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LIPSCHITZ CONTINUITY OF BLOCH TYPE MAPPINGS WITH RESPECT TO BERGMAN METRIC

Shaolin Chen and David Kalaj

Hengyang Normal University, College of Mathematics and Statistics
Hengyang, Hunan 421008, P. R. China; mathechen@126.com
and Soochow University, Suzhou, Jiangsu 215006, P. R. China

University of Montenegro, Faculty of Natural Sciences and Mathematics
Cetinjski put b. b. 81000 Podgorica, Montenegro; davidk@t-com.me

Abstract. In this paper, we establish the sharp estimate of the Lipschitz continuity with respect to the Bergman metric. The obtained results are the improvement and generalization of the corresponding results of Ghatage et al. [3] and Hosokawa et al. [4].

1. Introduction and main results

Let $\mathbf{C}^n$ denote the Euclidean space of complex dimension n. For $z = (z_1, \ldots, z_n) \in \mathbf{C}^n$, the conjugate of z, denoted by $\overline{z}$, is defined by $\overline{z} = (\overline{z}_1, \ldots, \overline{z}_n)$. For z and $w = (w_1, \ldots, w_n) \in \mathbf{C}^n$, we write

$$\langle z, w\rangle := z \cdot w = \sum_{k=1}^{n} z_k \overline{w}_k \quad \text{and} \quad |z| := \langle z, z\rangle^{1/2} = \left(\sum_{k=1}^{n} |z_k|^2\right)^{1/2}.$$

For $a = (a_1, \ldots, a_n) \in \mathbf{C}^n$, we set

$$\mathbf{B}^n(a, r) = \{z \in \mathbf{C}^n \colon |z - a| < r\}.$$

Also, we use $\mathbf{B}^n$ to denote the unit ball $\mathbf{B}^n(0, 1)$ and let $\mathbf{D} = \mathbf{B}^1$. In the following, we always treat $z \in \mathbf{C}^n$ as a column vector, that is, $n \times 1$ column matrix

$$z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}.$$

The class of all holomorphic functions from $\mathbf{B}^n$ into $\mathbf{C}^n$ is denoted by $H(\mathbf{B}^n, \mathbf{C}^n)$. Let $\mathrm{Aut}(\mathbf{B}^n)$ be the automorphism group consisting of all biholomorphic self mappings of the unit ball $\mathbf{B}^n$ (cf. [8]).

For $z \in \mathbf{B}^n$, let

$$B(z) = \frac{(1 - |z|^2)I + A(z)}{(1 - |z|^2)^2}$$

be *the Bergman matrix*, where I is the $n \times n$ identity matrix and

$$A(z) = \begin{pmatrix} z_1\overline{z}_1 & \cdots & z_1\overline{z}_n \\ & \vdots & \\ z_n\overline{z}_1 & \cdots & z_n\overline{z}_n \end{pmatrix}.$$

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For a smooth curve $\gamma\colon [0,1] \to \mathbf{B}^n$, let

$$\ell(\gamma) = \int_0^1 \langle B(\gamma(t))\gamma'(t), \gamma'(t)\rangle^{1/2}\, dt.$$

For any two points z and w in $\mathbf{B}^n$, let $\beta(z,w)$ be the infimum of the set consisting of all $\ell(\gamma)$, where γ is a piecewise smooth curve in $\mathbf{B}^n$ from z to w. Then we call β *the Bergman metric* in $\mathbf{B}^n$ (cf. [11]).

The *prenorm* $\|f\|_{\mathcal{P}(n,\alpha)}$ of $f \in H(\mathbf{B}^n, \mathbf{C}^n)$ is given by

$$\|f\|_{\mathcal{P}(n,\alpha)} = \sup_{z\in\mathbf{B}^n} D_f^{n,\alpha}(z),$$

where $\alpha > 0$ and

$$D_f^{n,\alpha}(z) = (1-|z|^2)^{\frac{\alpha(n+1)}{2n}} |\det f'(z)|^{\frac{1}{n}}.$$

For more details on prenorm, we refer to [6].

