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A HARDY–LITTLEWOOD THEOREM FOR BERGMAN SPACES

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Abstract. We study positive weight functions $\omega(z)$ on the unit disk $\mathbf{D}$ such that

$$\int_{\mathbf{D}} |f(z)|^p \omega(z)\, dA(z) < \infty$$

if and only if

$$\int_{\mathbf{D}} (1-|z|^2)^p |f'(z)|^p \omega(z)\, dA(z) < \infty,$$

where f is analytic on $\mathbf{D}$ and dA is area measure on $\mathbf{D}$. We obtain some conditions on ω that imply the equivalence above, and we apply our conditions to several important classes of weights that have appeared in the literature before.

1. Introduction

Let $\mathbf{D}$ be the open unit disk in the complex plane $\mathbf{C}$ and let $H(\mathbf{D})$ denote the space of all analytic functions on $\mathbf{D}$. For $p > 0$ and $\alpha > -1$ we consider the Bergman spaces

$$A^p_\alpha = L^p(\mathbf{D}, dA_\alpha) \cap H(\mathbf{D}),$$

where

$$dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha\, dA(z).$$

Here dA is area measure on $\mathbf{C}$ normalized so that $A(\mathbf{D}) = 1$.

It is well known that a function $f \in H(\mathbf{D})$ belongs to A^p_α if and only if the function $(1-|z|^2)f'(z)$ belongs to $L^p(\mathbf{D}, dA_\alpha)$. Moreover, we have

$$\int_{\mathbf{D}} |f(z)-f(0)|^p\, dA_\alpha(z) \sim \int_{\mathbf{D}} (1-|z|^2)^p |f'(z)|^p\, dA_\alpha(z)$$

for $f \in H(\mathbf{D})$. See [6, 14]. It is then natural to ask for conditions on finite positive Borel measures μ on $\mathbf{D}$ such that

$$\int_{\mathbf{D}} |f(z)-f(0)|^p\, d\mu(z) \sim \int_{\mathbf{D}} (1-|z|^2)^p |f'(z)|^p\, d\mu(z)$$

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for all $f \in H(\mathbf{D})$. Such estimates are usually called Hardy–Littlewood theorems; see [4, 5]. The problem above was studied in [1, 10] when $d\mu(z) = \omega(z)\,dA(z)$ and $\omega(z)$ is a positive radial function.

It is easy to see that the Hardy–Littlewood theorem above is false if no restriction is placed on μ. For example, if $f(z) = \log(1-z)$ and

$$\mu = \sum_{k=1}^{\infty} c_k \delta_{a_k},$$

where δ_a means the unit point-mass at the point a, $a_k = k/(k+1)$, and $\{c_k\}$ is any sequence of positive numbers satisfying

$$\sum_{k=1}^{\infty} c_k < \infty, \quad \sum_{k=1}^{\infty} c_k (\log(k+1))^p = \infty,$$

then we have

$$\int_{\mathbf{D}} |f(z) - f(0)|^p \, d\mu(z) = \sum_{k=1}^{\infty} c_k |f(a_k)|^p = \sum_{k=1}^{\infty} c_k (\log(k+1))^p = \infty.$$

On the other hand, we always have

$$(1-|z|^2)|f'(z)| = (1-|z|^2)/|1-z| \le 2, \quad z \in \mathbf{D}.$$

Thus

$$\int_{\mathbf{D}} (1-|z|^2)^p |f'(z)|^p \, d\mu(z) \le 2^p \mu(\mathbf{D}) = 2^p \sum_{k=1}^{\infty} c_k < \infty.$$

Therefore, for this particular choice of μ, there is no positive constant C such that

$$\int_{\mathbf{D}} |f(z) - f(0)|^p \, d\mu(z) \le C \int_{\mathbf{D}} (1-|z|^2)^p |f'(z)|^p \, d\mu(z)$$

for all $f \in H(\mathbf{D})$.

We will focus on finite positive Borel measures on $\mathbf{D}$ that are absolutely continuous with respect to area measure. More specifically, we consider the case when

$$d\mu(z) = \omega(z)\,dA(z),$$

where ω is a Lebesgue integrable, nonnegative weight function on $\mathbf{D}$. Once again, as opposed to the results in [1, 10], our weight functions here are not necessarily radial.

Let $\rho(z,w) = |z-w|/|1-\overline{z}w|$ denote the pseudo-hyperbolic distance between z and w in $\mathbf{D}$. Our main results are the following.

Theorem A. *If there exist constants $r \in (0,1)$ and $C > 0$ such that*

$$C^{-1} \le \frac{\omega(z_1)}{\omega(z_2)} \le C$$

for all z_1 and z_2 in $\mathbf{D}$ with $\rho(z_1, z_2) < r$, then there exists another constant $C > 0$ (independent of f and p) such that

$$\int_{\mathbf{D}} (1-|z|^2)^p |f'(z)|^p \omega(z)\,dA(z) \le C \int_{\mathbf{D}} |f(z) - f(0)|^p \omega(z)\,dA(z)$$

for all $f \in H(\mathbf{D})$ and $p > 0$.

Theorem B. *Suppose that there are constants $s_0 \in [-1, 0)$ and $t_0 \geq 0$ with the following property: for any $s > s_0$ and $t > t_0$ there exists a positive constant C such that*

$$\int_{\mathbf{D}} \frac{\omega(u)(1-|u|^2)^s \, dA(u)}{|1 - z\overline{u}|^{2+s+t}} \leq \frac{C\omega(z)}{(1-|z|^2)^t}$$

for all $z \in \mathbf{D}$. Then for any $p > 0$ there exists another positive constant C such that

$$\int_{\mathbf{D}} |f(z) - f(0)|^p \omega(z) \, dA(z) \leq C \int_{\mathbf{D}} (1-|z|^2)^p |f'(z)|^p \omega(z) \, dA(z)$$

for all $f \in H(\mathbf{D})$.

