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PROOF OF THE CONJECTURE OF KESKİN, ŞİAR AND KARAATLI

Rachid Boumahdi, Omar Kihel and Sukrawan Mavecha

Université des Sciences et de la Technologie Houari Boumediène
Laboratoire d'Arithmétique, Codage, Combinatoire et Calcul Formel
1611, El Alia, Alger, Algérie; r_boumehdi@esi.dz

Brock University, Department of Mathematics
Ontario, Canada L2S 3A1; okihel@brocku.ca

King Mongkut's Institute of Technology Ladkrabang
Faculty of Science, Department of Mathematics
Bangkok, 10520, Thailand; sukrawan.ta@kmitl.ac.th

Abstract. In this paper among other results, we will prove the conjecture of Keskin, Şiar and Karaatlı on the Diophantine equation $x^2 - kxy + y^2 - 2^n = 0$.

1. Introduction

There has been much recent interest in the Diophantine equation

$$x^2 - kxy + y^2 + lx = 0 \tag{1}$$

for different values of the integers k and l. Marlewski and Zarzycki [4] considered equation (1) for $l = 1$, and proved that equation (1) has no positive solutions for $l = 1$ and $k > 3$, but has an infinite number of solutions for $k = 3$ and $l = 1$. Keskin, in [1] considered equation (1) for $l = -1$ and proved that it has positive integer solutions for $k > 1$. Yuan and Hu [6] considred equation (1) with $l = 1, 2$ or 4 and determined the values of the integer k for which equation (1) has an infinite number of positive solutions. Expanding on the work of Yuan and Hu [6], Keskin et al. in [2] and [3] considered equation (1) for $l = \pm 2^r$ with r a positive integer. They explained that in order to determine when equation (1) with $l = -2^r$, has an infinite number of positive integer solutions, one needs only to determine when the Diophantine equation

$$x^2 - kxy + y^2 - 2^n = 0 \tag{2}$$

has an infinite number of positive integer solutions x and y for certain values of the non negative integer n. Similarly for $l = 2^r$ in equation (1), one needs only to consider the Diophantine equation

$$x^2 - kxy + y^2 + 2^n = 0. \tag{3}$$

Keskin et al. solved equation (2) and equation (3) for $0 \le n \le 10$, and formulated the following conjecture in [3].

Conjecture 1. (i) *Let n be an odd integer and $n > 2$. If $k > 2^n - 2$, then equation (2) has no positive integer solutions. If $k \le 2^n - 2$ and (2) has a solution, then k is even.*

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(ii) *Let n be an even integer. If $k > 2^n - 2$, then equation* (2) *has no positive odd integer solutions. If $k \le 2^n - 2$ and equation* (2) *has a positive odd integer solution, then k is even.*

In this paper, among other results, we will prove Conjecture 1 in Theorem 3.1, and prove Theorem 3.2 a result analogous to Conjecture 1.

2. Preliminary results

In this section, we will recall some results that we will need for the proof of our theorems. Let d be a positive integer which is not a perfect square and consider the Pell equation

$$x^2 - dy^2 = 1. \tag{4}$$

It is well known (cf. [5, p. 197]) that equation (4) always has a positive solution when $d \ge 2$. Consider all the solutions $x + y\sqrt{d}$ with positive x and y. Among these there is a least solution $x_1 + y_1\sqrt{d}$ in which x_1 and y_1 have their least positive values. The number $x_1 + y_1\sqrt{d}$ is called the fundamental solution, and all positive integer solutions to (4) are given by

$$x_n + y_n\sqrt{d} = \left(x_1 + y_1\sqrt{d}\right)^n \quad \text{with} \quad n \ge 1.$$

Let C be a nonzero integer, and consider the Diophantine equation

$$u^2 - dv^2 = C. \tag{5}$$

Suppose that $u + v\sqrt{d}$ is a solution to equation (5). If $x + y\sqrt{d}$ is any solution of equation (4), then

$$u' + v'\sqrt{d} = \left(u + v\sqrt{d}\right)\left(x + y\sqrt{d}\right) = ux + vyd + (yu + vx)\sqrt{d}$$

is also a solution of (5). The solution $u' + v'\sqrt{d}$ is said to be associated with the solution $u + v\sqrt{d}$. The set of all solutions associated with each other form a class of solutions of equation (5). Every class contains an infinity of solutions. We have the following lemmas.

