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# BOUNDARY BEHAVIOR OF THE QUASI-HYPERBOLIC METRIC

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**Abstract.** The precise behavior of the quasi-hyperbolic metric near a $\mathcal{C}^{1,1}$-smooth part of the boundary of a domain in $\mathbf{R}^n$ is obtained.

## 1. Introduction and results

Let $D$ be a proper subdomain of $\mathbf{R}^n$. Define the *quasi-hyperbolic metric* of $D$ by

$$h_D(a,b) = \inf_\gamma \int_\gamma \frac{\|du\|}{d_D(u)}, \quad a,b \in D,$$

where $\|\cdot\|$ is the Euclidean norm, $d_D = \operatorname{dist}(\cdot, \partial D)$ and the infimum is taken over all rectifiable curves $\gamma$ in $D$ joining $a$ to $b$. By [5, Lemma 1], the infimum is attained, and any extremal curve is called *quasi-hyperbolic geodesic* (for short, *geodesic*). It turns out that the geodesics are $\mathcal{C}^{1,1}$-smooth (see [8, Corollary 4.8]). The quasi-hyperbolic metric arises in the theory of quasi-conformal maps.

This paper is devoted to the boundary behavior of $h_D$. First, we point out the following general lower bound.

**Proposition 1.** [4, Lemma 2.6] *If $D$ is a proper subdomain of $\mathbf{R}^n$, then*

$$h_D(a,b) \geq 2 \log \frac{d_D(a) + d_D(b) + \|a-b\|}{2\sqrt{d_D(a) d_D(b)}}, \quad a,b \in D.$$

Observe that equality occurs if $n = 1$ (then $D$ is an open interval or ray).

From now, we assume that $n \geq 2$. Throughout the paper, we will say that $\zeta$ is a *$\mathcal{C}^\alpha$-smooth boundary point* of $D$ if and only if it admits a neighborhood in which $\partial D$ is $\mathcal{C}^\alpha$-smooth.

Recall that a $\mathcal{C}^1$-smooth boundary point $\zeta$ of a domain $D$ in $\mathbf{R}^n$ is said to be *Dini-smooth* if the inner unit normal vector $n$ to $\partial D$ near $\zeta$ is a Dini-continuous function. This means that there exists a neighborhood $U$ of $\zeta$ such that $\int_0^1 \frac{\omega(t)}{t}\, dt < +\infty$, where

$$\omega(t) = \omega(n, \partial D \cap U, t) := \sup\{\|n_x - n_y\| \colon \|x-y\| < t,\ x,y \in \partial D \cap U\}$$

is the respective modulus of continuity.

If $\int_0^1 \omega(t) \frac{\log t}{t}\, dt > -\infty$, then the point $\zeta$ is called *log-Dini smooth.*

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The following relations between different notions of smoothness are clear: $\mathcal{C}^{1,\varepsilon} \implies$ log-Dini $\implies$ Dini $\implies \mathcal{C}^1$.

**Theorem 2.** [9, Theorem 7] *Let $\zeta$ be a Dini-smooth boundary point of a domain $D$ in $\mathbf{R}^n$. Then for any constant $c > 1 + \frac{\sqrt{2}}{2}$ there exists a neighborhood $U$ of $\zeta$ such that*

$$h_D(a,b) \le 2\log\left(1 + \frac{c\|a-b\|}{\sqrt{d_D(a)d_D(b)}}\right), \quad a,b \in D \cap U.$$

Since $h_D$ is an inner metric, we get an upper bound of $h_D$, similar to the lower bound from Proposition 1.

**Corollary 3.** [9, Corollary 8] *Let $D$ be a Dini-smooth bounded domain in $\mathbf{R}^n$. Then there exists a constant $c > 0$ such that*

$$h_D(a,b) \le 2\log\left(1 + \frac{c\|a-b\|}{\sqrt{d_D(a)d_D(b)}}\right), \quad a,b \in D.$$

Set now

$$\begin{aligned} s_D(a,b) &= 2\sinh^{-1}\frac{\|a-b\|}{2\sqrt{d_D(a)d_D(b)}} \\ &= 2\log\frac{\|a-b\| + \sqrt{\|a-b\|^2 + 4d_D(a)d_D(b)}}{2\sqrt{d_D(a)d_D(b)}}, \quad a,b \in D. \end{aligned}$$

Note that $h_D = s_D$ if $D$ is a half-space in $\mathbf{R}^n$ (cf. [12, (2.8)]).

The following sharp result holds in the $\mathcal{C}^1$-smooth case.