Let $\mathcal{B}_{\mathcal{P}(n,\alpha)}$ be the class of all holomorphic mappings $f \in H(\mathbf{B}^n, \mathbf{C}^n)$ satisfying $\|f\|_{\mathcal{P}(n,\alpha)} < \infty$. We call $f \in \mathcal{B}_{\mathcal{P}(n,\alpha)}$ a Bloch type mappings. In particular, $\mathcal{B}_{\mathcal{P}(1,\alpha)}$ is the classical family of α-Bloch functions (cf. [10, 11]).

For $z, w \in \mathbf{D}$, the *pseudo-hyperbolic distance* is defined as

$$\rho(z,w) = \left|\frac{z-w}{1-\overline{w}z}\right|.$$

In particular, if $n = 1$, then, for $z, w \in \mathbf{D}$, $\tanh\beta(z,w) = \rho(z,w)$. In [3], Ghatage et al. showed that $D_f^{1,1}(z)$ is Lipschitz continuous with respect to the pseudo-hyperbolic metric, which is given in Theorem A below.

Theorem A. [3, Theorem 1] *Let $f \in \mathcal{B}_{\mathcal{P}(1,1)}$. Then, for all $z, w \in \mathbf{D}$,*

$$\left|D_f^{1,1}(z) - D_f^{1,1}(w)\right| \le C\|f\|_{\mathcal{P}(1,1)}\rho(z,w),$$

where $C = 3.31$.

We remark that the constant $C = 3.31$ in Theorem 1 is not sharp.

In [5], Hosokawa and Ohno proved

$$\left|D_f^{1,1}(z) - D_f^{1,1}(w)\right| \le 20\|f\|_{\mathcal{P}(1,1)}\rho(z,w), \tag{1}$$

and they used it to discussed the composition operators on the Bloch spaces (cf. [4, 5]). For more details on this topic, see [1, 2, 7, 9].

Let φ be a holomorphic mapping of $\mathbf{B}^n$ into $\mathbf{B}^n$. For all $f \in \mathcal{B}_{\mathcal{P}(n,\alpha)}$, let $C_\varphi\colon f \mapsto f\circ\varphi$ be a composition operator. As an application of Theorem A, Ghatage et al. [3] proved

Theorem B. [3, Theorem 2] *Let φ be a holomorphic mapping of $\mathbf{D}$ into $\mathbf{D}$. If for some constants $r \in (0, 1/4)$, and $\varepsilon > 0$, for each $w \in \mathbf{D}$, there is a point $z_w \in \mathbf{D}$ such that*

$$\rho(\varphi(z_w), w) < r, \quad \text{and} \quad \frac{1-|z_w|^2}{1-|\varphi(z_w)|^2}|\varphi'(z_w)| > \varepsilon,$$

then $C_\varphi\colon \mathcal{B}_{\mathcal{P}(1,1)} \to \mathcal{B}_{\mathcal{P}(1,1)}$ is bounded below.

In this paper, by using a different method, we generalize Theorems A and B to several dimensional case and obtain the sharp estimate of the Lipschitz constant with respect to the Bergman metric.

Theorem 1. *Let $f \in \mathcal{B}_{\mathcal{P}(n,1)}$. Then, for $z_1, z_2 \in \mathbf{B}^n$,*

$$\left|D_f^{n,1}(z_2) - D_f^{n,1}(z_1)\right| \leq M(n)\|f\|_{\mathcal{P}(n,1)}\left[\tanh\beta(z_1, z_2)\right]^{\frac{1}{n}}, \tag{2}$$

where $M(n) = (2+n)^{\frac{1}{2n}}\left(\frac{n+2}{n+1}\right)^{\frac{n+1}{2n}}$. Moreover, the constant $M(n)$ in (2) cannot be replaced by a smaller number.

The following result is an application of Theorem 1.

