(1-r)^{-\beta p_1'}\, dr\right)^{1/p_1'} \\
&\quad \times \left(\int_{\mathbf{B}} |x-y|^{\alpha_1-N} g(y)\, dy\right)^{1/p_1} \\
&\le C\left(\int_0^1 ||x|-r|^{(\alpha-\alpha_1/p_1)p_1'-1}(1-r)^{-\beta p_1'}\, dr\right)^{1/p_1'} \\
&\quad \times \left(\int_{\mathbf{B}} |x-y|^{\alpha_1-N} g(y)\, dy\right)^{1/p_1},
\end{aligned}$$

where $g(y) = (1-|y|)^{\beta p_1} f(y)^{p_1}$. Since $(\alpha-\alpha_1/p_1)p_1' > 0$, $-\beta p_1' + 1 > 0$ and $(\alpha-\alpha_1/p_1)p_1' - \beta p_1' + 1 < 1$ by our assumptions, the Riesz composition formula (see e.g. [15, p. 59]) yields

$$\int_{\mathbf{B}} |x-y|^{\alpha-N} f(y)\, dy \le C(1-|x|)^{(\alpha-\alpha_1/p_1)-\beta}\left(\int_{\mathbf{B}} |x-y|^{\alpha_1-N} g(y)\, dy\right)^{1/p_1},$$

as required. □

In view of Lemma 4.1 and Theorem 3.1, we obtain the following theorem.

Theorem 4.2. *Let $1 \le q < \infty$ and let $0 < \alpha p_1 - \alpha_1 < \beta p_1 < p_1 - 1$.*

(1) *Suppose $(\omega 4)$ holds for some $\varepsilon_0 > 0$ and α replaced by α_1 . If $0 < \varepsilon < \min\{\alpha_1, \varepsilon_0/(p-1)\}$ and*

$$N - \alpha_1 p + \varepsilon(p-1) - 1 < (N-1)/q < N - \alpha_1 p - \varepsilon(p-1),$$

then there exists a constant $C > 0$ such that

$$\begin{aligned}
&\liminf_{r\to 1-} (1-r)^{N-(\alpha_1+\varepsilon)p-(N-1)/q}\omega(1-r)^p \\
&\quad \times S_q(\Phi((1-r)^\varepsilon((1-r)^{-(\alpha-\alpha_1/p_1)+\beta} I_\alpha f)^{p_1}), r) \le C
\end{aligned}$$

for all nonnegative measurable functions f with $\|g\|_{M^{\Phi,\omega}(\mathbf{B})} \le 1$, where $g(y) = (1-|y|)^{\beta p_1} f(y)^{p_1}$.

(2) *If* $0 < \varepsilon < \alpha_1$ *and*

$$\max\{N - \alpha_1 p - \varepsilon(p-1), N - \alpha_1 p + \varepsilon(p-1) - 1\} < (N-1)/q < N - \alpha_1 p + \varepsilon(p-1),$$

then there exists a constant $C > 0$ *such that*

$$\liminf_{r\to 1-} \min\{(1-r)^{N-(\alpha_1+\varepsilon)p-(N-1)/q}\omega(1-r)^p, (1-r)^{-\varepsilon}\} \times S_q(\Phi((1-r)^{\varepsilon}((1-r)^{-(\alpha-\alpha_1/p_1)+\beta} I_\alpha f)^{p_1}), r) \le C$$

for all nonnegative measurable functions f *with* $\|g\|_{M^{\Phi,\omega}(\mathbf{B})} \le 1$, *where* $g(y) = (1-|y|)^{\beta p_1} f(y)^{p_1}$.

(3) *If* $0 < \varepsilon < \alpha_1$ *and* $(N-1)/q > N - \alpha_1 p + \varepsilon(p-1) > 0$, *then there exists a constant* $C > 0$ *such that*

$$\min\{(1-r)^{N-(\alpha_1+\varepsilon)p-(N-1)/q}\omega(1-r)^p, (1-r)^{-\varepsilon}\} \times S_q(\Phi((1-r)^{\varepsilon}((1-r)^{-(\alpha-\alpha_1/p_1)+\beta} I_\alpha f)^{p_1}), r) \le C$$

for all $1/2 < r < 1$ *and all nonnegative measurable functions* f *with* $\|g\|_{M^{\Phi,\omega}(\mathbf{B})} \le 1$, *where* $g(y) = (1-|y|)^{\beta p_1} f(y)^{p_1}$.

