

TOPOLOGICAL BARRIERS FOR LOCALLY HOMEOMORPHIC QUASIREGULAR MAPPINGS IN 3-SPACE

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Abstract. We construct a new type of locally homeomorphic quasiregular mappings in the 3-sphere and discuss their relation to the Lavrentiev problem, the Zorich map with an essential singularity at infinity, the Fatou's problem and a quasiregular analogue of domains of holomorphy in complex analysis. The construction of such mappings comes from our construction of non-trivial compact 4-dimensional cobordisms M with symmetric boundary components and whose interiors have complete 4-dimensional real hyperbolic structures. Such locally homeomorphic quasiregular mappings are defined in the 3-sphere S^3 as mappings equivariant with the standard conformal action of uniform hyperbolic lattices $\Gamma \subset \text{Isom } H^3$ in the unit 3-ball and its complement in S^3 and with its discrete representation $G = \rho(\Gamma) \subset \text{Isom } H^4$. Here G is the fundamental group of our non-trivial hyperbolic 4-cobordism $M = (H^4 \cup \Omega(G))/G$ and the kernel of the homomorphism $\rho: \Gamma \rightarrow G$ is a free group F_3 on three generators.

1. Introduction

Liouville's rigidity of spatial conformal geometry shows that conformal mappings in domains in $S^n = \mathbf{R}^n \cup \{\infty\}$, $n \geq 3$ are restrictions of Möbius transformations. However this rigidity no longer persists in quasiconformal geometry intensively studied since 1930s after its introduction by Gröszsch [19] and Lavrentiev [20]. First assertions reflecting spatial specifics in this quasiconformal geometry were made by Lavrentiev [21], on removability of some singularities of quasiconformal mappings and on locally homeomorphic mappings in \mathbf{R}^3 . Zorich's 1967 solution [35] of the last Lavrentiev's problem (the global homeomorphism theorem) shows that locally homeomorphic quasiregular mappings of \mathbf{R}^n , $n \geq 3$, into itself are homeomorphisms of \mathbf{R}^n , and thus quasiconformal mappings.

In addition to his famous proof of Lavrentiev's problem Zorich gave an example of a nonsurjective quasiregular mapping $\mathbf{R}^3 \rightarrow \mathbf{R}^3$ omitting the origin and having an essential singularity at infinity. This so-called Zorich map is a spatial analogue of the exponential function in \mathbf{C} and is based on Belinskii's construction of a quasiconformal mapping of a half space \mathbf{R}_+^3 onto a round solid cylinder. Due to the previous Zorich theorem, the branching of the map (along parallel lines orthogonally intersecting the boundary plane at integer points) cannot be avoided. In a general sense all quasiregular mappings topologically have a branched covering type. Namely by Reshetnyak's theorem, quasiregular mappings are (generalized) branched covers, that is, discrete and open mappings and hence local homeomorphisms modulo an

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exceptional set of (topological) codimension at least two. The intensive study of quasiregular mappings, especially after the mentioned Zorich results and conjectures in [35]–[39], resulted in a rich theory of quasiregular mappings which is a natural and beautiful generalization of the geometric aspects of the theory of holomorphic functions in the plane to higher dimensions. It is covered by several papers [9], [10], [11], [13], [16], [23], [33], [35]–[39] and a number of monographs—see [17], [18], [26], [29], [34].

Our motivation stems from three sides of the mentioned Lavrentiev–Zorich assertions: on locally homeomorphic spatial quasiregular (quasimeromorphic) mappings defined in the (almost) whole sphere $S^3 = \overline{\mathbf{R}^3}$, surjectivity of such mappings having the whole sphere S^3 as the image, and their essential singularities. Despite a relative rigidity of quasiregular mappings without branching, in Theorem 4.1 we present a new (flexible) way for constructions of such locally homeomorphic quasiregular mappings $S^3 \setminus S_* \rightarrow S^3$ defined in the sphere S^3 except S_* (a dense subset of the 2-sphere $S^2 \subset S^3$) and having the whole sphere S^3 as the image. The exceptional subset S_* of the 2-sphere $S^2 \subset S^3$ (or a quasi-sphere $S_q^2 \subset S^3$) is a countable orbit of a Cantor subset of zero 2-measure. It creates a barrier (of a topological nature) for continuous extension of our quasiregular mapping since points of S_* are essential singularities of our mapping having no radial limits.

The construction of such quasiregular mappings in $S^3 \setminus S_*$ having a dense subset $S_* \subset S^2 \subset S^3$ as their barrier is heavily based on our construction [7] of non-trivial compact 4-dimensional cobordisms M^4 with symmetric boundary components, which makes it absolutely necessary for understanding to repeat in Section 2 the construction of the corresponding discrete actions and their homomorphisms related to those symmetric 4-cobordisms. The interiors of these 4-cobordisms have complete 4-dimensional real hyperbolic structures and universally covered by the real hyperbolic space H^4 , while the boundary components of M^4 have (symmetric) 3-dimensional conformally flat structures obtained by deformations of the same hyperbolic 3-manifold whose fundamental group Γ is a uniform lattice in $\text{Isom } H^3$. Such conformal deformations of hyperbolic manifolds are well understood after their discovery in [3], see [6]. Nevertheless till recently such “symmetric” hyperbolic 4-cobordisms with described properties were unknown despite our well known constructions of non-trivial hyperbolic homology 4-cobordisms with very asymmetric boundary components—see [8] and [6]–[5]. In [7] we presented a method of constructing such non-trivial “symmetric” hyperbolic 4-cobordisms $M^4 = H^4/G$ whose fundamental groups $\pi_1(M^4)$ act discretely in the hyperbolic 4-space H^4 by isometries, $\pi_1(M^4) \cong G \subset \text{Isom } H^4$, and can be obtained from the hyperbolic 3-lattice $\Gamma \subset \text{Isom } H^3$ by a homomorphism $\rho: \Gamma \rightarrow G \subset \text{Isom } H^4$ with non-trivial kernel (in our construction such kernel of ρ is a free subgroup $F_3 \subset \Gamma$ on three generators). In Section 2 we present all necessary details of our construction of such “symmetric” hyperbolic 4-cobordisms and used discrete groups (Theorem 2.2 and Proposition 2.4). By using such “symmetric” hyperbolic 4-cobordisms, our locally homeomorphic quasiregular mappings F are defined in the complement $S^3 \setminus S^2$ of the 2-sphere $S^2 = \{x \in \mathbf{R}^3: |x| = 1\}$ as mappings equivariant with the standard conformal action of uniform hyperbolic lattices $\Gamma \subset \text{Isom } H^3$ in the unit 3-ball $B^3(0, 1) = \{x \in \mathbf{R}^3: |x| < 1\}$ and in the complement in S^3 to its closure and with the discrete representation $G = \rho(\Gamma) \subset \text{Isom } H^4$. In other words such Γ -equivariance of our quasiregular mappings F can be described as $F(\Gamma(x)) = \rho(\Gamma)(F(x)) = G(F(x))$, $x \in S^3 \setminus S^2$. Another essential element of our

construction is a direct building in Section 3 of the so called bending quasiconformal homeomorphisms between polyhedra which preserve combinatorial structure of polyhedra and their dihedral angles.

One may find a resemblance of our construction of locally homeomorphic quasiregular mappings F in Theorem 4.1 to constructions of Martio and Srebro [24], [25] and Tukia [32] of locally homeomorphic quasiregular mappings. However our quasiregular mappings are not automorphic with respect to any discrete Möbius group as are the mappings in those papers.

Our construction is also related to the well known open question (Fatou's problem) on the correct analogue for higher-dimensional quasiregular mappings of the Fatou's theorem [14] on radial limits of a bounded analytic function of the unit disc. Though in higher dimensions $n \geq 3$ it is not even known if there exists a bounded n -dimensional mapping of the unit ball without any radial limits, cf. [22], [33], there are several results concerning radial limits of mappings of the unit ball. The most recent progress is due to Rajala who in particular proved that radial limits exist for infinitely many points of the unit sphere, see [27] and references there for some earlier results in this direction. Considering restriction of our locally homeomorphic quasiregular mapping F in Theorem 4.1 to the unit ball we see that it is bounded, and its exceptional subset S_* of the boundary unit sphere is a countable orbit of a Cantor subset with Hausdorff dimension $\ln 5 / \ln 6 \approx 0.89822444$ (zero 2-measure). All points of S_* are essential singularities of our bounded locally homeomorphic quasiregular mapping having no radial limits.