Theorem 2. *Let $f \in \mathcal{B}_{\mathcal{P}(1,1)}$. Then, for $z \in \mathbf{D}$,*

$$(1-|z|^2)\left(\left|\frac{\partial}{\partial z}D_f^{1,1}(z)\right| + \left|\frac{\partial}{\partial \overline{z}}D_f^{1,1}(z)\right|\right) \leq \frac{3\sqrt{3}}{2}\|f\|_{\mathcal{P}(1,1)} \tag{3}$$

and

$$|f''(z)| \leq \frac{\left(2|z| + \frac{3\sqrt{3}}{2}\right)\|f\|_{\mathcal{P}(1,1)}}{(1-|z|^2)^2}. \tag{4}$$

Moreover, the extreme functions $f(z) = \pm 3\sqrt{3}z^2/4$ show that the estimates (3) *and* (4) *are sharp.*

Applying Theorem 1, we get the following result which is an improvement of Theorem A.

Theorem 3. *Let φ be a holomorphic mapping of $\mathbf{B}^n$ into $\mathbf{B}^n$. Suppose that there is constants $0 < r < \frac{1}{M(n)}\left(\frac{n+2}{1+n}\right)^{\frac{1}{n}}$ and $\varepsilon > 0$ such that, for each $w \in \mathbf{B}^n$, there is a point $z_w \in \mathbf{B}^n$ satisfying $\tanh\beta(\varphi(z_w), w) < r^n$ and $|\tau_\varphi(z_w)| > \varepsilon$, where*

$$\tau_\varphi(z_w) = \left(\frac{1-|z_w|^2}{1-|\varphi(z_w)|^2}\right)^{\frac{n+1}{2n}}|\det\varphi'(z_w)|^{\frac{1}{n}}$$

and $M(n)$ is defined as in Theorem 1. Then, for all $f \in \mathcal{B}_{\mathcal{P}(n,1)}$, there is a constant $k(n,r,\varepsilon) > 0$ depended only on r, ε and n such that

$$\|C_\varphi(f)\|_{\mathcal{P}(n,1)} \geq k(n,r,\varepsilon)\|f\|_{\mathcal{P}(n,1)}.$$

The proofs of Theorems 1-3 will be presented in Section 2.

2. Proofs of the main results

We begin the section by recalling the following results which play an important role in the proofs of Theorem 1.

Lemma C. [2, Lemma 1.1] *For $x \in [0,1]$, let*

$$\varphi(x) = x(1-x^2)^{\frac{\alpha(n+1)}{2}}\sqrt{\alpha(1+n)+1}\left[\frac{\alpha(n+1)+1}{\alpha(n+1)}\right]^{\frac{\alpha(n+1)}{2}}$$

and

$$a_0(\alpha) = \frac{1}{\sqrt{\alpha(1+n)+1}}.$$

Then φ is increasing in $[0, a_0(\alpha)]$, decreasing in $[a_0(\alpha), 1]$ and $\varphi(a_0(\alpha)) = 1$.

Theorem D. [2, Theorem 1.2] *Suppose that $f \in H(\mathbf{B}^n, \mathbf{C}^n)$ such that $\|f\|_{\mathcal{P}(n,\alpha)} = 1$ and $\det f'(0) = \lambda \in (0,1]$. Then, for all z with $|z| \leq \frac{a_0(\alpha)+m_\alpha(\lambda)}{1+a_0(\alpha)m_\alpha(\lambda)}$, we have*

$$|\det f'(z)| \geq \mathrm{Re}\left(\det f'(z)\right) \geq \frac{\lambda(m_\alpha(\lambda)-|z|)}{m_\alpha(\lambda)(1-m_\alpha(\lambda)|z|)^{\alpha(n+1)+1}}, \tag{5}$$

where $m_\alpha(\lambda)$ is the unique real root of the equation $\varphi(x) = \lambda$ in the interval $[0, a_0(\alpha)]$ and, φ and $a_0(\alpha)$ are defined as in Lemma C. Moreover, for all z with $|z| \leq \frac{a_0(\alpha)-m_\alpha(\lambda)}{1-a_0(\alpha)m_\alpha(\lambda)}$, we have

$$|\det f'(z)| \leq \frac{\lambda(m_\alpha(\lambda)+|z|)}{m_\alpha(\lambda)(1+m_\alpha(\lambda)|z|)^{\alpha(n+1)+1}}. \tag{6}$$

