

ALMOST ISOMETRIES BETWEEN TEICHMÜLLER SPACES

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Abstract. We prove that the Teichmüller space of hyperbolic surfaces with given boundary lengths equipped with the arc metric (resp. the Teichmüller metric) is almost isometric to the Teichmüller space of punctured surfaces equipped with the Thurston metric (resp. the Teichmüller metric).

1. Introduction

Let S be an oriented surface of genus g with n boundary components such that $n \geq 1$. The Euler characteristic of S is $\chi(S) = 2 - 2g - n$. Throughout this paper we assume that $\chi(S) < 0$. Recall that a *marked complex structure* on S is a pair (X, f) where X is a Riemann surface and $f: S \rightarrow X$ is an orientation preserving homeomorphism. Two marked complex structures (X, f) and (Y, h) are called *equivalent* if there is a conformal map homotopic to $f \circ h^{-1}$. Denote by $[X, f]$ the equivalence class of (X, f) . The set of equivalence classes of marked complex structures is the Teichmüller space denoted by $T_{g,n}$.

Let X be a Riemann surface with boundary. There exist two different hyperbolic metrics on X . One is of infinite area obtained from the Uniformization theorem, the other one is of finite area obtained from the restriction to X of the hyperbolic metric on its (Schottky) double such that each boundary component is a smooth simple closed geodesic (see §2.1). The second one is called the intrinsic metric on X . In this paper when we mention a hyperbolic metric on a surface with nonempty boundary we mean the second one. The correspondence between complex structure and hyperbolic metric provides another approach for the Teichmüller theory. Recall that a *marked hyperbolic surface* (X, f) is a hyperbolic surface X equipped with an orientation-preserving homeomorphism $f: S \rightarrow X$, where f maps each component of the boundary of S to a geodesic boundary of X . Two marked hyperbolic surfaces (X, f) and (Y, h) are called *equivalent* if there is an isometry homotopic to $f \circ h^{-1}$ relative to the boundary. The Teichmüller space $T_{g,n}$ is also the set of equivalence classes of marked hyperbolic surfaces. For simplicity, we will denote a point $[X, f]$ in $T_{g,n}$ by X , without explicit reference to the marking or to the equivalence relation.

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Let β_1, \dots, β_n be the boundary components of S . For any $\Lambda = (\lambda_1, \dots, \lambda_n) \in R_{\geq 0}^n$, let $T_{g,n}(\Lambda) \subset T_{g,n}$ be the set of the equivalence classes of marked hyperbolic metrics whose boundary components have hyperbolic lengths $(l(\beta_1), \dots, l(\beta_n)) = \Lambda$. In particular, $T_{g,n}(0)$ is the Teichmüller space of surfaces with n punctures. It is clear that $T_{g,n} = \bigcup_{\Lambda \in \mathbf{R}_+^n} T_{g,n}(\Lambda)$. Let $\Gamma = \{\gamma_1, \dots, \gamma_{3g-3+n}\}$ be a pants decomposition of S , i.e., the complement of Γ on S consists of $2g - 2 + n$ pairs of pants $\{R_i\}_{i=1}^{2g-2+n}$. Let μ be a set of disjoint simple closed curves whose restriction to any pair of pants R_i consists of three arcs, such that any two of the arcs are not free homotopic with respect to the boundary of R_i . The pair (Γ, μ) is called a *marking* of S . For any $X \in T_{g,n}$, let $(\mathcal{L}, \mathcal{T}, \Lambda)$ be the corresponding Fenchel–Nielsen coordinates with respect to the marking (Γ, μ) , where $\mathcal{L} = (l_1, \dots, l_{3g-3+n})$ represents the lengths of $\{\gamma_1, \dots, \gamma_{3g-3+n}\}$, $\mathcal{T} = (t_1, \dots, t_{3g-3+n})$ represents the twists along $\{\gamma_1, \dots, \gamma_{3g-3+n}\}$ and $\Lambda = (\lambda_1, \dots, \lambda_n)$ represents the lengths of the boundary components (for details about Fenchel–Nielsen coordinates we refer to [4]). The Fenchel–Nielsen coordinates induce a natural homeomorphism between Teichmüller spaces $T_{g,n}(\Lambda)$ and $T_{g,n}(0)$ in the following way:

$$\begin{aligned} \Phi_\Gamma: T_{g,n}(\Lambda) &\longrightarrow T_{g,n}(0), \\ (\mathcal{L}, \mathcal{T}, \Lambda) &\longmapsto (\mathcal{L}, \mathcal{T}, 0). \end{aligned}$$

The goal of this paper is to compare various metrics on the Teichmüller spaces $T_{g,n}(\Lambda)$ and $T_{g,n}(0)$ via the homeomorphism Φ_Γ .

Definition 1.1. Two metric spaces (X_1, d_1) and (X_2, d_2) are called *almost isometric* if there exist a map $f: X_1 \rightarrow X_2$, two positive constants A and B , such that both of the following two conditions hold.

(1) For any $x, y \in X_1$,

$$|d_2(f(x), f(y)) - d_1(x, y)| \leq B.$$

(2) For any $z \in X_2$, there exists $x \in X_1$ such that

$$d_2(z, f(x)) \leq A.$$

1.1. The Thurston metric and the arc metric. An *essential simple closed curve* on S is a simple closed curve which is not homotopic to a single point or a boundary component. An *essential arc* is a simple arc whose endpoints are on the boundary and which is not homotopic to any subarc of the boundary. Let $\mathcal{S}(S)$ be the set of homotopy classes of essential simple closed curves on S , $\mathcal{A}(S)$ be the set of homotopy classes of essential arcs on S , and $\mathcal{B}(S)$ be the set of homotopy classes of the boundary components.

For any $X_1, X_2 \in T_{g,n}(\Lambda)$, define

$$d_{Th}(X_1, X_2) := \log \sup_{[\alpha] \in \mathcal{S}(S)} \frac{l_{X_2}([\alpha])}{l_{X_1}([\alpha])}$$

and

$$d_A(X_1, X_2) := \log \sup_{[\alpha] \in \mathcal{A}(S)} \frac{l_{X_2}([\alpha])}{l_{X_1}([\alpha])}.$$

From the works [16] and [11], both d_{Th} and d_A are asymmetric metric on $T_{g,n}(\Lambda)$, which are called the *Thurston metric* and the *arc metric* respectively. Moreover, the

authors [11] observed that

$$d_A(X_1, X_2) = \log \sup_{[\alpha] \in \mathcal{A}(S) \cup \mathcal{B}(S) \cup \mathcal{S}(S)} \frac{l_{X_2}([\alpha])}{l_{X_1}([\alpha])}.$$

Our first result is the following.

Theorem 1.2. *($T_{g,n}(\Lambda), d_A$) and ($T_{g,n}(0), d_{Th}$) are almost isometric. More precisely, there is a constant C_1 depending on the surface S and boundary lengths Λ such that,*

$$|d_A(X_1, X_2) - d_{Th}(\Phi_\Gamma(X_1), \Phi_\Gamma(X_2))| \leq C_1.$$

Remark 1. Papadopoulos and Su [17] considered the case where Λ is close to zero, they showed that the constant C_1 in Theorem 1.2 will tend to zero if Λ tends to zero.

Proof of Theorem 1.2. To prove Theorem 1.2, it suffices to verify that they satisfy the two conditions in Definition 1.1. The first condition follows from Theorem 1.3 and Theorem 1.4. The second condition follows from the fact that Φ_Γ is a homeomorphism. \square

Theorem 1.3. *The arc metric and the Thurston metric are almost-isometric in $T_{g,n}(\Lambda)$. More precisely, there is a constant C_2 depending on the surfaces S and boundary lengths Λ such that,*

$$|d_A(X_1, X_2) - d_{Th}(X_1, X_2)| \leq C_2.$$

Remark 2. Liu, Papadopoulos, Su and Théret [11, Theorem 3.7] proved that d_{Th} and d_A are almost isometric on the ϵ_0 -relative ϵ -thick part of the Teichmüller space of surfaces with boundary. Later, Liu, Su and Zhong [12, Theorem 1.5] proved that the symmetrizations of these two metrics $d_L(X_1, X_2) := \max\{d_{Th}(X_1, X_2), d_{Th}(X_2, X_1)\}$ and $d_{AL}(X_1, X_2) := \max\{d_A(X_1, X_2), d_A(X_2, X_1)\}$ are almost isometric on the ϵ thick part of the Teichmüller space of surfaces with boundary.