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2. Non-trivial “symmetric” hyperbolic 4-cobordisms

Since the construction of the fundamental group $\pi_1(M^4) \cong G \subset \text{Isom } H^4$ of a non-trivial “symmetric” hyperbolic 4-cobordisms $M^4 = H^4/G$ acting discretely in the hyperbolic 4-space H^4 is very essential for our construction of a locally homeomorphic quasiregular mapping $F: S^3 \setminus S_* \rightarrow S^3$ having S_* in a quasi-sphere $S_q^2 \subset S^3$ as a barrier, we start with a detailed construction of such discrete group $G \subset \text{Isom } H^4$ and the corresponding discrete representation $\rho: \Gamma \rightarrow G$ of a uniform hyperbolic lattice $\Gamma \subset \text{Isom } H^3$ from our paper [7].

These discrete groups G and Γ negatively answer a conjecture: *If one had a hyperbolic 4-cobordism M^4 whose boundary components N_1 and N_2 are highly (topologically and geometrically) symmetric to each other it would be in fact an h -cobordism, possibly not trivial, i.e. not homeomorphic to the product of N_1 and the segment $[0, 1]$.*

Namely the boundary components N_1 and N_2 of $M^4 = M(G) = \{H^4 \cup \Omega(G)\}/G$ are covered by the discontinuity set $\Omega(G) \subset S^3$ of G with two connected components Ω_1 and Ω_2 , where the conformal action of $G = \rho(\Gamma)$ is symmetric and has contractible fundamental polyhedra P_1 and P_2 of the same combinatorial type allowing to realize them as a compact polyhedron P_0 in the hyperbolic 3-space, i.e. the dihedral angle data of these polyhedra satisfy the Andreev's conditions [1]. Nevertheless this geometric symmetry of boundary components of our hyperbolic 4-cobordism

$M(G)$) is not enough to ensure that the group $G = \pi_1(M^4)$ is quasi-Fuchsian and our 4-cobordism M is trivial.

Here a Fuchsian group $\Gamma \subset \text{Isom } H^3 \subset \text{Isom } H^4$ conformally acts in the 3-sphere $S^3 = \partial H^4$ and preserves a round ball $B^3 \subset S^3$ where it acts as a cocompact discrete group of isometries of H^3 . Due to the Sullivan structural stability (see Sullivan [30] for $n = 2$ and Apanasov [6], Theorem 7.2), the space of quasi-Fuchsian representations of a hyperbolic lattice $\Gamma \subset \text{Isom } H^3$ into $\text{Isom } H^4$ is an open connected component of the Teichmüller space of H^3/Γ or the variety of conjugacy classes of discrete representations $\rho: \Gamma \rightarrow \text{Isom } H^4$. Points in this (quasi-Fuchsian) component correspond to trivial hyperbolic 4-cobordisms $M(G)$ where the discontinuity set $\Omega(G) = \Omega_1 \cup \Omega_2 \subset S^3 = \partial H^4$ is the union of two topological 3-balls Ω_i , $i = 1, 2$, and $M(G)$ is homeomorphic to the product of N_1 and the closed interval $[0, 1]$.

To simplify the situation we may consider the hyperbolic 4-cobordisms $M(\rho(\Gamma))$ corresponding to uniform hyperbolic lattices $\Gamma \subset \text{Isom } H^3$ generated by reflections (or cobordisms related to their finite index subgroups). Natural inclusions of these lattices into $\text{Isom } H^4$ act at infinity $\partial H^4 = S^3$ as Fuchsian groups $\Gamma \subset \text{Möb}(3)$ preserving a round ball in the 3-sphere S^3 . In this case the above conjecture can be reformulated as the following question on the Möbius action of corresponding reflection groups $G = \rho(\Gamma) \subset \text{Isom } H^4$ on the 3-sphere $S^3 = \partial H^4$:

Question 2.1. Is any discrete Möbius group G generated by finitely many reflections with respect to spheres $S^2 \subset S^3$ and whose fundamental polyhedron $P(G) \subset S^3$ is the union of two contractible polyhedra $P_1, P_2 \subset S^3$ of the same combinatorial type (with equal corresponding dihedral angles) quasiconformally conjugate in the sphere S^3 to some Fuchsian group preserving a round ball $B^3 \subset S^3$?

Our construction of the mentioned discrete groups Γ and $G = \rho(\Gamma)$ gives a negative answer to this question and proves the following (see Apanasov [7]):

Theorem 2.2. *There exists a discrete Möbius group $G \subset \text{Möb}(3)$ on the 3-sphere S^3 generated by finitely many reflections such that:*

- (1) *Its discontinuity set $\Omega(G)$ is the union of two invariant components Ω_1, Ω_2 ;*
- (2) *Its fundamental polyhedron $P \subset S^3$ has two contractible components $P_i \subset \Omega_i$, $i = 1, 2$, having the same combinatorial type (of a compact hyperbolic polyhedron $P_0 \subset H^3$);*
- (3) *For the uniform hyperbolic lattice $\Gamma \subset \text{Isom } H^3$ generated by reflections in sides of the hyperbolic polyhedron $P_0 \subset H^3$ and acting on the sphere $S^3 = \partial H^4$ as a discrete Fuchsian group $i(\Gamma) \subset \text{Isom } H^4 = \text{Möb}(3)$ preserving a round ball B^3 (where $i: \text{Isom } H^3 \subset \text{Isom } H^4$ is the natural inclusion), the group G is its image under a homomorphism $\rho: \Gamma \rightarrow G$ but it is not quasiconformally (topologically) conjugate in S^3 to $i(\Gamma)$.*

Proof. For our construction of the desired Möbius group $G \subset \text{Möb}(3)$ generated by reflections it is enough to define its finite collection Σ of reflecting 2-spheres $S_i \subset S^3$, $1 \leq i \leq N$. As the first four spheres we consider mutually orthogonal spheres centered at the vertices of a regular tetrahedron in \mathbf{R}^3 . Let $B = \bigcup_{1 \leq i \leq 4} B_i$ be the union of the closed balls bounded by these four spheres, and let ∂B be its boundary (a topological 2-sphere) having four vertices which are the intersection points of four triples of our spheres. Applying a Möbius transformation in $S^3 \cong \mathbf{R}^3 \cup \{\infty\}$, we may assume that the first three spheres S_1, S_2 and S_3 correspond to the coordinate planes

$\{x \in \mathbf{R}^3: x_i = 0\}$, and $S_4 = S^2(0, R)$ is the round sphere of some radius $R > 0$ centered at the origin. The value of the radius R will be determined later.

On the topological 2-sphere ∂B with four vertices we consider a simple closed loop $\alpha \subset \partial B$ which does not contain any of our vertices and which symmetrically separates two pairs of these vertices from each other as the white loop does on the tennis ball shown in Figure 1. This white loop α can be considered as the boundary of a topological 2-disc σ embedded in the complement $D = S^3 \setminus B$ of our four balls. Our geometric construction needs a detailed description of such a 2-disc σ and its boundary loop $\alpha = \partial\sigma$ obtained as it is shown in Figure 3.



Figure 1. White loop separating two pairs of vertices on a tennis ball.

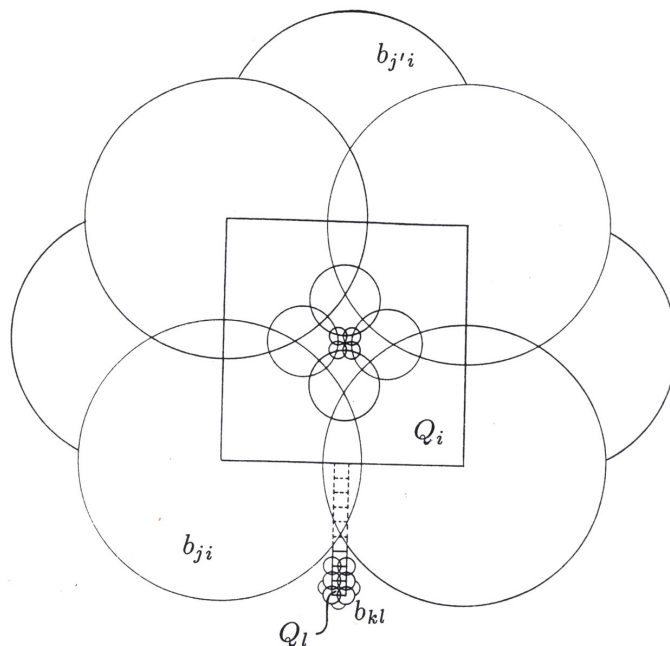


Figure 2. Big and small cube sizes and ball covering.