Lemma 2.1. *If $u + v\sqrt{d}$ is the fundamental solution of a class K of the equation*

$$u^2 - dv^2 = N$$

where N is a positive integer and if $x_1 + y_1\sqrt{d}$ is the fundamental solution of equation (4), *then we have the inequalities*

$$0 \le v \le \frac{y_1}{\sqrt{2(x_1+1)}}\sqrt{N}$$

and

$$0 < |u| \le \sqrt{\frac{1}{2}(x_1+1)N}.$$

Lemma 2.2. *If $u + v\sqrt{d}$ is the fundamental solution of a class K of the equation $u^2 - dv^2 = -N$, where N is a positive integer and if $x_1 + y_1\sqrt{d}$ is the fundamental solution of equation* (4), *then we have the inequalities*

$$0 < v \le \frac{y_1}{\sqrt{2(x_1-1)}}\sqrt{N}$$

and

$$0 \leq |u| \leq \sqrt{\frac{1}{2}\left(x_1 - 1\right) N}.$$

For the proof of Lemma 2.1 and Lemma 2.2, see [5].

3. New results

In this section, we will prove Conjecture 1 in Theorem 3.1, and in Theorem 3.2 a result that is analogous to Conjecture 1 for the Diophantine equation

$$x^2 - kxy + y^2 = -2^n.$$

If $k = 0$, then equation (2) has finitely many solutions and equation (3) has no solution. We suppose in the sequel that $k \neq 0$.

Theorem 3.1. *Conjecture 1 is true.*

Proof. (i) Let $n > 2$ be an odd integer. If (x, y) is a positive solution of equation (2), then clearly x and y have the same parity. If x and y are odd, then k is even. Let $x = 2^a X$ and $y = 2^b Y$ with X and Y odd. Since n is odd, it can be seen that $a = b$. Thus we get

$$X^2 - kXY + Y^2 = 2^{n-2a} = 2^r \quad \text{with } r \text{ odd.} \tag{6}$$

Hence k is clearly even. After a change of variables, equation (2) with n odd yields

$$u^2 - dv^2 = 2^n; \tag{7}$$

where $u = \left|x - \frac{k}{2}y\right|, y = v$ and $d = \frac{k^2}{4} - 1$.

Since k is even, then u and v are positive integers. If $k = 2$, equation (7) implies that $u^2 = 2^n$, which is impossible. Hence, $k > 2$, whereupon $d > 1$. The solution $\frac{k}{2} + \sqrt{\frac{k^2}{4} - 1}$ is the fundamental solution to the Diophantine equation

$$x^2 - dy^2 = 1, \quad \text{where} \quad d = \frac{k^2}{4} - 1.$$

If equation (2) has a positive solution with n an odd positive integer, then equation (7) has a positive solution. If $u + v\sqrt{d}$ is the fundamental solution of a class K of equation (7), then Lemma 2.1 implies that

$$0 \leq v \leq \frac{1}{\sqrt{2\left(\frac{k}{2} + 1\right)}}\sqrt{2^n}.$$

If $v = 0$, then equation (7) yields $u^2 = 2^n$, which is impossible. Therefore, $v \geq 1$ and the inequality above implies that $\sqrt{k+2} \leq \sqrt{2^n}$, i.e. $k \leq 2^n - 2$.

