Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 43, 2018, 89–107

BOUNDEDNESS OF VOLTERRA OPERATORS ON SPACES OF ENTIRE FUNCTIONS

Oscar Blasco

Universidad de Valencia, Departamento de Análisis Matemático
46100 Burjassot, Valencia, Spain; oblasco@uv.es

Abstract. In this paper we find some necessary and sufficient conditions on an entire function g for the Volterra operator $V_g(f)(z) = \int_0^z f(\xi)g'(\xi)\,d\xi$ to be bounded between different weighted spaces of entire functions $H_v^\infty(\mathbf{C})$ or Fock-type spaces $\mathcal{F}_p^\phi(\mathbf{C})$.

1. Introduction

Let Ω be the unit disc $\mathbf{D}$ or the complex plane $\mathbf{C}$ and, as usual, denote by $\mathcal{H}(\Omega)$ the space of holomorphic functions in Ω. Given $g \in \mathcal{H}(\Omega)$ the Volterra operator with symbol g, to be denoted by V_g, is defined by

$$V_g(f)(z) = \int_0^z f(\xi)g'(\xi)\,d\xi, \quad z \in \Omega,\ f \in \mathcal{H}(\Omega).$$

In the case $\Omega = \mathbf{D}$, this operator was first introduced by Pommerenke [20]. He showed that it is bounded on the Hardy space $H^2(\mathbf{D})$ if and only if $g \in BMOA$. A bit later the result was extended to $H^p(\mathbf{D})$ for any $1 \le p < \infty$ by Aleman and Siskakis [1, 4]. In particular, they showed that, for $1 \le p < \infty$,

$$\|V_g(f)\|_{H^p} \le C_p \|g\|_{BMOA} \|f\|_{H^p}, \quad f \in H^p(\mathbf{D}), \tag{1}$$

for a constant $C_p > 0$ depending only on p. The boundedness, compactness and other properties of V_g acting on spaces of holomorphic functions defined in the unit disc have been deeply studied (see [5] for weighted Bergman spaces, [6, 15] for weighted spaces of holomorphic functions $H_v^\infty(\mathbf{D})$ and [17, 19] for several other spaces). The reader is also referred to [2, 3] for different results concerning the spectra of the Volterra operator in some cases.

In this article we are only concerned with spaces of entire functions. Throughout the paper we write $\mathcal{P}$ for the space of polynomials (with the notation $u_n(z) = z^n$) and $\mathcal{H}_0(\mathbf{C})$ for the space of entire functions vanishing at the origin. For each $0 < p < \infty$, $0 < r < \infty$ and $f \in \mathcal{H}(\mathbf{C})$ we write $M_\infty(f,r) = \sup_{|z|=r} |f(z)|$ and $M_p(f,r) = \left(\int_0^{2\pi} |f(re^{it})|^p \frac{dt}{2\pi}\right)^{1/p}$.

Given $0 < p < \infty$ and a measurable function $\phi\colon (0,\infty) \to \mathbf{R}$, we denote by $\mathcal{F}_p^\phi(\mathbf{C})$ the space of entire functions f such that $\int_{\mathbf{C}} |f(z)|^p e^{-p\phi(|z|)}\,dm(z) < \infty$ and we write

$$\|f\|_{\mathcal{F}_p^\phi} = (2\pi)^{1/p} \left(\int_0^\infty M_p^p(f,r) r e^{-p\phi(r)}\,dr\right)^{1/p}.$$

https://doi.org/10.5186/aasfm.2018.4303
2010 Mathematics Subject Classification: Primary 47G10; Secondary 30H20, 47B07, 47B37, 47B38, 30D15, 30D20, 46E15.
Key words: Volterra operator, weighted spaces, entire function, Fock-type spaces.
The author is partially supported by the Project MTM2014-53009-P(MINECO Spain).

The classical Fock spaces $\mathcal{F}_p(\mathbf{C})$ correspond to $\phi(z) = \frac{|z|^2}{2}$.

For the limiting case $\mathcal{F}^\phi_\infty(\mathbf{C})$ we shall also use the standard notation $H^\infty_v(\mathbf{C})$ where $v(z) = e^{-\phi(|z|)}$, that is the space of entire functions f such that

$$\|f\|_{\mathcal{F}^\phi_\infty} = \|f\|_v = \sup_{r\geq 0} e^{-\phi(r)} M_\infty(f,r) < \infty.$$

As usual $H^0_v(\mathbf{C})$ denotes the subspace of $H^\infty_v(\mathbf{C})$ of functions such that $\lim_{|z|\to\infty} v(|z|)|f(z)| = 0$.

It is well known that we can change the values of ϕ or v in a bounded interval $[0, R_0]$ and even that we can replace ϕ for another weight φ being continuous and increasing so that $H^\infty_w(\mathbf{C}) = H^\infty_v(\mathbf{C})$ and $\mathcal{F}^\varphi_p(\mathbf{C}) = \mathcal{F}^\phi_p(\mathbf{C})$ with equivalent norms. Since we are only interested in spaces containing the polynomials, that is $\mathcal{P} \subset H^0_v(\mathbf{C})$ or $\mathcal{P} \subset \bigcap_{p>0} \mathcal{F}^\phi_p(\mathbf{C})$, we shall impose the following assumptions on the weights:

$$\lim_{r\to\infty} r^m v(r) = 0, \quad \forall m \in \mathbf{N}, \tag{2}$$

or

$$\int_0^\infty r^m e^{-p\phi(r)}\,dr < \infty, \quad \forall m \in \mathbf{N},\ \forall p > 0. \tag{3}$$

Due to the above considerations we introduce the following definition.

Definition 1.1. We write $\mathcal{W}$ for the class of functions $\phi\colon [0,\infty) \to \mathbf{R}$ which are continuous, increasing in $[r_\phi, \infty)$ for some $r_\phi > 0$ and for each $m \in \mathbf{N}$ satisfy

$$\sup_{r>0} r^m e^{-\phi(r)} < \infty. \tag{4}$$

Notice that conditions (2), (3) and (4) are in fact equivalent. Examples of weights in $\mathcal{W}$ to have in mind are $\varphi_{\alpha,\beta,\gamma}(r) = \beta r^\alpha - \gamma \log r$ for $\alpha, \beta > 0$ and $\gamma \geq 0$.

The study of the Volterra operator on certain spaces of entire functions was initiated by Constantin in [11]. She characterized continuity (and compactness) of V_g on the classical Fock spaces.

Theorem 1.1. [11, Theorem 1] *Let $0 < p, q < \infty$ and $0 \neq g \in \mathcal{H}_0(\mathbf{C})$.*

(i) *Case $0 < p \leq q$: V_g is bounded from $\mathcal{F}_p(\mathbf{C})$ into $\mathcal{F}_q(\mathbf{C})$ if and only if $g(z) = az^2 + bz$ for some $a, b \in \mathbf{C}$.*
(ii) *Case $q < p$: V_g is bounded from $\mathcal{F}_p(\mathbf{C})$ into $\mathcal{F}_q(\mathbf{C})$ if and only if $\frac{1}{q} - \frac{1}{p} < \frac{1}{2}$ and $g(z) = az$ for some $0 \neq a \in \mathbf{C}$.*

Later in collaboration with Peláez [12] the results were extended to a class of Fock-type spaces $\mathcal{F}^\phi_p(\mathbf{C})$ defined by certain smooth radial weights ϕ. In [12] certain class $\mathcal{I}$ of twice differentiable and rapidly increasing weights was introduced. This class includes examples such as $\phi(r) = r^\alpha$ for $\alpha > 2$, $\phi(r) = e^{\beta r}$ for $\beta > 0$ or $\phi(r) = e^{e^r}$. For weights in this class they obtained the complete characterization of the symbols g which produce bounded Volterra operators V_g acting from $\mathcal{F}^\phi_p(\mathbf{C})$ into $\mathcal{F}^\phi_q(\mathbf{C})$ (see [12, Theorem 3]). In particular for $p = q$ they showed that for $0 \neq g \in \mathcal{H}(\mathbf{C})$ and $\phi \in \mathcal{I}$, the Volterra operator V_g is bounded on $\mathcal{F}^\phi_p(\mathbf{C})$ if and only if

$$\sup_{z\in\mathbf{C}} \frac{|g'(z)|}{1 + \phi'(|z|)} < \infty. \tag{5}$$

Also they generalized Theorem 1.1 as follows:

Theorem 1.2. [12, Corollary 25] *Let* $0 < p, q < \infty$, $0 \neq g \in \mathcal{H}_0(\mathbf{C})$ *and* $\phi(r) = r^\alpha$ *with* $\alpha > 2$.

(i) *Case* $0 < p \leq q$ *and* $1+(\alpha-2)(1-\frac{1}{p}+\frac{1}{q}) \geq 0$: V_g *is bounded from* $\mathcal{F}_p^\phi(\mathbf{C})$ *into* $\mathcal{F}_q^\phi(\mathbf{C})$ *if and only if* g *is a polynomial with* $\deg(g) \leq 2 + (\alpha-2)(1-\frac{1}{p}+\frac{1}{q})$.

(ii) *Case* $q < p$: V_g *is bounded from* $\mathcal{F}_p^\phi(\mathbf{C})$ *into* $\mathcal{F}_q^\phi(\mathbf{C})$ *if and only if* $\frac{1}{q} - \frac{1}{p} < \frac{\alpha-1}{2}$ *and* g *is a polynomial with* $\deg(g) < \alpha - 2(1-\frac{1}{p}+\frac{1}{q})$.

The study for $\mathcal{F}_\infty^\phi(\mathbf{C}) = H_v^\infty(\mathbf{C})$ was considered by Bonet and Taskinen [9] for certain classes of radial weights v. We refer also the interested reader to [8, 11, 13] for results concerning the spectra of the Volterra operator in this setting. In [9] certain class of weights $\mathcal{J}$ (see conditions appearing in [9, Proposition 3.2]) was introduced. This class includes examples such as $\psi(r) = \beta r^\alpha - \gamma \log r - \delta \log(\log(1+r))$, for some $\alpha, \beta > 0, \gamma, \delta \in \mathbf{R}$, $\psi(r) = (\log(1+r))^{1+\epsilon} - \gamma \log r - \delta \log(\log(1+r))$, for some $\epsilon > 0, \gamma, \delta \in \mathbf{R}$ or, more generally twice differentiable weights satisfying certain conditions (see [9, Thm 3.6, Thm 3.7]). For such a class, using the notation $\tilde{v}(z)$ for the so-called associate weight of v (see [10]), they obtained (see [9, Theorem 3.4]) that for $0 \neq g \in \mathcal{H}(\mathbf{C})$, $v(z) = e^{-\phi(|z|)}$ and $w(z) = e^{-\psi(|z|)}$ with $\psi \in \mathcal{J}$, the boundedness of V_g from $H_v^\infty(\mathbf{C})$ into $H_w^\infty(\mathbf{C})$ is equivalent to the condition

$$\sup_{z\in\mathbf{C}} \frac{|g'(z)|w(z)}{\psi'(|z|)\tilde{v}(z)} < \infty. \tag{6}$$

As a consequence they established the following theorem.