The desired disc $\sigma \subset D = S^3 \setminus B$ can be described as the boundary in the domain D of the union of a finite chain of adjacent blocks Q_i (regular cubes) with disjoint interiors whose centers lie on the coordinate planes S_1 and S_2 and whose sides are parallel to the coordinate planes. This chain starts from the unit cube whose center lies in the second coordinate axis, in $e_2 \cdot \mathbf{R}_+ \subset S_1 \cap S_3$. Then our chain goes up through small adjacent cubes centered in the coordinate plane S_1 , at some point changes its direction to the horizontal one toward the third coordinate axis, where

it turns its horizontal direction by a right angle again (along the coordinate plane S_2), goes toward the vertical line passing through the second unit cube centered in $e_1 \cdot \mathbf{R}_+ \subset S_2 \cap S_3$, then goes down along that vertical line and finally ends at that second unit cube, see Figure 3. We will define the size of small cubes Q_i in our block chain and the distance of the centers of two unit cubes to the origin in the next step of our construction.

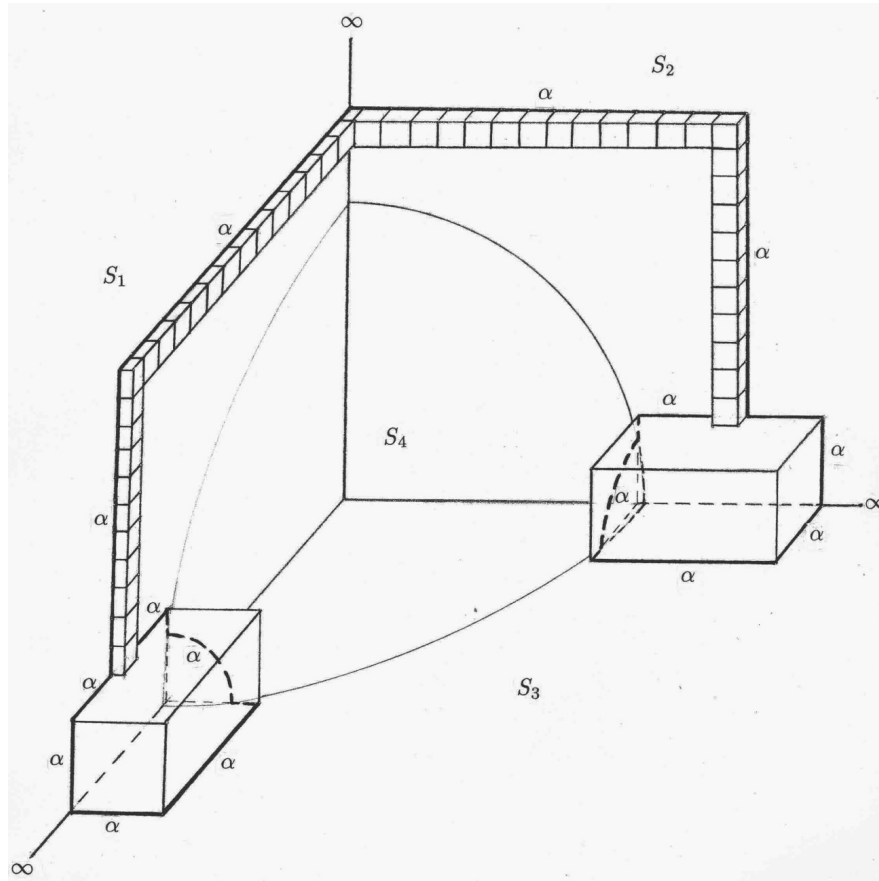


Figure 3. Configuration of blocks and the white loop $\alpha \subset \partial B$.

Let us consider one of our cubes Q_i , i.e. a block of our chain, and let f be its square side having a nontrivial intersection with our 2-disc $\sigma \subset D$. For that side f we consider spheres S_j centered at its vertices and having a radius such that each two spheres centered at the ends of an edge of f intersect each other with angle $\pi/3$. In particular, for the unit cubes such spheres have radius $\sqrt{3}/3$. From such defined spheres we select those spheres that have centers in our domain D and then include them in the collection Σ of reflecting spheres. Now we define the distance of the centers of our big (unit) cubes to the origin. It is determined by the condition that the sphere $S_4 = S^2(0, R)$ is orthogonal to the sphere $S_j \in \Sigma$ centered at the vertex of such a cube closest to the origin.

As in Figure 2, let f be a square side of one of our cubic blocks Q_i having a nontrivial intersection $f_\sigma = f \cap \sigma$ with our 2-disc $\sigma \subset D$. We consider a ring of four spheres S_i whose centers are interior points of f which lie outside of the four previously defined spheres S_j centered at vertices of f and such that each sphere S_i intersects two adjacent spheres S_{i-1} and S_{i+1} (we numerate spheres $S_i \bmod 4$) with angle $\pi/3$. In addition these spheres S_i are orthogonal to the previously defined ring of bigger spheres S_j , see Figure 2. From such defined spheres S_i we select those

spheres that have nontrivial intersections with our domain D outside the previously defined spheres S_j , and then include them in the collection Σ of reflecting spheres. If our side f is not the top side of one of the two unit cubes we add another sphere $S_k \in \Sigma$. It is centered at the center of this side f and is orthogonal to the four previously defined spheres S_i with centers in f , see Figure 2.

Now let f be the top side of one of the two unit cubes of our chain. Then, as before, we consider another ring of four spheres S_k . Their centers are interior points of f , lie outside of the four previously defined spheres S_i closer to the center of f and such that each sphere S_k intersects two adjacent spheres S_{k-1} and S_{k+1} (we numerate spheres $S_k \bmod 4$) with angle $\pi/3$. In addition these new four spheres S_k are orthogonal to the previously defined ring of bigger spheres S_i , see Figure 2. We note that the centers of these four new spheres S_k are vertices of a small square $f_s \subset f$ whose edges are parallel to the edges of f , see Figure 2. We set this square f_s as the bottom side of the small cubic box adjacent to the unit one. This finishes our definition of the family of twelve round spheres whose interiors cover the square ring $f \setminus f_s$ on the top side of one of the two unit cubes in our cube chain and tells us which two spheres among the four new defined spheres S_k were already included in the collection Σ of reflecting spheres (as the spheres $S_j \in \Sigma$ associated to small cubes in the first step).

This also defines the size of small cubes in our block chain. Now we can vary the remaining free parameter R (which is the radius of the sphere $S_4 \in \Sigma$) in order to make two horizontal rows of small blocks with centers in S_1 and S_2 , correspondingly, to share a common cubic block centered at a point in $e_3 \cdot \mathbf{R}_+ \subset S_1 \cap S_2$, see Figure 3.

The constructed collection Σ of reflecting spheres S_j bounding round balls B_j , $1 \leq j \leq N$, has the following properties:

- (1) The closure of our 2-disc $\sigma \subset D$ is covered by balls B_j : $\bar{\sigma} \subset \text{int} \bigcup_{j \geq 5}^N B_j$;
- (2) Any two spheres $S_j, S_{j'} \in \Sigma$ either are disjoint or intersect with angle $\pi/2$ or $\pi/3$;
- (3) The complement of all balls, $S^3 \setminus \bigcup_{j=1}^N B_j$ is the union of two contractible polyhedra P_1 and P_2 of the same combinatorial type.