Theorem 1.4. [16] *($T_{g,n}(\Lambda), d_{Th}$) and ($T_{g,n}(0), d_{Th}$) are almost isometric. More precisely, there is a constant C_3 depending on the surfaces S and boundary lengths Λ such that,*

$$|d_{Th}(X_1, X_2) - d_{Th}(\Phi_\Gamma(X_1), \Phi_\Gamma(X_2))| \leq C_3.$$

1.2. The Teichmüller metric. The arc metric and the Thurston metric describe the deformation of hyperbolic metric on the surface, while the Teichmüller metric describes the deformation of conformal structure (complex structure). Given two marked complex structures $[X_1, f_1]$ and $[X_2, f_2]$, the Teichmüller metric is defined by

$$d_T([X_1, f_1], [X_2, f_2]) = \frac{1}{2} \log \inf \{K(f) : f \text{ is isotopic to } f_1 \circ (f_2)^{-1}\},$$

where $K(f)$ represents the quasiconformal dilation of f .

For closed surfaces, Kerckhoff expressed the Teichmüller metric in terms of the extremal length of simple closed curves in the following way. For any X_1, X_2 in the Teichmüller space,

$$(1) \quad d_T(X_1, X_2) := \frac{1}{2} \sup_{[\alpha]} \log \frac{\text{Ext}_{X_2}([\alpha])}{\text{Ext}_{X_1}([\alpha])},$$

where the sup ranges over all essential simple closed curves on the surface.

For surfaces with boundary, Liu, Papadopoulos, Su and Th  ret ([10]) developed similar result. They described the Teichm  ller metric in terms of the extremal lengths of essential arcs and boundary components.

The theorem below is our second main result.

Theorem 1.5. *For small ϵ , $(\mathcal{T}_{g,n}(\epsilon), d_T)$ and $(\mathcal{T}_{g,n}(0), d_T)$ are almost isometric. More precisely, for any $X, Y \in \mathcal{T}_{g,b}(\epsilon)$,*

$$|d_T(X_1, X_2) - d_T(\Phi_\Gamma(X_1), \Phi_\Gamma(X_2))| \leq \log(n + 3).$$

Remark 3. The constant $\log(n + 3)$ is not optimal.

The organization of this paper is as following. In  2, we recall some basic concepts and facts. In  3, we prove Theorem 1.3. In  4, we prove Theorem 1.5. Finally, we collect a few questions in  5.

2. Preliminaries

2.1. (Schottky) double and Teichm  ller map. Let X be a Riemann surface with nonempty boundary. We can represent X as H/G , where H is the upper half plane and G is a torsion-free Fuchsian group of second kind. There is an infinite set \mathcal{I} of open intervals I on the extended real axis $\mathbf{R} \cup \infty$ such that G acts properly discontinuously on $H \cup J \cup L$ where L is the lower half plane and J is the union of all $I \in \mathcal{I}$. The surface $X^d := (H \cup J \cup L)/G$ is called the (Schottky) *double* of X and $X_q := L/G$ is called the *mirror image* of X . The restriction of the hyperbolic metric on X^d to X is called the *intrinsic metric* on X . It is clear that in the intrinsic metric each boundary component is a simple closed geodesic. The double of any essential geodesic arc on X is a simple closed geodesic on X^d .

An *admissible quadratic differential* on X is the restriction to X of a holomorphic quadratic differential q^d on X^d such that

- (1) at each puncture on X , q^d has at worst a first order pole,
- (2) ∂X is an q^d -horizontal line.

Note that the symmetry requires that the zeroes of q^d on ∂X have even order. Away from the zeroes of q^d , there is a local coordinate $\zeta = \xi + i\eta$ such that $q^d = d\zeta^2$. Let $\bar{\zeta} = K\xi + i\eta$, $0 < K < \infty$. $\bar{\zeta}$ defines a new Riemann surface $X_{q^d}^d$. The map $f: \zeta \mapsto \bar{\zeta}$ is called the *Teichm  ller map* from X^d to $X_{q^d}^d$ with initial quadratic differential q^d . The restriction of f to X is called the *Teichm  ller map* from X to \bar{X} with initial quadratic differential q . Given two marked Riemann surfaces $[X_1, f_1]$, $[X_2, f_2]$ with boundary, there is a unique Teichm  ller map $f: X_1 \rightarrow X_2$ homotopic to $f_2 \circ f_1^{-1}$ minimizing the quasiconformal dilation [1] such that the initial quadratic differential is an admissible quadratic differential on X .

2.2. Measured lamination. Given a hyperbolic surface X with nonempty geodesic boundary, a simple geodesic is one of the four types below:

- an essential simple closed geodesic;
- a geodesic boundary component;
- an essential geodesic arc;
- an infinite geodesic in the interior.

A *geodesic lamination* m on X is a closed subset of X consisting of mutually disjoint simple geodesics which are called leaves of this geodesic lamination. A *transverse invariant measure* μ of a geodesic lamination m is a Radon measure defined on every arc k transverse to the support of m such that μ is invariant with respect to

any homotopy of k relative to the leaves of m . A *measured geodesic lamination* is a lamination m endowed with a transverse invariant measure μ . For simplicity, we denote by μ the measured geodesic lamination (m, μ) . Each measured geodesic lamination μ induces a functional i_μ over $\mathcal{S}(S) \cup \mathcal{B}(S)$ in the following way:

$$i_\mu: \mathcal{S}(S) \cup \mathcal{B}(S) \longrightarrow \mathbf{R}_{\geq 0},$$

$$[\alpha] \longmapsto i(\mu, [\alpha]) := \inf_{\alpha' \in [\alpha]} \int_{\alpha'} d\mu.$$

Two measured geodesic laminations μ, μ' are said to be equivalent if $i(\mu, [\alpha]) = i(\mu', [\alpha])$ for any $[\alpha] \in \mathcal{S}(S) \cup \mathcal{B}(S)$. Denote by $\mathcal{ML}(X)$ the space of equivalence classes of measured geodesic laminations on X equipped with the topology that μ_n converges to μ if for any $[\alpha] \in \mathcal{S}(S) \cup \mathcal{B}(S)$, $i(\mu_n, [\alpha])$ converges to $i(\mu, [\alpha])$. Since there is a one-to-one correspondence between $\mathcal{ML}(X)$ and $\mathcal{ML}(X')$ for two different hyperbolic metrics X and X' , we denote by $\mathcal{ML}(S)$ the space of equivalence classes of measured geodesic lamination without pointing to any specific hyperbolic metric. Hubbard and Masur ([8]) proved that there is a homeomorphism between $\mathcal{ML}(S)$ and the space of the horizontal measured foliations of admissible quadratic differentials on X . For surfaces of finite type, $\mathcal{S}(S) \times \mathbf{R}_+$ is dense in $\mathcal{ML}(S)$ in this topology. But for the surfaces with boundary, this is no longer true. The simplest counterexample is an essential geodesic arc since it is not in the closure of $\mathcal{S}(S) \times \mathbf{R}_+$. Let $\mathcal{ML}_0(S)$ be a subset of $\mathcal{ML}(S)$ consisting of measured foliations whose leaves are either essential simple closed geodesics or infinite geodesics in the interior. It is clear that $\mathcal{S}(S) \times \mathbf{R}_+$ is dense in $\mathcal{ML}_0(S)$.

2.3. Extremal length. Let α be a simple closed curve or an essential arc, and X be a Riemann surface. A *conformal metric* on X is a metric which can be expressed as $\rho(z)|dz|$ locally. The extremal length of α on X is defined by

$$(2) \quad \text{Ext}_X(\alpha) := \sup_{\rho} \frac{l_{\rho}^2(\alpha)}{\text{Area}(\rho)},$$

where the sup ranges over all the conformal metrics on X , $\text{Area}(\rho)$ is the area of X endowed with the metric ρ , and $l_{\rho}(\alpha) := \inf_{\alpha' \in [\alpha]} \int_{\alpha'} \rho |dz|$. It is clear that

$$l_{a\rho}^2(\alpha) / \text{Area}(a\rho) = l_{\rho}^2(\alpha) / \text{Area}(\rho)$$

for any positive constant a . There exist a unique conformal metric up to scaling realizing the supremum which is called the *extremal metric* (see [18]). The extremal length is a conformal invariant. For surfaces without boundary, Kerckhoff extended the definition of extremal length from $\mathcal{S}(S) \times \mathbf{R}_+$ to $\mathcal{ML}(S)$. For surfaces with boundary, this extension also holds by considering the double X^d of X .