Theorem 1.3. [9, Corollary 3.11] *Let* $v(r) = e^{-\beta r^\alpha}$ *for* $\beta > 0$ *and* $\alpha \geq 1$ *and let* $0 \neq g \in \mathcal{H}_0(\mathbf{C})$. *Then* V_g *is bounded on* $H_v^\infty(\mathbf{C})$ *if and only* g *is a polynomial of* $\deg(g) \leq [\alpha]$, *where* $[a]$ *stands for the integer part of* $a > 0$.

Observe that $V_g = \mathcal{I}\mathcal{M}_{g'}$ where $\mathcal{M}_{g'}(f) = fg'$ and $\mathcal{I}(f)(z) = \int_0^z f(\xi)\,d\xi$. All the previous results are obtained analyzing the action of $\mathcal{M}_{g'}$ and $\mathcal{I}$ on the corresponding spaces independently, and using the equivalent definition of the norm of f in the spaces $H_v^\infty(\mathbf{C})$ and $\mathcal{F}_p^\phi(\mathbf{C})$ in terms of the derivative f' (see [9, Proposition 3.2]) or Littlewood–Paley formula (see [12, Theorem 10]) respectively.

In this paper we would like to attack the boundedness of the Volterra operator V_g (and certain modification of it) directly and not relying on the boundedness of the multiplication or differentiation operators independently. Note that the results in [12] do not apply to $\phi(r) = r^\alpha$ for $0 < \alpha \leq 2$ and not cover different weights ϕ and ψ and the results in [9] cover different weights but only for $p = q = \infty$. We shall present here some necessary and sufficient conditions for the boundedness of V_g from $\mathcal{F}_p^\phi(\mathbf{C})$ into $\mathcal{F}_q^\psi(\mathbf{C})$ for different parameters $0 < p, q \leq \infty$ and different weights ϕ and ψ belonging to $\mathcal{W}$, extending and providing some alternative proofs of some results in [9, 11, 12].

Besides the introduction the paper is divided into four sections. The first section contains some results on the class $\mathcal{W}$ while the second one is devoted to some preliminaries on the Volterra operator V_g and its modification $\tilde{V}_g(f)(z) = \frac{1}{z}\int_0^z f(\xi)Dg(\xi)\,d\xi$ where $Dg(z) = g(z) + zg'(z)$. The main contributions are in the last sections where some necessary and sufficient conditions for the boundedness of V_g and $\tilde{V}_g$ on weighted spaces of holomorphic functions and Fock-type spaces and their applications are provided. It will be shown (see Corollary 4.6) that the existence of a function $g \neq 0$ such that V_g is bounded $\mathcal{F}_p^\phi(\mathbf{C})$ into $\mathcal{F}_q^\psi(\mathbf{C})$ implies that V_{u_k} is also bounded $\mathcal{F}_p^\phi(\mathbf{C})$ into $\mathcal{F}_q^\psi(\mathbf{C})$ for all $k \in \mathbf{N}$ such that $g^{(k)}(0) \neq 0$. This forces some relationship between p,

q, ϕ and ψ. In particular we will show that there is no entire function $0 \neq g \in \mathcal{H}_0(\mathbf{C})$ such that V_g maps boundedly $H^\infty_{v_1}(\mathbf{C})$ into $H^\infty_{v_2}(\mathbf{C})$ for $v_i = e^{-\varphi_{\alpha_i,\beta_i,\gamma_i}}$ for $i = 1, 2$ whenever $\alpha_1 > \alpha_2$ or $\alpha_1 = \alpha_2$ and $\beta_1 > \beta_2$ or $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$ and $\alpha_1 - \gamma_2 + \gamma_1 < 1$ (this actually explains the restriction $\alpha \geq 1$ in Theorem 1.3). Moreover once such a function exists it must be a polynomial of degree less or equal than $\alpha_1 - \gamma_2 + \gamma_1$. In order to provide some sufficient conditions for the boundedness of V_g for different weights we shall introduce a function inspired by the so-called distortion function of ϕ considered in [12]. For each $0 < p < \infty$ and weight ϕ the authors considered the function $\psi_{p,\phi}(r) = \frac{\int_r^\infty se^{-p\phi(s)}ds}{(1+r)e^{-p\phi(r)}}, r \geq 0$, which was crucial to describe the norm of f in $\mathcal{F}^\phi_p(\mathbf{C})$ in terms of the derivative f'. We shall introduce for each pair (ϕ, ψ) of weights and $0 < p < \infty$ the function

$$H_{\psi,\phi,p}(r) = \begin{cases} e^{-\phi(r)}(\frac{1}{r}\int_r^\infty e^{-p\psi(s)}\,ds)^{-1/p}, & 0 < p \leq 1; \\ e^{-(\phi(r)+(p-1)\psi(r))}(\frac{1}{r}\int_r^\infty e^{-p\psi(s)}, ds)^{-1}, & 1 < p < \infty \end{cases}$$

which will play an important role in finding sufficient conditions on the boudedness of V_g. Namely we shall establish in Theorem 5.7 below that, for $0 < p < \infty$, $\phi, \psi \in \mathcal{W}$ and $g \in \mathcal{H}(\mathbf{C})$, the existence of a constant $A > 0$ such that

$$M_\infty(Dg, r) \leq AH_{\psi,\phi,p}(r), \quad r > 0. \tag{7}$$

implies that $\tilde{V}_g$ is bounded from $\mathcal{F}^\phi_p(\mathbf{C})$ into $\mathcal{F}^\psi_p(\mathbf{C})$. As a consequence one generalizes, at least for $p = q$, the results in [12] to a much wider class of weights.

2. Preliminaries on weights

We start by mentioning some classical families of weights. For each $\varepsilon, \alpha, \beta > 0$ and $\gamma \in \mathbf{R}$ the consider the weights ρ_ε and $\varphi_{\alpha,\beta,\gamma}$ given by

$$\rho_\varepsilon(r) = \big(\log(1+r)\big)^{1+\varepsilon}$$

and

$$e^{-\varphi_{\alpha,\beta,\gamma}(r)} = \min\{(1+r)^\gamma, r^\gamma\}e^{-\beta r^\alpha}$$

that is $\varphi_{\alpha,\beta,\gamma}(r) = \beta r^\alpha - \gamma\log(1+r)$ for $\gamma < 0$ and $\varphi_{\alpha,\beta,\gamma}(r) = \beta r^\alpha - \gamma \log r$ for $\gamma \geq 0$. It is easy to see that ρ_ε and $\varphi_{\alpha,\beta,\gamma}$ belong to $\mathcal{W}$.

The examples $\varphi_{\alpha,\beta,\gamma}$ can be obtained from a single one $\phi(r) = r$ using the following modifications:

$$\phi_\beta(r) = \phi(\beta r), \quad \beta > 0, \tag{8}$$

$$\phi^{(\alpha)}(r) = \phi(r^\alpha), \quad \alpha > 0, \tag{9}$$

$$e^{-\phi_{(\gamma)}(r)} = \min\{(1+r)^\gamma, r^\gamma\}e^{-\phi(r)}, \quad \gamma \in \mathbf{R}. \tag{10}$$

It is elementary to see that if ϕ belongs to $\mathcal{W}$ then $\phi_\beta, \phi^{(\alpha)}$ and $\phi_{(\gamma)}$ also belong to $\mathcal{W}$.

Definition 2.1. Let $0 < p < \infty$ and ϕ such that $\int_r^\infty e^{-p\phi(s)}ds < \infty$ for $r > 0$. We define, for $r > 0$,

$$\Phi_p(r) = -\frac{1}{p}\log\left(\frac{1}{r}\int_r^\infty e^{-p\phi(s)}\,ds\right), \tag{11}$$

or, equivalently $e^{-p\Phi_p(r)} = \frac{1}{r}\int_r^\infty e^{-p\phi(s)}\,ds$.

Lemma 2.1. *Let $0 < p < \infty$ and $\phi \in \mathcal{W}$. Then*

(i) $\Phi_p \in \mathcal{W}$,

(ii) *if* $\phi \in C^1(0,\infty)$ *and convex, then*

$$\sup_{r\geq 0} e^{\phi(r)-\Phi_p(r)} < \infty, \tag{12}$$

(iii) *if* $\phi(r) = \varphi_{\alpha,\beta,\gamma}$ *for some* $\alpha, \beta > 0$ *and* $\gamma \in \mathbf{R}$, *then*

$$\sup_{r>0} r^{\frac{\alpha}{p}} e^{\phi(r)-\Phi_p(r)} < \infty. \tag{13}$$

Proof. (i) Clearly $e^{-\Phi_p(r)}$ is decreasing and $\Phi_p(r)$ is increasing. Now for each $m \in \mathbf{N}$ with $mp > 1$ we have that

$$r^{pm} e^{-p\Phi_p(r)} = r^{pm-1} \int_r^\infty e^{-p\phi(s)}\, ds \leq \int_0^\infty s^{pm-1} e^{-p\phi(s)}\, ds < \infty.$$

This shows that $\Phi_p \in \mathcal{W}$.

(ii) Note that $r\phi'(r) \geq \phi'(1) = A$ for $r \geq 1$. Hence for $r \geq 1$

$$e^{-p\Phi_p(r)} = \frac{1}{r} \int_r^\infty e^{-p\phi(t)}\, dt \leq \frac{1}{A} \int_r^\infty \phi'(t) e^{-p\phi(t)}\, dt = \frac{1}{pA} e^{-p\phi(r)}.$$

Since $\sup_{0\leq r\leq 1} e^{\phi(r)-\Phi_p(r)} < \infty$ this gives (12).

(iii) We claim that for any $a \in \mathbf{R}$ there exists $C_a > 0$ so that

$$\int_r^\infty t^a e^{-t}\, dt \leq C_a r^a e^{-r}, \quad r > 0. \tag{14}$$

Of course the result holds true for $a \leq 0$ with $C_a = 1$. The case $a \in \mathbf{N}$ follows by induction and integration by parts. Now for $a > 0$ write $a = \lambda k_0 + (1-\lambda)k_1$ with $0 \leq \lambda \leq 1$ and $k_0, k_1 \in \mathbf{N} \cup \{0\}$, apply Hölder's inequality and the previous case to get (14). To show (13) we consider the cases $\gamma \geq 0$ and $\gamma < 0$ separately.

Case $\gamma \geq 0$: From (14) we have

$$\begin{aligned} e^{-\Phi_p(r)} &= \left(\frac{1}{r}\int_r^\infty t^{p\gamma} e^{-p\beta t^\alpha}\, dt\right)^{1/p} = C \left(\frac{1}{r}\int_{p\beta r^\alpha}^\infty s^{\frac{p\gamma+1}{\alpha}-1} e^{-s}\, dt\right)^{1/p} \\ &\leq C' r^{\gamma-\frac{\alpha}{p}} e^{-\beta r^\alpha} = C' r^{-\frac{\alpha}{p}} e^{-\phi(r)}. \end{aligned}$$

Case $\gamma < 0$: Arguing as above,

$$\begin{aligned} e^{-\Phi_p(r)} &= \left(\frac{1}{r}\int_r^\infty (1+t)^{p\gamma} e^{-p\beta t^\alpha}\, dt\right)^{1/p} \leq C \left(\frac{(1+r)^{p\gamma}}{r}\int_{p\beta r^\alpha}^\infty s^{\frac{1}{\alpha}-1} e^{-s}\, dt\right)^{1/p} \\ &\leq C(1+r)^\gamma r^{-\alpha/p} e^{-\beta r^\alpha} \leq r^{-\frac{\alpha}{p}} e^{-\phi(r)}. \end{aligned}$$

The proof is complete. □

Let us now consider a subclass of differentiable weights wide enough to include most of the classical weights.