(4) *If* $0 < \varepsilon < \alpha_1$ *and* $(\alpha_1 - \varepsilon)p + \varepsilon - N > 0$, *then there exists a constant* $C > 0$ *such that*

$$\min\{(1-r)^{N-(\alpha_1+\varepsilon)p}\omega(1-r)^p, (1-r)^{-\varepsilon}\} S_q(\Phi((1-r)^{\varepsilon}((1-r)^{-(\alpha-\alpha_1/p_1)+\beta} I_\alpha f)^{p_1}), r) \le C$$

for all $1/2 < r < 1$ *and all nonnegative measurable functions* f *with* $\|g\|_{M^{\Phi,\omega}(\mathbf{B})} \le 1$, *where* $g(y) = (1-|y|)^{\beta p_1} f(y)^{p_1}$.

When $\Phi(r) = r^p$ and $\omega(r) = 1$, we obtain the following corollary.

Corollary 4.3. *If* $1 \le q < \infty$, $0 < \alpha p_1 - \alpha_1 < \beta p_1 < p_1 - 1$ *and*

$$\frac{N - \alpha_1 p - 1}{N-1} < \frac{1}{q} < \frac{N - \alpha_1 p}{N-1},$$

then

$$\liminf_{r\to 1-} (1-r)^{N-(\alpha-\beta)pp_1-(N-1)/q} S_q((I_\alpha f)^{pp_1}, r) < \infty \tag{4.1}$$

for all nonnegative measurable functions f *such that*

$$\int_{\mathbf{B}} f(y)^{pp_1}(1-|y|)^{\beta p_1 p}\, dy < \infty.$$

5. Green potentials

Let $G(x,y)$ be a Green kernel on $\mathbf{B}$. When $N \ge 3$, there exists a constant $C > 0$ such that

$$C^{-1}\frac{(1-|x|)(1-|y|)}{|x-y|^{N-2}|x^*-y|^2} \le G(x,y) \le C\frac{(1-|x|)(1-|y|)}{|x-y|^{N-2}|x^*-y|^2} \le C\frac{(1-|x|)(1-|y|)}{|x-y|^N}$$

for $x, y \in \mathbf{B}$, where x^* is the inversion of x with respect to $S(0,1)$.

For $f \in L^1_{\mathrm{loc}}(\mathbf{B})$ and $x \in \mathbf{B}$, we write

$$\begin{aligned} Gf(x) &= \int_{\mathbf{B}} G(x,y)f(y)\,dy = \int_{B(x,(1-|x|)/2)} G(x,y)f(y)\,dy \\ &\quad + \int_{\{y\in\mathbf{B}\setminus B(x,(1-|x|)/2):1-|y|\le 1-|x|\}} G(x,y)f(y)\,dy \\ &\quad + \int_{\{y\in\mathbf{B}\setminus B(x,(1-|x|)/2):1-|y|> 1-|x|\}} G(x,y)f(y)\,dy \\ &= G_1(x)+G_2(x)+G_3(x). \end{aligned}$$

Lemma 5.1. *Let $1 \le q < \infty$.*

(1) *Suppose $\varepsilon > 0$ and*

$$(N-1)/q < N - \varepsilon(p-1).$$

Then there exists a constant $C > 0$ such that

$$S_q(\Phi((1-r)^{-1+\varepsilon}G_2), r) \le C(1-r)^{\varepsilon p - N + (N-1)/q}\omega(1-r)^{-p}$$

for all $1/2 < r < 1$ and nonnegative measurable functions f on $\mathbf{B}$ with $\|F\|_{M^{\Phi,\omega}(\mathbf{B})} \le 1$, where $F(y) = (1-|y|)f(y)$.