Therefore we can use the constructed collection Σ of reflecting spheres S_i to define a discrete group $G = G_\Sigma \subset \text{Möb}(3)$ generated by N reflections in spheres $S_j \in \Sigma$. The fundamental polyhedron $P = P_1 \cup P_2 \subset S^3$ for the action of this discrete reflection group G on the sphere S^3 is the union of two connected polyhedra P_1 and P_2 which are disjoint topological balls. So the discontinuity set $\Omega(G) \subset S^3$ of G consists of two invariant connected components Ω_1 and Ω_2 :

$$(2.1) \quad \Omega(G) = \bigcup_{g \in G} g(\bar{P}) = \Omega_1 \cup \Omega_2, \quad \Omega_i = \bigcup_{g \in G} g(\bar{P}_i), \quad i = 1, 2.$$

Lemma 2.3. *The splitting of the discontinuity set $\Omega \subset S^3$ of our discrete reflection group $G = G_\Sigma \subset \text{Möb}(3)$ into G -invariant components Ω_1 and Ω_2 in (2.1) defines a Heegaard splitting of the 3-sphere S^3 of infinite genus with ergodic word hyperbolic group G action on the separating boundary $\Lambda(G)$ which is quasi-self-similar in the sense of Sullivan.*

Proof. In fact, despite the contractibility of polyhedra P_1 and P_2 both components Ω_1 and Ω_2 are not simply connected and even are mutually linked. To show this it is enough to see that the union of the bounded polyhedron \bar{P}_1 (inside of our block chain) and its image $g_3(\bar{P}_1)$ under the reflection g_3 with respect to the plane S_3 has

a non-contractible loop β_1 which represents a non-trivial element of the fundamental group $\pi_1(\Omega_1)$. This loop is linked with the loop β_2 in the unbounded component Ω_2 which goes around $\bar{P}_1 \cup g_3(\bar{P}_1)$ and represents a non-trivial element of the fundamental group $\pi_1(\Omega_2)$.

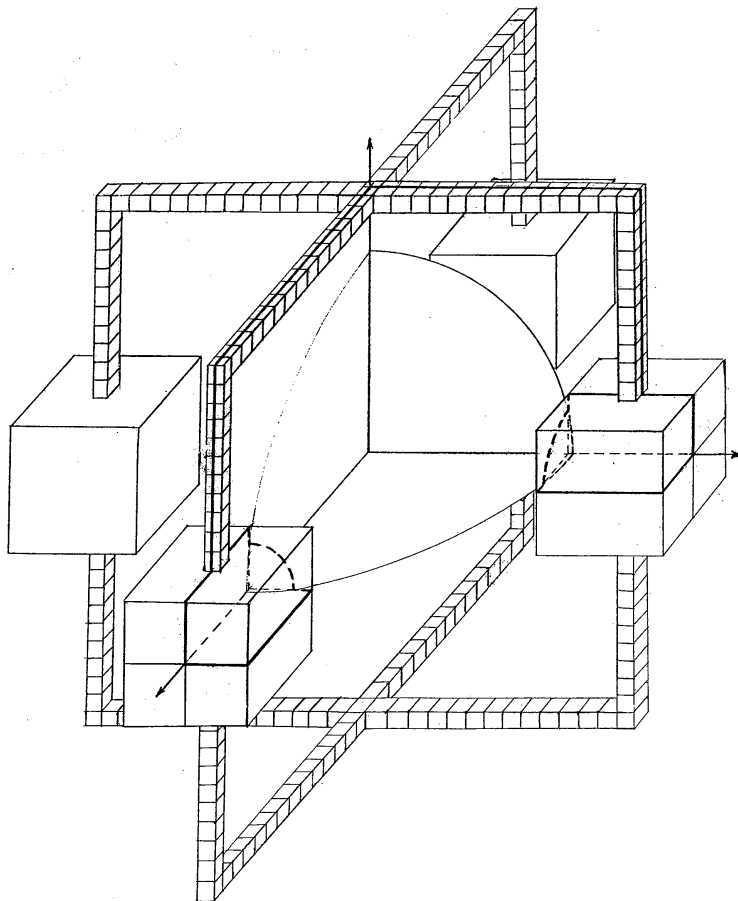


Figure 4. Handlebody obtained by the first 3 reflections of the cube chain.

This fact is illustrated by Figure 4 where one can see a handlebody obtained from our initial chain of building blocks in Figure 3 by the union of the images of this block chain by first generating reflections in the group G (in S_1, S_2 and S_3). Then our non-contractible loop $\beta_1 \subset \Omega_1$ lies inside of this handlebody in Figure 4 and is linked with the second loop $\beta_2 \subset \Omega_2$ which goes around one of the handles of the handlebody in Figure 4. The resulting handlebodies Ω_1 and Ω_2 are the unions of the corresponding images $g(\bar{P}_i)$ of the polyhedra \bar{P}_1 and \bar{P}_2 , so they have infinitely many mutually linked handles. Their fundamental groups $\pi_1(\Omega_1)$ and $\pi_1(\Omega_2)$ have infinitely many generators, and some of those generators correspond to the group G -images of the linked loops $\beta_1 \subset \Omega_1$ and $\beta_2 \subset \Omega_2$. The limit set $\Lambda(G)$ is the common boundary of Ω_1 and Ω_2 . Since the group $G \subset \text{Möb}(3)$ acts on the hyperbolic 4-space H^4 , $\partial H^4 = S^3$, as a convex cocompact isometry group, its action on the limit set $\Lambda(G)$ is ergodic. Moreover, the common boundary $\Lambda(G)$ of the handlebodies Ω_1 and Ω_2 is quasi-self-similar in the sense of Sullivan, that is each arbitrary small piece of $\Lambda(G)$ can be expanded to a standard size and then mapped into $\Lambda(G)$ by a K -quasi-isometry. More precisely, there are uniform constants K and r_0 such that, for any $x \in \Lambda(G)$ and for any ball $B(x, r)$ centered at x with radius r , $0 < r < r_0$, there

exists a K -quasi-isometric bijection f ,

$$(2.2) \quad f: \frac{\Lambda(G) \cap B(x, r)}{r} \hookrightarrow \Lambda(G)$$

which distorts distances in the interval between $1/K$ and K . In other words, the distortion of an unlimited “*microscoping*” (2.2) of the limit set $\Lambda(G)$ can be uniformly bounded, see Corollary 2.66 in Apanasov [6]. \square

To finish the proof of Theorem 2.2 we notice that the combinatorial type (with magnitudes of dihedral angles) of the bounded component P_1 of the fundamental polyhedron $P \subset S^3$ coincides with the combinatorial type of its unbounded component P_2 . Applying Andreev’s theorem on 3-dimensional hyperbolic polyhedra [1], one can see that there exists a compact hyperbolic polyhedron $P_0 \subset H^3$ of the same combinatorial type with the same dihedral angles ($\pi/2$ or $\pi/3$). So one can consider a uniform hyperbolic lattice $\Gamma \subset \text{Isom } H^3$ generated by reflections in sides of the hyperbolic polyhedron P_0 . This hyperbolic lattice Γ acts in the sphere S^3 as a discrete co-compact Fuchsian group $i(\Gamma) \subset \text{Isom } H^4 = \text{Möb}(3)$ (i.e. as the group $i(\Gamma) \subset \text{Isom } H^4$ where $i: \text{Isom } H^3 \subset \text{Isom } H^4$ is the natural inclusion) preserving a round ball B^3 and having its boundary sphere $S^2 = \partial B^3$ as the limit set. Obviously there is no self-homeomorphism of the sphere S^3 conjugating the action of the groups G and $i(\Gamma)$ because the limit set $\Lambda(G)$ is not a topological 2-sphere. So the constructed group G is not a quasi-Fuchsian group.