Moreover, the estimates of (5) *and* (6) *are sharp.*

Proof of Theorem 1. Without loss of generality, we assume that $\|f\|_{\mathcal{P}(n,1)} = 1$ and $D_f^{n,1}(z_2) \leq D_f^{n,1}(z_1)$. Let $\phi \in \mathrm{Aut}(\mathbf{B}^n)$ such that $\phi(0) = z_1$ and $w = \phi^{-1}(z_2)$. For $z \in \mathbf{B}^n$, set $g = f(\phi(z))$. By [11, Proposition 1.21], we have

$$\tanh\beta(z_1, z_2) = \tanh\beta\big(\phi^{-1}(z_1), \quad \phi^{-1}(z_2)\big) = \tanh\beta(0, w) = |w|.$$

Since

$$|\det\phi'(z)| = \left(\frac{1-|\phi(z)|^2}{1-|z|^2}\right)^{\frac{n+1}{2}},$$

we see that

$$|\det g'(0)|^{\frac{1}{n}} = |\det f'(\phi(0))|^{\frac{1}{n}}|\det\phi'(0)|^{\frac{1}{n}} = |\det f'(z_1)|^{\frac{1}{n}}(1-|z_1|^2)^{\frac{n+1}{2n}} \tag{7}$$

and

$$\begin{aligned}(1-|w|^2)^{\frac{n+1}{2n}}|\det g'(w)|^{\frac{1}{n}} &= (1-|w|^2)^{\frac{n+1}{2n}}|\det f'(\phi(w))|^{\frac{1}{n}}|\det\phi'(w)|^{\frac{1}{n}}\\ &= (1-|w|^2)^{\frac{n+1}{2n}}|\det f'(z_2)|^{\frac{1}{n}}\left(\frac{1-|\phi(w)|^2}{1-|w|^2}\right)^{\frac{n+1}{2n}}\\ &= |\det f'(z_2)|^{\frac{1}{n}}(1-|z_2|^2)^{\frac{n+1}{2n}}.\end{aligned} \tag{8}$$

Case 1. If $|\det g'(0)| = 0$, then it is obvious.

Case 2. Let $\det g'(0) = \lambda e^{i\theta}$, where $\lambda > 0$ and $\theta \in [0, 2\pi]$. Applying Theorem D (5) to $e^{-i\frac{\theta}{n}}g(z)$, for $|z| \leq \frac{a_0(1)+m_1(\lambda)}{1+a_0(1)m_1(\lambda)}$, we have

$$\mathrm{Re}\left(e^{-i\theta}\det g'(z)\right) \geq \frac{\lambda(m_1(\lambda)-|z|)}{m_1(\lambda)(1-m_1(\lambda)|z|)^{n+2}}, \tag{9}$$

where $m_1(\lambda)$ and $a_0(1)$ are defined as in Theorem D.

Subcase 2.1. $|w| \leq m_1(\lambda)$. It is easy to know that

$$m_1(\lambda) \leq \frac{a_0(1)+m_1(\lambda)}{1+a_0(1)m_1(\lambda)}. \tag{10}$$

On the other hand, by calculations, we have

$$(1-|w|^2)^{\frac{n+1}{2n}} \geq (1-|w|m_1(\lambda))^{\frac{n+1}{2n}} \geq (1-|w|m_1(\lambda))^{\frac{n+2}{n}}$$

and

$$m_1^{\frac{1}{n}}(\lambda) - |w|^{\frac{1}{n}} \leq \big(m_1(\lambda)-|w|\big)^{\frac{1}{n}},$$

which gives

$$m_1^{\frac{1}{n}}(\lambda) - \frac{(1-|w|^2)^{\frac{n+1}{2n}}(m_1(\lambda)-|w|)^{\frac{1}{n}}}{(1-m_1(\lambda)|w|)^{\frac{n+2}{n}}} \leq |w|^{\frac{1}{n}}. \tag{11}$$