(2) *Suppose $\varepsilon > 0$ and*

$$(N-1)/q > N - \varepsilon(p-1).$$

Then there exists a constant $C > 0$ such that

$$S_q(\Phi((1-r)^{-1+\varepsilon}G_2), r) \le C(1-r)^{\varepsilon}\omega(1-r)^{-p}$$

for all $1/2 < r < 1$ and nonnegative measurable functions f on $\mathbf{B}$ with $\|F\|_{M^{\Phi,\omega}(\mathbf{B})} \le 1$, where $F(y) = (1-|y|)f(y)$.

Proof. Let $\varepsilon > 0$ such that

$$\varepsilon(p-1) - N + (N-1)/q < 0.$$

For $1/2 < r = |x| < 1$, we have

$$\begin{aligned} G_2(x) &\le C\int_{\mathbf{B}} (1-|x|)(1-|y|)|x-y|^{-N} f_{2,x}(y)\,dy \\ &\le C(1-|x|)\int_{(1-|x|)/2}^{2} \left(\frac{1}{|B(x,t)|}\int_{B(x,t)} (1-|y|)f_{2,x}(y)\,dy\right) t^{-1}\,dt \\ &\le C(1-|x|)\int_{(1-|x|)/2}^{2} \left(\frac{1}{|B(x,t)|}\int_{B(x,t)} t^{\varepsilon}(1-|y|)f_{x,2}(y)\,dy\right) t^{-\varepsilon-1}\,dt, \end{aligned}$$

where $f_{2,x}(y) = f(y)\chi_{E_{2,x}}(y)$ with $E_{2,x} = \{y \in \mathbf{B}\setminus B(x,(1-|x|)/2)\colon 1-|y| \le 1-|x|\}$.

We have by Jensen's inequality and ($\Phi 3$)

$$\begin{aligned} &\Phi((1-|x|)^{-1+\varepsilon}G_2(x)) \\ &\le C(1-|x|)^{\varepsilon}\int_{(1-|x|)/2}^{2} \left(\frac{1}{|B(x,t)|}\int_{B(x,t)} \Phi(t^{\varepsilon}(1-|y|)f_{2,x}(y))\,dy\right) t^{-\varepsilon-1}\,dt \\ &\le C(1-|x|)^{\varepsilon}\int_{(1-|x|)/2}^{2} t^{\varepsilon p}\left(\frac{1}{|B(x,t)|}\int_{B(x,t)} \Phi((1-|y|)f_{2,x}(y))\,dy\right) t^{-\varepsilon-1}\,dt \\ &\le C(1-|x|)^{\varepsilon}\int_{\mathbf{B}} |x-y|^{\varepsilon(p-1)-N}\Phi((1-|y|)f_{2,x}(y))\,dy. \end{aligned}$$

Hence Minkowski's inequality, Lemma 2.2 and (Φ3) yield

$$\begin{aligned}
&S_q(\Phi((1-r)^{-1+\varepsilon}G_2), r) \\
&\le C(1-r)^{\varepsilon}\int_{\mathbf{B}} S_q(|\cdot - y|^{\varepsilon(p-1)-N}\chi_{E_{2,x}}(y), r)\Phi((1-|y|)f_{2,x}(y))\,dy \\
&\le C(1-r)^{\varepsilon}(1-r)^{\varepsilon(p-1)-N+(N-1)/q}\int_{\mathbf{B}\setminus B(0,r)} \Phi(F(y))\,dy \\
&\le C(1-r)^{\varepsilon p-N+(N-1)/q}\omega(1-r)^{-p}
\end{aligned}$$

since $\varepsilon(p-1)-N+(N-1)/q<0$, which gives assertion (1).

To show assertion (2), suppose $\varepsilon>0$ such that $\varepsilon(p-1)-N<0$ and $\varepsilon(p-1)-N+(N-1)/q>0$. Then we have by Lemma 2.2

$$\begin{aligned}
&S_q(\Phi((1-r)^{-1+\varepsilon}G_2), r) \\
&\le C(1-r)^{\varepsilon}\int_{\mathbf{B}} S_q(|\cdot - y|^{\varepsilon(p-1)-N}\chi_{E_{2,x}}(y), r)\Phi((1-|y|)f_{2,x}(y))\,dy \\
&\le C(1-r)^{\varepsilon}\int_{\mathbf{B}\setminus B(0,r)} \Phi(F(y))\,dy \le C(1-r)^{\varepsilon}\omega(1-r)^{-p}
\end{aligned}$$

since $\varepsilon(p-1)-N+(N-1)/q>0$.