The following lemmas will be used in this paper.

Lemma 2.1. *Suppose $X \in T_{g,n}(S)$. Let $\mu = \mu_1 + \mu_2 + \dots + \mu_k$ be a measured geodesic lamination where $\mu_j \in \mathbf{R}^+ \times (\mathcal{S}(S) \cup \mathcal{B}(S))$, $j = 1, 2, \dots, k$. Then*

$$\max_{1 \leq j \leq k} \{\text{Ext}_X(\mu_j)\} \leq \text{Ext}_X(\mu) \leq k^2 \max_{1 \leq j \leq k} \{\text{Ext}_X(\mu_j)\}.$$

Proof. Let ρ_i, ρ_μ be the extremal metrics of μ_i and μ respectively such that $\text{Area}(\rho_i) = \text{Area}(\rho_\mu) = 1$, $i = 1, 2, \dots, k$. Without loss of generality, we assume that $\text{Ext}_X(\mu_1) = \max_{1 \leq j \leq k} \{\text{Ext}_X(\mu_j)\}$. Then

$$\text{Ext}_X(\mu) \geq \frac{l_{\rho_1}^2(\mu)}{\text{Area}(\rho_1)} \geq \frac{l_{\rho_1}^2(\mu_1)}{\text{Area}(\rho_1)} = \text{Ext}_X(\mu_1).$$

On the other hand,

$$\begin{aligned} \text{Ext}_X(\mu) &= l_{\rho_\mu}^2(\mu) = (l_{\rho_\mu}(\mu_1) + l_{\rho_\mu}(\mu_2) + \cdots + l_{\rho_\mu}(\mu_k))^2 \\ &\leq (\sqrt{\text{Ext}_X(\mu_1)} + \sqrt{\text{Ext}_X(\mu_2)} + \cdots + \sqrt{\text{Ext}_X(\mu_k)})^2 \\ &\leq k^2 \max_{1 \leq j \leq k} \{\text{Ext}_X(\mu_j)\}. \end{aligned} \quad \square$$

Lemma 2.2. (Maskit [13]) *Let $Y \in \mathcal{T}_{g,n}(0)$ and α be a nontrivial simple closed curve, then*

(1) $l_X(\alpha)$ and $\text{Ext}_X(\alpha)$ goes to zero together, and

$$\lim_{l_X(\alpha) \rightarrow 0} l_X(\alpha)/\text{Ext}_X(\alpha) = \pi,$$

(2) $\frac{l_X(\alpha)}{\pi} \leq \text{Ext}_X(\alpha) \leq \frac{l_X(\alpha)}{2} e^{l_X(\alpha)/2}$.

Remark 4. The statements above also holds for $X \in \mathcal{T}_{g,n}(\Lambda)$ with $\Lambda \in \mathbf{R}_+^n$. In fact, suppose $X \in T_{g,n}(\Lambda)$, let X^d be the double of X . Let α be a simple closed curve on X and α^d be its double on X^d . Then $l_X(\alpha) = l_{X^d}(\alpha)/2$ and $\text{Ext}_X(\alpha) = \text{Ext}_{X^d}(\alpha)/2$.

2.4. Fenchel–Nielsen coordinates. Let R be a pair of pants with boundaries $\{\gamma_1, \gamma_2, \gamma_3\}$. Let a_1, a_2, a_3 be three geodesic arcs orthogonal to the boundaries (see Figure 1(a)). Choose an orientation for each boundary such that R is on the left. Let $\gamma_i: [0, 1] \rightarrow \gamma_i$ be a parametrization of γ_i with constant speed such that $\gamma_1([0, 1/2]), a_3, \gamma_2([0, 1/2]), a_1, \gamma_3([0, 1/2]), a_2$ consist a hexagon. We call this parametrization a *stand parametrization*. A homeomorphism f between two pairs of pants R, R' is called *boundary coherent* if $f \circ \gamma_i(s) = \gamma'_i(s)$ for $s \in [0, 1]$ and $i = 1, 2, 3$.

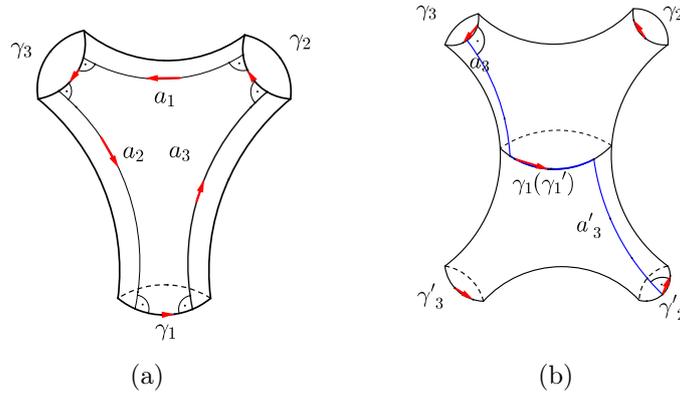


Figure 1.

For two pairs of pants R, R' , if $l(\gamma_i) = l(\gamma'_i)$ for some $i = 1, 2, 3$, we can paste R and R' along γ, γ' in the following way (see Figure 1(b))

$$\gamma_i(s) = \gamma'_i(t - s)$$

for some $t \in \mathbf{R}$. We say R and R' are pasted along γ_i with twist t .

Now we give an explanation for the Fenchel–Nielsen coordinates $(\mathcal{L}, \mathcal{T}, \Lambda)$. Let $\Gamma = \{\gamma_1, \cdots, \gamma_{3g-3+n}\}$ be a pants decomposition of X and $\{R_1, \cdots, R_{2g-2+n}\}$ be the corresponding $2g - 2 + n$ pairs of pants with stand parametrization. \mathcal{L} and Λ determine these $2g - 2 + n$ pairs of pants, and \mathcal{T} tells us how to paste these pairs of pants.

3. Proof of Theorem 1.3

Let $\gamma \in \mathcal{A}(S)$ be an arc connecting the boundary components β_i and β_j (β_i may equal to β_j). Then for any hyperbolic structure X , there is a unique geodesic in the relative homotopy class of γ , which is orthogonal to β_i, β_j at each endpoint. We still denote it by γ . It is not hard to see that a tubular neighborhood of $\beta_i \cup \beta_j \cup \gamma$ is a topological pair of pants. Let us call this pants determined by γ .

Proof of Theorem 1.3. It follows from the definitions that

$$d_{Th}(X_1, X_2) \leq d_A(X_1, X_2)$$

for any $X_1, X_2 \in T_{g,n}$.

To control the arc metric from above by the Thurston metric, it suffices to find an essential simple closed curve α for each essential arc $\gamma \in \mathcal{A}'(S) := \{\gamma \in \mathcal{A}(S) : l_{X_2}(\gamma) > l_{X_1}(\gamma)\}$ (α depends on γ) such that

$$\frac{l_{X_2}(\alpha)}{l_{X_1}(\alpha)} \geq C \frac{l_{X_2}(\gamma)}{l_{X_1}(\gamma)}$$

for some constant C which depends on the surface S and the boundary lengths Λ . We discuss for the two cases.

Case (1). γ connects two different boundary components β_i, β_j , see Figure 2(b). Then there is another simple closed curve $\alpha \in \mathcal{S}(S)$, such that β_i, β_j, α are the boundaries of the pants determined by γ .