By Lemma C and Theorem D, we have

$$m_1(\lambda)(1-m_1^2(\lambda))^{\frac{n+1}{2}}\big(M(n)\big)^n = \lambda, \tag{12}$$

which, together with (7), (8), (9), (10) and (11), implies that

$$\begin{aligned} D_f^{n,1}(z_1) - D_f^{n,1}(z_2) &= |\det g'(0)|^{\frac{1}{n}} - (1-|w|^2)^{\frac{n+1}{2n}} |\det g'(w)|^{\frac{1}{n}} \\ &= \lambda^{\frac{1}{n}} - (1-|w|^2)^{\frac{n+1}{2n}} |\det g'(w)|^{\frac{1}{n}} \\ &\le \lambda^{\frac{1}{n}} - \frac{(1-|w|^2)^{\frac{n+1}{2n}} \lambda^{\frac{1}{n}} (m_1(\lambda)-|w|)^{\frac{1}{n}}}{m_1^{\frac{1}{n}}(\lambda)(1-m_1(\lambda)|w|)^{\frac{n+2}{n}}} \\ &= \left(\frac{\lambda}{m_1(\lambda)}\right)^{\frac{1}{n}} \left[m_1^{\frac{1}{n}}(\lambda) - \frac{(1-|w|^2)^{\frac{n+1}{2n}} (m_1(\lambda)-|w|)^{\frac{1}{n}}}{(1-m_1(\lambda)|w|)^{\frac{n+2}{n}}} \right] \\ &\le \left(\frac{\lambda}{m_1(\lambda)}\right)^{\frac{1}{n}} |w|^{\frac{1}{n}} = M(n)\big(1-m_1^2(\lambda)\big)^{\frac{1}{n}} |w|^{\frac{1}{n}} \le M(n)|w|^{\frac{1}{n}}. \end{aligned}$$

Subcase 2.2. $|w| > m_1(\lambda)$. Then, by (7), (8) and (12)

$$\begin{aligned} D_f^{n,1}(z_1) - D_f^{n,1}(z_2) &= |\det g'(0)|^{\frac{1}{n}} - (1-|w|^2)^{\frac{n+1}{2n}} |\det g'(w)|^{\frac{1}{n}} \\ &\le \lambda^{\frac{1}{n}} = m_1^{\frac{1}{n}}(\lambda)\big(1-m_1^2(\lambda)\big)^{\frac{n+1}{2n}} M(n) < M(n)|w|^{\frac{1}{n}}. \end{aligned}$$

Now we prove the sharpness part. For any $\epsilon \in (0, M(n)]$, let

$$m_1^*(\lambda) = \min\left\{ \left[1 - \left(1 - \frac{\epsilon}{M(n)}\right)^{\frac{2n}{n+1}}\right]^{\frac{1}{2}}, a_0(1) \right\} \tag{13}$$

and

$$\lambda = m_1^*(\lambda)\left[1 - \big(m_1^*(\lambda)\big)^2\right]^{\frac{n+1}{2}} \big(M(n)\big)^n. \tag{14}$$

By (13), we get

$$M(n)\left[1 - (m_1^*(\lambda))^2\right]^{\frac{n+1}{2n}} \ge M(n) - \epsilon. \tag{15}$$

For $z \in \mathbf{B}^n$, let

$$f_\lambda(z) = \begin{pmatrix} \int_0^{z_1} \frac{\lambda(m_1^*(\lambda)-\xi)}{m_1^*(\lambda)(1-m_1^*(\lambda)\xi)^{n+2}}\, d\xi \\ z_2 \\ \vdots \\ z_n \end{pmatrix},$$

which gives that $\det f_\lambda'(0) = \lambda$.

Claim 1.

$$\|f_\lambda\|_{\mathcal{P}(n,1)} = 1. \tag{16}$$

Now we prove (16). For $z \in \mathbf{B}^n$, let

$$F(z) = \begin{pmatrix} -\frac{\big(M(n)\big)^n}{2} z_1^2 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}.$$

By Lemma C, we have

$$\|F\|_{\mathcal{P}(n,1)} = \sup_{z\in\mathbf{B}^n} D_F^{n,1}(z) = 1. \tag{17}$$

