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ESTIMATES FOR THE MAXIMAL CAUCHY INTEGRAL ON CHORD-ARC CURVES

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Abstract. We study the chord-arc Jordan curves that satisfy the Cotlar-type inequality $T_*(f) \lesssim M^2(Tf)$, where T is the Cauchy transform, T_* is the maximal Cauchy transform and M is the Hardy–Littlewood maximal function. Under the background assumption of asymptotic conformality we find a characterization of such curves in terms of the smoothness of a parametrization of the curve.

1. Introduction

Consider a homogeneous smooth Calderón–Zygmund operator in $\mathbf{R}^n$

$$Tf(x) = \text{p. v.} \int f(x-y)\, K(y)\, dy \equiv \lim_{\epsilon\to 0} T_\epsilon f(x), \quad x \in \mathbf{R}^n,$$

where T_ϵ is the truncation at level ϵ defined by

$$T_\epsilon f(x) = \int_{|y|>\epsilon} f(x-y) K(y)\, dy, \quad x \in \mathbf{R}^n,$$

and f is in $L^p(\mathbf{R}^n)$, $1 \le p < \infty$. Here the kernel K is of class C^∞ off the origin, homogeneous of order $-n$ and with zero integral on the unit sphere

$$\{x \in \mathbf{R}^n \colon |x| = 1\}.$$

Let T_* be the maximal singular integral

$$T_* f(x) = \sup_{\epsilon>0} |T_\epsilon f(x)|, \quad x \in \mathbf{R}^n.$$

A classical fact relating T_* and the standard Hardy–Littlewood maximal operator M is Cotlar's inequality, which reads

$$T_*(f)(x) \le C\left(M(Tf)(x) + M(f)(x)\right), \quad x \in \mathbf{R}^n. \tag{1.1}$$

Combining this with the L^p estimates $\|T(f)\|_p \le C\,\|f\|_p$ and $\|M(f)\|_p \le C\,\|f\|_p$, $1 < p < \infty$ one gets $\|T_*(f)\|_p \le C\,\|f\|_p$, $1 < p < \infty$.

It was discovered in [4] that if T is an even higher order Riesz transform, that is, if $K(x) = P(x)/|x|^{n+d}$, with P an even homogeneous polynomial of degree d, then one can get rid of the second term in the right hand side of (1.1), namely,

$$T_*(f)(x) \le C\, M(Tf)(x), \quad x \in \mathbf{R}^n. \tag{1.2}$$

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Hence $\|T_*(f)\|_p \le C\,\|T(f)\|_p$, $1 < p < \infty$, in this case. However, if T is an odd higher order Riesz transform, then (1.2) may fail and the right substitute turns out to be (see [3])

$$T_*(f)(x) \le C\,M^2(Tf)(x), \quad x \in \mathbf{R}^n, \tag{1.3}$$

where M^2 stands for the iteration of M.

Inequalities of the type (1.2) and (1.3) were first considered in relation to the David–Semmes problem (see [4],[3] and [9]) and later on were studied in the context of the Cauchy singular integral on Lipschitz graphs and C^1 curves by Girela-Sarríón in [2]. Let Γ be either a Lipschitz graph or a closed chord-arc curve in the plane, let T be the Cauchy Singular Integral and M the Hardy–Littlewood maximal operator, both with respect to the arc-length measure, and let T_* be the maximal Cauchy Integral. Precise definitions will be given below. Girela-Sarrión showed in [2] that the presence at a point z of the curve of a non-zero angle prevents (1.3), with x replaced by z, to hold. This agrees with the intuition that (1.3) should help in finding tangent lines, but suggests that it is a condition definetely stronger than the mere existence of tangents. It was also shown in [2] that if Γ is a closed C^1 curve with the property that the modulus of continuity $\omega(z,\delta)$ of the unit tangent vector satisfies

$$\omega(z,\delta) \le C\,\frac{1}{\log(\frac{1}{\delta})}, \quad z \in \Gamma, \quad \delta < 1/2, \tag{1.4}$$