When $\varepsilon>0$, $\varepsilon(p-1)-N\ge 0$ and $\varepsilon(p-1)-N+(N-1)/q>0$, taking $0<\delta<(N-1)/q$, we have

$$\begin{aligned}
&\Phi((1-|x|)^{-1+\varepsilon}G_2(x)) \\
&\le C(1-|x|)^{\varepsilon}\int_{(1-|x|)/2}^{2} t^{\varepsilon p}\left(\frac{1}{|B(x,t)|}\int_{B(x,t)} \Phi((1-|y|)f_{2,x}(y))\,dy\right)t^{-\varepsilon-1}\,dt \\
&\le C(1-|x|)^{\varepsilon}\int_{\mathbf{B}} |x-y|^{-\delta}\Phi((1-|y|)f_{2,x}(y))\,dy
\end{aligned}$$

and

$$\begin{aligned}
S_q(\Phi((1-r)^{-1+\varepsilon}G_2), r) &\le C(1-r)^{\varepsilon}\int_{\mathbf{B}} S_q(|\cdot - y|^{-\delta}\chi_{E_{2,x}}(y), r)\Phi(F(y))\,dy \\
&\le C(1-r)^{\varepsilon}\int_{\mathbf{B}\setminus B(0,r)} \Phi(F(y))\,dy \le C(1-r)^{\varepsilon}\omega(1-r)^{-p},
\end{aligned}$$

which completes the proof of assertion (2). □

Lemma 5.2. *Let $1\le q<\infty$.*

(1) *Suppose*

(ω5) $t^{\varepsilon_0-N+(N-1)/q}\omega(t)^{-p}$ *is almost decreasing on $(0,1)$ for some $\varepsilon_0>0$.*

Let $0<\varepsilon<\varepsilon_0/(p-1)$. Then there exists a constant $C>0$ such that

$$S_q(\Phi((1-r)^{-1+\varepsilon}G_3), r) \le C(1-r)^{\varepsilon p-N+(N-1)/q}\omega(1-r)^{-p}$$

for all $1/2<r<1$ and nonnegative measurable functions f on $\mathbf{B}$ with $\|F\|_{M^{\Phi,\omega}(\mathbf{B})}\le 1$, where $F(y)=(1-|y|)f(y)$.

(2) *Suppose $\varepsilon>0$ and*

$$(N-1)/q > N-\varepsilon(p-1).$$

Then there exists a constant $C>0$ such that

$$S_q(\Phi((1-r)^{-1+\varepsilon}G_3), r) \le C(1-r)^{\varepsilon}$$

for all $1/2 < r < 1$ *and nonnegative measurable functions* f *on* $\mathbf{B}$ *with* $\|F\|_{M^{\Phi,\omega}(\mathbf{B})} \le 1$, *where* $F(y) = (1-|y|)f(y)$.

Proof. First note from (ω5) and $0 < \varepsilon < \varepsilon_0/(p-1)$ that $t^{\varepsilon(p-1)-N+(N-1)/q}\omega(t)^{-p}$ is almost decreasing on $(0,1)$ and

$$\varepsilon(p-1) - N + (N-1)/q < 0.$$

In the same way as above, we obtain

$$S_q(\Phi((1-r)^{-1+\varepsilon}G_3), r) \le C(1-r)^{\varepsilon}\left(\int_{B(0,r)} (1-|y|)^{\varepsilon(p-1)-N+(N-1)/q}\Phi(F(y))\,dy + 1\right).$$

Let j_0 be the smallest integer such that $r \le 1 - 2^{-j_0-1}$. Note here that

$$\begin{aligned}
&\int_{B(0,r)} (1-|y|)^{\varepsilon(p-1)-N+(N-1)/q}\Phi(F(y))\,dy \\
&\le \sum_{j=0}^{j_0} \int_{A(0,1-2^{-j})} (1-|y|)^{\varepsilon(p-1)-N+(N-1)/q}\Phi(F(y))\,dy \\
&\le C\sum_{j=0}^{j_0} 2^{-j(\varepsilon(p-1)-N+(N-1)/q)} \int_{A(0,1-2^{-j})} \Phi(F(y))\,dy \\
&\le C\sum_{j=0}^{j_0} 2^{-j(\varepsilon(p-1)-N+(N-1)/q)}\omega(2^{-j})^{-p} \\
&\le C(1-r)^{\varepsilon(p-1)-N+(N-1)/q}\omega(1-r)^{-p}
\end{aligned}$$

by (ω5), which gives assertion (1).