For $a=(m_1^*(\lambda),0,\ldots,0)$ and $z\in\mathbf{B}^n$, set

$$\phi_a(z)=\left(\frac{m_1^*(\lambda)-z_1}{1-m_1^*(\lambda)z_1},\frac{\left[1-\left(m_1^*(\lambda)\right)^2\right]^{\frac{1}{2}}z_2}{m_1^*(\lambda)z_1-1},\ldots,\frac{\left[1-\left(m_1^*(\lambda)\right)^2\right]^{\frac{1}{2}}z_n}{m_1^*(\lambda)z_1-1}\right)\in\mathrm{Aut}(\mathbf{B}^n).$$

Then, by (14), we have

$$\begin{aligned}|\det f_\lambda'(z)|&=\frac{\left(M(n)\right)^n|m_1^*(\lambda)-z_1|\left[1-\left(m_1^*(\lambda)\right)^2\right]^{\frac{n+1}{2}}}{|1-m_1^*(\lambda)z_1|^{2+n}}\\&=\left|\det\left(F(\phi_a(z))\right)'\right|\\&=|\det F'(\phi_a(z))|\,|\det\phi_a'(z)|\\&=\frac{|\det F'(\phi_a(z))|\,(1-|\phi_a(z)|^2)^{\frac{n+1}{2}}}{(1-|z|^2)^{\frac{n+1}{2}}},\end{aligned}$$

which, together with (17), implies that

$$\begin{aligned}D_{f_\lambda}^{n,1}(z)&=(1-|z|^2)^{\frac{n+1}{2n}}\left|\det\left(F(\phi_a(z))\right)'\right|^{\frac{1}{n}}\\&=|\det F'(\phi_a(z))|^{\frac{1}{n}}\,(1-|\phi_a(z)|^2)^{\frac{n+1}{2n}}=D_F^{n,1}(\phi_a(z))\le 1.\end{aligned}\tag{18}$$

Hence (16) follows from (18). The proof of Claim 1 is finished.

Therefore, for $w'=(0,0,\ldots,0)$ and $w''=(m_1^*(\lambda),0,\ldots,0)$, we have

$$\tanh\beta(w',w'')=m_1^*(\lambda),$$

which, together with (14), (15) and (16), yields that

$$\begin{aligned}\left|D_{f_\lambda}^{n,1}(w')-D_{f_\lambda}^{n,1}(w'')\right|&=\lambda^{\frac{1}{n}}=M(n)\left(m_1^*(\lambda)\right)^{\frac{1}{n}}\left[1-(m_1^*(\lambda))^2\right]^{\frac{n+1}{2n}}\\&\ge\|f_\lambda\|_{\mathcal{P}(n,1)}\left[\tanh\beta(w',w'')\right]^{\frac{1}{n}}\left(M(n)-\epsilon\right).\end{aligned}$$

The above inequality shows that the constant $M(n)$ is sharp. The proof of this theorem is complete. □

Proof of Theorem 2. For $z=x+iy\in\mathbf{D}$, let $w=z+re^{i\theta}$. Then, by Theorem 1, we have

$$\begin{aligned}\Lambda_f(z)&=\max_{\theta\in[0,2\pi]}\left[\lim_{r\to0^+}\left(\frac{\left|D_f^{1,1}(z)-D_f^{1,1}(w)\right|}{r}\cdot\frac{r}{\rho(z,w)}\right)\right]\\&=(1-|z|^2)\max_{\theta\in[0,2\pi]}\left|\frac{\partial}{\partial x}D_f^{1,1}(z)\cos\theta+\frac{\partial}{\partial y}D_f^{1,1}(z)\sin\theta\right|\\&=\frac{(1-|z|^2)}{2}\left(\left|\frac{\partial}{\partial x}D_f^{1,1}(z)+i\frac{\partial}{\partial y}D_f^{1,1}(z)\right|+\left|\frac{\partial}{\partial x}D_f^{1,1}(z)-i\frac{\partial}{\partial y}D_f^{1,1}(z)\right|\right)\\&=(1-|z|^2)\left(\left|\frac{\partial}{\partial z}D_f^{1,1}(z)\right|+\left|\frac{\partial}{\partial\overline{z}}D_f^{1,1}(z)\right|\right)\le\frac{3\sqrt{3}}{2}\|f\|_{\mathcal{P}(1,1)},\end{aligned}$$

where

$$\Lambda_f(z)=\max_{\theta\in[0,2\pi]}\left(\lim_{r\to0^+}\frac{\left|D_f^{1,1}(z)-D_f^{1,1}(w)\right|}{\rho(z,w)}\right).$$