One can construct a natural homomorphism $\rho: \Gamma \rightarrow G$, $\rho \in \mathcal{R}_3(\Gamma)$, between these two Gromov hyperbolic groups $G \subset \text{Isom } H^4$ and $\Gamma \subset \text{Isom } H^3$ defined by the correspondence between sides of the hyperbolic polyhedron $P_0 \subset H^3$ and reflecting spheres S_i in the collection Σ bounding the fundamental polyhedra P_1 and P_2 . \square

Proposition 2.4. *The homomorphism $\rho \in \mathcal{R}_3(\Gamma)$, $\rho: \Gamma \rightarrow G$, in Theorem 2.2 is not an isomorphism. Its kernel $\ker(\rho) = \rho^{-1}(e_G)$ is a free rank 3 subgroup $F_3 \triangleleft \Gamma$.*

Proof. The homomorphism ρ cannot be an isomorphism since its kernel $\rho^{-1}(e_G)$ is not trivial, $\rho^{-1}(e_G) \neq \{e_\Gamma\}$. In fact this kernel is a free rank 3 group $F_3 = \langle x, y, z \rangle$ generated by three hyperbolic translations $x, y, z \in \Gamma$. The first hyperbolic translation $x = a_1 b_1$ in H^3 is the composition of reflections a_1 and b_1 in two disjoint hyperbolic planes $H_1, H'_1 \subset H^3$ containing those two 2-dimensional faces of the hyperbolic polyhedron P_0 that correspond to two sides of the polyhedron P_1 which are disjoint parts of the sphere S_4 . The second hyperbolic translation $y = a_2 b_2$ in H^3 is the composition of reflections a_2 and b_2 in two disjoint hyperbolic planes $H_2, H'_2 \subset H^3$ containing those two 2-dimensional faces of the hyperbolic polyhedron P_0 that correspond to two sides of the polyhedron P_1 which are disjoint parts of the sphere S_3 . And the third generator z is a hyperbolic translation in H^3 which is a_1 -conjugate of y , $z = a_1 y a_1$. The fact that these hyperbolic 2-planes H_1 and H'_1 (correspondingly, the 2-planes H_2 and H'_2) are disjoint follows from Andreev’s result [2] on sharp angled hyperbolic polyhedra. Restricting our homomorphism ρ to the subgroup of Γ generated by reflections $a_1, a_2, b_1, b_2 \in \Gamma$, we can formulate its properties as the following statement in combinatorial group theory:

Lemma 2.5. *Let $A = \langle a_1, a_2 \mid a_1^2, a_2^2, (a_1 a_2)^2 \rangle \cong B = \langle b_1, b_2 \mid b_1^2, b_2^2, (b_1 b_2)^2 \rangle \cong C = \langle c_1, c_2 \mid c_1^2, c_2^2, (c_1 c_2)^2 \rangle \cong \mathbf{Z}_2 \times \mathbf{Z}_2$, and let $\varphi: A * B \rightarrow C$ be a homomorphism of the free product $A * B$ into C such that $\varphi(a_1) = \varphi(b_1) = c_1$ and $\varphi(a_2) = \varphi(b_2) = c_2$. Then the kernel $\ker(\varphi) = \varphi^{-1}(e_C)$ of φ is a free rank 3 subgroup $F_3 \triangleleft A * B$ generated by elements $x = a_1 b_1$, $y = a_2 b_2$ and $z = a_1 a_2 b_2 a_1 = a_1 y a_1$.*

Proof. It is obvious that $K_0 = \langle x, y, z \rangle$ is a subgroup in $\ker(\varphi)$. From the definition of φ on the generators of $A * B$ it is also clear that $\ker(\varphi)$ is $\langle\langle x, y \rangle\rangle$, the normal closure of elements $x = a_1b_1$ and $y = a_2b_2$. Therefore in order to prove that $K_0 = \ker(\varphi)$ it is enough to show that K_0 contains all elements which are conjugate in $A * B$ to x and y .

As any element $w \in A * B$ is a product of generators of $A * B$, the conjugation by any such w may be regarded as a consequent conjugation by the generators of $A * B$. So, it is enough to prove that K_0 contains any element conjugate to x, y, z by a_1, a_2, b_1, b_2 .

In fact it is easy to verify that

$$\begin{aligned} a_1^{-1}xa_1 &= x^{-1}, & a_1^{-1}ya_1 &= z, & a_1^{-1}za_1 &= y, \\ a_2^{-1}xa_2 &= zxy^{-1}, & a_2^{-1}ya_2 &= y^{-1}, & a_2^{-1}za_2 &= z^{-1}, \\ b_1^{-1}xb_1 &= x^{-1}, & b_1^{-1}yb_1 &= x^{-1}zx, & b_1^{-1}zb_1 &= x^{-1}yx, \\ b_2^{-1}xb_2 &= y^{-1}zx, & b_2^{-1}yb_2 &= y^{-1}, & b_2^{-1}zb_2 &= y^{-1}z^{-1}y. \end{aligned}$$

Now we should show that the elements x, y and z form a free basis for K_0 . Let us check that any reduced word $w(x, y, z)$ represents a nontrivial element of $A * B$. By a ‘‘letter’’ we mean any of symbols $x^{\pm 1}, y^{\pm 1}, z^{\pm 1}$. We claim that for the element g represented by a reduced word $w(x, y, z)$ the following holds: the last syllable of g written in the normal form is always equal to the last syllable of the last letter of the word w (and so is nontrivial) except for the case when the last letter is x and the preceding letter is z (cf. [28], §4.1). In this case the last syllable equals b_1b_2 (and so it is also nontrivial). Besides that, if the last letter of w is z then the two last syllables of w are the ones of z .

This statement can be easily verified by induction on the length of $w(x, y, z)$. So it obviously implies nontriviality of the element g represented by $w(x, y, z)$. \square

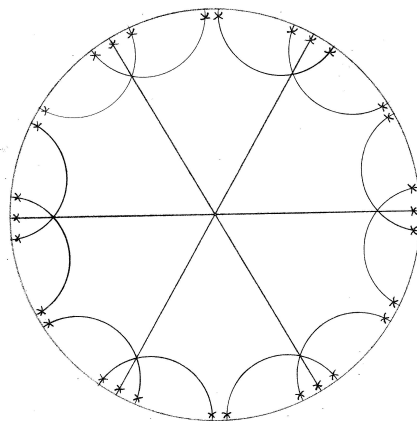


Figure 5. Cayley graph of the free group F_3 .

Now the claim that $\ker(\rho) \subset \Gamma$ is a free rank 3 subgroup $F_3 = \langle x, y, z \rangle$ generated by our hyperbolic translations $x, y, z \in \text{Isom } H^3$ follows directly from Lemma 2.5, which completes the proof of Proposition 2.4. \square

Therefore the configuration of reflecting spheres $S_j \subset \Sigma$ shows that one can deform our discrete co-compact Fuchsian group $i(\Gamma) \subset \text{Isom } H^4 = \text{M\"ob}(3)$ preserving a round ball $B^3 \subset S^3$ into the group $G \subset \text{Isom } H^4$ by continuously moving two pairs of reflecting 2-spheres of the Fuchsian group $i(\Gamma)$ corresponding to the pairs of hyperbolic planes $H_1, H'_1 \subset H^3$ and $H_2, H'_2 \subset H^3$ into the reflecting spheres S_4 and S_3 while keeping all dihedral angles unchanged.

Remark 2.6. A simple but important observation is that in our construction we can change the unit round ball $B(0, 1) \subset S^3$ to any quasiball $B_q \subset S^3$ bounded by a quasisphere S_q where our hyperbolic lattice $\Gamma \subset \text{Isom } H^3$ acts conformally and cocompactly. This action is a quasi-Fuchsian group action corresponding to a point in the space of quasi-Fuchsian representations of $\Gamma \subset \text{Isom } H^3$ into $\text{Isom } H^4$ (an open connected component of the Teichmüller space of conformally flat structures on H^3/Γ), cf. [3, 6].

3. Bending homeomorphisms between polyhedra

In this section we construct quasiconformal homeomorphisms $\phi_1: P_1 \rightarrow P_0$ and $\phi_2: P_2 \rightarrow \widehat{P_0}$ between components P_i , $i = 1, 2$, of the fundamental polyhedron $P \subset \Omega(G) \subset S^3$ for the group G and the corresponding components P_0 and $\widehat{P_0}$ of the fundamental polyhedron for conformal action in S^3 of our hyperbolic lattice $\Gamma \subset \text{Isom } H^3$ from Theorem 2.2. These mappings ϕ_i are compositions of finitely many elementary “bending homeomorphisms”, map faces to faces, and preserve the combinatorial structure of the polyhedra and their corresponding dihedral angles.