Assertion (2) is proved as in the proof of Lemma 2.7 (2). □

Theorem 5.3. *Let* $1 \le q < \infty$.

(1) *Suppose* (ω5) *holds for some* $\varepsilon_0 > 0$. *If* $0 < \varepsilon < \min\{1, \varepsilon_0/(p-1)\}$ *and*

$$N - 2p - 1 + \varepsilon(p-1) < (N-1)/q < N - 2p - \varepsilon(p-1),$$

then there exists a constant $C > 0$ *such that*

$$\liminf_{r\to 1-} (1-r)^{N-2p-(N-1)/q+\varepsilon p}\omega(1-r)^p S_q(\Phi((1-r)^{1-\varepsilon}Gf), r) \le C$$

for all nonnegative measurable functions f *with* $\|F\|_{M^{\Phi,\omega}(\mathbf{B})} \le 1$, *where* $F(y) = (1-|y|)f(y)$.

(2) *Suppose* (ω5) *holds for some* $\varepsilon_0 > 0$. *If* $0 < \varepsilon < \min\{1, \varepsilon_0/(p-1)\}$, $2p - N - \varepsilon(p-1) < 0$ *and*

$$N - 2p - \varepsilon(p-1) < (N-1)/q < N - \varepsilon(p-1),$$

then there exists a constant $C > 0$ *such that*

$$(1-r)^{N-2p-(N-1)/q+\varepsilon p}\omega(1-r)^p S_q(\Phi((1-r)^{1-\varepsilon}Gf), r) \le C$$

for all $1/2 < r < 1$ *and nonnegative measurable functions* f *with* $\|F\|_{M^{\Phi,\omega}(\mathbf{B})} \le 1$, *where* $F(y) = (1-|y|)f(y)$.

(3) *If* $0 < \varepsilon < 1$, $2p - N < \varepsilon(p-1)$ *and* $(N-1)/q > N - \varepsilon(p-1)$, *then there exists a constant* $C > 0$ *such that*

$$(1-r)^{-2(1-\varepsilon)p-\varepsilon} S_q(\Phi((1-r)^{1-\varepsilon}Gf), r) \le C$$

for all $1/2 < r < 1$ and all nonnegative measurable functions f with $\|F\|_{M^{\Phi,\omega}(\mathbf{B})} \le 1$, where $F(y) = (1-|y|)f(y)$.

Proof. Let f be a nonnegative measurable function in $M^{\Phi,\omega}(\mathbf{B})$. For $x \in \mathbf{B}$, write

$$Gf(x) = G_1(x) + G_2(x) + G_3(x)$$

as before.

Let $0 < \varepsilon < \min\{1, \varepsilon_0/(p-1)\}$ such that

$$-1 < (2-\varepsilon)p + \varepsilon - N + (N-1)/q < (2+\varepsilon)p - \varepsilon - N + (N-1)/q < 0.$$

Set

$$d = \varepsilon(p-1) - (N-1)/q + N - 2p.$$

Then $0 < d < 1$. Since $\varepsilon(p-1) - N + (N-1)/q < \varepsilon_0 - N + (N-1)/q \le 0$ by $(\omega 5)$, one notes by Lemmas 5.1 (1) , 5.2 (1) and $(\Phi 3)$ that

$$\begin{aligned}
&(1-r)^{N-2p+\varepsilon p-(N-1)/q}\omega(1-r)^p S_q(\Phi((1-r)^{1-\varepsilon}G_2), r)\\
&\le C(1-r)^{N-2p+\varepsilon p-(N-1)/q}\omega(1-r)^p (1-r)^{2(1-\varepsilon)p} S_q(\Phi((1-r)^{-1+\varepsilon}G_2), r)\\
&\le C(1-r)^{N-\varepsilon p-(N-1)/q}\omega(1-r)^p S_q(\Phi((1-r)^{-1+\varepsilon}G_2), r) \le C
\end{aligned}$$

and

$$(1-r)^{N-2p+\varepsilon p-(N-1)/q}\omega(1-r)^p S_q(\Phi((1-r)^{1-\varepsilon}G_3), r) \le C.$$