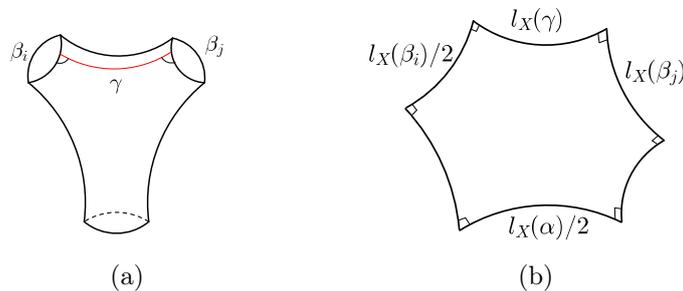


Figure 2.

For any $X \in T_{g,n}(L)$ with $l_X(\beta_i) = \lambda_i$, $l_X(\beta_j) = \lambda_j$, we have

$$(3) \quad \cosh(l_X(\gamma)) = \frac{\cosh \frac{\lambda_i}{2} \cosh \frac{\lambda_j}{2} + \cosh \frac{l_X(\alpha)}{2}}{\sinh \frac{\lambda_i}{2} \sinh \frac{\lambda_j}{2}}.$$

Let $\lambda = \max_{1 \leq i, j \leq n} \left\{ \sinh \frac{\lambda_i}{2} \sinh \frac{\lambda_j}{2}, \frac{\cosh \frac{\lambda_i}{2} \cosh \frac{\lambda_j}{2} + 1}{\sinh \frac{\lambda_i}{2} \sinh \frac{\lambda_j}{2}} \right\}$. Therefore

$$\frac{e^{\frac{l_X(\alpha)}{2}}}{2\lambda} \leq \cosh(l_X(\gamma)) \leq \lambda e^{\frac{l_X(\alpha)}{2}}.$$

On the other hand,

$$\frac{e^{l_X(\gamma)}}{2} \leq \cosh(l_X(\gamma)) \leq e^{l_X(\gamma)},$$

So we get

$$-2 \log 2\lambda \leq l_X(\alpha) - 2l_X(\gamma) \leq 2 \log 2\lambda.$$

Let $K = \log 2\lambda$.

- If $l_X(\gamma) \geq K$, then

$$1 \leq \frac{l_X(\alpha)}{l_X(\gamma)} \leq 3.$$

- If $l_X(\gamma) \leq K$, set $r_0 = \min_{1 \leq i, j \leq n} \cosh^{-1} \left(\frac{\cosh \frac{\lambda_i}{2} \cosh \frac{\lambda_j}{2}}{\sinh \frac{\lambda_i}{2} \sinh \frac{\lambda_j}{2}} \right)$, it follows from (3) that

$$l_X(\gamma) \geq r_0,$$

and

$$l_X(\alpha) \leq x_0 := 2 \max_{1 \leq i, j \leq n} \cosh^{-1} \left(K \sinh \frac{\lambda_i}{2} \sinh \frac{\lambda_j}{2} - \cosh \frac{\lambda_i}{2} \cosh \frac{\lambda_j}{2} \right).$$

For any $\gamma \in \mathcal{A}'(S) := \{\gamma \in \mathcal{A}(S) : l_{X_2}(\gamma) > l_{X_1}(\gamma)\}$, we consider the following situations.

- If $l_{X_2}(\gamma) \geq l_{X_1}(\gamma) \geq K$, then

$$\frac{l_{X_2}(\alpha)}{l_{X_1}(\alpha)} \geq \frac{l_{X_2}(\gamma)}{3l_{X_1}(\gamma)}.$$

- If $l_{X_2}(\gamma) \geq K \geq l_{X_1}(\gamma)$, then

$$\frac{l_{X_2}(\alpha)}{l_{X_1}(\alpha)} \geq \frac{l_{X_2}(\gamma) l_{X_1}(\gamma)}{l_{X_1}(\gamma) x_0} \geq \frac{l_{X_2}(\gamma) r_0}{l_{X_1}(\gamma) x_0}.$$

- If $K \geq l_{X_2}(\gamma) \geq l_{X_1}(\gamma)$, then

$$\frac{l_{X_2}(\alpha)}{l_{X_1}(\alpha)} \geq 1 \geq \frac{l_{X_2}(\gamma) r_0}{l_{X_1}(\gamma) K}.$$

Let $C_1 = \max\{\frac{1}{3}, \frac{r_0}{x_0}, \frac{r_0}{K}\}$, we have

$$\log \frac{l_{X_2}(\alpha)}{l_{X_1}(\alpha)} \geq C_1 \frac{l_{X_2}(\gamma)}{l_{X_1}(\gamma)}.$$

Case (2). γ connects β_i to itself for some boundary component β_i , see Figure 3(a). Then there exist another two simple closed curves $\alpha, \delta \in \mathcal{C}(S)$, such that β_i, α, δ are the boundaries of the pants determined by γ .

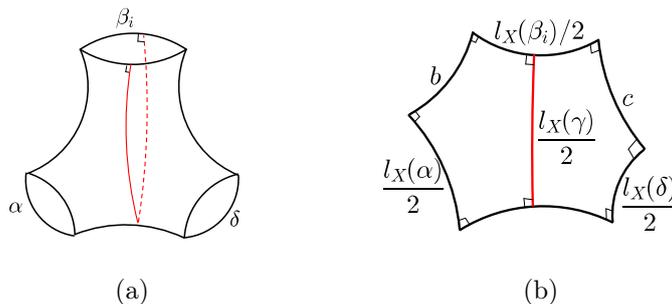


Figure 3.

Hence

$$\begin{aligned} \cosh^2\left(\frac{l_X(\gamma)}{2}\right) &= \sinh^2 \frac{l_X(\alpha)}{2} \sinh^2 b = \sinh^2 \frac{l_X(\alpha)}{2} (\cosh^2 b - 1) \\ &= \sinh^2 \frac{l_X(\alpha)}{2} \left[\left(\frac{\cosh \frac{l_X(\alpha)}{2} \cosh \frac{\lambda_i}{2} + \cosh \frac{l_X(\delta)}{2}}{\sinh \frac{l_X(\alpha)}{2} \sinh \frac{\lambda_i}{2}} \right)^2 - 1 \right] \\ &= \frac{\left[\cosh \frac{l_X(\delta)}{2} + \cosh \left(\frac{l_X(\alpha)}{2} + \frac{\lambda_i}{2} \right) \right] \left[\cosh \frac{l_X(\delta)}{2} + \cosh \left(\frac{l_X(\alpha)}{2} - \frac{\lambda_i}{2} \right) \right]}{\sinh^2 \frac{\lambda_i}{2}}. \end{aligned}$$

On the other hand,

$$e^{-\frac{\lambda_i}{2}} \cosh \frac{l_X(\alpha)}{2} \leq \cosh \left(\frac{l_X(\alpha)}{2} \pm \frac{\lambda_i}{2} \right) \leq e^{\frac{\lambda_i}{2}} \cosh \frac{l_X(\alpha)}{2}.$$

From the discussions above, we get

$$\frac{e^{-\frac{\lambda_i}{2}}}{\sinh \frac{\lambda_i}{2}} \leq \frac{\cosh \frac{l_X(\gamma)}{2}}{\left(\cosh \frac{l_X(\delta)}{2} + \cosh \frac{l_X(\alpha)}{2} \right)} \leq \frac{e^{\frac{\lambda_i}{2}}}{\sinh \frac{\lambda_i}{2}}.$$

Then

$$\log \frac{e^{-\frac{\lambda_i}{2}}}{2 \sinh \frac{\lambda_i}{2}} \leq l_X(\gamma) - \max\{l_X(\delta), l_X(\alpha)\} \leq \log \frac{4e^{\frac{\lambda_i}{2}}}{\sinh \frac{\lambda_i}{2}}.$$

Note that

$$\cosh \frac{l_X(\alpha)}{2} \leq \sinh \frac{l_X(\beta_i)}{2} \sinh \frac{l_X(\gamma)}{2}.$$

Therefore

$$l_X(\gamma) \geq 2 \sinh^{-1} \frac{1}{\sinh \frac{l_X(\beta_i)}{2}} = 2 \sinh^{-1} \frac{1}{\sinh \frac{\lambda_i}{2}}.$$

The same as the discussion in Case 1, we have

$$\max\left\{ \frac{l_{X_2}(\alpha)}{l_{X_1}(\alpha)}, \frac{l_{X_2}(\delta)}{l_{X_1}(\delta)} \right\} \geq \frac{\max\{l_{X_2}(\delta), l_{X_2}(\alpha)\}}{\max\{l_{X_1}(\delta), l_{X_1}(\alpha)\}} \geq C_2 \frac{l_{X_2}(\gamma)}{l_{X_1}(\gamma)}$$

for some constant C_2 .