On the other hand,

$$\Big|\frac{\partial}{\partial z}D_f^{1,1}(z)\Big| + \Big|\frac{\partial}{\partial \overline{z}}D_f^{1,1}(z)\Big| = \left|-\overline{z}|f'(z)| + \frac{f''(z)\overline{f'(z)}}{2|f'(z)|}(1-|z|^2)\right| + \left|-z|f'(z)| + \frac{\overline{f''(z)}f'(z)}{2|f'(z)|}(1-|z|^2)\right| \geq |f''(z)|(1-|z|^2) - 2|z||f'(z)|,$$

which, together with (19), gives that

$$|f''(z)|(1-|z|^2) \leq 2|z||f'(z)| + \Big|\frac{\partial}{\partial z}D_f^{1,1}(z)\Big| + \Big|\frac{\partial}{\partial \overline{z}}D_f^{1,1}(z)\Big| \leq \frac{2|z|\|f\|_{\mathcal{P}(1,1)}}{1-|z|^2} + \frac{\frac{3\sqrt{3}}{2}\|f\|_{\mathcal{P}(1,1)}}{1-|z|^2} = \frac{\left(2|z| + \frac{3\sqrt{3}}{2}\right)\|f\|_{\mathcal{P}(1,1)}}{(1-|z|^2)}.$$

The proof of this theorem is complete. □

Proof of Theorem 3. Without loss of generality, we assume that $\|f\|_{\mathcal{P}(n,1)} = 1$. For $z \in \mathbf{B}^n$, it follows from Theorem 1 that there is a point $w \in \mathbf{B}^n$ such that

$$D_f^{n,1}(w) > 1 - \sigma$$

and

$$\big|D_f^{n,1}(w) - D_f^{n,1}(z)\big| \leq \left[M(n)\left(\frac{n+1}{n+2}\right)^{\frac{1}{n}} + \sigma\right]\big[\tanh\beta(w,z)\big]^{\frac{1}{n}},$$

where

$$\sigma = \frac{1 - rM(n)\left(\frac{n+1}{n+2}\right)^{\frac{1}{n}}}{2(1+r)} \quad \text{and} \quad r < \frac{1}{M(n)}\left(\frac{n+2}{1+n}\right)^{\frac{1}{n}}.$$

By the assumption, there is a point z_w such that

$$\big[\tanh\beta(\varphi(z_w),w)\big]^{\frac{1}{n}} < r < \frac{1}{M(n)}\left(\frac{n+2}{1+n}\right)^{\frac{1}{n}}$$

and $|\tau_\varphi(z_w)| > \varepsilon$, which imply that

$$\begin{aligned} D_{f_\lambda}^{n,1}(\varphi(z_w)) &\geq D_{f_\lambda}^{n,1}(w) - \left[M(n)\left(\frac{n+1}{n+2}\right)^{\frac{1}{n}} + \sigma\right]\big[\tanh\beta(\varphi(z_w),w)\big]^{\frac{1}{n}} \\ &\geq 1 - \sigma - \left[M(n)\left(\frac{n+1}{n+2}\right)^{\frac{1}{n}} + \sigma\right] r \\ &= 1 - rM(n)\left(\frac{n+1}{n+2}\right)^{\frac{1}{n}} - (1+r)\sigma \\ &= \frac{1 - rM(n)\left(\frac{n+1}{n+2}\right)^{\frac{1}{n}}}{2} > 0. \end{aligned} \tag{19}$$

By (19), we conclude that

$$\|C_\varphi(f)\|_{\mathcal{P}(n,1)} \geq D_{f_\lambda}^{n,1}(\varphi(z_w))|\tau_\varphi(z_w)| > k(n,r,\varepsilon),$$

where

$$k(n,r,\varepsilon)=\frac{\left[1-rM(n)\left(\frac{n+1}{n+2}\right)^{\frac{1}{n}}\right]\varepsilon}{2}.$$

The proof of this theorem is compete. □

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