Definition 2.2. Let us denote $\mathcal{W}_0$ the collection of continuous functions $\phi\colon [0,\infty) \to \mathbf{R}$ such that $\phi \in C^1([r_\phi,\infty))$ for some $r_\phi \geq 0$ and

$$\lim_{r\to\infty} r\phi'(r) = \infty. \tag{15}$$

Note that the classical examples $\varphi_{\alpha,\beta,\gamma}$ and ρ_ε belong to $\mathcal{W}_0$ for any $\epsilon, \alpha, \beta > 0$ and $\gamma \in \mathbf{R}$.

Lemma 2.2. $\mathcal{W}_0 \subset \mathcal{W}$.

Proof. Let $\phi \in \mathcal{W}_0$. Then $\phi'(r) > 0$ in some interval (R, ∞) and for each $m \in \mathbf{N}$, L'Hospital's rule gives $\lim_{r\to\infty} \frac{\phi(r)-m\log r}{m\log r} = \infty$. In particular $\lim_{r\to\infty}(\phi(r) - m\log r) = \infty$. Hence (2) holds and then $\phi \in \mathcal{W}$. □

Proposition 2.3. *Let $0 < p < \infty$ and let ϕ be differentiable with $\phi'(r) > 0$ for $r > 0$. Then*

$$\phi \in \mathcal{W}_0 \iff \Phi_p \in \mathcal{W}_0 \iff \lim_{r\to\infty} e^{\phi(r)-\Phi_p(r)} = 0.$$

Proof. Differentiating in the formula $e^{-p\Phi_p(r)} = \frac{1}{r}\int_r^\infty e^{-p\phi(s)}ds$ one has that $pr\Phi_p'(r) = e^{p(\Phi_p(r)-\phi(r))} + 1$. Now use L'Hospital's rule to obtain

$$\lim_{r\to\infty} pr\Phi_p'(r) = \lim_{r\to\infty} \frac{re^{-p\phi(r)}}{\int_r^\infty e^{-p\phi(s)}\,ds} + 1 = p\lim_{r\to\infty} r\phi'(r).$$

Thus both equivalences are shown. □

Let us give a notation to the sequence of the norms of u_k in the space $\mathcal{F}_p^\phi$ for any weight $\phi \in \mathcal{W}$ and $0 < p \le \infty$.

Definition 2.3. Let $0 < p \le \infty$, $\phi \in \mathcal{W}$ and $k \in \mathbf{N}\cup\{0\}$. We define

$$C_k(\phi, p) = \left(\int_0^\infty r^{pk+1}e^{-p\phi(r)}\,dr\right)^{1/p} = (2\pi)^{-1/p}\|u_k\|_{\mathcal{F}_p^\phi}, \tag{16}$$

$$C_k(\phi,\infty) = \sup_{0<r<\infty} r^k e^{-\phi(r)} = \|u_k\|_{\mathcal{F}_\infty^\phi}. \tag{17}$$

Next result is immediate and left to the reader.

Example 2.1. Let $\alpha, \beta, p > 0$, $\gamma \ge 0$ and $\phi = \varphi_{\alpha,\beta,\gamma}$. Then

$$C_k(\phi,\infty) = (\alpha\beta)^{-\frac{k+\gamma}{\alpha}}(k+\gamma)^{\frac{k+\gamma}{\alpha}}e^{-\frac{k+\gamma}{\alpha}} \tag{18}$$

and

$$C_k^p(\phi,p) = \frac{(p\beta)^{-\frac{pk+2+p\gamma}{\alpha}}}{\alpha}\Gamma\left(\frac{pk+2+p\gamma}{\alpha}\right). \tag{19}$$

Remark 2.1. For $0 < p, p_1, p_2 < \infty$, $k_1, k_2, k \in \mathbf{N}\cup\{0\}$ and $\phi, \psi \in \mathcal{W}$ we have

$$C_{k_1+k_2}(\phi+\psi,p) \le \min\{C_{k_1}(\phi,p)C_{k_2}(\psi,\infty), C_{k_2}(\phi,p)C_{k_1}(\psi,\infty)\}, \tag{20}$$

$$C_k(\phi,p_3) \le C_k(\phi,p_1)C_k(\phi,p_2), \quad \frac{1}{p_3} = \frac{1}{p_1}+\frac{1}{p_2}, \tag{21}$$

$$C_k(\phi,p_2) \le C_k(\phi,p_1)^{p_1/p_2}C_k(\phi,\infty)^{1-p_1/p_2}, \quad p_1 < p_2. \tag{22}$$

Lemma 2.4. *Let $\phi \in \mathcal{W}$ and $0 < p \le \infty$. Then the sequences $\Big((C_0^{-1}(\phi,p)C_k(\phi,p))^{1/k}\Big)_k$ and $\Big(C_{k+1}(\phi,p)/C_k(\phi,p)\Big)_k$ are increasing with*

$$\lim_k \frac{C_{k+1}(\phi,p)}{C_k(\phi,p)} = \lim_{k\to\infty} C_k^{1/k}(\phi,p) = \infty.$$

Proof. Case $p = \infty$: Since $e^{-\phi(r)/k} \le e^{-\phi(r)/(k+1)}$ for all $r > 0$ and $k \in \mathbf{N}$ then obviously $(C_k(\phi,\infty)^{1/k})_k$ is increasing. Let us show that $(\frac{C_{k+1}(\phi,\infty)}{C_k(\phi,\infty)})_k$ is also increasing. Since $k = \frac{1}{2}(k-1)+\frac{1}{2}(k+1)$, we have that

$$C_k(\phi,\infty) = \sup_{r>0} r^{\frac{(k-1)}{2}}e^{-\frac{\phi(r)}{2}}r^{\frac{(k+1)}{2}}e^{-\frac{\phi(r)}{2}} \le C_{k-1}(\phi,\infty)^{1/2}C_{k+1}(\phi,\infty)^{1/2}$$

and then $C_{k+1}(\phi,\infty)/C_k(\phi,\infty)$ is increasing. Finally, using now that $C_k(\phi,\infty)^{1/k} \leq C_{k+1}(\phi,\infty)^{1/(k+1)}$ we have $\frac{C_{k+1}(\phi,\infty)}{C_k(\phi,\infty)} \geq C_{k+1}(\phi,\infty)^{1/(k+1)}$. Hence $\lim_{k\to\infty} \frac{C_{k+1}(\phi,\infty)}{C_k(\phi,\infty)} = \lim_{k\to\infty} C_k(\phi,\infty)^{1/k} = \infty$.

Case $0 < p < \infty$: Applying Cauchy–Schwarz we have

$$\left(\int_0^\infty r^{pk+p+1} e^{-p\phi(r)}\, dr\right)^2 \leq \left(\int_0^\infty r^{pk+2p+1} e^{-p\phi(r)}\, dr\right)\left(\int_0^\infty r^{pk+1} e^{-p\phi(r)}\, dr\right).$$

This shows that $C_{k+1}(\phi,p)^2 \leq C_{k+2}(\phi,p)C_k(\phi,p)$. Thus $C_{k+1}(\phi,p)/C_k(\phi,p)$ is increasing. Now consider the measure $d\mu_p(r) = C_0(\phi,p)^{-p} r e^{-p\phi(r)}\, dr$ defined in $\mathbf{R}^+$. Of course, $\mu_p(\mathbf{R}^+) = 1$ and $(C_0(\phi,p)^{-1}C_k(\phi,p))^{1/k} = \|u_1\|_{L^{pk}(\mathbf{R}^+,d\mu_p)}$ where $u_1(r) = r$. This gives $(C_0(\phi,p)^{-1}C_{k_1}(\phi,p))^{1/k_1} \leq (C_0(\phi,p)^{-1}C_{k_2}(\phi,p))^{1/k_2}$ whenever $k_1 \leq k_2$. In particular $(C_0(\phi,p)^{-1}C_k(\phi,p))^{1/k}$ is increasing. Now taking into account that

$$\frac{C_{k+1}(\phi,p)}{C_k(\phi,p)} \geq (C_0^{-1}(\phi,p)C_{k+1}(\phi,p))^{1/(k+1)} = \|u_1\|_{L^{p(k+1)}(\mu_p)}$$

we conclude that $\lim_{k\to\infty} \frac{C_{k+1}}{C_k} \geq \lim_{k\to\infty} \|u_1\|_{L^{p(k+1)}(\mu_p)} = \|u_1\|_{L^\infty(\mu_p)} = \infty$. The proof is complete. □

Lemma 2.5. *Let $\phi \in \mathcal{W}$ and $0 < p \leq \infty$. Then*

$$C_k(\phi,p) \leq C_{k_1}(\phi,p)^{\frac{k_2-k}{k_2-k_1}} C_{k_2}(\phi,p)^{\frac{k-k_1}{k_2-k_1}}, \quad k_1 \leq k \leq k_2. \tag{23}$$

In particular $C_k^2(\phi,p) \leq C_{2k}(\phi,p)C_0(\phi,p)$ for all $k \in \mathbf{N}$.

Proof. Let us denote $\mathrm{M}_k = C_k(\phi,\infty)$ and $\mathrm{C}_k = C_k(\phi,p)$ for $0 < p < \infty$. We start with the case $p = \infty$. For each $k, k_1, k_2 \in \mathbf{N}$ such that $\frac{1}{k} = \frac{\theta}{k_1} + \frac{1-\theta}{k_2}$, we obviously have $\mathrm{M}_k^{1/k} \leq \mathrm{M}_{k_1}^{\theta/k_1}\mathrm{M}_{k_2}^{(1-\theta)/k_2}$. Hence for each $k_1 \leq k \leq k_2$, choosing $\theta = \frac{k_1}{k}\frac{k_2-k}{k_2-k_1}$ one obtains $\mathrm{M}_k \leq \mathrm{M}_{k_1}^{\frac{k_2-k}{k_2-k_1}}\mathrm{M}_{k_2}^{\frac{k-k_1}{k_2-k_1}}$.

For $0 < p < \infty$, arguing as in the previous lemma we can write for $k_1 \leq k \leq k_2$ and $\frac{1}{pk} = \frac{\theta}{pk_1} + \frac{1-\theta}{pk_2}$ that

$$\|u_1\|_{L^{pk}(\mathbf{R}^+,d\mu_p)} \leq \|u_1\|^{\theta}_{L^{pk_1}(\mathbf{R}^+,d\mu_p)} \|u_1\|^{1-\theta}_{L^{pk_2}(\mathbf{R}^+,d\mu_p)}.$$

Now (23) follows since $\theta = \frac{k_1}{k}\frac{k_2-k}{k_2-k_1}$ and $1-\theta = \frac{k_2}{k}\frac{k-k_1}{k_2-k_1}$.

Finally selecting $k_1 = 0$ and $k_2 = 2k$ one gets $\mathrm{M}_k^2 \leq \mathrm{M}_{2k}\mathrm{M}_0$ and $\mathrm{C}_k^2 \leq \mathrm{C}_{2k}\mathrm{C}_0$. □

Remark 2.2. The conditions appearing in Lemmas 2.4 and 2.5 are closely related to the ones appearing when defining the Denjoy–Carleman classes (see for instance [16]).