Theorem C. *Suppose $\sigma > 0$ and μ is a positive Borel measure such that*

$$\int_{\mathbf{D}} (1-|z|^2)^\sigma \, d\mu(z) < \infty.$$

Let

$$\omega_\sigma(z) = \int_{\mathbf{D}} (1 - |\varphi_z(w)|^2)^\sigma \, d\mu(w),$$

where $\varphi_z(w) = (z-w)/(1-\overline{z}w)$. Then

(a) *ω_σ satisfies the condition in Theorem A.*
(b) *ω_σ satisfies the condition in Theorem B if and only if $\sigma < 2$.*
(c) *For $\sigma \geq 2$ and $p > 0$ we can find a measure μ such that the induced weight function ω_σ has the following property: there is NO positive constant C satisfying*

$$\int_{\mathbf{D}} |f(z) - f(0)|^p \omega_\sigma(z) \, dA(z) \leq C \int_{\mathbf{D}} (1-|z|^2)^p |f'(z)|^p \omega_\sigma(z) \, dA(z)$$

for all $f \in H(\mathbf{D})$.

Thus our condition in Theorem B is fine enough to detect the cut-off point $\sigma = 2$ for the weights ω_σ. Note that the weight functions $\omega_\sigma(z)$ were used in [3, 2] to study Dirichlet type spaces and their corresponding Möbius invariant counterparts.

We will also apply the conditions in Theorems A and B to several other classes of weight functions, including a class of positive harmonic weights and a class of positive subharmonic weights.

2. Raising the order of derivative

Recall that, for z and w in $\mathbf{D}$,

$$\rho(z, w) = \left| \frac{z-w}{1-\overline{z}w} \right|$$

is the pseudo-hyperbolic metric between z and w. Thus

$$D(a, r) = \{z \in \mathbf{D} \colon \rho(z, a) < r\}$$

is the pseudo-hyperbolic disk centered at a with radius r. It is well known that $D(a, r)$ is actually a Euclidean disk whose Euclidean center and radius are given by

$$\frac{1-r^2}{1-r^2|a|^2}\, a, \quad \frac{1-|a|^2}{1-r^2|a|^2}\, r.$$

See any of the books [5, 6, 14].

Lemma 1. *For any $r \in (0,1)$ there exists a positive constant $C = C(r)$ such that*

$$C^{-1} \le \frac{|1 - z\overline{u}|}{|1 - w\overline{u}|} \le C$$

for all z and w in $\mathbf{D}$ with $\rho(z,w) < r$ and all $u \in \overline{\mathbf{D}}$.

Proof. Fix two points z and w with $\rho(z,w) < r$ and let $a = \varphi_z(w)$. Then $|a| < r$. For any $u \in \overline{\mathbf{D}}$ we have

$$\begin{aligned}\frac{1 - z\overline{u}}{1 - w\overline{u}} &= \frac{1 - z\overline{u}}{1 - \varphi_z(a)\overline{u}} = (1 - z\overline{u}) / \left[1 - \frac{z - a}{1 - \overline{z}a}\,\overline{u}\right] \\ &= \frac{(1 - z\overline{u})(1 - \overline{z}a)}{1 - z\overline{u} - a(\overline{z} - \overline{u})} = \frac{1 - \overline{z}a}{1 - a\overline{\varphi_z(u)}}.\end{aligned}$$

Since $|a| < r$, the numerator and the denominator here are both bounded above and bounded below, which proves the desired estimates. $\square$

Theorem 2. *Suppose $p > 0$ and ω is a non-negative function in $L^1(\mathbf{D}, dA)$. If there exist two constants $r \in (0,1)$ and $C > 0$ such that*

$$C^{-1}\omega(z_2) \le \omega(z_1) \le C\omega(z_2) \tag{1}$$

for all z_1 and z_2 satisfying $\rho(z_1, z_2) < r$, then there exists another positive constant C such that

$$\int_{\mathbf{D}} (1 - |z|^2)^p |f'(z)|^p \omega(z)\, dA(z) \le C \int_{\mathbf{D}} |f(z) - f(0)|^p \omega(z)\, dA(z) \tag{2}$$

for all $f \in H(\mathbf{D})$.

Proof. Let $f \in H(\mathbf{D})$. It follows from the subharmonicity of $|f|^p$ that there exists a positive constant C_1 such that

$$|f(\zeta)|^p \le C_1 \int_{|z|<r} |f(z)|^p\, dA(z)$$

for all ζ with $|\zeta| \le r/2$. By Cauchy's formula,

$$f'(0) = \frac{1}{2\pi i} \int_{|\zeta| = r/2} \frac{f(\zeta)\, d\zeta}{\zeta^2}.$$

It follows that

$$|f'(0)| \le \frac{2}{r} \max_{|\zeta| = r/2} |f(\zeta)|.$$

Combining this with our earlier estimate for $|f(\zeta)|$, we obtain another positive constant C_2 such that

$$|f'(0)|^p \le C_2 \int_{|u|<r} |f(u)|^p\, dA(u).$$

Replace f by $f \circ \varphi_z$, where

$$\varphi_z(u) = \frac{z - u}{1 - \overline{z}u},$$

and make an obvious change of variables. We obtain

$$\begin{aligned}(1 - |z|^2)^p |f'(z)|^p &\le C_2 \int_{|u|<r} |f \circ \varphi_z(u)|^p\, dA(u) \\ &= C_2 (1 - |z|^2)^2 \int_{D(z,r)} |f(u)|^p \frac{dA(u)}{|1 - \overline{z}u|^4}.\end{aligned}$$