**Proposition 4.** [9, Proposition 6(a)] *If $\zeta$ is a $\mathcal{C}^1$-smooth boundary point of a domain $D$ in $\mathbf{R}^n$, then*

$$\lim_{\substack{a,b\to\zeta \\ a\ne b}} \frac{h_D(a,b)}{s_D(a,b)} = 1.$$

Since the proof of this proposition is not long, we shall include it for completeness.

**Corollary 5.** [9, Proposition 6(b) and p. 3] *If $D$ is a $\mathcal{C}^1$-smooth bounded domain in $\mathbf{R}^n$, then*

$$q_D(a,b) = \begin{cases} \frac{h_D(a,b)}{s_D(a,b)}, & a,b \in D,\ a \ne b, \\ 1, & \textit{otherwise}, \end{cases}$$

*is a continuous function on $\mathbf{R}^n \times \mathbf{R}^n$.*

The main goal of this paper is to prove the following result related to Proposition 4.

**Theorem 6.** *If $\zeta$ is a $\mathcal{C}^{1,1}$-smooth boundary point of a domain $D$ in $\mathbf{R}^n$, then*

$$\lim_{a,b\to\zeta} (h_D(a,b) - s_D(a,b)) = 0.$$

Note that Theorem 6 and Proposition 4 say the same only if $s_D$ and $1/s_D$ are bounded.

The assumption about regularity in Theorem 6 can be weakened in the plane.

**Proposition 7.** *If $\zeta$ is a log-Dini smooth boundary point of a domain $D$ in $\mathbf{R}^2$, then*

$$\lim_{a,b\to\zeta} (h_D(a,b) - s_D(a,b)) = 0.$$

The above results imply the following optimal version of Theorem 2.

**Corollary 8.** *Let $\zeta$ be a $\mathcal{C}^{1,1}$-smooth boundary point of a domain $D$ in $\mathbf{R}^n$ or $\zeta$ be a log-Dini smooth boundary point of a domain $D$ in $\mathbf{R}^2$. Then for any constant $c > 1$ there exists a neighborhood $U$ of $\zeta$ such that*

$$h_D(a,b) \le 2\log\left(1 + \frac{c\|a-b\|}{\sqrt{d_D(a)d_D(b)}}\right), \quad a,b \in D \cap U.$$

The rest of the paper is organized as follows: Section 2 contains the proofs of Propositions 4, 7 and Corollary 8. Section 3 contains the proof of Theorem 6. It should be mentioned that the three proofs use different flattening maps. Section 4 contains the proof of a result analogous to Corollary 8 for the Kobayashi distance.

## 2. Proofs of Propositions 4, 7 and Corollary 8

*Proof of Proposition 4.* After translation and rotation, we may assume that $\zeta = 0$ and that there is a neighborhood $U$ of 0 such that

$$D' := D \cap U = \{x \in U \colon r(x) := x_1 + f(x') > 0\},$$

where points of $\mathbf{R}^n$ are denoted by $x = (x_1, x')$, with $x' \in \mathbf{R}^{n-1}$, and $f$ is a $\mathcal{C}^1$-smooth function in $\mathbf{R}^n$ with $f(0) = 0$ and $\nabla f(0) = 0$.

Let $c > 1$ and $\theta(x) = (r(x), x')$. We may shrink $U$ such that

$$c^{-1}\|x-y\| \le \|\theta(x) - \theta(y)\| \le c\|x-y\|, \quad x,y \in U. \tag{1}$$

Choose now a neighborhood $V \subset U$ of 0 such that $d_{D'} = d_D$ on $D \cap V$. The regularity of $D$ implies that it is a *uniform domain* near $\zeta$ in the sense of [5]. Using, for example, [5, Corollary 2], one can find a neighborhood $W \subset V$ of 0 such that any geodesic joining points in $\tilde{D} = D \cap W$ is contained in $D \cap V$. Then $h_D = h_{D'}$ on $\tilde{D}^2$.

Set $\mathbf{R}^n_+ = \{x \in \mathbf{R}^n \colon x_1 > 0\}$. Using the above arguments, we may shrink $W$ such that $h_{\mathbf{R}^n_+} = h_{\theta(D')}$ on $(\theta(\tilde{D}))^2$.