Combing Case(1) and Case(2) we know that for any arc $\gamma \in \mathcal{A}'(S)$, we can find a simple closed curve $\alpha' \in \mathcal{S}(S)$, such that

$$\frac{l_{X_2}(\alpha')}{l_{X_1}(\alpha')} \geq C \frac{l_{X_2}(\gamma)}{l_{X_1}(\gamma)},$$

where $C = \max\{C_1, C_2\}$. Consequently,

$$d_A(X_1, X_2) - \log C \leq d_{Th}(X_1, X_2) \leq d_A(X_1, X_2). \quad \square$$

4. Proof of Theorem 1.5

As we mentioned in the introduction, Liu, Papadopoulos, Su and Th eret [10] described the Teichm uller metric on the Teichm uller space of surfaces with boundary via the extremal lengths of essential arcs and the boundary components. Follow the idea in [10], we get the following approximation.

Proposition 4.1. *For small ϵ , $X, Y \in \mathcal{T}_{g,n}(\epsilon)$, we have*

$$\left| d_T(X, Y) - \frac{1}{2} \sup_{[\alpha] \in \mathcal{S}(S)} \left| \log \frac{\text{Ext}_Y(\alpha)}{\text{Ext}_X(\alpha)} \right| \right| \leq \log(n+2).$$

Proof. Let X^d, Y^d be the double of X, Y respectively. It follows from (1) that

$$\begin{aligned} d_T(X, Y) &= d_T(X^d, Y^d) \\ &= \frac{1}{2} \sup_{\alpha \in \mathcal{ML}(X^d)} \left| \log \frac{\text{Ext}_{Y^d}(\alpha)}{\text{Ext}_{X^d}(\alpha)} \right| \geq \frac{1}{2} \sup_{\alpha \in \mathcal{S}(S) \cup \mathcal{B}(S)} \left| \log \frac{\text{Ext}_Y(\alpha)}{\text{Ext}_X(\alpha)} \right|. \end{aligned}$$

It remains to prove the other direction. Let q^d be the initial quadratic differential associated to the Teichmüller map between X^d and Y^d . Let h^d be the horizontal measured lamination of q^d . Then

$$d_T(X, Y) = d_T(X^d, Y^d) = \frac{1}{2} \left| \log \frac{\text{Ext}_{Y^d}(h^d)}{\text{Ext}_{X^d}(h^d)} \right|.$$

Recall that ∂X is a q^d -horizontal line (see §2.1), h^d can be decomposed as $h^d = \sum_{i=1}^n a_i \beta_i + \mu$, where $a_i \geq 0$, β_i is a measured lamination represented by a boundary component of X , and μ is an interior measured geodesic lamination, i.e., $a_i \beta_i \in \mathbf{R}_{\geq 0} \times \mathcal{B}(S)$, $\mu \in \mathcal{ML}_0(S)$. Since $\mathbf{R}^+ \times \mathcal{S}(S)$ is dense in $\mathcal{ML}_0(S)$, there exists $\{c_i \delta_i\}_{i=1}^\infty \subset \mathbf{R}^+ \times \mathcal{S}(S)$ such that $c_i \delta_i \rightarrow \mu$ as $i \rightarrow \infty$. Therefore,

$$d_T(X, Y) = \lim_{j \rightarrow \infty} \frac{1}{2} \left| \log \frac{\text{Ext}_{Y^d}(\sum_{i=1}^n a_i \beta_i + c_j \delta_j)}{\text{Ext}_{X^d}(\sum_{i=1}^n a_i \beta_i + c_j \delta_j)} \right|.$$

For each $j \geq 1$, let $\mu_j \in \{a_1 \beta_1, a_2 \beta_2, \dots, a_n \beta_n, c_j \delta_j\}$ such that

$$\text{Ext}_{Y^d}(\mu_j) = \max_{1 \leq k \leq n} \{\text{Ext}_{Y^d}(a_k \beta_k), \text{Ext}_{Y^d}(c_j \delta_j)\}.$$

It follows from Lemma 2.1 that

$$\begin{aligned} d_T(X, Y) &\leq \frac{1}{2} \lim_{j \rightarrow \infty} \left| \log \frac{(n+1)^2 \text{Ext}_Y(\mu_j)}{\text{Ext}_X(\mu_j)} \right| \\ &\leq \frac{1}{2} \sup_{\alpha \in \mathcal{S}(S)} \left| \log \frac{\text{Ext}_Y(\alpha)}{\text{Ext}_X(\alpha)} \right| + \frac{1}{2} \sup_{\beta \in \mathcal{B}(S)} \left| \log \frac{\text{Ext}_Y(\beta)}{\text{Ext}_X(\beta)} \right| + \log(n+1) \\ &\leq \frac{1}{2} \sup_{\alpha \in \mathcal{S}(S)} \left| \log \frac{\text{Ext}_Y(\alpha)}{\text{Ext}_X(\alpha)} \right| + \log(n+2) \end{aligned}$$

where we use the first result of Lemma 2.2 in the last inequality. \square

Next, we estimate the extremal lengths. Let $X \in T_{g,n}(0)$, $\{p_1, \dots, p_n\}$ be the punctures of X . It is well known that every puncture has a cusp neighbourhood consisting of horocycles of length less than 1 (see [4], for example). Let $\{D_1^\epsilon, \dots, D_n^\epsilon\}$ be the corresponding cuspidal neighborhoods with boundary lengths ϵ , $\text{Cusp}_\epsilon(X) := \bigcup_{1 \leq i \leq n} D_i^\epsilon$ and $X_\epsilon := X \setminus \text{Cusp}_\epsilon(X)$. The following proposition is key to prove Theorem 1.5.

Proposition 4.2. *For small ϵ , there is a constant C_ϵ such that for any $\alpha \in \mathcal{S}(S)$ and any $X \in \mathcal{T}_{g,n}(0)$,*

$$1 \leq \frac{\text{Ext}_{X_\epsilon}(\alpha)}{\text{Ext}_X(\alpha)} \leq C_\epsilon.$$

Moreover, $C_\epsilon \rightarrow 1$ as $\epsilon \rightarrow 0$.

Remark 5. Minsky [14, Lemma 8.4] proved a similar result by a different method without the description of C_ϵ when ϵ tends to zero.

Proof. Before proving the proposition, we make some conventions. For any simple closed curve α , denote by $L_\rho(\alpha)$ the length of α under the metric ρ and $l_\rho(\alpha)$ the length of the geodesic representative of α under the metric ρ . Since $X_\epsilon \subset X$, it follows from the definition that $\text{Ext}_{X_\epsilon}(\alpha) \geq \text{Ext}_X(\alpha)$.

Let $\{p_1, \dots, p_n\}$ be the punctures of X and $\{D_1^\epsilon, \dots, D_n^\epsilon\}$ be the corresponding cuspidal neighborhoods with boundary lengths ϵ . Recall that each puncture p_i has a cuspidal neighbourhood D_i^1 with boundary length 1 such that $D_i^1 \cap D_j^1 = \emptyset$ for $i \neq j$. Let G_i be the infinite cyclic group generated by a simple closed curve which is homotopic to p_i . Let $D^* = \{w: 0 < |w| < 1\}$ be the punctured unit disc equipped with the hyperbolic metric $\rho = |dw|/(|w| \log |w|^{-1})$. Let $\pi_i: D^* \mapsto X$ be a covering map such that the fundamental group of D^* corresponds to G_i and that X coincides with the push-forward of ρ . In this setting, D_i^ϵ is conformal to the punctured disc $D_{R(\epsilon)}^* = \{w: 0 < |w| < R(\epsilon)\}$ where $R(\epsilon) = \exp(-2\pi/\epsilon)$. It is clear that $R(\epsilon) < 1/2R(1)$ for small ϵ .

The remaining of the proof will be split into two cases.