then (1.3) holds with $x \in \mathbf{R}^n$ replaced by $z \in \Gamma$. Observe that condition (1.4) quantifies the absence of corners in a curve for which (1.3) holds. In this paper we study the validity of inequality (1.3) in the context of chord-arc curves. A chord-arc curve is a rectifiable Jordan curve Γ in the plane with the property that there exists a positive constant C such that, given any two points $z_1, z_2 \in \Gamma$ one has

$$l(z_1,z_2) \le C\,|z_1 - z_2|,$$

where $l(z_1,z_2)$ is the length of the shortest arc in Γ joining z_1 and z_2. Equivalently Γ is a bilipschitz image of the unit circle (see [5], Theorem 7.9). Then Γ can be parametrized by a periodic function $\gamma\colon \mathbf{R} \to \Gamma$ of period T satisfying the bilipschitz condition

$$\frac{1}{L}\,|x-y| \le |\gamma(x) - \gamma(y)| \le L\,|x-y|, \quad x,y \in \mathbf{R}, \quad |x-y| \le \frac{T}{2}, \tag{1.5}$$

for some positive constant L. We say, by slightly abusing language, that γ is a bilipschitz parametrization of Γ. One can take, for instance, the T-periodic extension of the arc-length parametrization of Γ with T being the length of Γ.

One can easily define the maximal Hardy–Littlewood operator and the Cauchy Integral on a chord-arc curve. Given $z \in \Gamma$ let $t \in \mathbf{R}$ be such that $z = \gamma(t)$. Set

$$\Gamma_{z,r} := \gamma(\{\tau \colon |\tau - t| < r\}).$$

One should look at $\Gamma_{z,r}$ as "balls" of radius r centered at z. Indeed, owing to the bilipschitz condition (1.5), each $\Gamma_{z,r}$ contains and is contained in a disc in Γ of radius comparable to r, for $r < T$. It will be more convenient to work with $\Gamma_{z,r}$ than with the euclidean discs $D(z,r) \cap \Gamma$, where $D(z,r)$ stands for the planar disc of center z and radius r.

Denote by μ the arc-length measure on Γ. For $f \in L^1(\Gamma,\mu)$ and $z \in \Gamma$, we define the Hardy–Littlewood maximal function on the curve Γ as

$$Mf(z) := \sup_{r>0} \frac{1}{\mu(\Gamma_{z,r})} \int_{\Gamma_{z,r}} |f| \, d\mu.$$

The Cauchy Integral is defined as

$$Tf(z) = \text{p. v.} \frac{1}{\pi i} \int_\Gamma \frac{1}{w-z} f(w)\, dw \equiv \lim_{\epsilon\to 0} T_\epsilon f(z), \quad z \in \Gamma,$$

where

$$T_\epsilon f(z) = \frac{1}{\pi i} \int_{\Gamma\setminus\Gamma_{z,\epsilon}} \frac{f(w)}{w-z}\, dw$$

is the truncated Cauchy Integral at level ϵ. The maximal Cauchy Integral is

$$T_* f(z) := \sup_{\epsilon>0} \big|T_\epsilon f(z)\big|.$$

Our aim is to investigate under what conditions on Γ one has the inequality

$$T_* f(z) \le C\, M^2(Tf)(z), \quad z \in \Gamma, \quad f \in L^2(\Gamma,\mu),$$

where C is a positive constant. Since we know that angles prevent the above inequality to hold, we need to require on Γ a condition that excludes them. One such a condition is asymptotic conformality. Given two points $z_1, z_2 \in \Gamma$ let $A(z_1,z_2)$ be the arc in Γ joining the two points and having smallest diameter (there is only one if the two points are sufficiently close). The Jordan curve Γ is said to be asymptotically conformal if, given a positive number δ there exists a positive ϵ, so that for any two points $z_1, z_2 \in \Gamma$ satisfying $|z_1 - z_2| < \epsilon$ one has

$$|z_1 - z| + |z_2 - z| \le (1+\delta)|z_1 - z_2|, \quad z \in A(z_1,z_2).$$

Our main result reads as follows.

Theorem. *Let T be the Cauchy Integral on an asymptotically conformal chord-arc curve Γ and let γ be a bilipschitz parametrization of Γ. Then the estimate*

$$T_*(f)(z) \le C\, M^2(Tf)(z), \quad z \in \Gamma, \quad f \in L^2(\Gamma,\mu), \tag{1.6}$$

holds if and only if there exists $C > 0$ such that

$$\big|\gamma(x+\epsilon) + \gamma(x-\epsilon) - 2\gamma(x)\big| \le C \frac{\epsilon}{|\log \epsilon|}, \tag{1.7}$$

for each ϵ satisfying $0 < \epsilon < T$ and for each $x \in \mathbf{R}$.