On the other hand, (1) implies that (cf. [12, Exercise 3.17])

$$c^{-2}h_{D'}(z,w) \le h_{\theta(D')}(\theta(z), \theta(w)) \le c^2 h_{D'}(z,w), \quad z,w \in D'.$$

Let $z, w \in \tilde{D}$. Then

$$c^{-2}h_D(z,w) \le h_{\mathbf{R}^n_+}(\theta(z), \theta(w)) \le c^2 h_D(z,w).$$

Using (1) again, we get that

$$\begin{aligned} h_{\mathbf{R}^n_+}(\theta(z), \theta(w)) &= 2\sinh^{-1}\frac{\|\theta(z) - \theta(w)\|}{2\sqrt{r_D(z)r_D(w)}} \le 2\sinh^{-1}\frac{c^2\|z-w\|}{2\sqrt{d_D(z)d_D(w)}} \\ &\le c^2 s_D(z,w). \end{aligned}$$

We obtain in the same way that

$$h_{\mathbf{R}^n_+}(\theta(z), \theta(w)) \ge c^{-2}s_D(z,w).$$

So

$$c^{-4}h_D(z,w) \le s_D(z,w) \le c^4 h_D(z,w)$$

which implies the desired result. □

*Proof of Proposition 7.* We may find a neighborhood $U$ of $\zeta$ such that $D \cap U$ is a bounded simply connected log-Dini smooth domain. Using an argument from the previous proof, we may replace $D$ by $D \cap U$.

The Kellogg–Warschawski theorem (cf. [11, Theorem 3.5]) implies that there exists a conformal map $\tilde f$ from the unit disc $\mathbf{D}$ to $D$ which extends to a $\mathcal{C}^1$-diffeomorphism between $\overline{\mathbf{D}}$ to $\overline{D}$ such that $\tilde f(\zeta) = 1$ and

$$|\tilde f'(z) - \tilde f'(w)| \le \tilde\omega^*(|z-w|), \quad z, w \in \mathbf{D},$$

where $\tilde\omega^*(s) = \int_0^s \frac{\tilde\omega(t)}{t}\,dt + s\int_s^{+\infty} \frac{\tilde\omega(t)}{t^2}\,dt$ $(s \ge 0)$ and $\tilde\omega\colon \mathbf{R}^+ \to \mathbf{R}^+$ is a bounded continuous function with $\int_0^1 \tilde\omega(t)\frac{\log t}{t}\,dt > -\infty$.

Then $f(z) = \tilde f\left(\frac{1-z}{1+z}\right)$ maps conformally $\mathbf{R}^2_+$ onto $D$ and

$$|f'(z) - f'(w)| \le \omega^*(|z-w|), \quad z, w \in G = \mathbf{R}^2_+ \cap \mathbf{D},$$

where $\omega^*$ is defined from $\omega$ in the same way as $\tilde\omega^*$.

The equality

$$f(w) - f(z) - f'(z)(w-z) = (w-z)\int_0^1 (f'(z+t(w-z)) - f'(z))\,dt$$

implies that

$$|f(w) - f(z) - f'(z)(w-z)| \le |w-z|\omega^*(|w-z|)$$

(since $\omega^*$ is an increasing function). It follows that

(2) $$|d_D(f(z)) - |f'(z)|d_{\mathbf{R}^2_+}(z)| \le d_{\mathbf{R}^2_+}(z)\omega^*(d_{\mathbf{R}^2_+}(z)), \quad z \in G.$$

Since $D$ is a uniform domain, there exists a neighborhood $V$ of $\zeta$ such that any geodesic $\gamma$ joining points $a = f(\alpha)$ and $b = f(\beta)$ in $D \cap V$ is contained in $f(G)$. It follows by (2) that one may find a constant $C > 0$ (independent of $a$ and $b$) such that

$$h_{\mathbf{R}^2_+}(\alpha,\beta) \le \int_{f^{-1}\circ\gamma} \frac{|du|}{d_{\mathbf{R}^2_+}(u)} \le \int_\gamma \frac{|dv|}{d_D(v)} + C\int_\gamma \frac{\omega^*(d_D(v))}{d_D(v)}|dv|.$$

The first summand is equal to $h_D(a,b)$.

We claim that the second summand tends to 0 as $a, b \to \zeta$. Indeed, denote by $t$ the natural parameter of $\gamma$ by arc length and by $l = l(\gamma)$ the Euclidean length of $\gamma$. Since $D$ is a uniform domain, then [5, Corollary 2] provides a constant $c > 0$ (independent of $a$ and $b$) such that $c \cdot l \le |a-b|$ and $d_D(\gamma(t)) \ge c \cdot \max\{t, l-t\}$. Using that $\frac{\omega^*(s)}{s}$ is a decreasing function, we get

$$\int_\gamma \frac{\omega^*(d_D(v))}{d_D(v)}|dv| \le \frac{2}{c}\int_0^{cl/2} \frac{\omega^*(t)}{t}\,dt.$$

It is easy to check the log-Dini condition for $\omega$ is is equivalent to the fact that the last integral tends to 0 as $l \to 0$ which implies our claim.