First we observe that to each cube Q_j , $1 \leq j \leq m$, used in the previous section for our construction of the group G (see Figure 3 and Figure 4), we may associate a round ball B_j centered at the center of the cube Q_j and such that its boundary sphere is orthogonal to the reflection spheres S_i from our generating family Σ whose centers are at vertices of the cube Q_j . In particular for the unit cubes Q_1 and Q_m , the reflection spheres S_i centered at their vertices have radius $\sqrt{3}/3$, so the balls B_1 and B_m (whose boundary spheres are orthogonal to those corresponding reflection spheres S_i) should have radius $\sqrt{5}/12$. Also we add another extra ball $B^3(0, R)$ (which we consider as two balls B_0 and B_{m+1}) whose boundary is the reflection sphere $S^2(0, R) = S_4 \in \Sigma$ centered at the origin and orthogonal to the closest reflection spheres S_i centered at vertices of two unit cubes Q_1 and Q_m . Our different enumeration of this ball will be used when we consider different faces of our fundamental polyhedron P_1 lying on that reflection sphere S_4 .

Now for each cube Q_j , $1 \leq j \leq m$, we may associate a discrete subgroup $G_j \subset G \subset \text{Möb}(3) \cong \text{Isom } H^4$ generated by reflections in the spheres $S_i \in \Sigma$ associated to that cube Q_j - see our construction in Theorem 2.2. One may think about such a group G_j as a result of quasiconformal bending deformations (see [6], Chapter 5) of a discrete Möbius group preserving the round ball B_j associated to the cube Q_j (whose center coincides with the center of the cube Q_j). As the first step in such deformations, let us define two quasiconformal “bending” self-homeomorphisms of S^3 , f_1 and f_{m+1} , preserving the balls B_1, \dots, B_m and the set of their reflection spheres S_i , $i \neq 4$, and transferring ∂B_0 and ∂B_{m+1} into 2-spheres orthogonally intersecting ∂B_1 and ∂B_m along round circles b_1 and b_{m+1} , respectively.

To construct the bending f_1 (f_{m+1} is similar), we may assume that the balls B_0 and B_1 are half-spaces with boundary planes ∂B_0 and ∂B_1 and such that $b_1 = \{x \in \mathbf{R}^3: x_1 = x_2 = 0\}$ is their intersection line. From our construction of the group G , we have that the dihedral angle of the intersection $B_0 \cap B_1$ has a magnitude α , $0 < \alpha < \pi/2$, and there exists a dihedral angle $V_1 \subset \mathbf{R}^3$ with the edge b_1 and magnitude 2ζ , where $0 < \zeta < \pi/4$ and $\alpha < \pi - 2\zeta$, such that V_1 contains all the reflection spheres in Σ disjoint from b_1 . Let us assume the natural complex structure in the orthogonal to b_1 plane $\mathbf{R}^2 = \{x \in \mathbf{R}^3: x_3 = 0\}$. Then the quasiconformal homeomorphism $f_1: S^3 \rightarrow S^3$ is described by its restriction to this plane $\mathbf{C} = \mathbf{R}^2$

(where $-\pi < \arg z \leq \pi$ is the principal value of the argument of $z \in \mathbf{C}$) as follows, see Figure 6:

$$(3.1) \quad f_1(z) = \begin{cases} z & \text{if } |\arg z| \geq \pi - \zeta, \\ z \exp\left(i\left(\frac{\pi}{2} - \alpha\right)\right) & \text{if } |\alpha - \arg z| \leq \zeta, \\ z \exp\left(i\left(\frac{\pi}{2} - \alpha\right)\left(1 - \frac{\arg z - \zeta}{\pi - 2\zeta}\right)\right) & \text{if } \alpha + \zeta < \arg z < \pi - \zeta, \\ z \exp\left(i\left(\frac{\pi}{2} - \alpha\right)\left(1 + \frac{\zeta + \arg z}{\pi - 2\zeta}\right)\right) & \text{if } \zeta - \alpha < \arg z < \alpha - \zeta. \end{cases}$$

We remark that $f_1 = \text{id}$ in V_1 and hence it is the identity on all reflection spheres $S_i \in \Sigma$ disjoint from $b_1 = \partial B_0 \cap \partial B_1$. Also all spheres $S_k \in \Sigma$ intersecting b_1 and the exterior dihedral angles of their intersections with other spheres S_i are still invariant with respect to f_1 .

In the next steps in our bending deformations, for two adjacent cubes Q_{j-1} and Q_j , let us denote $G_{j-1,j} \subset G$ the subgroup generated by reflections with respect to the spheres $S_i \subset \Sigma$ centered at common vertices of these cubes. This subgroup preserves the round circle $b_j = b_{j-1,j} = \partial B_{j-1} \cap \partial B_j$. This shows that our group G is a result of the so called "block-building construction" (see [6], Section 5.4) from the block groups G_j by sequential amalgamated products:

$$(3.2) \quad G = G_1 *_{G_{1,2}} G_2 *_{G_{2,3}} \cdots *_{G_{j-2,j-1}} G_{j-1} *_{G_{j-1,j}} G_j *_{G_{j,j+1}} \cdots *_{G_{m-1,m}} G_m.$$

Then the chain of these building balls $\{B_j\}, 1 \leq j \leq m$, contains the bounded polyhedron $P_1 \subset \Omega_1$, and the unbounded polyhedron $P_2 \subset \Omega_2$ is inside of the chain of the balls $\{\widehat{B}_j, 1 \leq j \leq m\}$, which are the complements in S^3 to the balls B_j .

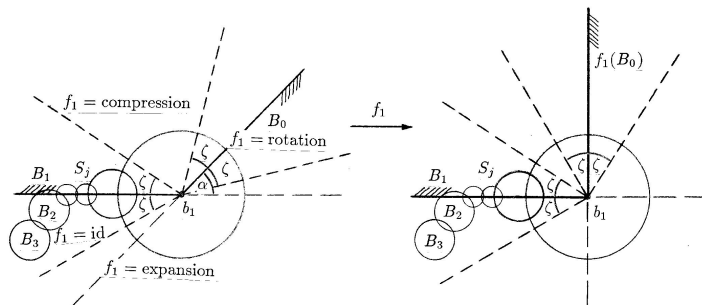


Figure 6. Elementary bending homeomorphism f_1 .

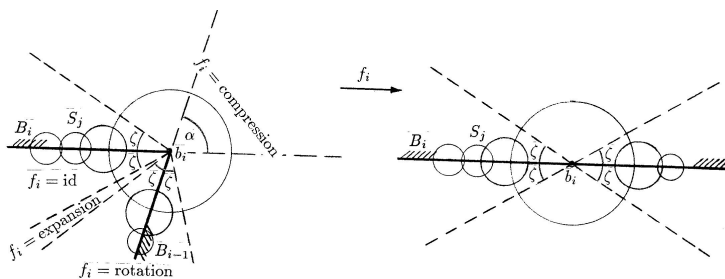


Figure 7. Elementary bending homeomorphism f_i .

Now for each pair of balls B_{i-1} and B_i with the common boundary circle $b_i = \partial B_{i-1} \cap \partial B_i, 1 \leq i \leq m$, we construct a quasi-conformal bending homeomorphism f_i that transfers $B_i \cup B_{i-1}$ onto the ball B_i and which is conformal in dihedral ζ_i -neighborhoods of the spherical disks $\partial B_i \setminus \overline{B_{i-1}}$ and $\partial B_{i-1} \setminus \overline{B_i}$. Namely, let B_i and B_{i-1} be half-spaces whose boundary planes ∂B_i and ∂B_{i-1} contain the origin and

intersect along the third coordinate axis $b_i = \{x \in \mathbf{R}^3 : x_1 = x_2 = 0\}$ at an angle α , $0 < \alpha < \pi$, and let ζ be a fixed number such that $0 < \zeta < \pi/2$ and $0 < \alpha < \pi - 2\zeta$. Assuming the natural complex structure in the plane $\mathbf{R}^2 = \{x \in \mathbf{R}^3 : x_3 = 0\}$, we define the quasi-conformal elementary bending homeomorphism f_i by its restriction to the plane $\mathbf{C} = \mathbf{R}^2$ (see Figure 7), where

$$(3.3) \quad f_i(z) = \begin{cases} z & \text{if } |\arg z| \geq \pi - \zeta, \\ z \exp(i(\pi - \alpha)) & \text{if } |\pi - \alpha - \arg z| \leq \zeta, \\ z \exp\left(i(\pi - \alpha)\left(1 - \frac{\arg z - \zeta}{\pi - 2\zeta}\right)\right) & \text{if } \alpha - \pi + \zeta < \arg z < \pi - \zeta, \\ z \exp\left(i(\pi - \alpha)\left(1 + \frac{\zeta + \arg z}{\pi - 2\zeta}\right)\right) & \text{if } \zeta - \pi < \arg z < \alpha - \pi - \zeta. \end{cases}$$

Note that in each i -th step, $2 \leq i \leq m$, we reduce the number of balls B_j in our chain by one. The constructed quasiconformal homeomorphisms f_i have the properties:

- (1) $f_i = \text{id}$ in a neighborhood of reflection spheres from our collection Σ that are disjoint from the circle b_i and intersect some balls B_j , $j \geq i$.
- (2) The composition $f_{m+1}f_i f_{i-1} \cdots f_2 f_1$ transfers all spheres from Σ to spheres orthogonal to the boundary spheres of some balls B_j , $i \leq j \leq m$, where all intersection angles between these spheres do not change.