Case 1. $\text{Ext}_{X_\epsilon}(\alpha) \leq \sqrt{\epsilon}$. It is clear that $\text{Ext}_X(\alpha) \leq \text{Ext}_{X_\epsilon}(\alpha) \leq \sqrt{\epsilon}$. Let ϕ be the quadratic differential on X whose horizontal measured foliation h_ϕ is equivalent to α . Denote by $|\phi|$ the induced flat metric on X , then $|\phi|$ is the extremal metric of α , i.e.,

$$(4) \quad \text{Ext}_X(\alpha) = \frac{l_{|\phi|}^2(\alpha)}{\|\phi\|},$$

where $\|\phi\| = \int_X |\phi|$ and $l_{|\phi|}(\alpha)$ is the length of the geodesic homotopic to α under $|\phi|$. To estimate $\text{Ext}_{X_\epsilon}(\alpha)$, we need to estimate the length of ∂D_i^ϵ , denoted by $L_{|\phi|}(\partial D_i^\epsilon)$, under the flat metric $|\phi|$. Recall that ϕ has a simple pole at p_i , it has the following expression in $D_{R(1)}^*$,

$$\phi(w) dw^2 = (\psi(w)/w) dw^2,$$

where ψ is holomorphic. Note that $|\psi(w)|$ is subharmonic and $\int_0^{2\pi} |\psi(re^{i\theta})| d\theta$ is an increasing function of r . For simplicity, set $R_1 = R(\epsilon)$ and $R_2 = R(1) = \exp(-2\pi)$, then

$$(5) \quad \begin{aligned} L_{|\phi|}(\partial D_i^\epsilon) &= \int_0^{2\pi} \sqrt{\frac{|\psi(R_1 e^{i\theta})|}{R_1}} R_1 d\theta \leq \left(2\pi \int_0^{2\pi} |\psi(R_1 e^{i\theta})| d\theta \right)^{1/2} \\ &\leq \left(\frac{2\pi}{R_2 - R_1} \int_{R_1}^{R_2} dr \int_0^{2\pi} |\psi(re^{i\theta})| d\theta \right)^{1/2} \\ &\leq \left(\frac{2\pi}{R_2 - R_1} \int_{D_{R_2}^*} |\phi| r dr d\theta \right)^{1/2} \leq \left(\frac{4\pi}{R_2} \|\phi\| \right)^{1/2}. \end{aligned}$$

Cutting X along the critical leaves of h_ϕ , we get a cylinder A_α . Let $A_{\epsilon,\alpha} \subset A_\alpha$ be the maximal cylinder whose core curve is homotopic to α and which is contained in X_ϵ (see Figure 4). Denote by H_α and $H_{\epsilon,\alpha}$ the heights of A_α and $A_{\epsilon,\alpha}$ respectively. Then $H_\alpha = \sqrt{(\text{Ext}_X(\alpha))^{-1} \|\phi\|}$ and

$$H_{\epsilon,\alpha} \geq H_\alpha - 2 \max_{1 \leq i \leq n} L_{|\phi|}(\partial D_i^\epsilon) \geq H_\alpha - \left(\frac{16\pi}{R_2} \|\phi\| \right)^{1/2}.$$

Hence

$$\frac{\text{Ext}_{X_\epsilon}(\alpha)}{\text{Ext}_X(\alpha)} \leq \frac{\text{Ext}_{A_{\epsilon,\alpha}}(\alpha)}{\text{Ext}_{A_\alpha}(\alpha)} = \frac{H_\alpha}{H_{\epsilon,\alpha}} \leq 1 + \sqrt{32\pi e^{2\pi}} \epsilon^{1/4}.$$

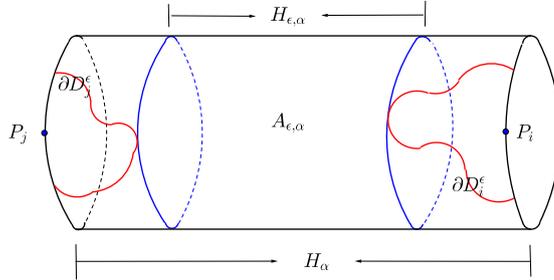


Figure 4.

Case 2. $\text{Ext}_{X_\epsilon}(\alpha) \geq \sqrt{\epsilon}$. Let ϕ_ϵ be the quadratic differential on X_ϵ whose horizontal measured foliation h_{ϕ_ϵ} is equivalent to α . Denote by $|\phi_\epsilon|$ the induced flat metric on X_ϵ , then $|\phi_\epsilon|$ is the extremal metric of α . Note that $\partial X_\epsilon = \bigcup_{i=1}^n \partial D_i^\epsilon$ is contained in the critical leaves of h_{ϕ_ϵ} , so ∂D_i^ϵ is a geodesic under the metric $|\phi_\epsilon|$. Hence

$$L_{|\phi_\epsilon|}(\partial D_i^\epsilon) \leq \sqrt{\text{Ext}_{X_\epsilon}(\partial D_i^\epsilon) \|\phi_\epsilon\|}.$$

Let $A_i = D_i^1 \setminus D_i^\epsilon$, then A_i is conformal to the annulus $D_{R(1)}^* \setminus D_{R(\epsilon)}^* = \{w : R(\epsilon) < |w| < R(1)\}$. Therefore

$$\text{Ext}_{X_\epsilon}(\partial D_i^\epsilon) \leq \text{Ext}_{A_i}(\partial D_i^\epsilon) = 2\pi \left(\log \frac{R(1)}{R(\epsilon)}\right)^{-1} = (1/\epsilon - 1)^{-1} \leq 2\epsilon.$$

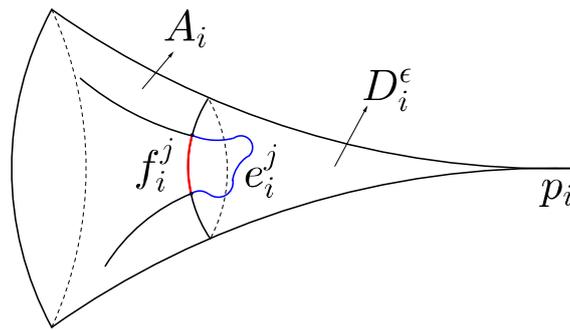


Figure 5.

On the other hand, $|\phi_\epsilon|$ defines a conformal metric ρ_ϵ on X , which coincides with $|\phi_\epsilon|$ on X_ϵ and vanishes elsewhere. For any simple closed curve α , set $e_i := \alpha \cap D_i^\epsilon$. Let e_i^j be a component of e_i (see Figure 5). $D_i^\epsilon \setminus e_i^j$ has two components, one is homeomorphic to a disc, denoted by E_i^j and the other is homeomorphic to a punctured disc. Let $f_i^j = \partial E_i^j \setminus e_i^j$. It follows that $f_i^j \subset \partial D_i^\epsilon$ and $f_i^j \cap f_i^k = \emptyset$ if $j \neq k$. We construct a new simple closed curve α' from α via replacing e_i^j by f_i^j . It is clear

that α' is homotopic to α and that α' is contained in X_ϵ . Then

$$\begin{aligned} L_{\rho_\epsilon}(\alpha) &\geq L_{|\phi_\epsilon|} \left(\alpha \setminus \left(\bigcup e_i^j \right) \right) = L_{|\phi_\epsilon|}(\alpha') - L_{|\phi_\epsilon|} \left(\bigcup f_i^j \right) \\ &\geq L_{|\phi_\epsilon|}(\alpha') - \sum_{i=1}^n L_{|\phi_\epsilon|}(\partial D_i^\epsilon) \geq \sqrt{\text{Ext}_{X_\epsilon}(\alpha) \|\phi_\epsilon\|} - n\sqrt{2\epsilon \|\phi_\epsilon\|} \\ &\geq (1 - n\sqrt{2}\epsilon^{1/4}) \sqrt{\text{Ext}_{X_\epsilon}(\alpha) \|\phi_\epsilon\|}, \end{aligned}$$

where we use $\text{Ext}_{X_\epsilon}(\alpha) \geq \sqrt{\epsilon}$ in the last inequality. As a consequence,

$$\text{Ext}_X(\alpha) \geq \frac{(l_{\rho_\epsilon}(\alpha))^2}{\text{Area}(\rho_\epsilon)} = \frac{(\inf_{\alpha' \in \mathcal{S}(S)} L_{\rho_\epsilon}(\alpha'))^2}{\text{Area}(\rho_\epsilon)} \geq (1 - 2n\pi\sqrt{2}\epsilon^{1/4})^2 \text{Ext}_{X_\epsilon}(\alpha),$$

where α' ranges over every simple closed curve homotopic to α .