Hence

$$\liminf_{a,b\to\zeta}(h_D(a,b) - h_{\mathbf{R}^2_+}(\alpha,\beta)) \ge 0.$$

The opposite inequality

$$\limsup_{a,b\to\zeta}(h_D(a,b) - h_{\mathbf{R}^2_+}(\alpha,\beta)) \le 0$$

follows in the same way by taking the geodesic joining $\alpha$ and $\beta$.

Using (2), we have that

(3) $$\lim_{\substack{a,b\to\zeta\\ a\ne b}} \frac{|a-b|}{2\sqrt{d_D(a)d_D(b)}} \cdot \frac{2\sqrt{d_{\mathbf{R}^2_+}(\alpha)d_{\mathbf{R}^2_+}(\beta)}}{|\alpha-\beta|} = 1.$$

Since $h_{\mathbf{R}^2_+} = s_{\mathbf{R}^2_+}$ and $\sinh^{-1} qt < \log q + \sinh^{-1} t$ for $q > 1$, $t > 0$, then

$$\lim_{a,b\to\zeta} (s_D(a,b) - h_{\mathbf{R}^2_+}(\alpha,\beta)) = 0$$

which completes the proof. □

*Proof of Corollary 8.* We may assume that $c = 2c' - 1 \in (1,3]$. By Proposition 4, Theorem 6 and Proposition 7, one may find a neighborhood $U$ of $\zeta$ such that for $a, b \in D \cap U$,

$$h_D(a,b) \le c' s_D(a,b), \quad h_D(a,b) \le s_D(a,b) + \log c'.$$

Then the result follows by the inequalities $\sinh^{-1} \dfrac{t}{2} < \log(1+t)$ $(t > 0)$, $(1+t)^{c'} < 1 + ct$ $(0 < t < 1)$ and $c'(1+t) < 1 + ct$ $(t > 1)$. □

## 3. Proof of Theorem 6

Theorem 6 will follow from Propositions 9 and 11 below.

For convenience, we assume that $D$ is a domain in $\mathbf{R}^{n+1}$ $(n \ge 1)$. We first localize the problem. We choose local coordinates so that $\zeta = 0$ and $T_0\partial D = \{0\} \times \mathbf{R}^n$. Denote points in $\mathbf{R}^{n+1}$ by $\bar{x} = (x_0, x) \in \mathbf{R} \times \mathbf{R}^n$. We also write $\mathbf{R}^{n+1}_+ = \{\bar{x} \in \mathbf{R}^{n+1} \colon x_0 > 0\}$.

There are a ball $\mathcal{U} \subset \mathbf{R}^{n+1}$ centered at $(0,0)$ and a function $f \in \mathcal{C}^{1,1}(\mathcal{U} \cap \mathbf{R}^n, \mathbf{R})$ such that $f(0) = 0$ and $Df(0) = 0$ and

$$D \cap \mathcal{U} = \{\bar{x} \in \mathcal{U} \colon x_0 > f(x)\}. \tag{4}$$

By shrinking the radius of $\mathcal{U}$ further we may assume that the projection which to $\bar{x} \in \mathcal{U} \cap D$ associates $\pi(\bar{x})$, the closest point in $\partial D$ is well-defined, and that $\mathcal{U} \subset \pi^{-1}(\mathcal{U} \cap D)$ (see [1, Lemma 4.11], or the proof of Lemma 10 (1) below).

**Proposition 9.** $\liminf_{a,b\to 0} (h_D(a,b) - s_D(a,b)) \ge 0$.

We can define a map $\varphi$ on $\mathcal{U}$ by

$$\varphi(\bar{x}) = (f(x), x) + x_0 n_x,$$

where $n_x$ is the inward unit normal to $\partial D$ at the point $(f(x), x)$.