One should recall that condition (1.7) implies that γ is differentiable almost everywhere in the ordinary sense and the derivative is a function of vanishing mean oscillation (see [11]). Therefore, for chord arc curves satisfying the background assumption of asymptotical conformality, inequality (1.6) is equivalent to the precise form of differentiability described in terms of second order differences in (1.7). Also notice that if γ is the arc-length parametrization of a C^1 curve, (1.4) implies (1.7), so that the Theorem generalizes Girela-Sarrión's result.

In Section 2 we prove a couple of Lemmas which allow to express condition (1.6) in an equivalent form in terms of a function related to the geometry of Γ. Section 3 is devoted to take care of a technical question, namely, that it is enough to estimate truncations at small enough levels. In Section 4 we prove the Theorem by means of three lemmas, one on them making the connection between the function carrying the geometrical information and the second difference condition (1.7). In Section 5 we present an example of a spiraling domain that enjoys the equivalent conditions in the Theorem but whose boundary is not of class C^1. This example justifies the efforts made in order to extend the condition (1.4) to a less regular case since new geometric behaviors can be detected.

Our terminology and notation are standard. We let C denote a constant independent of the relevant variables under consideration and which may vary at each occurrence. The notation $A \lesssim B$ means that there exists a constant $C > 0$ such that $A \leq CB$. We write $A \gtrsim B$ if $B \lesssim A$. The disc centered at z of radius r is denoted by $D(z, r)$.

2. Two preliminary lemmas

The beginning of the proof follows the ideas of [2], so that we will be rather concise. Given a function $f \in L^1(\Gamma, \mu)$ we denote by $m_{\Gamma_{z,\epsilon}}(f) = \fint_{\Gamma_{z,\epsilon}} f(w)\, d\mu(w)$ the mean of f on $\Gamma_{z,\epsilon}$ with respect to the arc length measure μ. We let $K_{z,\epsilon}$ denote the Cauchy kernel truncated at the point z at level ϵ, that is,

$$K_{z,\epsilon}(w) = \frac{1}{\pi\, i} \frac{1}{w - z}\, \chi_{\Gamma \setminus \Gamma_{z,\epsilon}}(w), \quad w \in \Gamma.$$

Set $g_{z,\epsilon} = T(K_{z,\epsilon})$ and let $N > 1$ be a big number to be chosen later. Following [2, p. 673] we obtain the identity

$$-T_\epsilon f(z) = I_\epsilon + II_\epsilon + III_\epsilon,$$

where

$$I_\epsilon := \int_{\Gamma_{z,N\epsilon}} Tf(w) \left(g_{z,\epsilon}(w) - m_{\Gamma_{z,N\epsilon}}(g_{z,\epsilon})\right) dw,$$

$$II_\epsilon := m_{\Gamma_{z,N\epsilon}}(g_{z,\epsilon}) \int_{\Gamma_{z,N\epsilon}} Tf(w)\, dw$$

$$III_\epsilon := \int_{\Gamma \setminus \Gamma_{z,N\epsilon}} Tf(w) g_{z,\epsilon}(w)\, dw. \tag{2.1}$$

Following closely the argument in [2] one can prove that

$$|I_\epsilon| \leq C\, M^2(Tf)(z),$$
$$|II_\epsilon| \leq C\, M(Tf)(z),$$

where the constant C does not depend on the choice of N. Since clearly $M(g) \leq M^2(g)$ for any g, we are left with the task of estimating III_ϵ. The next lemma provides an expression for III_ϵ in terms of a function encoding the smoothness of Γ. To state the lemma first we need to clarify the definition of a branch of the logarithm of $w - z$, as a function of w with $z \in \Gamma$ fixed, in an appropriate region.