Finally, renormalizing our last ball B_m as the unit ball $B(0,1)$, we define our desired quasiconformal homeomorphism $\phi_1 : P_1 \rightarrow P_0$ as the restriction of the composition $f_{m+1}f_m f_{m-1} \cdots f_2 f_1$ of our bending homeomorphisms f_j on the fundamental polyhedron $P_1 \subset \Omega_1$. Similarly (working with the balls \widehat{B}_j) we define the second quasiconformal homeomorphism $\phi_2 : P_2 \rightarrow \widehat{P}_0$. Both mappings preserve the combinatorial structure of the polyhedra and their dihedral angles.

4. Locally homeomorphic quasiregular mappings

Now we apply results of the previous Section 3 to define our quasiregular mapping F from $S^3 \setminus S_*$ onto S^3 .

Theorem 4.1. *Let the uniform hyperbolic lattice $\Gamma \subset \text{Isom } H^3$ and its discrete representation $\rho : \Gamma \rightarrow G \subset \text{Isom } H^4$ with the kernel as a free subgroup $F_3 \subset \Gamma$ be as in Theorem 2.2. Then there is a locally homeomorphic quasiregular mapping $F : S^3 \setminus S_* \rightarrow S^3$ whose all singularities lie in an exceptional subset S_* of the unit sphere $S^2 \subset S^3 = \mathbf{R}^3 \cup \{\infty\}$ and form a dense in S^2 Γ -orbit of a Cantor subset with Hausdorff dimension $\ln 5 / \ln 6 \approx 0.89822444$. These (essential) singularities create a barrier for F in the sense that at any such point $x \in S_*$ the map F does not have radial limits on either side of $S^2 \subset S^3$.*

Proof. First we define our locally homeomorphic quasiregular mapping F in the complement $S^3 \setminus S^2$ of the unit sphere, $F : S^3 \setminus S^2 \rightarrow \Omega(G) \subset S^3$, $F(\infty) = \infty$.

We recall that the discrete group $G = \rho(\Gamma) \subset \text{Isom } H^4 \cong \text{Möb}(3)$ constructed in Section 2 (Theorem 2.2) has its discontinuity set $\Omega(G) \subset S^3$, and its conformal and discontinuous action in $\Omega(G)$ has $P = P_1 \cup P_2$ as the fundamental polyhedron. Its symmetric connected components P_1 and P_2 constructed in Section 2 have the combinatorial type of the convex 3-dimensional hyperbolic polyhedron P_0 fundamental for our hyperbolic lattice $\Gamma \subset \text{Isom } H^3$ (for its conformal action in the unit ball $B^3(0,1)$). Let \widehat{P}_0 be the symmetric image of $P_0 \subset B^3(0,1)$ with respect to the reflection in the unit sphere $S^2 = \partial B^3$. The polyhedron $P_0 \cup \widehat{P}_0$ is a fundamental polyhedron (with

two connected components which are convex in the induced hyperbolic metrics) for conformal and discontinuous action of our hyperbolic lattice Γ in $S^3 \setminus S^2$.

In the previous Section 3 we have constructed quasiconformal homeomorphisms $\phi_1^{-1}: P_0 \rightarrow P_1$ and $\phi_2^{-1}: \widehat{P_0} \rightarrow P_2$. These two homeomorphisms map polyhedral sides of the polyhedra fundamental for the Γ -action to the corresponding sides of the polyhedra fundamental for the G -action and preserve combinatorial structures of polyhedra as well as their dihedral angles. Equivariantly extending these homeomorphisms, we define a quasiregular mapping $F: S^3 \setminus S^2 \rightarrow \Omega(G)$:

$$(4.1) \quad F(x) = \begin{cases} \rho(\gamma) \circ \phi_1^{-1} \circ \gamma^{-1}(x) & \text{if } |x| < 1, x \in \gamma(P_0), \gamma \in \Gamma, \\ \rho(\gamma) \circ \phi_2^{-1} \circ \gamma^{-1}(x) & \text{if } |x| > 1, x \in \gamma(\widehat{P_0}), \gamma \in \Gamma. \end{cases}$$

Since the initial quasiconformal homomorphisms ϕ_1^{-1} and ϕ_2^{-1} preserve combinatorial structures of polyhedra and their dihedral angles, the tessellations of $\Omega(\Gamma) = S^3 \setminus S^2$ and $\Omega(G) \subset S^3$ by corresponding Γ - and G -images of fundamental polyhedra of the reflection groups Γ and G around all sides of polyhedra including their edges and vertices are perfectly similar. This implies that our quasiregular mapping F defined by (4.1) is locally homeomorphic and $F(\infty) = \infty$.

It follows from Lemma 2.3 that the limit set $\Lambda(G) \subset S^3$ of the group $G \subset \text{Möb}(3)$ defines a Heegard splitting of infinite genus of the 3-sphere S^3 into two connected components Ω_1 and Ω_2 of the discontinuity set $\Omega(G)$. The action of G on the limit set $\Lambda(G)$ is an ergodic word hyperbolic action (quasi-self-similar in the sense of Sullivan, see [6], Cor. 2.66). For this ergodic action the set of fixed points of loxodromic elements $g \in G$ (conjugate to similarities in \mathbf{R}^3) is dense in $\Lambda(G)$. Preimages $\gamma \in \Gamma$ of such loxodromic elements $g \in G$ for our homomorphism $\rho: \Gamma \rightarrow G$ are loxodromic elements in Γ with two fixed points $p, q \in \Lambda(\Gamma) = S^2$, $p \neq q$. This and Tukia's arguments of the group completion (see [31] and [6], Section 4.6) show that our mapping F can be continuously extended to the set of fixed points of such elements $\gamma \in \Gamma$, $F(\text{Fix}(\gamma)) = \text{Fix}(\rho(\gamma))$. The sense of this continuous extension is that if $\gamma \in \Gamma$ is a loxodromic preimage of a loxodromic element $g \in G$, $\rho(\gamma) = g$, and if $x \in S^3 \setminus S^2$ tends to its fixed points p or q along the hyperbolic axis of γ (in $B(0, 1)$ or in its complement $\widehat{B(0, 1)}$) (i.e. radially) then $\lim_{|x| \rightarrow 1} F(x)$ exists and equals to the corresponding fixed point of the loxodromic element $g = \rho(\gamma) \in G$. In that sense one can say that the limit set $\Lambda(G)$ (the common boundary of the connected components $\Omega_1, \Omega_2 \subset \Omega(G)$) is the F -image of points in the unit sphere $S^2 \subset S^3$. So the mapping F is onto the whole sphere S^3 .

Nevertheless not all loxodromic elements $\gamma \in \Gamma$ in the hyperbolic lattice $\Gamma \subset \text{Isom } H^3$ have their images $\rho(\gamma) \in G$ as loxodromic elements. Proposition 2.4 shows that $\ker \rho \cong F_3$ is a free subgroup on three generators in the lattice Γ , and all elements $\gamma \in F_3$ are loxodromic. Now we look at radial limits $\lim_{x \rightarrow p} F(x)$ when x radially tends to a fixed point $p \in S^2$ of this loxodromic element $\gamma \in F_3 \subset \Gamma$.