By (1), there exists another positive constant C_3 such that

$$(1-|z|^2)^p|f'(z)|^p\omega(z) \le C_3(1-|z|^2)^2\int_{D(z,r)}\frac{|f(u)|^p\omega(u)\,dA(u)}{|1-\overline{z}u|^4}$$

for all $z\in\mathbf{D}$. It follows from this and Fubini's theorem that

$$\begin{aligned} I =: &\int_{\mathbf{D}}(1-|z|^2)^p|f'(z)|^p\omega(z)\,dA(z)\\ &\le C_3\int_{\mathbf{D}}(1-|z|^2)^2\,dA(z)\int_{D(z,r)}\frac{|f(u)|^p\omega(u)\,dA(u)}{|1-\overline{z}u|^4}\\ &= C_3\int_{\mathbf{D}}(1-|z|^2)^2\,dA(z)\int_{\mathbf{D}}\frac{|f(u)|^p\chi_{D(z,r)}(u)\omega(u)\,dA(u)}{|1-\overline{z}u|^4}\\ &= C_3\int_{\mathbf{D}}|f(u)|^p\omega(u)\,dA(u)\int_{\mathbf{D}}\frac{(1-|z|^2)^2\chi_{D(z,r)}(u)\,dA(z)}{|1-\overline{z}u|^4}. \end{aligned}$$

It is clear that, for characteristic functions, we have

$$\chi_{D(z,r)}(u)=\chi_{D(u,r)}(z).$$

Thus

$$I\le C_3\int_{\mathbf{D}}|f(u)|^p\omega(u)\,dA(u)\int_{D(u,r)}\frac{(1-|z|^2)^2\,dA(z)}{|1-\overline{z}u|^4}.$$

Since the area of $D(u,r)$ is comparable to $(1-|u|^2)^2$ (for any fixed $r\in(0,1)$) and

$$(1-|z|^2)^2\sim|1-\overline{z}u|^2\sim(1-|u|^2)^2$$

for $z\in D(u,r)$ (see Lemma 1), we can find another positive constant C_4 such that

$$I\le C_4\int_{\mathbf{D}}|f(u)|^p\omega(u)\,dA(u).$$

Replacing f by $f-f(0)$, we obtain (2). □

Corollary 3. *Suppose $p>0$, ω is a non-negative function in $L^1(\mathbf{D},dA)$, $f\in H(\mathbf{D})$, and n is any positive integer. If ω satisfies condition (1), then there exists a positive constant C such that*

$$\int_{\mathbf{D}}(1-|z|^2)^{pn}|f^{(n)}(z)|^p\omega(z)\,dA(z)\le C\int_{\mathbf{D}}|f(z)|^p\omega(z)\,dA(z)$$

for all $f\in H(\mathbf{D})$.

Proof. It follows from Lemma 1 again that, for any $t\ge 0$, the function $(1-|z|^2)^t\omega(z)$ satisfies condition (1) whenever ω does. The desired result then follows from repeatedly applying Theorem 2 above. □

3. Lowering the order of derivative

In this section we consider the problem of lowering the order of derivative. The problem is much more complex than the corresponding problem for raising the order of derivative.

Lemma 4. *Suppose $s>-1$, t is real, and*

$$I(z)=\int_{\mathbf{D}}\frac{(1-|w|^2)^s\,dA(w)}{|1-z\overline{w}|^{2+s+t}},\quad z\in\mathbf{D}.$$

(a) *If $t<0$, then $I(z)\sim 1$ for $z\in\mathbf{D}$.*

(b) *If* $t=0$, *then* $I(z)\sim\log[2/(1-|z|^2)]$ *for* $z\in\mathbf{D}$.
(c) *If* $t>0$, *then* $I(z)\sim 1/(1-|z|^2)^t$ *for* $z\in\mathbf{D}$.

Proof. This is well known. See [6, 14] for example. □

The following embedding theorem will be critical for us when we deal with L^p integrals in the case $0<p\le 1$.

Lemma 5. *If* $0<p\le 1$ *and* $\alpha>-1$, *then* $A^p_\alpha\subset A^1_\beta$, *where*

$$\beta=\frac{2+\alpha}{p}-2,$$

and the inclusion mapping from A^p_α *to* A^1_β *is a bounded linear operator.*

Proof. See [6, 14] for example. □

Our next result uses a condition that has appeared in the literature several times before. See Chapter 3 of [12] for example.

Theorem 6. *Suppose* $p>0$, $\omega\in L^1(\mathbf{D},dA)$ *is non-negative, and there exist* $t_0\ge 0$ *and* $s_0\in[-1,0)$ *with the following property: for any* $t>t_0$ *and* $s>s_0$ *there is a positive constant* $C=C(t,s)$ *such that*

$$\int_{\mathbf{D}}\frac{\omega(u)(1-|u|^2)^s\,dA(u)}{|1-z\overline{u}|^{2+s+t}}\le\frac{C\omega(z)}{(1-|z|^2)^t}\tag{3}$$

for all $z\in\mathbf{D}$. *Then there exists another positive constant* C *such that*

$$\int_{\mathbf{D}}|f(z)-f(0)|^p\omega(z)\,dA(z)\le C\int_{\mathbf{D}}(1-|z|^2)^p|f'(z)|^p\omega(z)\,dA(z)\tag{4}$$

for all $f\in H(\mathbf{D})$.

Proof. It follows from condition (3) that, for any $t>t_0$, we can find a positive constant c such that $\omega(z)\ge c(1-|z|^2)^t$ for all $z\in\mathbf{D}$. This implies that if the right-hand side of (4) is finite (otherwise there is nothing to prove) then $f(z)$ and $f'(z)$ have polynomial growth near the unit circle, that is, their growth rate does not exceed $(1-|z|^2)^{-k}$ for some positive integer k.

Fix some sufficiently large t (whose exact range will be specified later). Since $f'(z)$ has polynomial growth near the boundary (see previous paragraph), we have

$$f(z)-f(0)=\int_{\mathbf{D}}\frac{f'(u)(1-|u|^2)^t\,dA(u)}{\overline{u}(1-z\overline{u})^{1+t}},\quad z\in\mathbf{D}.$$

This is a well-known reproducing formula and can be found in [6, 14] for example. By [14, Lemma 4.26], there exists a positive constant C_1 such that

$$|f(z)-f(0)|\le C_1\int_{\mathbf{D}}\frac{|f'(u)|(1-|u|^2)^t\,dA(u)}{|1-z\overline{u}|^{1+t}}$$

for all $z\in\mathbf{D}$.