**Lemma 10.** (1) *There exists a ball $\mathcal{U}_0 \subset \mathcal{U}$ centered at $0$ such that $\varphi|_{\mathcal{U}_0}$ is a bilipschitz homeomorphism and for any $\bar{x} \in \mathcal{U}_0 \cap \mathbf{R}^{n+1}_+$,*

$$d_D\varphi(\bar{x}) = \|\varphi(\bar{x}) - (f(x), x)\| = x_0.$$

(2) *Furthermore, if $f \in \mathcal{C}^\alpha(\mathcal{U} \cap \mathbf{R}^n, \mathbf{R})$, for some $\alpha \ge 2$, then $\varphi|_{\mathcal{U}_0}$ is a $\mathcal{C}^{\alpha-1}$-diffeomorphism, and there exists a ball $\mathcal{U}_1 \subset \mathcal{U}_0$ centered at $0$ and a constant $C > 0$ such that for any $\bar{x} \in \mathcal{U}_1 \cap \mathbf{R}^{n+1}_+$ and any vector $v \in \mathbf{R}^{n+1}$,*

$$\|D\varphi(\bar{x}) \cdot v\| \ge (1 - Cx_0)\|v\|,$$

*where $D\varphi(\bar{x})$ stands for the differential of $\varphi$ taken at the point $\bar{x}$.*

(3) *In the general case where $f \in \mathcal{C}^{1,1}(\mathcal{U} \cap \mathbf{R}^n, \mathbf{R})$, then there exists a $C > 0$ such that for any $\mathcal{C}^1$ curve $\gamma \colon [t_1, t_2] \longrightarrow \mathcal{U}_1 \cap \mathbf{R}^{n+1}_+$, $\varphi \circ \gamma$ is rectifiable and for any $F \in \mathcal{C}([t_1, t_2], \mathbf{R}_+)$,*

$$\int_{t_1}^{t_2} F(t)|d\varphi \circ \gamma(t)| \ge \int_{t_1}^{t_2} F(t)|d\gamma(t)| - C \int_{t_1}^{t_2} F(t) d_D(\gamma(t))|d\gamma(t)|.$$

*Proof.* Part (1) of the lemma is classical (see [1, Theorem 4.8]). The main point is to prove that the domain has positive *reach,* that is to say that there exists $\delta > 0$ such that if $x \in D$ and $d_D(x) < \delta$, then this distance is attained at a single point, which will be the intersection of $\partial D$ and the unique normal line to it containing $x$ (see [1]). In other words, for $x \in \mathcal{U}$ well chosen and $x_0 < \delta$, $\varphi$ is one-to-one.

We quickly recall the proof. Suppose $\|\nabla f(x) - \nabla f(x')\| \le L\|x - x'\|$ for $(0, x)$, $(0, x') \in \mathcal{U}_1$, then, taking without loss of generality the projection to $\partial D$ to be $(0, 0)$, for some $\theta \in (0, 1)$,

$$\begin{aligned}\|(y_0, 0) - (f(x), x)\|^2 &= y_0^2 - 2y_0 \nabla f(\theta x) \cdot x + f(x)^2 + \|x\|^2 \\ &\ge y_0^2 + \|x\|^2 - 2y_0 L\|x\|^2 > y_0^2\end{aligned}$$

for $y_0 < 1/2L$ and $x \neq 0$.

Notice that a lemma in [6, Appendix], explained in detail in [7], shows that even though $n_x$ can only be expected to be continuous with bounded derivatives, and in general of class $\mathcal{C}^{\alpha-1}$ when $\varphi \in \mathcal{C}^\alpha$, the function $\bar{x} \mapsto d_D(\bar{x})$ has the same regularity as $\varphi$.

We now prove part (2). Let $(e_0, e_1, \dots, e_n)$ be the standard basis of $\mathbf{R}^{n+1}$. Let $\tilde{e}_j = \frac{\partial f}{\partial x_j}(x) e_0 + e_j$, for $1 \le j \le n$. They form a basis of the tangent space to $\partial D$ at $(x, f(x))$ and $\langle n_x, \tilde{e}_j \rangle = 0$ for $1 \le j \le n$. Then $D\varphi(\bar{x}) \cdot e_0 = n_x$, and $D\varphi(\bar{x}) \cdot e_j = \tilde{e}_j + x_0 \frac{\partial n_x}{\partial x_j}$, for $1 \le j \le n$.