Let $C_\epsilon = \max\{(1 - 2n\pi\sqrt{2}\epsilon^{1/4})^{-2}, 1 + \sqrt{32\pi e^{2\pi}\epsilon^{1/4}}\}$. Combining Case 1 and Case 2, we get

$$\frac{\text{Ext}_{X_\epsilon}(\alpha)}{\text{Ext}_X(\alpha)} \leq C_\epsilon$$

for any $\alpha \in \mathcal{C}_0(S)$. Moreover, $C_\epsilon \rightarrow 1$ as $\epsilon \rightarrow 0$. □

Corollary 4.3. *For small ϵ , there is a constant C_ϵ such that for any interior measured laminations μ and any $X \in \mathcal{T}_{g,n}(0)$,*

$$1 \leq \frac{\text{Ext}_{X_\epsilon}(\mu)}{\text{Ext}_X(\mu)} \leq C_\epsilon.$$

Moreover, $C_\epsilon \rightarrow 1$ as $\epsilon \rightarrow 0$.

The last step of the proof is to quasiconformally embed $X \in T_{g,n}(\epsilon)$ into $\Phi(X) \in T_{g,n}(0)$ in some nice way. We need the following theorem due to Buser, Makover, Muetzel and Silhol [5].

Theorem 4.4. [5, Theorem 2.1] *Let $l_1, l_2 > 0$, $0 < \epsilon < 1/2$, and set $\epsilon^* = \frac{2}{\pi}\epsilon$. Let $Y_{l_1, l_2, \epsilon}$ be a pair of pants with boundary length l_1, l_2, ϵ , and set $Y_{l_1, l_2, 0}^{\epsilon^*} = Y_{l_1, l_2, 0} \setminus \text{Cusp}_{\epsilon^*}(Y_{l_1, l_2, 0})$. Then there exists a boundary coherent (see §2.4 for the definition) quasiconformal homeomorphism*

$$\phi: Y_{l_1, l_2, \epsilon} \rightarrow Y_{l_1, l_2, 0}^{\epsilon^*}$$

with dilation $q_\phi \leq 1 + 2\epsilon^2$.

Remark 6. Under the assumptions of Theorem 4.4, it is clear that there exists a boundary coherent quasiconformal homeomorphism $\phi: Y_{l_1, \epsilon_1, \epsilon_2} \rightarrow Y_{l_1, 0, 0}^{\epsilon_1^*, \epsilon_2^*}$ with dilation $q_\phi \leq (1 + 2\epsilon_1^2)(1 + 2\epsilon_2^2)$.

Proof of Theorem 1.5. The second condition in Definition 1.1 follows from the fact that Φ is a homeomorphism. It remains to verify the first condition. Let $X \in T_{g,n}(\epsilon)$ and $\Phi(X) \in T_{g,n}(0)$. It follows from Theorem 4.4 that there exists a quasiconformal homeomorphism g_1 from X_1 to $X_{1, \epsilon^*} := \Phi(X_1) \setminus \text{Cusp}_{\epsilon^*}$ (resp. g_2 from X_2 to $X_{2, \epsilon^*} := \Phi(X_2) \setminus \text{Cusp}_{\epsilon^*}$) with dilation $K(g_i) \leq \prod_{j=1}^n (1 + 2\epsilon_j^2)$, $i = 1, 2$. This can be obtained in the following way. Let $\{R_1, \dots, R_{2g-2+n}\}$ be the $2g-2+n$ pairs of pants associated to the pants decomposition Γ . If $\partial R_i \cap \partial X \neq \emptyset$, Let $h_1: R_1 \rightarrow R_{1, \epsilon^*}$ be the map obtained from Theorem 4.4, otherwise let $h_i: R_i \rightarrow R_i$ be the identity map. Gluing $\{h_i\}_{i=1}^{2g-2+n}$ via the Fenchel–Nielsen coordinates, we obtained the desired

maps. Hence, for any interior simple closed curve $\alpha \in \mathcal{S}(S)$,

$$\frac{1}{K(g_1)} \leq \frac{\text{Ext}_{X_1}(\alpha)}{\text{Ext}_{X_{1,\epsilon^*}}(\alpha)} \leq K(g_1), \quad \frac{1}{K(g_2)} \leq \frac{\text{Ext}_{X_2}(\alpha)}{\text{Ext}_{X_{2,\epsilon^*}}(\alpha)} \leq K(g_2).$$

Combining with Proposition 4.2, we have

$$\frac{1}{C'_\epsilon} \leq \frac{\text{Ext}_{X_1}(\alpha)}{\text{Ext}_{\Phi(X_1)}(\alpha)} \leq C'_\epsilon C_\epsilon, \quad \frac{1}{C'_\epsilon} \leq \frac{\text{Ext}_{X_2}(\alpha)}{\text{Ext}_{\Phi(X_2)}(\alpha)} \leq C'_\epsilon C_\epsilon,$$

where $C'_\epsilon = \prod_{j=1}^n (1 + 2\epsilon_j^2)$ and C_ϵ is the constant in Proposition 4.2.

Now the theorem follows from the Kerckhoff's formula on $T_{g,n}(0)$, and Proposition 4.1. □

5. Further study and questions

5.1. Nielsen extension. Let X be a hyperbolic surface with geodesic boundary. The *infinite Nielsen extension* X_∞ of X is a punctured surface (see [3]). For any $\Lambda \in \mathbf{R}_+^n$, we can define a map $\Psi: T_{g,n}(\Lambda) \rightarrow T_{g,n}(0)$ which associate the infinite Nielsen extension X_∞ to any $X \in T_{g,n}(\Lambda)$. It is natural to ask the following question.

Question 1. Given $\epsilon \in \mathbf{R}_+^n$, is $\Psi: T_{g,n}(\epsilon) \rightarrow T_{g,n}(0)$ an almost isometry?

Unlike the Fenchel–Nielsen map Φ_Γ , we do not know whether Ψ is a homeomorphism. But for small ϵ , Ψ is indeed an almost isometry.

Theorem 5.1. *For small ϵ , $\Psi: T_{g,n}(\epsilon) \rightarrow T_{g,n}(0)$ is an almost isometry with respect to the Teichmüller metric.*

Proof. The theorem follows from Proposition 5.2 and Proposition 5.5. □

Proposition 5.2. *For $X, Y \in \mathcal{T}_{g,n}(\epsilon)$,*

$$|d_T(X, Y) - d_T(X_0, Y_0)| \leq \log(n + 3),$$

where $X_0 = \Psi(X), Y_0 = \Psi(Y)$.

Proof. It follows from Proposition 4.1 and Proposition 5.3. □

Proposition 5.3. *Given $\Lambda = (\lambda_1, \dots, \lambda_n) \in \mathbf{R}_+^n$. For $X \in \mathcal{T}_{g,n}(\Lambda)$, let $X_0 \in T_{g,n}(0)$ be the infinite Nielsen extension of X . There is a constant C_Λ such that for any $\alpha \in \mathcal{S}(S)$ and any $X \in \mathcal{T}_{g,n}(\Lambda)$,*

$$1 \leq \frac{\text{Ext}_X(\alpha)}{\text{Ext}_{X_0}(\alpha)} \leq C_\Lambda.$$

Moreover, $C_\Lambda \rightarrow 0$ as $\Lambda \rightarrow 0$.

Proof. Since $X \subset X_0$, $\text{Ext}_X(\alpha) \geq \text{Ext}_{X_0}(\alpha)$. For the right inequality, we distinguish two cases.