3. Preliminaries on the Volterra operator

Given $g \in \mathcal{H}(\mathbf{C})$ we denote by $\mathcal{M}_g$, $\mathcal{D}$ and $\mathcal{I}$ the multiplication, differentiation and integration operators respectively, i.e. for $f \in \mathcal{H}(\mathbf{C})$ we have

$$\mathcal{M}_g(f)(z) = g(z)f(z), \quad \mathcal{D}f(z) = f'(z), \quad \mathcal{I}f(z) = \int_0^z f(\xi)\, d\xi.$$

Of course $\mathcal{I}(\mathcal{H}(\mathbf{C})) = \mathcal{H}_0(\mathbf{C})$, $Id_{\mathcal{H}(\mathbf{C})} = \mathcal{D}\mathcal{I}$ and $Id_{\mathcal{H}_0(\mathbf{C})} = \mathcal{I}\mathcal{D}$ where Id_X stands for the identity operator acting on X. We denote by S and S^{-1} the shift and backwards shift operators defined by

$$S^{-1}f(z) = \frac{f(z)-f(0)}{z} = \sum_{n=0}^{\infty} a_{n+1}u_n, \quad Sf(z) = zf(z) = \sum_{n=1}^{\infty} a_{n-1}u_n,$$

for each $f = \sum_{n=0}^{\infty} a_n u_n \in \mathcal{H}(\mathbf{C})$. Using the notation $P_m(f)$ for the Taylor polynomial of degree m and $R_m f = f - P_{m-1}(f)$ for the remainder of degree m we have

$$S^m f(z) = z^m f(z) = \sum_{k=m}^{\infty} a_{k-m} z^k, \quad S^{-m} f(z) = \sum_{k=0}^{\infty} a_{k+m} z^k = \frac{R_m f(z)}{z^m}.$$

This gives that $S^m S^{-m} f = R_m f$ and $S^{-m} S^m f = f$ for $m \in \mathbf{N}$.

Since $\mathcal{P} \subset \mathcal{F}_p^{\phi}(\mathbf{C})$ we have that $f \in \mathcal{F}_p^{\phi}(\mathbf{C})$ if and only if $R_m f \in \mathcal{F}_p^{\phi}(\mathbf{C})$ for any $m \in \mathbf{N}$, $0 < p \leq \infty$ and $\phi \in \mathcal{W}$. Note that $\|S^m f\|_{\mathcal{F}_p^{\phi}} = \|f\|_{\mathcal{F}_p^{\phi_{(m)}}}$ for each $0 < p \leq \infty$ where $\phi_{(m)}$ was defined by $e^{-\phi_{(m)}(r)} = r^m e^{-\phi(r)}$.

Lemma 3.1. *Let $m \in \mathbf{N}$, $1 \leq p \leq \infty$ and $\phi \in \mathcal{W}$. Then*

$$\|S^{-m} f\|_{\mathcal{F}_p^{\phi_{(m)}}} \leq (m+1)\|f\|_{\mathcal{F}_p^{\phi}}$$

for $f = \sum_{k=0}^{\infty} a_k u_k \in \mathcal{F}_p^{\phi}(\mathbf{C})$.

Proof. For each $k \in \mathbf{N} \cup \{0\}$, $r > 0$ and $p \geq 1$ we have $|a_k| r^k \leq M_1(f,r) \leq M_p(f,r)$. Thus

$$|a_k| C_k(\phi, p) \leq (2\pi)^{-1/p} \|f\|_{\mathcal{F}_p^{\phi}}, \quad k \in \mathbf{N} \cup \{0\}. \tag{24}$$

Therefore $\|P_{m-1}(f)\|_{\mathcal{F}_p^{\phi}} \leq m\|f\|_{\mathcal{F}_p^{\phi}}$, $\|R_m(f)\|_{\mathcal{F}_p^{\phi}} \leq (m+1)\|f\|_{\mathcal{F}_p^{\phi}}$ and

$$\|S^{-m}(f)\|_{\mathcal{F}_p^{\phi_{(m)}}} = \|R_m(f)\|_{\mathcal{F}_p^{\phi}} \leq (m+1)\|f\|_{\mathcal{F}_p^{\phi}}.$$

This finishes the proof. □

As mentioned in the introduction the *Volterra operator with symbol g* is defined by the formula

$$V_g(f)(z) = \mathcal{I}\mathcal{M}_{\mathcal{D}g}(z) = z\int_0^1 f(tz) g'(tz)\, dt, \quad z \in \mathbf{C}, \tag{25}$$

for each $f \in \mathcal{H}(\mathbf{C})$.

Note that $V_g = 0$ for any constant function g and that also $V_g(f) \in \mathcal{H}_0(\mathbf{C})$ for any $f \in \mathcal{H}(\mathbf{C})$. We shall consider the following modification to avoid these restrictions. For each $f, g \in \mathcal{H}(\mathbf{C})$ we write

$$\tilde{V}_g(f)(z) = \frac{1}{z}\int_0^z f(\xi) Dg(\xi)\, d\xi, \quad z \in \mathbf{C}, \tag{26}$$

where $D = \mathcal{D}S$, that is $Df(z) = \sum_{n=0}^{\infty}(n+1)a_n z^n = zf'(z) + f(z)$.

Denoting $I = S^{-1}\mathcal{I}$, we have for $f = \sum_{n=0}^{\infty} a_n u_n$ that

$$If(z) = \sum_{n=0}^{\infty} \frac{a_n}{(n+1)} z^n = \frac{1}{z}\int_0^z f(\xi)\, d\xi$$

and we obtain that $\tilde{V}_g = I\mathcal{M}_{Dg}$. In this way $\tilde{V}_g$ is well defined for $g \in \mathcal{H}(\mathbf{C})$ and takes values in $\mathcal{H}(\mathbf{C})$. Moreover, for each $f, g \in \mathcal{H}(\mathbf{C})$

$$\tilde{V}_g(f) = S^{-1} V_{Sg}(f), \quad V_g(f) = S\tilde{V}_{S^{-1}g}(f). \tag{27}$$

Since V_g is continuous (in the topology of the uniform convergence on compact sets) from $\mathcal{H}(\mathbf{C})$ into $\mathcal{H}_0(\mathbf{C})$ and the map given by $g \to V_g$ is linear and continuous from $\mathcal{H}_0(\mathbf{C})$ into the space of continuous linear operators, using (27) similar results hold for $\tilde{V}_g$. Next result is immediate from the definitions.

Lemma 3.2. *Let $0 < p, q \leq \infty$, $\phi, \psi \in \mathcal{W}$ and $g \in \mathcal{H}(\mathbf{C})$. Then V_g is bounded from $\mathcal{F}_p^{\phi}(\mathbf{C})$ into $\mathcal{F}_q^{\psi}(\mathbf{C})$ if and only if $\tilde{V}_{S^{-1}g}$ is bounded from $\mathcal{F}_p^{\phi}(\mathbf{C})$ into $\mathcal{F}_q^{\psi(1)}(\mathbf{C})$.*

Other expressions for the operators above are given as follows:

Lemma 3.3. *Let $f, g \in \mathcal{H}(\mathbf{C})$ with $f = \sum_{m=0}^{\infty} b_m u_m$ and $g = \sum_{n=0}^{\infty} a_n u_n$. Then*

$$V_g(f)(z) = \sum_{j=1}^{\infty} \frac{1}{j} \left(\sum_{n+m=j} n a_n b_m \right) z^j, \tag{28}$$

$$\tilde{V}_g(f)(z) = \sum_{j=0}^{\infty} \frac{1}{j+1} \left(\sum_{n+m=j} (n+1) a_n b_m \right) z^j. \tag{29}$$

Proof. The proof is straightforward using

$$\begin{aligned} V_g(f)(z) &= \sum_{n=1}^{\infty} n a_n z^n \left(\int_0^1 f(zs) s^{n-1}\, ds \right) = \sum_{n=1}^{\infty} n a_n z^n \left(\sum_{m=0}^{\infty} \frac{b_m}{n+m} z^m \right) \\ &= \sum_{j=1}^{\infty} \frac{1}{j} \left(\sum_{n+m=j} n a_n b_m \right) z^j. \end{aligned}$$

The other formula follows from (27). □

Remark 3.1. From (28) and (29) we obtain for any $f, g \in \mathcal{H}(\mathbf{C})$ and $k \in \mathbf{N}$,

$$V_g(u_0) = g - g(0), \quad \tilde{V}_g(u_0) = g, \quad V_{u_0}(f) = 0, \quad \tilde{V}_{u_0}(f) = If,$$

$$V_g(u_k) = u_k \sum_{n=1}^{\infty} \frac{n a_n}{n+k} u_n, \quad \tilde{V}_g(u_k) = u_k \sum_{n=0}^{\infty} \frac{(n+1) a_n}{n+k+1} u_n, \tag{30}$$

and

$$V_{u_k}(f) = k u_k \sum_{n=0}^{\infty} \frac{b_n}{n+k} u_n, \quad \tilde{V}_{u_k}(f) = (k+1) u_k \sum_{n=0}^{\infty} \frac{b_n}{n+k+1} u_n. \tag{31}$$

Let us reformulate the boundedness of $\tilde{V}_g$ acting on $\mathcal{F}_2^{\phi}(\mathbf{C})$. Note that for each $f = \sum_{m=0}^{\infty} b_m u_m$ we can write

$$\|f\|_{\mathcal{F}_2^{\phi}} = \frac{1}{\sqrt{2\pi}} \left(\sum_{m=0}^{\infty} |b_m|^2 C_m^2(\phi, 2) \right)^{1/2}. \tag{32}$$

Proposition 3.4. *Let $\phi, \psi \in \mathcal{W}$ and $g \in \mathcal{H}(\mathbf{C})$ with $g = \sum_{n=0}^{\infty} a_n u_n$. Then $\tilde{V}_g$ maps $\mathcal{F}_2^{\phi}(\mathbf{C})$ into $\mathcal{F}_2^{\psi}(\mathbf{C})$ if and only if the matrix $A = (a(m,j))_{m,j=0}^{\infty}$ given by*

$$a(m,j) = \begin{cases} \frac{j-m+1}{j+1} a_{j-m} \frac{C_j(\psi,2)}{C_m(\phi,2)}, & m \leq j; \\ 0, & 0 \leq j < m. \end{cases}$$

defines a bounded operator on $\ell^2(\mathbf{N} \cup \{0\})$.