We first consider the case $p=1$ and assume that t is large enough so that $t-1>t_0$. Then by Fubini's theorem and condition (3) with $s=0$, there exists another positive constant $C_2>0$ such that

$$\begin{aligned}I_1=:&\int_{\mathbf{D}}|f(z)-f(0)|\omega(z)\,dA(z)\\ &\le C_1\int_{\mathbf{D}}\omega(z)\,dA(z)\int_{\mathbf{D}}\frac{|f'(u)|(1-|u|^2)^t\,dA(u)}{|1-z\overline{u}|^{1+t}}\end{aligned}$$

$$= C_1 \int_{\mathbf{D}} |f'(u)|(1-|u|^2)^t \, dA(u) \int_{\mathbf{D}} \frac{\omega(z)\, dA(z)}{|1-z\overline{u}|^{2+(t-1)}}$$
$$\le C_2 \int_{\mathbf{D}} (1-|u|^2)|f'(u)|\omega(u)\, dA(u).$$

This proves the desired result for $p = 1$.

Next we assume that $p > 1$ with $1/p + 1/q = 1$. We also assume that t is large enough so that $p(t-1) > t_0$. Write $1 + t = \sigma_1 + \sigma_2$, where

$$\frac{2}{q} < \sigma_2 < \frac{2}{q} - \frac{s_0}{p}. \tag{5}$$

We have

$$|f(z) - f(0)| \le C_1 \int_{\mathbf{D}} \frac{(1-|u|^2)^t |f'(u)|\, dA(u)}{|1-z\overline{u}|^{\sigma_1}|1-z\overline{u}|^{\sigma_2}}$$
$$\le C_1 \left[\int_{\mathbf{D}} \frac{(1-|u|^2)^{pt}|f'(u)|^p\, dA(u)}{|1-z\overline{u}|^{p\sigma_1}}\right]^{\frac{1}{p}} \left[\int_{\mathbf{D}} \frac{dA(u)}{|1-z\overline{u}|^{q\sigma_2}}\right]^{\frac{1}{q}}.$$

By (5), we have $q\sigma_2 > 2$. It follows from Lemma 4 that there exists another positive constant C_2 such that

$$|f(z) - f(0)| \le \frac{C_2}{(1-|z|^2)^{(q\sigma_2-2)/q}} \left[\int_{\mathbf{D}} \frac{(1-|u|^2)^{pt}|f'(u)|^p\, dA(u)}{|1-z\overline{u}|^{p\sigma_1}}\right]^{\frac{1}{p}}$$
$$= \frac{C_2}{(1-|z|^2)^{\sigma_2-2+(2/p)}} \left[\int_{\mathbf{D}} \frac{(1-|u|^2)^{pt}|f'(u)|^p\, dA(u)}{|1-z\overline{u}|^{p\sigma_1}}\right]^{\frac{1}{p}}.$$

Therefore,

$$|f(z) - f(0)|^p \le C_2^p (1-|z|^2)^{2p-p\sigma_2-2} \int_{\mathbf{D}} \frac{(1-|u|^2)^{pt}|f'(u)|^p\, dA(u)}{|1-z\overline{u}|^{p\sigma_1}}.$$

It follows from this and Fubini's theorem that the integral

$$\int_{\mathbf{D}} |f(z) - f(0)|^p \omega(z)\, dA(z)$$

does not exceed

$$C_2^p \int_{\mathbf{D}} (1-|u|^2)^{pt}|f'(u)|^p\, dA(u) \int_{\mathbf{D}} \frac{\omega(z)(1-|z|^2)^{2p-p\sigma_2-2}\, dA(z)}{|1-z\overline{u}|^{p\sigma_1}}.$$

By (5), we have $s_0 < 2p - p\sigma_2 - 2 < 0$. Since

$$p\sigma_1 = 2 + (2p - p\sigma_2 - 2) + p(t-1)$$

and $p(t-1) > t_0$, it follows from (3) that there exists another positive constant C_3 such that

$$\int_{\mathbf{D}} |f(z) - f(0)|^p \omega(z)\, dA(z) \le C_3 \int_{\mathbf{D}} \frac{(1-|u|^2)^{pt}|f'(u)|^p \omega(u)\, dA(u)}{(1-|u|^2)^{p(t-1)}}$$
$$= C_3 \int_{\mathbf{D}} (1-|u|^2)^p |f'(u)|^p \omega(u)\, dA(u).$$

This proves the case $1 < p < \infty$.

Finally, we assume that $0 < p < 1$. In this case, we also assume that t is large enough so that $p(t+1) > 2 > 1-p$. We can then write

$$t = \frac{2+\sigma}{p} - 2, \quad \sigma > -1.$$

By Lemma 5, there exists another positive constant C_4 such that

$$\begin{aligned}
|f(z) - f(0)| &\le C_1 \int_{\mathbf{D}} \frac{(1-|u|^2)^t |f'(u)|\, dA(u)}{|1-\overline{z}u|^{1+t}} \\
&= C_1 \int_{\mathbf{D}} \left| \frac{f'(u)}{(1-\overline{z}u)^{1+t}} \right| (1-|u|^2)^{\frac{2+\sigma}{p}-2}\, dA(u) \\
&\le C_4 \left[\int_{\mathbf{D}} \left| \frac{f'(u)}{(1-\overline{z}u)^{1+t}} \right|^p (1-|u|^2)^{\sigma}\, dA(u) \right]^{\frac{1}{p}}.
\end{aligned}$$

Therefore,

$$|f(z) - f(0)|^p \le C_4^p \int_{\mathbf{D}} \frac{(1-|u|^2)^{\sigma} |f'(u)|^p\, dA(u)}{|1-\overline{z}u|^{p(1+t)}}.$$