Next we see that $G_1(x) \le \int_{B(x,(1-|x|)/2)} |x-y|^{2-N} f(y)\,dy$, and hence

$$(1-|x|)G_1(x) \le C\int_{B(x,(1-|x|)/2)} |x-y|^{2-N}(1-|y|)f(y)\,dy.$$

By Lemma 2.4 (1) with $\alpha = 2$, we obtain

$$S_q(\Phi(\omega(1-r)(1-r)^{1-\varepsilon}G_1), r) \le C(1-r)^{-\varepsilon}\int_{A(0,r)} |r-|y||^{-d} g(y)\,dy,$$

where $g(y) = \Phi(\omega(1-|y|)F(y))$ with $F(y) = (1-|y|)f(y)$. Therefore we establish by $(\Phi 4)$

$$\begin{aligned}
&(1-r)^{N-2p+\varepsilon p-(N-1)/q}\omega(1-r)^p S_q(\Phi((1-r)^{1-\varepsilon}G_1), r)\\
&\le C(1-r)^{N-2p+\varepsilon p-(N-1)/q} S_q(\Phi(\omega(1-r)(1-r)^{1-\varepsilon}G_1), r)\\
&\le Ct^d \int_{A(0,r)} |r-|y||^{-d} g(y)\,dy.
\end{aligned}$$

In view of Lemma 2.5, we can find a sequence $\{r_j\}$ of positive numbers such that $1 - 2^{-j+1} < r_j < 1 - 2^{-j}$ and

$$\sup_j\, (1-r_j)^{N-2p+\varepsilon p-(N-1)/q}\omega(1-r_j)^p S_q(\Phi((1-r_j)^{1-\varepsilon}G_1), r_j) \le C,$$

which proves assertion (1).

To show assertion (2), suppose $0 < \varepsilon < \min\{1, \varepsilon_0/(p-1)\}$, $2p - N - \varepsilon(p-1) < 0$ and

$$\varepsilon(p-1) + N - 2p < (N-1)/q < N - \varepsilon(p-1).$$

Then, for $1/2 < r < 1$, we see from Lemmas 5.1 (1), 5.2 (1) and $(\Phi 3)$ that

$$(1-r)^{N-2p+\varepsilon p-(N-1)/q}\omega(1-r)^p S_q(\Phi((1-r)^{1-\varepsilon}G_2), r) \le C$$

and

$$(1-r)^{N-2p+\varepsilon p-(N-1)/q}\omega(1-r)^p S_q(\Phi((1-r)^{1-\varepsilon}G_3),r) \le C,$$

as above. By Lemma 2.4 (2) with $\alpha = 2$, we obtain

$$\begin{aligned}&(1-r)^{N-2p+\varepsilon p-(N-1)/q}\omega(1-r)^p S_q(\Phi((1-r)^{1-\varepsilon}G_1),r)\\ &\le C(1-r)^{N-2p+\varepsilon p-(N-1)/q} S_q(\Phi(\omega(1-r)(1-r)^{1-\varepsilon}G_1),r) \le C,\end{aligned}$$

which proves assertion (2).

For a proof of (3), suppose $0 < \varepsilon < 1$, $2p - N < \varepsilon(p-1)$ and $(N-1)/q > N - \varepsilon(p-1)$. Then Lemmas 5.1 (2), 5.2 (2) yield

$$\begin{aligned}&(1-r)^{-2(1-\varepsilon)p-\varepsilon} S_q(\Phi((1-r)^{1-\varepsilon}G_2),r)\\ &\le C(1-r)^{-2(1-\varepsilon)p-\varepsilon}\omega(1-r)^p(1-r)^{2(1-\varepsilon)p} S_q(\Phi((1-r)^{-1+\varepsilon}G_2),r)\\ &\le C(1-r)^{-\varepsilon}\omega(1-r)^p S_q(\Phi((1-r)^{-1+\varepsilon}G_2),r) \le C\end{aligned}$$