Proof. Using (32) and (29) we obtain

$$\|\tilde{V}_g(f)\|_{\mathcal{F}_2^\psi} = \frac{1}{\sqrt{2\pi}} \sup_{\|(\gamma_j)\|_2=1} \left| \sum_{j=0}^\infty \frac{1}{j+1} \left(\sum_{m=0}^j (j-m+1) a_{j-m} b_m \right) C_j(\psi,2)\gamma_j \right|$$
$$= \frac{1}{\sqrt{2\pi}} \sup_{\|(\gamma_j)\|_2=1} \left| \sum_{m=0}^\infty \left(\sum_{j=m}^\infty \frac{j-m+1}{j+1} a_{j-m} \frac{C_j(\psi,2)}{C_m(\phi,2)} \gamma_j \right) C_m(\phi,2) b_m \right|.$$

Hence

$$\|\tilde{V}_g\| = \sup_{\|(\gamma_j)\|_2=1} \left(\sum_{m=0}^\infty \left| \sum_{j=0}^\infty a(m,j)\gamma_j \right|^2 \right)^{1/2}.$$

This gives the result. □

The analysis of V_g for $g \in \mathcal{P}$ actually depends only on the integration operator. Let us denote by V_k and $\tilde{V}_k$ the operators V_{u_k} and $\tilde{V}_{u_k}$ for $k \in \mathbf{N} \cup \{0\}$. Hence from (31) we obtain

$$V_0 = 0, \quad V_1 = \mathcal{I}, \quad V_k = k\mathcal{I}S^{k-1}, \quad k \in \mathbf{N}, \tag{33}$$

and

$$\tilde{V}_0 = I, \quad \tilde{V}_k = (k+1)IS^k, \quad k \in \mathbf{N}. \tag{34}$$

In particular $V_k = S\tilde{V}_{k-1}$ for $k \in \mathbf{N}$. A simple consequence of Proposition 3.4 gives the following particular case.

Corollary 3.5. *Let $k \in \mathbf{N} \cup \{0\}$ and $\phi, \psi \in \mathcal{W}$. Then $\tilde{V}_k$ maps $\mathcal{F}_2^\phi(\mathbf{C})$ into $\mathcal{F}_2^\psi(\mathbf{C})$ if and only if*

$$\sup_{m\geq 0} \frac{C_{m+k}(\psi,2)}{(m+k+1)C_m(\phi,2)} < \infty.$$

The following reformulations are elementary and left to the reader.

Lemma 3.6. *Let $k \in \mathbf{N}$, $\phi, \psi \in \mathcal{W}$ and $0 < p, q \leq \infty$. The following statements are equivalent:*

(i) $V_k\colon \mathcal{F}_p^\phi(\mathbf{C}) \to \mathcal{F}_q^\psi(\mathbf{C})$ *is bounded.*

(ii) $\tilde{V}_{k-1}\colon \mathcal{F}_p^\phi(\mathbf{C}) \to \mathcal{F}_q^{\psi(1)}(\mathbf{C})$ *is bounded.*

(iii) $\mathcal{I}\colon \mathcal{F}_p^{\phi(k-1)}(\mathbf{C}) \to \mathcal{F}_q^\psi(\mathbf{C})$ *is bounded.*

(iv) $I\colon \mathcal{F}_p^{\phi(k-1)}(\mathbf{C}) \to \mathcal{F}_q^{\psi(1)}(\mathbf{C})$ *is bounded.*

4. On necessary conditions for the boundedness

Taking into account that $V_g(u_0) = g - g(0)$ the first condition for V_g to map $\mathcal{F}_p^\phi(\mathbf{C})$ into $\mathcal{F}_q^\psi(\mathbf{C})$ is that $g \in \mathcal{F}_q^\psi(\mathbf{C})$. In particular we have the following trivial necessary condition.

Proposition 4.1. *Let $0 < p, q \leq \infty$, $\phi, \psi \in \mathcal{W}$ and $0 \neq g \in \mathcal{H}_0(\mathbf{C})$. If $V_g\colon \mathcal{F}_p^\phi(\mathbf{C}) \to \mathcal{F}_q^\psi(\mathbf{C})$ is bounded, then there exists a constant $A > 0$ such that*

$$M_\infty(g,r) \leq AK_{\psi,q}(r), \quad r > 0, \tag{35}$$

where

$$K_{\psi,q}(z) = \sum_{k=0}^\infty C_k(\psi,q)^{-1} z^k, \quad z \in \mathbf{C}. \tag{36}$$

Proof. Using (24) for $V_g(u_0) = g(z) = \sum_{n=1}^{\infty} b_n z^n$ we obtain

$$M_\infty(g,r) \le \sum_{n=0}^{\infty} |a_n| r^n \le (2\pi)^{-1/q} \sum_{n=0}^{\infty} C_n(\psi,q)^{-1} \|V_g(u_0)\|_{\mathcal{F}_q^\psi} r^n$$
$$\le (2\pi)^{1/p-1/q} \|V_g\| C_0(\phi,p) (\sum_{n=0}^{\infty} C_n(\psi,q)^{-1} r^n).$$

This shows (35). □

Let us find a necessary condition for the boundedness of V_g from $\mathcal{F}_p^\phi(\mathbf{C})$ into $\mathcal{F}_q^\psi(\mathbf{C})$ in the case $\mathcal{F}_q^\psi(\mathbf{C}) \subseteq \mathcal{F}_p^\phi(\mathbf{C})$.

Proposition 4.2. *Let $0 < p, q \le \infty$, $\phi, \psi \in \mathcal{W}$ such that $\mathcal{F}_q^\psi(\mathbf{C}) \subseteq \mathcal{F}_p^\phi(\mathbf{C})$ and $0 \ne g \in \mathcal{H}_0(\mathbf{C})$. If $V_g \colon \mathcal{F}_p^\phi(\mathbf{C}) \to \mathcal{F}_q^\psi(\mathbf{C})$ is bounded then there exists $A > 0$ such that*

$$M_\infty(g,r) \le A\phi(r), \quad r > 0. \tag{37}$$

Proof. Let $A_0 = \max\{1, \|u_0\|_{\mathcal{F}_p^\phi}\}$ and $C = \|\operatorname{Id}\|_{\mathcal{F}_q^\psi(\mathbf{C}) \to \mathcal{F}_p^\phi(\mathbf{C})}$. We observe that $V_g(u_0) = g \in \mathcal{F}_q^\psi(\mathbf{C})$. Hence $g \in \mathcal{F}_p^\phi(\mathbf{C})$ and $\|g\|_{\mathcal{F}_p^\phi} \le C\|g\|_{\mathcal{F}_q^\psi} \le C\|V_g\|A_0$. Since $V_g(g) = \frac{g^2}{2}$ we also obtain

$$\left\|\frac{g^2}{2}\right\|_{\mathcal{F}_p^\phi} \le C\left\|\frac{g^2}{2}\right\|_{\mathcal{F}_q^\psi} \le C^2\|V_g\|^2\|u_0\|_{\mathcal{F}_p^\phi} \le (C\|V_g\|A_0)^2.$$

This allows to iterate the procedure to obtain $\frac{g^n}{n!} \in \mathcal{F}_p^\phi(\mathbf{C})$ and $\|\frac{g^n}{n!}\|_{\mathcal{F}_p^\phi} \le (C\|V_g\|A_0)^n$.

Recall that $\mathcal{F}_p^\phi(\mathbf{C})$ is a $\tilde{p}$-Banach space for $\tilde{p} = \min\{p, 1\}$. Hence if $\sum_n \|f_n\|_{\mathcal{F}_p^\phi}^{\tilde{p}} < \infty$ implies that $\sum_n f_n \in \mathcal{F}_p^\phi$. Therefore choosing $K > C\|V_g\|A_0$ we conclude that $\sum_{n=0}^{\infty} \frac{\beta_n g^n}{K^n n!} \in \mathcal{F}_p^\phi(\mathbf{C})$ for any sequence of complex numbers with $\sup_n |\beta_n| \le 1$.

In particular, choosing $\beta_n = 1$ for all $n \ge 0$ we obtain $e^{g/K} \in \mathcal{F}_p^\phi(\mathbf{C})$. Therefore $\int_{\mathbf{C}} e^{-p(\phi(|z|) - \frac{\Re g(z)}{K})} dm(z) < \infty$ and $\sup_{z\in\mathbf{C}} e^{-\phi(|z|) + \frac{\Re g(z)}{K}} < \infty$ in the cases $p < \infty$ and $p = \infty$ respectively. In both cases one gets $\Re(g(z)) \le K\phi(|z|) + C$. Selecting β_n as $(-1)^n, i^n$ and $(-i)^n$ one concludes that $|g(z)| \le A\phi(|z|)$ for some constant $A > 0$ and the proof is complete. □

A simple consequence of Proposition 4.2 is the following corollary.

Corollary 4.3. *Let $0 < p \le \infty$, $\phi(r) = \varphi_{\alpha,\beta,\gamma}$ for some $\alpha, \beta > 0$, $\gamma \in \mathbf{R}$ and $g \in \mathcal{H}_0(\mathbf{C})$.*

(i) *Case $0 < \alpha < 1$: $V_g \colon \mathcal{F}_p^\phi(\mathbf{C}) \to \mathcal{F}_p^\phi(\mathbf{C})$ is bounded if and only if $g = 0$.*
(ii) *Case $\alpha \ge 1$: If $V_g \colon \mathcal{F}_p^\phi(\mathbf{C}) \to \mathcal{F}_p^\phi(\mathbf{C})$ is bounded then $g \in \mathcal{P}$ and $1 \le \deg(g) \le \alpha$.*

Let us now show that boundedness of V_g or $\tilde{V}_g$ between spaces $H_v^\infty(\mathbf{C})$ or $\mathcal{F}_p^\phi(\mathbf{C})$ forces certain a priori conditions on the weights.

Proposition 4.4. *Let $0 < p, q \le \infty$, $\phi, \psi \in \mathcal{W}$, $g(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}(\mathbf{C})$ and define $\Lambda = \{n \colon a_n \ne 0\}$. If $\tilde{V}_g \colon \mathcal{F}_p^\phi(\mathbf{C}) \to \mathcal{F}_q^\psi(\mathbf{C})$ is bounded and $k \in \Lambda$, then $\tilde{V}_k \colon \mathcal{F}_p^\phi(\mathbf{C}) \to \mathcal{F}_q^\psi(\mathbf{C})$ is also bounded and $\|\tilde{V}_k\| \le \frac{\|\tilde{V}_g\|}{|a_k|}$. In particular, $I \colon \mathcal{F}_p^\phi(\mathbf{C}) \to \mathcal{F}_q^\psi(\mathbf{C})$ is bounded whenever $g(0) \ne 0$.*

Proof. Let $k \in \Lambda$. We have

$$(k+1)a_k w^k = \int_0^{2\pi} Dg(e^{i\theta}w)e^{-ik\theta}\frac{d\theta}{2\pi}, \quad w \in \mathbf{C},$$

and therefore

$$\begin{aligned}\tilde{V}_k f(z) &= \frac{1}{z}\int_0^z f(w)(k+1)w^k\,dw = \frac{1}{a_k}\frac{1}{z}\int_0^z f(w)\left(\int_0^{2\pi} Dg(e^{i\theta}w)e^{-ik\theta}\frac{d\theta}{2\pi}\right)dw \\ &= \frac{1}{a_k}\int_0^{2\pi}\left(\frac{1}{z}\int_0^z f(w)Dg(e^{i\theta}w)\,dw\right)e^{-ik\theta}\frac{d\theta}{2\pi}.\end{aligned}$$

Hence, making the change of variable $e^{i\theta}w = w'$ and denoting $f_{e^{-i\theta}}(z) = f(e^{-i\theta}z)$, we have

$$\tilde{V}_k f(z) = \frac{1}{a_k}\int_0^{2\pi} \tilde{V}_g(f_{e^{-i\theta}})(e^{i\theta}z)e^{-ik\theta}\frac{d\theta}{2\pi}.$$