By Fubini's theorem and (3) with $s = 0$, we obtain another positive constant C_5 such that

$$\begin{aligned}
I_p =&: \int_{\mathbf{D}} |f(z) - f(0)|^p \omega(z)\, dA(z) \\
&\le C_4^p \int_{\mathbf{D}} (1-|u|^2)^{\sigma} |f'(u)|^p\, dA(u) \int_{\mathbf{D}} \frac{\omega(z)\, dA(z)}{|1-\overline{z}u|^{p(1+t)}} \\
&\le C_5 \int_{\mathbf{D}} \frac{(1-|u|^2)^{\sigma} |f'(u)|^p \omega(u)\, dA(u)}{(1-|u|^2)^{p(1+t)-2}} \\
&= C_5 \int_{\mathbf{D}} (1-|u|^2)^p |f'(u)|^p \omega(u)\, dA(u).
\end{aligned}$$

This completes the proof of the theorem. □

Corollary 7. *Suppose $p > 0$, $\omega \in L^1(\mathbf{D}, dA)$ is non-negative, $f \in H(\mathbf{D})$, and n is a positive integer. If ω satisfies condition (3), then*

$$\int_{\mathbf{D}} (1-|z|^2)^{np} |f^{(n)}(z)|^p \omega(z)\, dA(z) < \infty$$

implies that

$$\int_{\mathbf{D}} |f(z)|^p \omega(z)\, dA(z) < \infty.$$

Proof. If ω satisfies condition (3), then for any positive integer k the weight function

$$\omega_k(z) = (1-|z|^2)^k \omega(z)$$

also satisfies condition (3). In fact, if t_0 is the number in (3) for the weight ω and if $t > t_0 + k$, then for any $s > -1$ we have

$$\begin{aligned}
\int_{\mathbf{D}} \frac{\omega_k(u)(1-|u|^2)^s\, dA(u)}{|1-z\overline{u}|^{2+s+t}} &= \int_{\mathbf{D}} \frac{\omega(u)(1-|u|^2)^{s+k}\, dA(u)}{|1-z\overline{u}|^{2+(s+k)+(t-k)}} \\
&\le \frac{C\omega(z)}{(1-|z|^2)^{t-k}} = \frac{C\omega_k(z)}{(1-|z|^2)^t}.
\end{aligned}$$

This shows that ω_k satisfies condition (3) as well. The desired result then follows from repeatedly applying Theorem 6. $\square$

Combining Corollaries 3 and 7, we obtain the following Hardy–Littlewood theorem as the first main result of the paper.

Theorem 8. *Suppose $p > 0$, ω is a non-negative function in $L^1(\mathbf{D}, dA)$, $f \in H(\mathbf{D})$, and n is a positive integer. If ω satisfies both (1) and (3), then*

$$\int_{\mathbf{D}} |f(z)|^p \omega(z)\, dA(z) < \infty$$

if and only if

$$\int_{\mathbf{D}} (1 - |z|^2)^{pn} |f^{(n)}(z)|^p \omega(z)\, dA(z) < \infty.$$

Moreover, the quantity

$$\int_{\mathbf{D}} \left| f(z) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!}\, z^k \right|^p \omega(z)\, dA(z)$$

is comparable to the quantity

$$\int_{\mathbf{D}} (1 - |z|^2)^{pn} |f^{(n)}(z)|^p\, \omega(z)\, dA(z).$$

Alternatively, the quantities

$$\int_{\mathbf{D}} |f(z)|^p \omega(z)\, dA(z)$$

and

$$\sum_{k=0}^{n-1} |f^{(k)}(0)| + \int_{\mathbf{D}} (1 - |z|^2)^{pn} |f^{(n)}(z)|^p \omega(z)\, dA(z)$$

are comparable.

In view of Theorem 8, it is now natural to ask which weight functions ω satisfy conditions (1) and (3). We will consider several classes of weight functions and determine exactly when they satisfy conditions (1) and (3). But first, for any $\alpha > -1$, the weight function

$$\omega(z) = (1 - |z|^2)^\alpha$$

clearly satisfies both (1) and (3). This follows from Lemmas 1 and 4.

4. Examples of weights

For any $\sigma > 0$ and any positive Borel measure μ on $\mathbf{D}$ satisfying

$$\int_{\mathbf{D}} (1 - |w|^2)^\sigma\, d\mu(w) < \infty,$$

we consider the weight function

$$\omega_\sigma(z) = \int_{\mathbf{D}} (1 - |\varphi_z(w)|^2)^\sigma\, d\mu(w).$$

These weights were considered in [3, 2] and were used there to study certain weighted Dirichlet spaces and the associated Möbius invariant function spaces.

We will see that the properties of these weights depend on whether $\sigma < 2$ or $\sigma \geq 2$. Note that ω_σ is superharmonic when $\sigma \leq 1$. When $\sigma > 1$, the function ω_σ is usually neither superharmonic nor subharmonic. This can be seen from the formula

$$\frac{\partial^2 \omega_\sigma}{\partial z \partial \overline{z}} = \sigma \int_{\mathbf{D}} (1 - |\varphi_w(z)|^2)^{\sigma-2}(\sigma|\varphi_w(z)|^2 - 1)|\varphi_w'(z)|^2 \, d\mu(w).$$

Our analysis depends on the following estimate which has become more and more useful in recent years.

Lemma 9. *Suppose $s > -1$, $t > 0$, $r > 0$, and $t - s > 2 > r - s$. Then*

$$\int_{\mathbf{D}} \frac{(1-|z|^2)^s \, dA(z)}{|1 - z\overline{w}|^t |1 - z\overline{a}|^r} \sim \frac{1}{(1-|w|^2)^{t-s-2}|1 - w\overline{a}|^r}$$

for w and a in $\mathbf{D}$.

Proof. See [7, 9, 12]. □

Proposition 10. *The weight function ω_σ satisfies condition (1) for all $\sigma > 0$, and it satisfies condition (3) when $0 < \sigma < 2$.*

Proof. Condition (1) follows from Lemma 1.