Given $v = \sum_0^n v_j e_j$,

$$D\varphi(\bar{x}) \cdot v = \left( v_0 n_x + \sum_1^n v_j \tilde{e}_j \right) + x_0 \sum_1^n v_j \frac{\partial n_x}{\partial x_j} =: V_1 + V_0.$$

Clearly, $\|V_0\| = O(x_0)\|v\|$. By the orthogonality of $n_x$ to the tangent space,

$$\begin{aligned}\|V_1\|^2 &= v_0^2 + \left\| \sum_1^n v_j \tilde{e}_j \right\|^2 = v_0^2 + \left\| \sum_1^n v_j e_j + \left( \sum_1^n v_j \frac{\partial f}{\partial x_j}(x) \right) e_0 \right\|^2 \\ &= v_0^2 + \sum_1^n v_j^2 + \left| \sum_1^n v_j \frac{\partial f}{\partial x_j}(x) \right|^2 \ge \|v\|^2.\end{aligned}$$

In the case where $f \in \mathcal{C}^{1,1}$, then $\varphi \circ \gamma$ is only a Lipschitz map. By Rademacher's theorem (see e.g. [2, Theorem 3.1.6]), it is almost everywhere differentiable and the fundamental theorem of calculus holds. We then perform the same calculation as in case (2), where the integrands are defined a.e. □

*Proof of Proposition 9.* Using Lemma 10, the proof repeats the second part of the proof of Proposition 7. Suppose that $\zeta = 0$ and that the domain $D$ is given by a local representation as above. We may assume that the points $a, b \in D$ are in a small enough neighborhood of 0 so that the geodesic $\gamma$ which joins them is entirely contained in the range of invertibility of $\varphi$ and Lemma 10 holds; we write $a = \varphi(\bar{\alpha})$, $b = \varphi(\bar{\beta})$, $\gamma = \varphi(\tilde{\gamma})$, where $\tilde{\gamma}$ is an arc in $\mathbf{R}_+^{n+1}$. Then

$$h_D(a, b) = \int_\gamma \frac{\|du\|}{d_D(u)} \ge \int_{\tilde{\gamma}} \frac{\|dv\|}{d_{\mathbf{R}_+^{n+1}}(v)} - C \cdot l(\tilde{\gamma}) \ge h_{\mathbf{R}_+^{n+1}}(\bar{\alpha}, \bar{\beta}) - C'\|\bar{\alpha} - \bar{\beta}\|,$$

where $C' > 0$ is a constant independent of $a$ and $b$. Note that $h_{\mathbf{R}^{n+1}_+} = s_{\mathbf{R}^{n+1}_+}$. Since the differential of $\varphi$ at $\bar{x}$ tends to the identity as $x \to 0$, it follows that

$$\lim_{a,b\to\zeta} (s_{\mathbf{R}^{n+1}_+}(\bar{\alpha}, \bar{\beta}) - s_D(a,b)) = 0$$

which completes the proof. □

**Proposition 11.** $\limsup_{a,b\to 0} (h_D(a,b) - s_D(a,b)) \le 0$.

The proof is similar to that of Proposition 9, using a modification of the map $\varphi$ which depends on $a$ and $b$.

*Proof.* We again assume that $a, b \in D$, and the geodesic connecting them, all lie in a neighborhood of $\zeta$ small enough so that any point in it has a unique closest point on $\partial D$. Let $a', b'$ be the respective closest points. We take new coordinates (and obtain a new function $f$) so that $a' = 0$ (instead of $\zeta = 0$ as in the proof of Proposition 9) and

$$D \cap \mathcal{U} = \{\bar{x} \in \mathcal{U} \colon x_0 > f(x_1, \dots, x_n)\}.$$

We may also assume that $b'_2 = \dots = b'_n = 0$. Shrinking the radius $r$ of $\mathcal{U}$, we may replace $x_1$ by $\sigma_1(x_1)$ such that for $\sigma = (f(\sigma_1, 0, \dots, 0), \sigma_1, 0, \dots, 0)$ one has $\|\sigma'\| = 1$ (in other words, $\sigma$ is parametrized by arc length). Note that $r$ can be chosen independently of $a$ and $b$. Let $\ell$ be the length of the curve $\sigma$ from $a'$ to $b'$, so that $\sigma(0) = a'$, $\sigma(\ell) = b'$.

Consider the map $\varphi$ from $\mathbf{R}^2_+$ (near 0) to $D$ defined by

$$\varphi(x_0, x_1) = \sigma(x_1) + x_0 n_{\sigma(x_1)},$$

where $n_{\sigma(x_1)}$ is the inward unit normal to $\partial D$ at the point $\sigma(x_1)$. Then $d_D(\varphi(\bar{x})) = x_0$ if $x_0$ is small enough, and if $\alpha = (d_D(a), 0)$ and $\beta = (d_D(b), \ell)$, we have $\varphi(\alpha) = a$, $\varphi(\beta) = b$.