For a group Γ with a finite set $\Sigma = \{\gamma_1, \dots, \gamma_k\}$ of generators we consider its Cayley graph $K(\Gamma, \Sigma)$, i.e. a 1-complex whose set of vertices is Γ and such that $a, b \in \Gamma$ are joined by an edge if and only if $a = b\gamma^{\pm 1}$ for some $\gamma \in \Sigma$. Since our Γ is a co-compact lattice acting in the hyperbolic space, we may define an embedding φ of its Cayley graph $K(\Gamma, \Sigma)$ in the hyperbolic space H^3 (model in the unit ball $B(0, 1)$ or in its complement $\widehat{B(0, 1)}$). For a point $0 \in H^3$ not fixed by any $\gamma \in \Gamma \setminus \{1\}$, vertices $\gamma \in K(\Gamma, \Sigma)$ are mapped to $\gamma(0)$, and edges joining vertices $a, b \in K(\Gamma, \Sigma)$ are mapped to the hyperbolic geodesic segments $[a(0), b(0)]$. In other words, $\varphi(K(\Gamma, \Sigma))$

is the graph that is dual to the tessellation of H^3 by polyhedra $\gamma(P_0)$ (or $\widehat{P_0}$), $\gamma \in \Gamma$. Obviously, the map φ is a Γ -equivariant proper embedding: for any compact $C \subset H^3$, its pre-image $\varphi^{-1}(\varphi(K(\Gamma, \Sigma)) \cap C)$ is compact. Moreover this embedding is a pseudo-isometry (see [12] and [6], Theorem 4.35):

Theorem 4.2. *Let $\Gamma \subset \text{Isom } H^n$ be a convex co-compact group. Then the map $\varphi: K(\Gamma, \Sigma) \hookrightarrow H^n$ is a pseudo-isometry of the word metric $(*, *)$ on $K(\Gamma, \Sigma)$ and the hyperbolic metric d , that is, there are positive constants K and K' such that*

$$(4.2) \quad (a, b)/K \leq d(\varphi(a), \varphi(b)) \leq K \cdot (a, b)$$

for all $a, b \in K(\Gamma, \Sigma)$ satisfying one of the following two conditions: either $(a, b) \geq K'$ or $d(\varphi(a), \varphi(b)) \geq K'$.

Theorem 4.2 implies (see [6], Theorem 4.38) that the limit set of any convex-cocompact group $\Gamma \subset \text{Möb}(n)$ can be identified with its group completion $\overline{\Gamma}$, $\overline{\Gamma} = \overline{K(\Gamma, \Sigma)} \setminus K(\Gamma, \Sigma)$. Namely there exists a continuous and Γ -equivariant bijection $\varphi_\Gamma: \overline{\Gamma} \rightarrow \Lambda(\Gamma)$.

Now for the kernel subgroup $F_3 = \ker \rho \subset \Gamma \subset \text{Isom } H^3$ and for the pseudo-isometric embedding φ from Theorem 4.2, we consider its Cayley subgraph in $\varphi(K(\Gamma, \Sigma)) \subset H^3$ which is a tree—see Figure 5. Since the limit set of $\ker \rho = F_3 \subset \Gamma$ corresponds to the ‘boundary at infinity’ $\partial_\infty F_3$ of $F_3 \subset \Gamma$ (the group completion $\overline{F_3}$), it is a closed Cantor subset of the unit sphere S^2 with Hausdorff dimension $\ln 5 / \ln 6 \sim 0.89822444$ (zero measure).

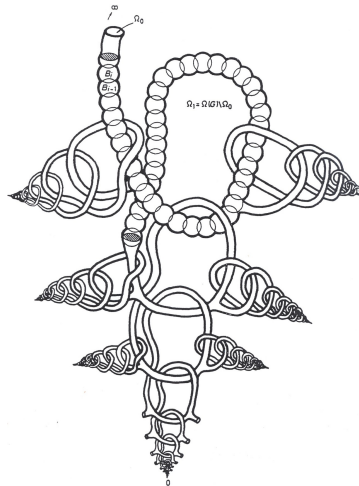


Figure 8. Locally inextensible wild embedding of a closed ball into \mathbf{R}^3 .

Remark 4.3. The most common construction of the used Cantor set $\partial_\infty F_3$ (similar to the original Cantor’s ternary construction of a perfect set that is nowhere dense) follows from the Figure 5. It can be obtained by removing one middle sixth of a line segment and repeating this for remaining five subsegments. This gives its Hausdorff dimension $\ln 5 / \ln 6$.

The Γ -orbit $\Gamma(\Lambda(F_3))$ of our Cantor set is a dense subset S_* of $S^2 = \Lambda(\Gamma)$ because of density in the limit set $\Lambda(\Gamma)$ of the Γ -orbit of any limit point. In particular we have such dense Γ -orbit $\Gamma(\{p, q\})$ of fixed points p and q of a loxodromic element $\gamma \in F_3 \subset \Gamma$ (the images of p and q are fixed points of Γ -conjugates of such loxodromic elements $\gamma \in F_3 \subset \Gamma$).

On the other hand let $x \in l_\gamma$ where l_γ is the hyperbolic axis of an element $\gamma \in F_3 \subset \Gamma$ (or its Γ -conjugate). Here the hyperbolic axes are either in $B(0, 1)$ or in its complement $\overline{B(0, 1)}$. Denoting d_γ the translation distance of γ , we have that any segment $[x, \gamma(x)] \subset l_\gamma$ is mapped by our quasiregular mapping F to a non-trivial closed loop $F([x, \gamma(x)]) \subset \Omega(G) = \Omega_1 \cup \Omega_2$, inside of a handle of the mutually linked handlebodies Ω_1 or Ω_2 (similar to the loops $\beta_1 \subset \Omega_1$ and $\beta_2 \subset \Omega_2$ constructed in the proof of Lemma 2.3). Therefore when $x \in l_\gamma$ radially tends to a fixed point p (in $\text{fix}(\gamma) \in S^2$) of such element γ , its image $F(x)$ goes along that closed loop $F([x, \gamma(x)]) \subset \Omega(G)$ because $F(\gamma(x)) = \rho(\gamma)(F(x)) = F(x)$. Immediately it implies that the radial limit $\lim_{x \rightarrow p} F(x)$ does not exist. This shows that fixed points of any element $\gamma \in F_3 \subset \Gamma$ (or its conjugate) are essential (topological) singularities of our quasiregular mapping F . So our quasiregular mapping F has no continuous extension to the subset $S_* \subset S^2$ (from both sides of the unit sphere S^2 in S^3) which is a dense subset of the unit sphere $S^2 \subset S^3$. \square

Remark 4.4. In terms of the holomorphic function theory of several complex variables, both components of the complement $S^3 \setminus S^2$ play the role of the so called domain of holomorphy for the constructed in Theorem 4.1 locally homeomorphic quasiregular mapping F . Obviously instead of the sphere $S^2 \subset S^3$ one may consider any quasi-sphere $S_q^2 \subset S^3$ which is the image of S^2 under a quasiconformal homeomorphism of S^3 . In other words, the complement $S^3 \setminus S_q^2$ consisting of two quasi-balls has the same property of domain of holomorphy for a locally homeomorphic quasiregular mapping F constructed in Theorem 4.1, and the quasi-sphere $S_q^2 \subset S^3$ is a barrier for it with a dense subset of essential singularities.

Remark 4.5. The constructed in Theorem 4.1 barrier S^2 for our locally homeomorphic quasiregular mapping F in the 3-space has completely different nature from the topological barrier S^2 for the quasisymmetric embedding $f: \overline{B(0, 1)} \hookrightarrow \mathbf{R}^3$ of the closed unit ball $\overline{B(0, 1)}$ into \mathbf{R}^3 constructed in [4]. That topological barrier for the embedding f was due to wild knottings of the boundary topological sphere $f(S^2) \subset \mathbf{R}^3$ on its dense subset. Figure 8 demonstrates the topological nature of this wild knotting when the fundamental group of the complement $\mathbf{R}^3 \setminus f(\overline{B(0, 1)})$ is infinitely generated near wild knotting points.

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