and

$$(1-r)^{-2(1-\varepsilon)p-\varepsilon} S_q(\Phi((1-r)^{1-\varepsilon}G_3),r) \le C$$

for all $1/2 < r < 1$. Further we see from Lemma 2.4 (2) with $\alpha = 2$ that

$$\begin{aligned}&(1-r)^{-2(1-\varepsilon)p-\varepsilon} S_q(\Phi((1-r)^{1-\varepsilon}G_1),r)\\ &\le C(1-r)^{-2(1-\varepsilon)p-\varepsilon} S_q(\Phi((1-r)^{1-\varepsilon}\omega(1-r)G_1),r)\\ &\le C(1-r)^{\varepsilon(p-1)-N+(N-1)/q} \le C\end{aligned}$$

since

$$(N-1)/q > N - \varepsilon(p-1) > N - 2p + \varepsilon(p-1)$$

by $\varepsilon < 1 < p/(p-1)$. Hence we obtain assertion (3). □

We can prove the following result in the same way as Corollary 3.7.

Theorem 5.4. *Let* $1 \le q < \infty$ *and* f *be a nonnegative measurable function such that* $F \in M_0^{\Phi,\omega}(\mathbf{B})$, *where* $F(y) = (1-|y|)f(y)$.

(1) *Suppose* (ω5) *holds for some* $\varepsilon_0 > 0$. *If* $0 < \varepsilon < \min\{1, \varepsilon_0/(p-1)\}$ *and*

$$N - 2p - 1 + \varepsilon(p-1) < (N-1)/q < N - 2p - \varepsilon(p-1),$$

then

$$\liminf_{r\to 1-} (1-r)^{N-2p-(N-1)/q+\varepsilon p}\omega(1-r)^p S_q(\Phi((1-r)^{1-\varepsilon}Gf),r) = 0.$$

(2) *Suppose* (ω5) *holds for some* $\varepsilon_0 > 0$. *If* $0 < \varepsilon < \min\{1, \varepsilon_0/(p-1)\}$, $2p - N - \varepsilon(p-1) < 0$ *and*

$$N - 2p - \varepsilon(p-1) < (N-1)/q < N - \varepsilon(p-1),$$

then

$$\lim_{r\to 1-} (1-r)^{N-2p-(N-1)/q+\varepsilon p}\omega(1-r)^p S_q(\Phi((1-r)^{1-\varepsilon}Gf),r) = 0.$$

(3) *If* $0 < \varepsilon < 1$, $2p - N < \varepsilon(p-1)$ *and* $(N-1)/q > N - \varepsilon(p-1)$, *then*

$$\lim_{r\to 1-} (1-r)^{-2(1-\varepsilon)p-\varepsilon} S_q(\Phi((1-r)^{1-\varepsilon}Gf),r) = 0.$$

Remark 5.5. Gardiner [4] proved that for a Green potential $G\mu$ in $\mathbf{B}$

(1) when $(N-1)/(N-2) \le q < (N-1)/(N-3)$,

$$\liminf_{r\to 1-} (1-r)^{N-1-(N-1)/q} S_q(G\mu, r) = 0;$$

(2) when $1 \le q < (N-1)/(N-2)$,

$$\lim_{r\to 1-} (1-r)^{N-1-(N-1)/q} S_q(G\mu, r) = 0.$$

To obtain this result, we need modify Theorem 5.3 as in Corollary 3.7.

6. Monotone functions

A continuous function u is said to be monotone in Ω in the sense of Lebesgue [7], if for every relatively compact subdomain G of Ω we have

$$\max_{\bar{G}} u = \max_{\partial G} u \quad \text{and} \quad \min_{\bar{G}} u = \min_{\partial G} u.$$

For monotone functions, see Koskela–Manfredi–Villamor [6], Manfredi–Villamor [9, 10], the first author [14, 15], Villamor–Li [24] and Vuorinen [25, 26].

Theorem 6.1. *Let $p_1 > N-1$ and $p_1 \le q < \infty$. Suppose*

(ω6) $t^{\varepsilon_0-(N-p_1-1)/p_1+(N-1)/q}\Phi^{-1}\left(t^{-1}\omega(t)^{-p}\right)^{1/p_1}$ *is almost decreasing in* $(0,1)$ *for some* $\varepsilon_0 > 0$.