**Lemma 12.** *There exist a neighborhood $U$ of $\zeta$, a neighborhood $V$ of 0 and a constant $C > 0$ such that for any $a, b \in D \cap U$ and $\bar{x} \in \mathbf{R}^2_+ \cap V$ and any vector $v \in \mathbf{R}^2$, then $\alpha, \beta \in V$ and*

$$\|D\varphi(\bar{x}) \cdot v\| \le (1 + Cx_0)\|v\|.$$

*Proof.* As in the proof of Lemma 10 (2), in the $\mathcal{C}^2$-smooth case,

$$D\varphi(\bar{x}) \cdot e_0 = n_{\sigma(x_1)}, \quad D\varphi(\bar{x}) \cdot e_1 = \sigma'(x_1) + x_0 \frac{\partial n_{\sigma(x_1)}}{\partial x_1}.$$

Because $\|\sigma'\| = 1$ and is tangent to $\partial D$, $(\sigma'(x), n_x)$ form an orthonormal system, so that $D\psi(\bar{x})$ differs from a linear isometric embedding by a term bounded by $\left\| \frac{\partial n_{\sigma(x_1)}}{\partial x_1} \right\| x_0$.

Geometric considerations show that $\left\| \frac{\partial n_{\sigma(x_1)}}{\partial x_1} \right\| \le \frac{1}{R}$ whenever there exist two balls $B_1, B_2$ of radius $R$, tangent to each side of $\partial D$ at $\sigma(x_1)$. The argument in the proof of Lemma 10 (1) shows there exists $\delta > 0$ (depending only on the neighborhood $\mathcal{U}_0$ mentioned in that lemma) such that there exist two such balls of radius $\delta$ at each point in $\mathcal{U}_0 \cap \partial D$.

As in the proof of Lemma 10 (3), the $\mathcal{C}^{1,1}$-smooth case follows by applying Rademacher's theorem. □

The proof of Proposition 11 can be finished similarly to that of Proposition 9. Let $\gamma$ be the geodesic joining $\alpha$ to $\beta$ in $\mathbf{R}^2_+$. Let $U, V$ be as in Lemma 12. Shrinking

$V$ if needed so that $\varphi(V) \subset U$, we have $d_D(\varphi(u)) = d_{\mathbf{R}^2_+}(u)$ for any $u \in \gamma$. Since $\varphi \circ \gamma$ is a curve joining $a$ to $b$ in $D$, using Lemma 12, we get

$$h_D(a,b) \le \int_0^\ell \frac{\|D\varphi(\gamma(t)) \cdot \gamma'(t)\|}{d_D(\varphi \circ \gamma(t))}\, dt \le h_{\mathbf{R}^2_+}(\alpha,\beta) + Cl(\gamma)$$
$$< s_{\mathbf{R}^2_+}(\alpha,\beta) + C\pi\|\alpha - \beta\|$$

(here $\pi$ is the Ludolphine number, not the projection). The differential of $\varphi$ is close to a linear isometric embedding of $\mathbf{R}^2$ in $\mathbf{R}^{n+1}$ and hence we have the asymptotic relation (3) and

$$\lim_{a,b\to\zeta} (s_{\mathbf{R}^2_+}(\alpha,\beta) - s_D(a,b)) = 0,$$

which completes the proof. □

## 4. An upper estimate for the Kobayashi distance

Let $D$ be a domain in $\mathbf{C}^n$. The Kobayashi (pseudo) distance $k_D$ is obtained from the Lempert function

$$l_D(a,b) = \inf\{\tanh^{-1}|\alpha| \colon \exists \varphi \in \mathcal{O}(\mathbf{D}, D) \text{ with } \varphi(0) = a,\ \varphi(\alpha) = b\}, \quad a,b \in D.$$

The Lempert function does not always satisfy the triangle inequality, but setting

$$k_D(a,b) := \inf\left\{\sum_{j=0}^{m-1} l_D(a_j, a_{j+1}) \colon a_j \in D,\ a_0 = a,\ a_m = b,\ m \ge 1\right\},$$

one does obtain a (pseudo) distance, which is the largest that is dominated by $l_D$.

Recall that $k_D$ is the integrated form of the Kobayashi (pseudo) metric

$$\kappa_D(a; X) = \inf\{|\alpha| \colon \exists \varphi \in \mathcal{O}(\mathbf{D}, D) \text{ with } \varphi(0) = a,\ \alpha\varphi'(0) = X\}, \quad a \in D,\ X \in \mathbf{C}^n.$$

Note that Theorem 2 and Proposition 7 (even in the Dini-smooth case) hold for $2k_D$ instead of $h_D$ (see [9, Theorem 7] and [10, Proposition 6]). Moreover, the following result corresponds to Proposition 4.