In particular,

$$\|\tilde{V}_k f\|_{\mathcal{F}_q^\psi} \le \frac{1}{|a_k|}\int_0^{2\pi} \|\tilde{V}_g(f_{e^{-i\theta}})(e^{i\theta}z)\|_{\mathcal{F}_q^\psi}\frac{d\theta}{2\pi}, \quad 1 \le q \le \infty,$$

and

$$\|\tilde{V}_k f\|^q_{\mathcal{F}_q^\psi} \le \frac{1}{|a_k|^q}\int_0^{2\pi} \|\tilde{V}_g(f_{e^{-i\theta}})(e^{i\theta}z)\|^q_{\mathcal{F}_q^\psi}\frac{d\theta}{2\pi}, \quad 0 < q < 1.$$

This gives, taking into account that $\|f_{e^{-i\theta}}\|_{\mathcal{F}_p^\phi} = \|f\|_{\mathcal{F}_p^\phi}$ for any radial weight, the estimate $\|\tilde{V}_k f\|_{\mathcal{F}_q^\psi} \le \frac{\|\tilde{V}_g\|}{|a_k|}\|f\|_{\mathcal{F}_p^\phi}$ and the proof is complete. □

Corollary 4.5. *Let $0 < p, q \le \infty$, $\phi, \psi \in \mathcal{W}$ and $0 \ne g \in \mathcal{H}(\mathbf{C})$. If $\tilde{V}_g\colon \mathcal{F}_p^\phi(\mathbf{C}) \to \mathcal{F}_q^\psi(\mathbf{C})$ is bounded then there exists $k \in \mathbf{N}$ and $A_k > 0$ such that*

$$C_{n+k}(\psi, q) \le A_k(n+1)C_n(\phi, p), \quad n \ge 0.$$

In particular,

$$IK_{\phi,p}(r) \le A_k S^{-k} K_{\psi,q}(r), \quad r > 0,$$

where $K_{\phi,p}$ stands for the kernel given in (36).

Proof. Since $0 \ne g$ there exists $k \in \Lambda$, that is $a_k \ne 0$. Due to Proposition 4.4 and the fact $\tilde{V}_k(u_n) = (k+1)I(u_{n+k}) = \frac{k+1}{n+k+1}u_{n+k}$ we have $\frac{k+1}{n+k+1}\|u_{n+k}\|_{\mathcal{F}_q^\psi} \le \|\tilde{V}_k\|\|u_n\|_{\mathcal{F}_q^\psi}$. In particular, for all $n \in \mathbf{N}$,

$$\|u_{n+k}\|_{\mathcal{F}_q^\psi} \le \|\tilde{V}_k\|(n+1)\|u_n\|_{\mathcal{F}_q^\psi}.$$

This shows that

$$IK_{\phi,p}(r) = \sum_{n=0}^{\infty}\frac{r^n}{(n+1)C_n(\phi,q)} \le A_k\sum_{n=0}^{\infty}\frac{r^n}{C_{n+k}(\psi,q)} = A_k S^{-k}K_{\psi,q}(r),$$

and the proof is complete. □

Corollary 4.6. *Let $0 < p, q \le \infty$, $\phi, \psi \in \mathcal{W}$ and let $g(z) = \sum_{n=1}^\infty a_n z^n \in \mathcal{H}_0(\mathbf{C})$ such that $V_g\colon \mathcal{F}_p^\phi(\mathbf{C}) \to \mathcal{F}_q^\psi(\mathbf{C})$ is bounded. Then $V_k\colon \mathcal{F}_p^\phi(\mathbf{C}) \to \mathcal{F}_q^\psi(\mathbf{C})$ is also bounded for each k such that $g^{(k)}(0) \ne 0$. Moreover, the estimate $\|V_k\| \le \frac{k!}{|g^{(k)}(0)|}\|V_g\|$ holds. In particular, $\mathcal{I}\colon \mathcal{F}_p^\phi(\mathbf{C}) \to \mathcal{F}_q^\psi(\mathbf{C})$ is bounded whenever $g'(0) \ne 0$.*

Proof. Recall that due to Lemma 3.2 we have that $\tilde{V}_{S^{-1}g}\colon \mathcal{F}_p^{\phi}(\mathbf{C}) \to \mathcal{F}_q^{\psi_{(1)}}(\mathbf{C})$ where $e^{-\psi_{(1)}(r)} = re^{-\psi(r)}$. Therefore invoking Proposition 4.4 and Lemma 3.6 we obtain that $V_k\colon \mathcal{F}_p^{\phi}(\mathbf{C}) \to \mathcal{F}_q^{\psi}(\mathbf{C})$ whenever $g^{(k)}(0) \neq 0$ and the corresponding estimate in norm holds. □

Corollary 4.7. *Let $\alpha_i, \beta_i > 0$ and $\gamma_i \geq 0$ for $i = 1, 2$, $v(r) = e^{-\varphi_{\alpha_1,\beta_1,\gamma_1}(r)}$ and $w(r) = e^{-\varphi_{\alpha_2,\beta_2,\gamma_2}(r)}$ and $0 \neq g \in \mathcal{H}_0(\mathbf{C})$. Assume that $V_g\colon H_v^{\infty}(\mathbf{C}) \to H_w^{\infty}(\mathbf{C})$ is bounded. Then either $\alpha_1 < \alpha_2$ or $\alpha_1 = \alpha_2$ and $\beta_1 \leq \beta_2$ or $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$ and $\gamma_2 \leq \gamma_1 + \alpha_1 - 1$. Moreover, in the case $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$ and $\delta = \alpha_1 - \gamma_2 + \gamma_1 \geq 1$, then $g \in \mathcal{P}$ with $\deg(g) \leq \delta$.*

Proof. Due to Corollary 4.6 we have that V_k is bounded from $H_v^{\infty}(\mathbf{C})$ into $H_w^{\infty}(\mathbf{C})$ for all $k \in \mathbf{N}$ such that $g^{(k)}(0) \neq 0$. Since $V_k(u_n) = \frac{k}{n+k}u_{n+k}$ we have

$$C_{n+k}(\varphi_{\alpha_2,\beta_2,\gamma_2}, \infty) \leq \|V_k\| \frac{n+k}{k} C_n(\varphi_{\alpha_1,\beta_1,\gamma_1}, \infty), \quad n \in \mathbf{N}.$$

Now take into account Example 2.1 to obtain for all $n \in \mathbf{N}$

$$(\alpha_2\beta_2)^{-\frac{k+n+\gamma_2}{\alpha_2}}(k+n+\gamma_2)^{\frac{k+n+\gamma_2}{\alpha_2}} e^{-\frac{k+n+\gamma_2}{\alpha_2}} \leq \|V_k\| \frac{n+k}{k} (\alpha_1\beta_1)^{-\frac{n+\gamma_1}{\alpha_1}} (n+\gamma_1)^{\frac{n+\gamma_1}{\alpha_1}} e^{-\frac{n+\gamma_1}{\alpha_1}}.$$

Hence there exists $C > 0$ such that

$$n^{n(\frac{1}{\alpha_2}-\frac{1}{\alpha_1})} \leq C(\alpha_2\beta_2 e)^{\frac{n}{\alpha_2}}(\alpha_1\beta_1 e)^{-\frac{n}{\alpha_1}} n^{1-\frac{k+\gamma_2}{\alpha_2}+\frac{\gamma_1}{\alpha_1}}, \quad \forall n \in \mathbf{N}.$$

This implies that $\alpha_1 \leq \alpha_2$.

In the case $\alpha_1 = \alpha_2$ the inequality becomes $\left(\frac{\beta_1}{\beta_2}\right)^{\frac{n}{\alpha_1}} \leq Cn^{1-\frac{k+\gamma_2-\gamma_1}{\alpha_1}}$ for all $n \in \mathbf{N}$. This gives $\beta_1 \leq \beta_2$.

Finally in the case $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$ we would have $n^{\frac{k+\gamma_2-\gamma_1}{\alpha_1}-1} \leq C$ for all $n \in \mathbf{N}$. This implies $\frac{k+\gamma_2-\gamma_1}{\alpha_1} \leq 1$. This gives, in particular, $\gamma_2 \leq \gamma_1 + \alpha_1 - 1$.

To finish the proof notice that $g^{(k)}(0) \neq 0$ implies $k \leq \alpha_1 - \gamma_2 + \gamma_1$ which implies that $g \in \mathcal{P}$ with $\deg(g) \leq \alpha_1 - \gamma_2 + \gamma_1$. □

5. On sufficient conditions for the boundedness

Let us start presenting some sufficient conditions for the operators V_g and $\tilde{V}_g$ to be bounded from $\mathcal{F}_p^{\phi}(\mathbf{C})$ into $\mathcal{F}_p^{\psi}(\mathbf{C})$ for any $1 \leq p \leq \infty$ and for general weights.

Proposition 5.1. *Let $\phi, \psi \in \mathcal{W}$ and $g \in \mathcal{H}(\mathbf{C})$. Let us write $g_r(z) = g(rz)$ for $r > 0$ and set*

$$A(\phi, \psi) = \sup_{r>0} e^{\phi(r)-\psi(r)} \|g_r\|_{BMOA} \tag{38}$$

and

$$B(\phi, \psi) = \sup_{r>0} e^{\phi(r)-\psi(r)} \|(g_r)'\|_{H^1}. \tag{39}$$

(i) *If $A(\phi, \psi) < \infty$, then both $\tilde{V}_g$ and V_g are bounded from $\mathcal{F}_p^{\phi}(\mathbf{C})$ into $\mathcal{F}_p^{\psi}(\mathbf{C})$ for any $1 \leq p < \infty$.*

(ii) *If $B(\phi, \psi) < \infty$, then both $\tilde{V}_g$ and V_g are bounded from $\mathcal{F}_{\infty}^{\phi}(\mathbf{C})$ into $\mathcal{F}_{\infty}^{\psi}(\mathbf{C})$.*

Proof. (i) Let $1 \le p < \infty$ and set $A(\phi, \psi) = A$. Since $(g_r)'(w) = rg'(rw)$ for each $r > 0$ and $|w| < 1$, we have

$$V_g(f)(rw) = \int_0^w f_r(\xi)(g_r)'(\xi)\, d\xi.$$

Hence, using the estimate (1) we have

$$M_p(V_g(f), r) \le C_p \|g_r\|_{BMOA} M_p(f, r), \quad r > 0. \tag{40}$$

Since $\tilde{V}_g = S^{-1} V_{Sg}$, $(Sg)_r = rS(g_r)$ and $\|Sg_r\|_{BMOA} = \|g_r\|_{BMOA}$, we also have

$$M_p(\tilde{V}_g(f), r) = \frac{1}{r} M_p(V_{Sg}(f), r) \le C_p \|g_r\|_{BMOA} M_p(f, r).$$

Therefore, we conclude that

$$\begin{aligned}\max\{\|V_g(f)\|_{\mathcal{F}_p^\psi}, \|\tilde{V}_g(f)\|_{\mathcal{F}_p^\psi}\} &\le 2\pi C_p^p \int_0^\infty M_p^p(f, r)\|g_r\|_{BMOA}^p r e^{-p\psi(r)} dr \\ &\le 2\pi C_p^p A^p \int_0^\infty M_p^p(f, r) r e^{-p\phi(r)} dr = C_p^p A^p \|f\|_{\mathcal{F}_p^\phi}^p.\end{aligned}$$

(ii) Let $p = \infty$ and set $B(\phi, \psi) = B$. Without loss of generality we can assume that $g \in \mathcal{H}_0(\mathbf{C})$. Hence $g(z) = z \int_0^1 g'(zt)dt$ and thus $M_1(g, r) \le rM_1(g', r) = \|(g_r)'\|_{H^1}$. In particular,

$$M_1(Dg, r) \le rM_1(g', r) + M_1(g, r) \le 2rM_1(g', r) = 2\|(g_r)'\|_{H^1}. \tag{41}$$

Hardy's inequality (see [14]) gives for $f(z) = \sum_{n=0}^\infty a_n z^n$

$$M_\infty(If, r) \le \sum_{n=0}^\infty \frac{|a_n| r^n}{n+1} \le C_0 M_1(f, r), \quad r > 0.$$

Therefore,

$$\begin{aligned}M_\infty(\tilde{V}_g f, r) &\le C_0 M_1((Dg)f, r) \le C_0 \|f\|_{\mathcal{F}_\infty^\phi(\mathbf{C})} e^{\phi(r)} M_1(Dg, r) \\ &\le 2BC_0 \|f\|_{\mathcal{F}_\infty^\phi(\mathbf{C})} e^{\psi(r)}.\end{aligned}$$

This gives the boundedness of $\tilde{V}_g$ from $\mathcal{F}_\infty^\phi(\mathbf{C})$ into $\mathcal{F}_\infty^\psi(\mathbf{C})$.