To verify condition (3), we fix some $\sigma \in (0, 2)$. We take $t_0 = \sigma$ and fix some $s_0 \in (-1, 0)$ such that $\sigma < 2 + s_0$. For $t > t_0$ and $s > s_0$, we use Fubini's theorem to obtain

$$\begin{aligned} I_\sigma =: & \int_{\mathbf{D}} \frac{\omega_\sigma(z)(1-|z|^2)^s \, dA(z)}{|1 - a\overline{z}|^{2+s+t}} \\ = & \int_{\mathbf{D}} \frac{(1-|z|^2)^{s+\sigma} \, dA(z)}{|1 - a\overline{z}|^{2+s+t}} \int_{\mathbf{D}} \frac{(1-|w|^2)^\sigma \, d\mu(w)}{|1 - z\overline{w}|^{2\sigma}} \\ = & \int_{\mathbf{D}} (1-|w|^2)^\sigma \, d\mu(w) \int_{\mathbf{D}} \frac{(1-|z|^2)^{s+\sigma} \, dA(z)}{|1 - a\overline{z}|^{2+s+t}|1 - z\overline{w}|^{2\sigma}}. \end{aligned}$$

It follows from Lemma 9 and the assumptions $t > \sigma$ and $s > s_0$ that there exists a positive constant C such that

$$\begin{aligned} I_\sigma \leq & \, C \int_{\mathbf{D}} \frac{(1-|w|^2)^\sigma \, d\mu(w)}{(1-|a|^2)^{t-\sigma}|1 - a\overline{w}|^{2\sigma}} \\ = & \frac{C}{(1-|a|^2)^t} \int_{\mathbf{D}} \frac{(1-|w|^2)^\sigma (1-|a|^2)^\sigma}{|1 - a\overline{w}|^{2\sigma}} \, d\mu(w) \\ = & \frac{C\omega_\sigma(a)}{(1-|a|^2)^t}. \end{aligned}$$

This completes the proof of the proposition. □

We are going to show that the assumption $\sigma < 2$ in the proposition above is best possible. In other words, when $\sigma \geq 2$, we can construct a counterexample of μ such that the induced weight function ω_σ does not satisfy condition (3). Thus our condition (3) is sensitive enough to detect the changing behaviour of ω_σ at the critical point $\sigma = 2$.

Lemma 11. *Suppose $p > 0$ and $r \in (0, 1)$. There exists a positive constant C such that*

$$|f(z)|^p \leq \frac{C}{(1-|z|^2)^2} \int_{D(z,r)} |f(w)|^p \, dA(w)$$

for all $f \in H(\mathbf{D})$ and $z \in \mathbf{D}$.

Proof. This is well known. See [6, 14] for example. □

Proposition 12. *If $p > 0$ and $\sigma \geq 2$, then there exists a positive Borel measure μ on $\mathbf{D}$ such that its induced weight function ω_σ belongs to $L^1(\mathbf{D}, dA)$ but there is no positive constant C with*

$$\int_{\mathbf{D}} |f(z) - f(0)|^p \omega_\sigma(z)\, dA(z) \leq C \int_{\mathbf{D}} (1 - |z|^2)^p |f'(z)|^p \omega_\sigma(z)\, dA(z) \tag{6}$$

for all $f \in H(\mathbf{D})$. Consequently, ω_σ does not satisfy condition (3).

Proof. The definition of $\omega_\sigma(z)$ only requires

$$\int_{\mathbf{D}} (1 - |z|^2)^\sigma\, d\mu(z) < \infty.$$

When this condition is satisfied and $\sigma > 2$, we deduce from Lemma 4 that

$$\begin{aligned}
\int_{\mathbf{D}} \omega_\sigma(z)\, dA(z) &= \int_{\mathbf{D}} (1 - |z|^2)^\sigma\, dA(z) \int_{\mathbf{D}} \frac{(1 - |w|^2)^\sigma\, d\mu(w)}{|1 - z\overline{w}|^{2\sigma}} \\
&= \int_{\mathbf{D}} (1 - |w|^2)^\sigma\, d\mu(w) \int_{\mathbf{D}} \frac{(1 - |z|^2)^\sigma\, dA(z)}{|1 - z\overline{w}|^{2\sigma}} \\
&\sim \int_{\mathbf{D}} \frac{(1 - |w|^2)^\sigma\, d\mu(w)}{(1 - |w|^2)^{2\sigma - \sigma - 2}} \\
&= \int_{\mathbf{D}} (1 - |w|^2)^2\, d\mu(w).
\end{aligned}$$

Therefore, for $\sigma > 2$ and for any positive Borel measure μ with

$$\int_{\mathbf{D}} (1 - |w|^2)^2\, d\mu(w) < \infty,$$

the weight function ω_σ is not only well defined but also belongs to $L^1(\mathbf{D}, dA)$.

For $f \in H(\mathbf{D})$ let us write

$$I(f) = \int_{\mathbf{D}} |f(z)|^p \omega_\sigma(z)\, dA(z)$$

and

$$J(f) = \int_{\mathbf{D}} (1 - |z|^2)^p |f'(z)|^p \omega_\sigma(z)\, dA(z).$$

Fix some $r \in (0, 1)$ and use Fubini's theorem. We obtain

$$\begin{aligned}
I(f) &= \int_{\mathbf{D}} |f(z)|^p\, dA(z) \int_{\mathbf{D}} (1 - |\varphi_z(w)|^2)^\sigma\, d\mu(w) \\
&= \int_{\mathbf{D}} d\mu(w) \int_{\mathbf{D}} |f(z)|^p (1 - |\varphi_z(w)|^2)^\sigma\, dA(z) \\
&\geq \int_{\mathbf{D}} d\mu(w) \int_{D(w,r)} |f(z)|^p (1 - |\varphi_z(w)|^2)^\sigma\, dA(z) \\
&\geq (1 - r^2)^\sigma \int_{\mathbf{D}} d\mu(w) \int_{D(w,r)} |f(z)|^p\, dA(z).
\end{aligned}$$

It follows from Lemma 11 that there exists a positive constant $c = c(r) > 0$ such that

$$I(f) \geq c \int_{\mathbf{D}} (1 - |w|^2)^2 |f(w)|^p\, d\mu(w).$$

Thus $I(f - f(0)) \leq CJ(f)$ would imply that there is another positive constant C such that

$$\int_{\mathbf{D}} |f(z) - f(0)|^p (1 - |z|^2)^2 \, d\mu(z) \leq C \int_{\mathbf{D}} (1 - |z|^2)^p |f'(z)|^p \omega_\sigma(z) \, dA(z) \tag{7}$$

for all $f \in H(\mathbf{D})$.