**Proposition 13.** [9, Proposition 5(a)] *If $\zeta$ is a $\mathcal{C}^1$-smooth boundary point of a domain $D$ in $\mathbf{C}^n$, then*

$$\limsup_{\substack{a,b\to\zeta \\ a\ne b}} \frac{2k_D(a,b)}{h_D(a,b)} \le 1.$$

It turns out that Corollary 8 also holds for $2k_D$ instead of $h_D$. This gives the optimal version of [3, Proposition 2.5] in the $\mathcal{C}^{1,1}$-smooth case.

**Proposition 14.** *Let $\zeta$ be a $\mathcal{C}^{1,1}$-smooth boundary point of a domain $D$ in $\mathbf{C}^n$ or $\zeta$ be a log-Dini smooth boundary point of a domain $D$ in $\mathbf{C}$. Then for any constant $c > 1$ there exists a neighborhood $U$ of $\zeta$ such that*

$$k_D(a,b) \le \log\left(1 + \frac{c\|a-b\|}{\sqrt{d_D(a)d_D(b)}}\right), \quad a,b \in D \cap U.$$

*Proof.* Having in mind Corollary 8, it is enough to show that

$$\limsup_{\substack{a,b\to\zeta \\ a\ne b}} \frac{2k_D(a,b) - h_D(a,b)}{\|a-b\|} < +\infty.$$

Since $k_D$ is the integrated form of $\kappa_D$ and the lengths of the quasi-hyperbolic geodesics joining points in $D$ near $\zeta$ are bounded up to a multiplicative constant by the Euclidean distances between the points, the last inequality will be a consequence of the following one:

$$\limsup_{\substack{a\to\zeta\\ \|X\|=1}}\left(2\kappa_D(a;X)-\frac{1}{d_D(a)}\right)<+\infty.$$

To see this, note that there exists an $r>0$ such that any $a\in D$ near $\zeta$ is contained in a (unique) ball $\mathbf{B}_n(\tilde{a},r)\subset D$ with $r-\|a-\tilde{a}\|=d_D(a)$ (the inner ball condition). It remains to use that for such an $a$ and $\|X\|=1$ one has that

$$\kappa_D(a;X)\le\kappa_{\mathbf{B}_n(\tilde{a},r)}(a;X)\le\frac{r}{r^2-\|a-\tilde{a}\|^2}<\frac{1}{2d_D(a)}+\frac{1}{4r}.\qquad\square$$

## References

[1] Federer, H.: Curvature measures. - Trans. Amer. Math. Soc. 93, 1959, 418–491.

[2] Federer, H.: Geometric measure theory. - Grundlehren Math. Wiss. 153, Springer, Berlin, 1969.

[3] Forstnerič, F., and J.-P. Rosay: Localization ot the Kobayashi metric and the boundary continuity of proper holomorphic mappings. - Math. Ann. 279, 1987, 239–252.

[4] Gehring, F. W., K. Hag, and O. Martio: Quasihyperbolic geodesics in John domains. - Math. Scand. 65, 1989, 75–92.

[5] Gehring, F. W., and B. G. Osgood: Uniform domains and the quasi-hyperbolic metric. - J. Anal. Math. 36, 1979, 50–74.

[6] Gilbarg, D., and N. Trudinger: Elliptic partial differential equations of second order. - Springer Verlag, Berlin, 1977.

[7] Krantz, S. G., and H. R. Parks: Distance to $\mathcal{C}^k$ hypersurfaces. - J. Differential Equations 40, 1981, 116–120.

[8] Martin, G. J.: Quasiconformal and bi-Lipschitz homeomorphisms, uniform domains and the quasihyperbolic metric. - Trans. Amer. Math. Soc. 292, 1985, 169–191.

[9] Nikolov, N., and L. Andreev: Estimates of the Kobayashi and quasi-hyperbolic distances. - Ann. Mat. Pura Appl. 196, 2017, 43–50.

[10] Nikolov, N., M. Trybuła, and L. Andreev: Boundary behavior of invariant functions on planar domains. - Complex Var. Elliptic Equ. 61, 2016, 1064–1072.

[11] Pommerenke, Ch.: Boundary behaviour of conformal maps. - Grundlehren Math. Wiss. 299, Springer, Berlin, 1992.

[12] Vuorinen, M.: Conformal geometry and quasiregular mappings. - Lecture Notes in Math. 1319, Springer, Berlin, 1988.

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