To handle the case V_g we use that $M_1(D(S^{-1}g), r) = M_1(g', r)$. Arguing as above, we have

$$M_\infty(V_g f, r) = rM_\infty(\tilde{V}_{S^{-1}g} f, r) \le C\|f\|_{\mathcal{F}_\infty^\phi(\mathbf{C})} e^{\phi(r)} \|(g_r)'\|_{H^1}$$

and the result follows with the same argument. □

Proposition 5.2. *Let $\phi, \psi \in \mathcal{W}$ where ψ is differentiable with $\psi'(t) > 0$ for $t > 0$ and $g \in \mathcal{H}_0(\mathbf{C})$. Set*

$$B_1(\phi, \psi) = \sup_{r>0} \frac{e^{\phi(r) - \psi(r)}}{r\psi'(r)} \|(g_r)'\|_{H^\infty}. \tag{42}$$

If $B_1(\phi, \psi) < \infty$, then both $\tilde{V}_g$ and V_g are bounded from $H_v^\infty(\mathbf{C})$ to $H_w^\infty(\mathbf{C})$, where $v(z) = e^{-\phi(|z|)}$ and $w(z) = e^{-\psi(|z|)}$.

Proof. Let $B_1(\phi, \psi) = B_1$. Arguing as in (41) we obtain that $M_\infty(Dg, r) \leq 2\|(g_r)'\|_{H^\infty}$. Now for $|z| = r$ we can estimate

$$\begin{aligned}|\tilde{V}_g(f)(z)| &\leq \int_0^1 |f(zt)||Dg(zt)|\,dt \leq \int_0^1 M_\infty(f, rt)M_\infty(Dg, rt)\,dt\\ &\leq \frac{1}{r}\int_0^r M_\infty(f,t)M_\infty(Dg,t)\,dt = 2\|f\|_v\frac{1}{r}\int_0^r e^{\phi(t)}\|(g_t)'\|_{H^\infty}\,ds\\ &\leq 2B_1\|f\|_v\frac{1}{r}\int_0^r t\psi'(t)e^{\psi(t)}\,ds = 2B_1\|f\|_v(e^{\psi(r)} - e^{\psi(0)}) \leq 2B_1\|f\|_v e^{\psi(|z|)}.\end{aligned}$$

This completes the proof for $\tilde{V}_g$. The case V_g follows similarly using that

$$M_\infty(D(S^{-1}g), r) = M_\infty(g', r).$$ □

Let us apply the previous result to polynomials, in particular for $V_k = V_{u_k}$.

Corollary 5.3. *Let $v(z) = e^{-\varphi_{\alpha,\beta,\gamma}}$ for some $\beta > 0$, $\gamma \in \mathbf{R}$ and $\alpha \geq 1$ and let $g \in \mathcal{H}_0(\mathbf{C})$. Then the following statements are equivalent:*

(i) *$V_g\colon H_v^\infty(\mathbf{C}) \to H_v^\infty(\mathbf{C})$ is bounded.*

(ii) *$g \in \mathcal{P}$ and $1 \leq \deg(g) \leq [p]$.*

Proof. (i) $\Longrightarrow$ (ii) This is the case $p = \infty$ in Corollary 4.3.

(ii) $\Longrightarrow$ (i). It suffices to show that V_k is bounded on $H_v^\infty(\mathbf{C})$ for $1 \leq k \leq \alpha$. Now for each $1 \leq k \leq [\alpha]$ we have

$$\lim_{r\to\infty}\frac{\varphi'_{\alpha,\beta,\gamma}(r)}{r^{k-1}} = \begin{cases}\alpha\beta, & k = \alpha;\\ \infty, & k < \alpha.\end{cases}$$

We can then apply Proposition 5.2 for $\phi = \psi = \varphi_{\alpha,\beta,\gamma}$ and $g = u_k$ to finish the proof. □

Let us get now some conditions depending on p for the boundedness on $\mathcal{F}_p^\phi(\mathbf{C})$. We shall use the following result.

Lemma 5.4. *Let $0 < p < \infty$, $\phi \in \mathcal{W}$. If $f \in \mathcal{F}_p^{\Phi_p}(\mathbf{C})$, then $I(f) \in \mathcal{F}_p^\phi(\mathbf{C})$.*

Proof. Using that $If(z) = \int_0^1 f(zt)\,dt$, for any $0 < p < \infty$ we obtain

$$M_p^p(I(f), r) \leq \int_0^1 M_p^p(f, rt)\,dt \leq \frac{1}{r}\int_0^r M_p^p(f,t)\,dt.$$

Therefore,

$$\begin{aligned}\|I(f)\|^p_{\mathcal{F}_p^\phi} &\leq C\int_0^\infty\left(\int_0^r M_p^p(f,t)\,dt\right)e^{-p\phi(r)}dr \leq C\int_0^\infty M_p^p(f,t)\left(\int_t^\infty e^{-p\phi(r)}dr\right)dt\\ &\leq C\int_0^\infty M_p^p(f,t)te^{-p\Phi_p(t)}\,dt = C\|f\|^p_{\mathcal{F}_p^{\Phi_p}}.\end{aligned}$$

The proof is now complete. □

Proposition 5.5. *Let $0 < p < \infty$, $\phi, \psi \in \mathcal{W}$ and set*

$$A_1(\phi, \psi, p) = \sup_{r>0} e^{\phi(r) - \Psi_p(r)}M_\infty(Dg, r). \tag{43}$$

If $A_1(\phi, \psi, p) < \infty$, then $\tilde{V}_g$ is bounded from $\mathcal{F}_p^\phi(\mathbf{C})$ into $\mathcal{F}_p^\psi(\mathbf{C})$.

Proof. Let $A_1(\phi,\psi,p)=A_1$. Using Lemma 5.4 and recalling that $re^{-p\Psi_p(r)}=\int_r^\infty e^{-p\psi(s)}ds$ we have

$$\begin{aligned}\|\tilde V_g(f)\|^p_{\mathcal{F}_p^\psi} &= \|IM_{Dg}f\|^p_{\mathcal{F}_p^\psi}\le C\|M_{Dg}f\|^p_{\mathcal{F}_p^{\Psi_p}}\\ &\le C\int_0^\infty M_p^p(f,r)M_\infty^p(Dg,r)\left(\int_r^\infty e^{-p\psi(s)}ds\right)dr\\ &\le CA_1^p\int_0^\infty M_p^p(f,r)re^{-p\phi(r)}dr.\end{aligned}$$

The proof is finished. □

We can actually weaken the condition (43) in the case $p>1$ using the following modification of the p-distortion functions.

Definition 5.1. Let $\psi,\phi\in\mathcal{W}$ and $0<p<\infty$. We define

$$H_{\psi,\phi,p}(r)=e^{\Psi_p(r)-\phi(r)},\quad 0<p\le 1,$$

and

$$H_{\psi,\phi,p}(r)=\frac{re^{-(\phi(r)+(p-1)\psi(r))}}{\int_r^\infty e^{-p\psi(s)}ds},\quad 1<p<\infty.$$

In particular, $H_{\phi,\phi,p}(r)=e^{\max\{p,1\}(\Phi_p(r)-\phi(r))}$.

Remark 5.1. Note that for $p\ge 1$ we can write

(44) $$H_{\psi,\phi,p}(r)=e^{\Psi_p(r)-\phi(r)}e^{(p-1)(\Psi_p(r)-\psi(r))}=e^{p(\Psi_p(r)-\psi(r))}e^{\psi(r)-\phi(r)}.$$

In particular, due to (ii) in Lemma 2.1 if ψ is differentiable and convex, then

$$e^{\Psi_p(r)-\phi(r)}\le CH_{\psi,\phi,p}(r),\quad r>R,$$

and for $\psi\in\mathcal{W}_0$, from Proposition 2.3, one has

$$e^{\psi(r)-\phi(r)}\le CH_{\psi,\phi,p}(r),\quad r>R.$$

We shall use the following general fact.

Lemma 5.6. *Let $1\le p<\infty$, let $U,W\colon(0,\infty)\to(0,\infty)$ be measurable functions with $W\in L^1((0,\infty))$ and let $G\colon[0,\infty)\to\mathbf{R}^+$ be a continuous function. Assume that there exists $C>0$ such that*

(45) $$G(r)\le C\left(\frac{1}{r}\int_r^\infty W(t)\,dt\right)^{-1}U^{1/p}(r)W^{1/p'}(r),\quad r>0.$$

Then

(46) $$\int_0^\infty\left(\frac{1}{r}\int_0^r F(t)G(t)\,dt\right)^p rW(r)\,dr\le C\int_0^\infty F^p(r)rU(r)\,dr$$

for any continuous function $F\colon[0,\infty)\to\mathbf{R}^+$.

Proof. For $p=1$ condition (45) becomes $G(t)(\int_t^\infty W(r)dr)\le CtU(t)$ for $t>0$ and the result follows from Fubini's theorem.