(ii) Let n be a positive even integer, and suppose that (x, y) is a solution to $x^2 - kxy + y^2 = 2^n$. If x and y are odd, then clearly k is even. Hence equation (2) yields $u^2 - dv^2 = 2^n$, where $u = \left|x - \frac{k}{2}y\right|, v = y$ and $d = \frac{k^2}{4} - 1$. Since $k \neq 0$, $k \geq 2$, and d is a non negative integer. Clearly, $d = 0$ implies $k = 2$. The fact that $k > 2^n - 2$ implies that $k + 2 > 2^n$. For $k = 2$, we have that $4 > 2^n$, which is impossible because n is even. Hence, $d > 1$, since $d \neq 1$. Lemma 2.1 implies that

$$0 \leq v \leq \frac{1}{\sqrt{2\left(\frac{k}{2} + 1\right)}}\sqrt{2^n}$$

since the solution $\frac{k}{2}+\sqrt{\frac{k^2}{4}-1}$ is the fundamental solution to $x^2-dy^2=1$, where $d=\frac{k^2}{4}-1$. If $v=0$, then equation (7) yields $u=2^{n/2}$ and all solutions in the same class as $(2^{n/2},0)$ are even. Hence, we suppose $v\geq 1$, and the last inequality implies that $\sqrt{2\left(\frac{k}{2}+1\right)}\leq\sqrt{2^n}$, i.e. $k\leq 2^n-2$. □

Theorem 3.2. (i) *Let n be an odd integer and $n>2$. If $k>2^n+2$, then the equation $x^2-kxy+y^2=-2^n$ has no positive integer solutions. If $k\leq 2^n+2$, and the equation $x^2-kxy+y^2=-2^n$ has a solution, then k is even.*

(ii) *Let n be an nonzero even integer. If $k>2^n+2$, then the equation $x^2-kxy+y^2=-2^n$ has no positive odd integer solution. If $k\leq 2^n+2$ and the equation $x^2-kxy+y^2=-2^n$ has a positive odd integer solution, then k is even and 2 divides exactly k.*

Proof. (i) Let n be a positive odd integer and $n>2$. Using the same reasoning as in the proof of Theorem 3.1, without loss of generality, we can suppose that the solutions x and y to (1) are odd. Hence k is even. Again, the same method in the proof of Theorem 3.1 and Lemma 2.2 implies that

$$1\leq v\leq\frac{1}{\sqrt{2\left(\frac{k}{2}-1\right)}}\sqrt{2^n}$$

whereupon, $\sqrt{k-2}\leq\sqrt{2^n}$, i.e. $k\leq 2^n+2$.

(ii) Suppose that n is even and that the equation $x^2-kxy+y^2=-2^n$ has a positive integer solution. Then clearly k is even because $n\geq 1$ (the case $n=0$ has been settled in [2]). Again the same method as in the proof of Theorem 3.1 and Lemma 2.2 implies that $k\leq 2^n+2$ and k even. If (x,y) is an odd solution to equation (1), then taking $x^2-kxy+y^2=-2^n$ modulo 4 implies that 2 divides exactly k. □

Remark 3.1. It was proved in [3] that the Diophantine equation $x^2-kxy+y^2=2^n$ with $k=2^n-2$ has infinitely many solutions and in [2] that the Diophantine equation $x^2-kxy+y^2=-2^n$ with $k=2^n+2$ has infinitely many solutions. Hence the bounds of k in Theorem 3.1 and Theorem 3.2 are sharp.

Theorem 3.3. (i) *Let $n>2$ be an odd integer and p a prime such that $\left(\frac{2}{p}\right)=-1$. If equation (2) has a positive solution, then $\frac{k}{2}\not\equiv\pm 1\mod p$. In particular, k is a multiple of 3.*

(ii) *Let $n>2$ an odd integer, and p a prime such that $\left(\frac{2}{p}\right)=1$. If equation (3) has a positive solution, then $\frac{k}{2}\not\equiv\pm 1\mod p$.*

Proof. (i) If $n>2$ is an odd integer and equation (2) has a positive solution, then the proof of Theorem 3.1 implies that k is even and the Diophantine equation $u^2-(\frac{k^2}{4}-1)v^2=2^n$ is solvable. Hence if p is an odd prime such that $\left(\frac{2}{p}\right)=-1$, then $\frac{k}{2}\not\equiv\pm 1\mod p$. By taking $p=3$, we obtain that k is a multiple of 3.

(ii) The proof of (ii) is similar to (i) and will be omitted. □

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