Now consider the measure

$$\mu = \sum_{k=1}^{\infty} c_k \delta_{a_k}, \quad a_k = k/(k+1),$$

where the positive sequence $\{c_k\}$ is chosen to satisfy

$$\sum_{k=1}^{\infty} \frac{c_k}{(k+1)^2} < \infty, \quad \sum_{k=1}^{\infty} \frac{c_k}{(k+1)^2} (\log(k+1))^p = \infty. \tag{8}$$

For this choice of μ and the function $f(z) = \log(1 - z)$, the left-hand side of (7) dominates the sum

$$\sum_{k=1}^{\infty} \frac{c_k}{(k+1)^2} (\log(k+1))^p \,,$$

while the right-hand side of (7) is dominated by

$$\sum_{k=1}^{\infty} \frac{c_k}{(k+1)^2}.$$

Thus the conditions in (8) show that (7) is not possible, which proves the desired result when $\sigma > 2$.

When $\sigma = 2$, the definition of ω_σ requires that

$$\int_{\mathbf{D}} (1 - |z|^2)^2 \, d\mu(z) < \infty,$$

and an examination of an earlier argument shows that the condition $\omega_\sigma \in L^1(\mathbf{D}, dA)$ is equivalent to

$$\int_{\mathbf{D}} (1 - |z|^2)^2 \log \frac{2}{1 - |z|^2} \, d\mu(z) < \infty. \tag{9}$$

Therefore, in this case, if μ satisfies condition (9), then the induced weight function ω_σ is well defined and belongs to $L^1(\mathbf{D}, dA)$. In particular, if we define

$$\mu = \sum_{k=1}^{\infty} c_k \delta_{a_k}, \quad a_k = k/(k+1),$$

where the positive sequence $\{c_k\}$ is chosen to satisfy

$$\sum_{k=1}^{\infty} \frac{c_k}{(k+1)^2} \log(k+1) < \infty, \quad \sum_{k=1}^{\infty} \frac{c_k}{(k+1)^2} (\log(k+1))^p = \infty, \quad p > 1,$$

then the induced weight function ω_σ is well defined and belongs to $L^1(\mathbf{D}, dA)$, but condition (9) shows that the inequality in (7) is not possible for $p > 1$. Thus we have shown that condition (6) is not possible for $p > 1$.

Finally, we observe that if condition (6) holds for some p, then replacing f by f^2 gives

$$\int_{\mathbf{D}} |f(z)|^{2p} \omega_\sigma(z) \, dA(z) \leq C \int_{\mathbf{D}} |f(z)|^p |f'(z)|^p (1 - |z|^2)^p \omega_\sigma(z) \, dA(z)$$

for all $f \in H(\mathbf{D})$ with $f(0)=0$, where C is 2^p times the original constant. It follows from the Cauchy-Schwarz inequality that

$$\left[\int_{\mathbf{D}} |f(z)|^{2p}\omega_\sigma(z)\,dA(z)\right]^2$$

is less than or equal to C^2 times

$$\int_{\mathbf{D}} |f(z)|^{2p}\omega_\sigma(z)\,dA(z)\int_{\mathbf{D}} |f'(z)|^{2p}(1-|z|^2)^{2p}\omega_\sigma(z)\,dA(z).$$

Cancel the common factor from both sides. We obtain

$$\int_{\mathbf{D}} |f(z)|^{2p}\omega_\sigma(z)\,dA(z)\le C^2\int_{\mathbf{D}} |f'(z)|^{2p}(1-|z|^2)^{2p}\omega_\sigma(z)\,dA(z)$$

for all $f\in H(\mathbf{D})$ with $f(0)=0$. For general f, we replace f in the inequality above by $f-f(0)$. Then

$$\int_{\mathbf{D}} |f(z)-f(0)|^{2p}\omega_\sigma(z)\,dA(z)\le C^2\int_{\mathbf{D}} |f'(z)|^{2p}(1-|z|^2)^{2p}\omega_\sigma(z)\,dA(z)$$

for all $f\in H(\mathbf{D})$. Repeating this argument k times, we obtain

$$\int_{\mathbf{D}} |f(z)-f(0)|^{2^kp}\omega_\sigma(z)\,dA(z)\le C_k\int_{\mathbf{D}} |f'(z)|^{2^kp}(1-|z|^2)^{2^kp}\omega_\sigma(z)\,dA(z)$$

for all $f\in H(\mathbf{D})$. When k is large enough, we will have $2^kp>1$. This together with what was already proved in the previous paragraph shows that, for any $p>0$, there is a positive Borel measure μ such that the inequality in (6) does not hold for all $f\in H(\mathbf{D})$. This completes the proof of the proposition. □

Next we consider a class of positive harmonic weight functions. Thus for a finite positive Borel measure ν on the unit circle $\mathbf{T}$ we define

$$\omega_\nu(z)=\int_{\mathbf{T}}\frac{1-|z|^2}{|\zeta-z|^2}\,d\nu(\zeta).$$

Motivated by his study of cyclic analytic two-isometries, Richer [11] introduced a Dirichlet type space $\mathcal{D}_\nu$ induced by the weight function ω_ν. More specifically, a function $f\in H(\mathbf{D})$ belongs to $\mathcal{D}_\nu$ if $f'\in L^2(\mathbf{D},\omega_\nu\,dA)$. The $\mathcal{D}_\nu$ spaces have attracted a lot of attention in recent years. In particular, the higher order derivative characterization of $\mathcal{D}_\nu$ was proved in [8] using Schur's theorem. As a consequence of our Theorem 8 and the following proposition, we will obtain an alternative proof of the higher order derivative characterization of $\mathcal{D}_\nu$ without using Schur's theorem. Note that the method based on Schur's theorem is valid for $L^p(\mathbf{D},\omega_\nu\,dA)\cap H(\mathbf{D})$ only when $p>1$, while our approach is valid for all $p>0$.