Assume $p>1$. For each $R,\varepsilon>0$ integrating by parts we have

$$\int_\varepsilon^R\left(\frac{1}{r}\int_0^r F(t)G(t)\,dt\right)^p rW(r)\,dr$$

$$
\begin{aligned}
&= \left(\int_0^{\varepsilon} F(t)G(t)\,dt\right)^p \left(\int_{\varepsilon}^{\infty} \frac{W(t)}{t^{p-1}}\,dt\right) - \left(\int_0^{R} F(t)G(t)\,dt\right)^p \left(\int_{R}^{\infty} \frac{W(t)}{t^{p-1}}\,dt\right) \\
&\quad + p\int_0^R \left(\int_0^r F(t)G(t)\,dt\right)^{p-1} F(r)G(r)\left(\int_r^{\infty} \frac{W(t)}{t^{p-1}}\,dt\right) dr \\
&\le \left(\varepsilon\int_0^{\infty} W(t)\,dt\right)\left(\frac{1}{\varepsilon}\int_0^{\varepsilon} F(t)G(t)\,dt\right)^p \\
&\quad + p\int_0^{\infty} \left(\int_0^r F(t)G(t)\,dt\right)^{p-1} F(r)G(r)\left(\int_r^{\infty} \frac{W(t)}{t^{p-1}}dt\right) dr.
\end{aligned}
$$

Now passing to the limit as $R \to \infty$ and $\varepsilon \to 0$, applying (45) and Hölder's inequality we have

$$
\begin{aligned}
&\int_0^{\infty} \left(\frac{1}{r}\int_0^r F(t)G(t)\,dt\right)^p rW(r)\,dr \\
&\le p\int_0^{\infty} \left(\int_0^r F(t)G(t)\,dt\right)^{p-1} F(r)G(r)\left(\int_r^{\infty} \frac{W(t)}{t^{p-1}}\,dt\right) dr \\
&\le p\int_0^{\infty} \left(\frac{1}{r}\int_0^r F(t)G(t)\,dt\right)^{p-1} F(r)G(r)\left(\int_r^{\infty} W(t)\,dt\right) dr \\
&\le C\left(\int_0^{\infty} \left(\frac{1}{r}\int_0^r F(t)G(t)\,dt\right)^p rW(r)\,dr\right)^{1/p'} \\
&\quad \cdot \left(\int_0^{\infty} F(r)^p G(r)^p W^{1-p}(r) r \left(\frac{1}{r}\int_r^{\infty} W(t)\,dt\right)^p dr\right)^{1/p} \\
&\le C\left(\int_0^{\infty} \left(\frac{1}{r}\int_0^r F(t)G(t)\,dt\right)^p rW(r)\,dr\right)^{1/p'} \left(\int_0^{\infty} F^p(r) rU(r)\,dr\right)^{1/p}.
\end{aligned}
$$

This implies (46) and the proof is then complete. □

Theorem 5.7. *Let $0 < p < \infty$, $\phi, \psi \in \mathcal{W}$ and $g \in \mathcal{H}(\mathbf{C})$. If there exists $A > 0$ such that*

$$M_{\infty}(Dg, r) \le AH_{\psi,\phi,p}(r), \quad r > 0, \tag{47}$$

then $\tilde{V}_g$ is bounded from $\mathcal{F}_p^{\phi}(\mathbf{C})$ into $\mathcal{F}_p^{\psi}(\mathbf{C})$.

Proof. The case $0 < p \le 1$ was shown in Proposition 5.5.

Let us assume now that $1 < p < \infty$. Writing $\tilde{V}_g(f)(z) = \int_0^1 f(zt)Dg(zt)\,dt$ we have for $0 < r < \infty$ and $\theta \in [0, 2\pi)$,

$$|\tilde{V}_g(f)(re^{i\theta})| \le \int_0^1 |f(re^{i\theta}t)| M_{\infty}(Dg, rt)\,dt.$$

Using vector-valued Minkowski's inequality we have

$$M_p(\tilde{V}_g(f), r) \le \frac{1}{r}\int_0^r M_p(f, t) M_{\infty}(Dg, t)\,dt. \tag{48}$$

Let $U(r) = e^{-p\phi(r)}$ and $W(r) = e^{-p\psi(r)}$ and observe that

$$H_{\psi,\phi,p}(r) = \left(\frac{1}{r}\int_r^{\infty} W(t)\,dt\right)^{-1} U^{1/p}(r) W^{1/p'}(r).$$

Consider now $F(t) = M_p(f,t)$ and $G(t) = M_\infty(Dg,t)$ and notice that (47) together with (48) allow us to apply Lemma 5.6 to obtain

$$\int_0^\infty M_p^p(\tilde{V}_g(f),r)re^{-p\psi(r)}dr \le C\int_0^\infty M_p^p(f,r)re^{-p\phi(r)}dr.$$

This finishes the proof. □

We can now extend the condition in Proposition 5.2 also for boundedness in Fock-type spaces, at least for convex functions ψ.

Corollary 5.8. *Let $0 < p < \infty$ and let $\psi \in \mathcal{W}$ be differentiable and convex in $(0,\infty)$. If $g \in \mathcal{H}(\mathbf{C})$ satisfies*

$$\sup_{r>0} \frac{e^{\phi(r)-\psi(r)}M_\infty(Dg,r)}{r\psi'(r)} = A < \infty, \tag{49}$$

then $\tilde{V}_g\colon \mathcal{F}_p^\phi(\mathbf{C}) \to \mathcal{F}_p^\psi(\mathbf{C})$ is bounded.

Proof. First observe that

$$r\psi'(r)e^{-p\Psi_p(r)} \le \int_r^\infty \psi'(s)e^{-p\psi(s)}ds \le \frac{1}{p}e^{-p\psi(r)}, \quad r > 0.$$

Hence assumption (49) gives

$$M_\infty(Dg,r) \le \frac{A}{p}e^{(1-p)(\psi(r)-\Psi_p(r))}e^{\Psi_p(r)-\phi(r)}.$$

Hence, according to (44) we obtain the condition (47) in the case $p \ge 1$. On the other hand, for $0 < p \le 1$ due to part (ii) in Lemma 2.1 to know that $\sup_{r>0} e^{\psi(r)-\Psi_p(r)} < \infty$. Hence $M_\infty(Dg,r) \le Ke^{\Psi_p(r)-\phi(r)} = KH_{\psi,\phi,p}(r)$. The result now follows from Theorem 5.7. □

Corollary 5.9. *Let $0 < p < \infty$, $\phi(r) = \varphi_{\alpha,\beta,\gamma}(r)$ for $\beta > 0$, $\gamma \ge 0$ and $\alpha \ge 1$ and let $g \in \mathcal{H}_0(\mathbf{C})$. Then the following statements are equivalent:*

(i) *$V_g\colon \mathcal{F}_p^\phi(\mathbf{C}) \to \mathcal{F}_p^\phi(\mathbf{C})$ is bounded.*

(ii) *$g \in \mathcal{P}$ and $1 \le \deg(g) \le [\alpha]$.*

Proof. (i) $\Longrightarrow$ (ii) This was shown in Corollary 4.3.

(ii) $\Longrightarrow$ (i). Let $1 \le k \le [\alpha]$ and let us show that $V_k\colon \mathcal{F}_p^\phi(\mathbf{C}) \to \mathcal{F}_p^\phi(\mathbf{C})$ is bounded, or equivalently $\tilde{V}_{k-1}\colon \mathcal{F}_p^\phi(\mathbf{C}) \to \mathcal{F}_p^{\phi(1)}(\mathbf{C})$ is bounded. From Proposition 5.7 it suffices to see that (47) holds for $g(z) = z^{k-1}$. Recall that $H^{-1}_{\psi,\phi,p}(r) = e^{p(\psi(r)-\Psi_p(r))}e^{\phi(r)-\psi(r)}$ for $p \ge 1$ and $H^{-1}_{\psi,\phi,p}(r) = e^{\phi(r)-\Psi_p(r)}$ for $0 < p < 1$. Hence, in particular for $\psi = \phi_{(1)} = \varphi_{\alpha,\beta,\gamma+1}$ we have $\phi(r) - \psi(r) = \log(r)$, we obtain, invoking (iii) in Lemma 2.1, that

$$H^{-1}_{\psi,\phi,p}(r) \le Cr^{-\alpha+1}, \quad r > 0. \tag{50}$$

This gives

$$\sup_{r\ge 1} H^{-1}_{\psi,\phi,p}(r)M_\infty(Du_{k-1},r) \le C_k \sup_{r\ge 1} r^{k-\alpha} < \infty.$$

The proof is now complete. □

Acknowledgements. I wish to thank J. Bonet for his comments on the first draft of the paper. I am also grateful to the referee for his/her careful reading.

References

[1] Aleman, A.: A class of integral operators on spaces of analytic functions. - Topics in complex analysis and operator theory 330, Univ. Malaga, Malaga, 2007.

[2] Aleman, A., and O. Constantin: Spectra of integration operators on weighted Bergman spaces. - J. Anal. Math. 109, 2009, 199–231.

[3] Aleman, A., and J. A. Peláez: Spectra of integration operators and weighted square functions. - Indiana Univ. Math. J. 61, 2012, 1–19.

[4] Aleman, A., and A. G. Siskakis: An integral operator on H^p. - Complex Var. Theory Appl. 28, 1995, 149–158.

[5] Aleman, A., and A. G. Siskakis: Integration operators on Bergman spaces. - Indiana Univ. Math. J. 46, 1997, 337–356.

[6] Basallote, M., M. D. Contreras, C. Hernández-Mancera, M. J. Martín, and P. J. Paúl: Volterra operators and semigroups in weighted Banach spaces of analytic functions. - Collect. Math. 65, 2014, 233–249.

[7] Blasco, O., and A. Galbis: On Taylor coefficient of entire functions integrable against exponential weights. - Math. Nachr. 223, 2001, 5–21.

[8] Bonet, J.: The spectrum of Volterra operators on weighted spaces of entire functions. - Quart. J. Math. 66, 2015, 799–807.

[9] Bonet, J., and J. Taskinen: A note on Volterra operators on weighted Banach spaces of entire functions. - Math. Nachr. 288:11-12, 2015, 1216–1225.

[10] Bierstedt, K. D., J. Bonet, and J. Taskinen: Associated weights and spaces of holomorphic functions. - Studia Math. 127, 1998, 137–168.

[11] Constantin, O.: A Volterra-type integration operator on Fock spaces. - Proc. Amer. Math. Soc. 140:12, 2012, 4247–4257.

[12] Constantin, O., and J. A. Peláez: Integral operators, embedding theorems and a Littlewood–Paley formula on weighted Fock spaces. - J. Geom. Anal. 26:2, 2016, 1109–1154.

[13] Constantin, O., and A. M. Persson: The spectrum of Volterra-type integration operators on generalized Fock spaces. - Bull. London Math. Soc. 47:6, 2015, 958–963.

[14] Duren, P.: Theory of Hardy spaces. - Academic Press, New York-London, 1970.

[15] Hu, Z.: Extended Cesaro operators on mixed-norm spaces. - Proc. Amer. Math. Soc. 131:7, 2003, 2171–2179.

[16] Komatsu, H.: Ultradistributions I, structure theorems and a characterization. - J. Fac. Sci. Tokyo (IA) 20, 1973, 25–105.

[17] Pau, J., and J. A. Peláez: Embedding theorems and integration operators on Bergman spaces with rapidly decreasing weights. - J. Funct. Anal. 259, 2010, 2727–2756.

[18] Pavlović, M. and J. A. Peláez: An equivalence for weighted integrals of an analytic function and its derivative. - Math. Nachr. 281:11, 2008, 1612–1623.

[19] Peláez, J. A., and J. Rättyä: Weighted Bergman Spaces induced by rapidly decreasing weights. - Mem. Amer. Math. Soc. 227, 2014.

[20] Pommerenke, Ch.: Schlichte Funktionen un analytische Functionen von beschränkter mittlerer Oszilation. - Comment. Math. Helv. 52, 1977, 591–602.

[21] Tung, J.: Fock spaces. - PhD dissertation, University of Michigan, 2005.

[22] Zhu, K.: Operator theory in function spaces. Second Edition. - Math. Surveys Monogr. 138, Amer. Math. Soc. Providence, Rhode Island, 2007.

[23] Zhu, K.: Analysis on Fock spaces. - Springer-Verlag, New York, 2012.

Received 16 February 2017 • Accepted 28 April 2017