Then there exists a constant $C > 0$ such that

$$\limsup_{r\to 1-}(1-r)^{(N-p_1-1)/p_1-(N-1)/q}\Phi^{-1}\left((1-r)^{-1}\omega(1-r)^{-p}\right)^{-1/p_1} S_q(u,r) \le C$$

for all monotone functions u on $\mathbf{B}$ such that $\|h\|_{M^{\Phi,\omega}(\mathbf{B})} \le 1$, where $h(y) = |\nabla u(y)|^{p_1}$.

For a proof of Theorem 6.1, we need the following result, which gives an essential tool in treating monotone functions.

Lemma 6.2. (cf. [9, 10, 15]) *Let $p_1 > N-1$. If u is a monotone Sobolev function on $B(x_0, 2r)$, then*

$$(6.1) \quad |u(x)-u(y)|^{p_1} \le M r^{p_1-N}\int_{B(x_0,2r)} |\nabla u(z)|^{p_1}\,dz \quad \text{whenever } x,\ y \in B(x_0,r).$$

Lemma 6.2 is a consequence of Sobolev's theorem, so that the restriction $p_1 > N-1$ is needed; for a proof of Lemma 6.2, see for example [9] or [15, Theorem 5.2, Chap. 8].

Now we are ready to prove Theorem 6.1, along the same lines as in the proof of [17, Theorem 2].

Proof. Let u be a monotone function on $\mathbf{B}$ such that $\|h\|_{M^{\Phi,\omega}(\mathbf{B})} \le 1$ with $p_1 > N-1$, where $h(y) = |\nabla u(y)|^{p_1}$. Let $r_j = 2^{-j-1}$ and $t_j = 1-r_{j-1}$ for $j = 1, 2, \ldots$. Using (6.1), we obtain from the proof of [17, Theorem 2] that

$$\begin{aligned}&|S_q(u,t_j) - S_q(u,t_{j+m})|\\ &\le C\sum_{\ell=j}^{j+m} r_\ell^{-(N-p_1-1)/p_1+(N-1)/q}\left(r_\ell^{-1}\int_{B(0,1-r_\ell)\setminus B(0,1-3r_\ell)} |\nabla u(y)|^{p_1}\,dy\right)^{1/p_1}.\end{aligned}$$

Hence, we have by (ω6) and (Φ3)

$$\begin{aligned}&|S_q(u,t_j) - S_q(u,t_{j+m})|\\ &\le C\sum_{\ell=j}^{j+m} r_\ell^{-(N-p_1-1)/p_1+(N-1)/q}\Phi^{-1}\left(r_\ell^{-1}\int_{B(0,1-r_\ell)\setminus B(0,1-3r_\ell)} \Phi\left(|\nabla u(y)|^{p_1}\right)\,dy\right)^{1/p_1}\end{aligned}$$

$$\leq C \sum_{\ell=j}^{j+m} r_\ell^{-(N-p_1-1)/p_1+(N-1)/q} \Phi^{-1}\left(r_\ell^{-1}\omega(r_\ell)^{-p}\right)^{1/p_1}$$
$$\leq C r_{j+m}^{-(N-p_1-1)/p_1+(N-1)/q} \Phi^{-1}\left(r_{j+m}^{-1}\omega(r_{j+m})^{-p}\right)^{1/p_1}.$$

If $t_j \leq r < 1$, then we take m such that $t_{j+m-1} \leq r < t_{j+m}$ and establish

$$|S_q(u,t_j) - S_q(u,r)| \leq C(1-r)^{-(N-p_1-1)/p_1+(N-1)/q} \Phi^{-1}\left((1-r)^{-1}\omega(1-r)^{-p}\right)^{1/p_1}.$$

Therefore it follows from (ω6) that

$$\limsup_{r\to 1-} (1-r)^{(N-p_1-1)/p_1-(N-1)/q} \Phi^{-1}\left((1-r)^{-1}\omega(1-r)^{-p}\right)^{-1/p_1} S_q(u,r) \leq C,$$

as required. □

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