Proposition 13. *Let ν be a finite positive Borel measure on the unit circle $\mathbf{T}$. Then ω_ν satisfies both conditions (1) and (3).*

Proof. By [8, Lemma 2.2], if $s>-1$ and $t>s+3$, then there exists a positive constant C such that

$$\int_{\mathbf{D}}\frac{(1-|z|^2)^s}{|1-\overline{u}z|^t}\omega_\nu(z)\,dA(z)\le\frac{C\,\omega_\nu(u)}{(1-|u|^2)^{t-s-2}}$$

for all $u\in\mathbf{D}$. Thus ω_ν satisfies condition (3). It follows from Lemma 1 that ω_ν also satisfies condition (1).

Recall that the pseudo-hyperbolic disk is actually a Euclidean disk and ω_ν is a positive harmonic function on $\mathbf{D}$. Thus the classical Harnack inequality also yields condition (1) for ω_ν. $\square$

We can also consider the following weights which are generalizations of positive harmonic functions,

$$\omega^\sigma(z) = \int_{\mathbf{T}} \left(\frac{1-|z|^2}{|1-ze^{-it}|^2}\right)^\sigma d\nu(t),$$

where ν is a finite positive Borel measure on the unit circle $\mathbf{T}$ and $\sigma > 0$. If $\sigma \geq 1$, then ω^σ is a subharmonic function. It follows easily from Fubini's theorem and Lemma 4 that the weight function ω^σ belongs to $L^1(\mathbf{D}, dA)$ if and only if $\sigma < 2$.

Proposition 14. *Suppose* $0 < \sigma < 2$. *Then* ω^σ *satisfies both conditions (1) and (3).*

Proof. Once again, it follows from Lemma 1 that each ω^σ satisfies condition (1). To verify condition (3) for ω^σ, let

$$t_0 = \sigma, \quad s_0 = \max(-1, \sigma - 2).$$

If $s > s_0$ and $t > t_0$, then we use Fubini's theorem and Lemma 9 to find a positive constant C such that

$$\begin{aligned}
\int_{\mathbf{D}} \frac{\omega^\sigma(u)(1-|u|^2)^s\, dA(u)}{|1-z\overline{u}|^{2+s+t}} &= \int_{\mathbf{D}} \frac{(1-|u|^2)^{s+\sigma}\, dA(u)}{|1-z\overline{u}|^{2+s+t}} \int_{\mathbf{T}} \frac{d\nu(t)}{|1-\overline{u}e^{it}|^{2\sigma}} \\
&= \int_{\mathbf{T}} d\nu(t) \int_{\mathbf{D}} \frac{(1-|u|^2)^{s+\sigma}\, dA(u)}{|1-z\overline{u}|^{2+s+t}|1-ue^{-it}|^{2\sigma}} \\
&\leq C \int_{\mathbf{T}} \frac{d\nu(t)}{(1-|z|^2)^{t-\sigma}|1-ze^{-it}|^{2\sigma}} \\
&= \frac{C\, \omega^\sigma(z)}{(1-|z|^2)^t}
\end{aligned}$$

for all $z \in \mathbf{D}$. This proves the desired result. $\square$

5. Further remarks

The results of the previous section show that the general problem of characterizing weight functions $\omega(z)$ such that

$$\int_{\mathbf{D}} |f(z) - f(0)|^p \omega(z)\, dA(z) \sim \int_{\mathbf{D}} (1-|z|^2)^p |f'(z)|^p \omega(z)\, dA(z)$$

is highly non-trivial. It is still an open problem.

In the introduction we illustrated with an example that there are finite positive Borel measures μ on $\mathbf{D}$ such that

$$\int_{\mathbf{D}} |f(z) - f(0)|^p\, d\mu(z) \sim \int_{\mathbf{D}} (1-|z|^2)^p |f'(z)|^p\, d\mu(z)$$

is false. It is also easy to produce counterexamples in the form of absolutely continuous measures. In fact, recall that the multiplier algebra of the Bergman space A^p is H^∞, and H^∞ is strictly contained in the Bloch space. Thus there exist a function f in the Bloch space and a function $g \in A^p$ such that $fg \notin A^p$. Let $\omega(z) = |g(z)|^p$. Then we have

$$\int_{\mathbf{D}} |f(z)|^p \omega(z)\, dA(z) = \infty,$$

while

$$\int_{\mathbf{D}} (1-|z|^2)^p |f'(z)|^p \omega(z)\, dA(z) < \infty.$$

Thus the Hardy–Littlewood theorem we are seeking is not valid for $\omega(z) = |g(z)|^p$.

The weights we considered in the previous section have all appeared in the literature before. The interested reader can also look at weight functions of the following form,

$$\omega(z) = (1-|z|^2)^a \int_{\mathbf{D}} \frac{(1-|w|^2)^b\, d\mu(w)}{|1-z\overline{w}|^c},$$

where a, b, and c are real parameters. Similarly, we can also consider weight functions of the form

$$\omega(z) = \int_{\mathbf{T}} \frac{(1-|z|^2)^a\, d\nu(t)}{|1-ze^{-it}|^b},$$

where a and b are real parameters.

Finally, we mention that all results obtained in the paper can easily be generalized to the open unit ball. The preliminary estimates necessary for the high dimensional results can be found in [13, 7]. We leave the routine details to the interested reader.

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