Case 1. $\text{Ext}_X(\alpha) \leq 4n^2\lambda e^{\lambda/2}$, where $\lambda = \max_{1 \leq i \leq n} \lambda_i$. By Lemma 2.2 and Proposition 5.4, there are constants ϵ_0, c_1, c_2 such that if $\text{Ext}_X(\alpha) < \epsilon_0$, then $c_1 \leq \frac{\text{Ext}_X(\alpha)}{\text{Ext}_{X_0}(\alpha)} \leq c_2$ for any $\alpha \in \mathcal{S}(S)$. If $\epsilon_0 \leq \text{Ext}_X(\alpha) \leq 4n^2\lambda e^{\lambda/2}$, then $\frac{\epsilon_0}{4n^2\lambda e^{\lambda/2}} \leq \frac{\text{Ext}_X(\alpha)}{\text{Ext}_{X_0}(\alpha)} \leq \frac{4n^2\lambda e^{\lambda/2}}{\epsilon_0}$.

Case 2. $\text{Ext}_X(\alpha) \geq 4n^2\lambda e^{\lambda/2}$, where $\lambda = \max_{1 \leq i \leq n} \lambda_i$. Applying the method used in Case 2 in the proof of Proposition 4.2, we get

$$\text{Ext}_X(\alpha) \leq 4\text{Ext}_{X_0}(\alpha).$$

The second part follows directly from Proposition 4.2. □

Proposition 5.4. [6] Given $\Lambda = (\lambda_1, \dots, \lambda_n) \in \mathbf{R}_+^n$ and $\lambda = \max_{1 \leq i \leq n} \lambda_i$. For $X \in \mathcal{T}_{g,n}(\Lambda)$, let X_∞ be the infinite Nielsen extension of X . Let α be a simple closed curve. If α is homotopic to one of the boundary components, $l_{X_\infty}(\alpha) = 0$. Otherwise $k_\infty l_X(\alpha) < l_{X_\infty}(\alpha) < l_X(\alpha)$, where $k_\infty = \prod_{i=1}^\infty [1 - (2/\pi) \tan^{-1}(2 \sinh \lambda/2^i)]$.

Proposition 5.5. For small $\epsilon \in \mathbf{R}^n$, $\Psi: T_{g,n}(\epsilon) \rightarrow T_{g,n}(0)$ is almost surjective, i.e., there is a constant C_ϵ such that $T_{g,n}(0)$ is contained in the C_ϵ neighbourhood of $\Psi(T_{g,n}(\epsilon))$.

Proof. For any $X \in T_{g,n}(0)$, let $\{D_1^{\epsilon_1}, \dots, D_n^{\epsilon_n}\}$ be the cuspidal neighbourhoods with boundary lengths ϵ . Let $\hat{X}_\epsilon := X \setminus (\bigcup_{1 \leq k \leq n} D_k^{\epsilon_k})$. It is clear that X is the infinite Nielsen extension of \hat{X}_ϵ . Let $X_\epsilon = \Phi_\Gamma^{-1}(X) \in T_{g,n}(\epsilon)$ be the preimage of X under the map Φ_Γ , X_ϵ^0 the infinite Nielsen extension of X_ϵ . By Theorem 4.4, there is a K_ϵ -quasiconformal map between \hat{X}_ϵ and X_ϵ . Combining with Proposition 5.2, we have

$$d_T(X, X_\epsilon^0) \leq d_T(\hat{X}_\epsilon, X_\epsilon) + \log(n + 3) \leq K_\epsilon + \log(n + 3).$$

Since $X \in T_{g,n}(0)$ is an arbitrary point, this means that Ψ is almost surjective for small ϵ , which verifies the second condition in Definition 1.1. □

5.2. Improving Theorem 1.5. In Theorem 1.5, we assume that the boundary component has small boundary length. We ask the following questions.

Question 2. Does Theorem 1.5 still hold if we remove the condition that ϵ is small ?

Let R, R' be two pairs of pants such that $\partial R = \{\gamma_1, \gamma_2, \gamma_3\}$ and $\partial R' = \{\gamma'_1, \gamma'_2, \gamma'_3\}$. Assume that $l(\gamma_1) = l(\gamma'_1)$, $l(\gamma_2) = l(\gamma'_2)$, $l(\gamma_3) = l_3$ and $l(\gamma'_3) = l'_3$. One possible way to answer Question 2 is to find a boundary coherent quasiconformal map $f: R \rightarrow R'$ with quasiconformal dilation only depends on l_3, l'_3 .

Question 3. Can we replace the constant $\log(n+3)$ in Theorem 1.5 by a constant $C(\epsilon)$ such that $C(\epsilon) \rightarrow 0$ if $\epsilon \rightarrow 0$?

5.3. Infinite type surfaces. A surface is of *infinite type* if it has infinite genus or infinite boundary boundary component or infinite punctures. In [2], the authors studied the Fenchel–Nielsen coordinates of the Teichmüller space of infinite type surfaces. In [9], the authors studied the length spectrum metric and the Teichmüller metric on the Teichmüller space of infinite type surfaces.

Question 4. Study similar questions for the Teichmüller space of infinite type surfaces. More formally, whether Theorem 1.2 and Theorem 1.5 are still true if the surface in consideration has infinite genus?

References

- [1] ABIKOFF, W.: The real analytic theory of Teichmüller space. - Lecture Notes in Math. 820, Springer, Berlin, 1980.
- [2] ALESSANDRINI, D., LIU, A. PAPADOPOULOS, W. SU, and Z. SUN: On Fenchel–Nielsen coordinates on Teichmüller spaces of surfaces of infinite type. - Ann. Acad. Sci. Fenn. Math. 36:2, 2011, 621–659.
- [3] BERS, L.: Nielsen extensions of Riemann surfaces. - Ann. Acad. Sci. Fenn. Ser. A I Math. 2, 1976, 29–34.

- [4] BUSER, P.: Geometry and spectra of compact Riemann surfaces. - Modern Birkhäuser Classics, Reprint of the 1992 edition, Birkhäuser Boston, Inc., Boston, MA, 2010.
- [5] BUSER, P., E. MAKOVER, B. MUETZEL, and R. SILHOL: Quasiconformal embeddings of Y -pieces. - *Comput. Methods Funct. Theory* 14:2-3, 2014, 431–452.
- [6] HALPERN, N.: Some remarks on Nielsen extensions of Riemann surfaces. - *Michigan Math. J.* 30:1, 1983, 65–68.
- [7] HALPERN, N.: A theorem on infinite Nielsen extensions. - *Proc. Amer. Math. Soc.* 97:3, 1986, 459–462.
- [8] HUBBARD, J., and H. MASUR: Quadratic differentials and foliations. - *Acta Math.* 142:3-4, 1979, 221–274.
- [9] LIU, L., and A. PAPADOPOULOS: Some metrics on Teichmüller spaces of surfaces of infinite type. - *Trans. Amer. Math. Soc.* 363:8, 2011, 4109–4134.
- [10] LIU, L., A. PAPADOPOULOS, W. SU, and G. THÉRET: Length spectra and the Teichmüller metric for surfaces with boundary. - *Monatsh. Math.* 161:3, 2010, 295–311.
- [11] LIU, L., A. PAPADOPOULOS, W. SU, and G. THÉRET: On length spectrum metrics and weak metrics on Teichmüller spaces of surfaces with boundary. - *Ann. Acad. Sci. Fenn. Math.* 35:1, 2010, 255–274.
- [12] LIU, L., W. SU, and Y. ZHONG: On metrics defined by length spectra on Teichmüller spaces of surfaces with boundary. - *Ann. Acad. Sci. Fenn. Math.* 40:2, 2015, 617–644.
- [13] MASKIT, B.: Comparison of hyperbolic and extremal lengths. - *Ann. Acad. Sci. Fenn. Ser. A I Math.* 10, 1985, 381–386.
- [14] MINSKY, Y. N.: Harmonic maps, length, and energy in Teichmüller space. - *J. Differential Geom.* 35:1, 1992, 151–217.
- [15] MINSKY, Y. N.: Extremal length estimates and product regions in Teichmüller space. - *Duke Math. J.* 83:2, 1996, 249–286.
- [16] PAN, H.: On finite marked length spectral rigidity of hyperbolic cone surfaces and the Thurston metric. - *Geom. Dedicata* (to appear).
- [17] PAPADOPOULOS, A., and W. SU: Thurston’s metric on Teichmüller space and the translation lengths of mapping classes. - *arXiv:1509.06499*.
- [18] STREBEL, K.: Quadratic differentials. - *Ergeb. Math. Grenzgeb.* (3), Springer-Verlag, Berlin, 1984.