Given $z \in \Gamma$ let Δ_z be a curve connecting z and ∞ in the unbounded component of $\mathbf{C} \setminus \Gamma$. Such curves exist and indeed we will construct a special one in Section 4 (under the additional assumption of asymptotic conformality). Hence $\mathbf{C} \setminus \Delta_z$ is a simply connected domain containing $\Gamma \setminus \{z\}$ and so there exists in $\mathbf{C} \setminus \Delta_z$ a branch of $\log(w - z)$. In particular, if $z = \gamma(x)$ for some $x \in \mathbf{R}$, the expressions $\log(\gamma(x + \epsilon) - \gamma(x))$ and $\log(\gamma(x - \epsilon) - \gamma(x))$ make sense for $0 < \epsilon < T$.

Lemma 1. *Let Γ be a chord-arc curve and γ a bilipschitz parametrization of Γ. Let $z \in \Gamma$ and let x be a real number such that $\gamma(x) = z$. Then for almost every*

$w \in \Gamma \backslash \Gamma_{z,N\epsilon}$ we have

$$T(K_{z,\epsilon})(w) = \frac{1}{\pi^2(z-w)}\big[F(x,\epsilon) + G_{z,\epsilon}(w)\big],$$

where

$$F(x,\epsilon) = \log(\gamma(x+\epsilon) - \gamma(x)) - \log(\gamma(x-\epsilon) - \gamma(x)) + \pi i$$

and

$$|G_{z,\epsilon}(w)| \le \frac{C\,\epsilon}{|z-w|}. \tag{2.2}$$

Proof. Take $w \in \Gamma\backslash\Gamma_{z,N\epsilon}$. Then

$$\begin{aligned} T(K_{z,\epsilon})(w) &= -\frac{1}{\pi^2}\lim_{\delta\to 0}\int_{\Gamma\backslash(\Gamma_{w,\delta}\cup\Gamma_{z,\epsilon})} \frac{1}{(\zeta - z)(\zeta - w)}\,d\zeta \\ &= -\frac{1}{\pi^2}\frac{1}{w-z}\lim_{\delta\to 0}\int_{\Gamma\backslash(\Gamma_{w,\delta}\cup\Gamma_{z,\epsilon})}\left(\frac{1}{\zeta - w} - \frac{1}{\zeta - z}\right)d\zeta. \end{aligned}$$

Let $y \in \mathbf{R}$ with $\gamma(y) = w$. Then the latest integral in the above formula is

$$\begin{aligned} &\log(\gamma(y-\delta) - \gamma(y)) - \log(\gamma(x+\epsilon) - \gamma(y)) + \log(\gamma(x-\epsilon) - \gamma(y)) \\ &- \log(\gamma(y+\delta) - \gamma(y)) - \Big(\log(\gamma(y-\delta) - \gamma(x)) \\ &- \log(\gamma(x+\epsilon) - \gamma(x)) + \log(\gamma(x-\epsilon) - \gamma(x)) - \log(\gamma(y+\delta) - \gamma(x))\Big). \end{aligned}$$

Assume that γ is differentiable at the point y and the derivative $\gamma'(y)$ does not vanish. Then we have that

$$\lim_{\delta\to 0}\Big(\log(\gamma(y-\delta) - \gamma(y)) - \log(\gamma(y+\delta) - \gamma(y))\Big) = \pi i,$$

because the curve Δ_w lies in the unbounded component of $\mathbf{C}\setminus\Gamma$, and then to the right hand side of Γ, oriented according to the parametrization γ. Taking limit as δ goes to 0 we obtain

$$\begin{aligned} T(K_{z,\epsilon})(w) = -\frac{1}{\pi^2}\frac{1}{w-z}\Big(&\big(\log(\gamma(x+\epsilon) - \gamma(x)) - \log(\gamma(x-\epsilon) - \gamma(x)) + \pi i\big) \\ &- \big(\log(\gamma(x+\epsilon) - \gamma(y)) - \log(\gamma(x-\epsilon) - \gamma(y))\big)\Big). \end{aligned}$$

Define

$$G_{z,\epsilon}(w) = \log(\gamma(x-\epsilon) - \gamma(y)) - \log(\gamma(x+\epsilon) - \gamma(y)).$$

It remains to show the decay inequality (2.2). According to the choice of Δ_w we have a well defined branch of $\log(\gamma(x+t) - w)$, $-\epsilon < t < \epsilon$. Thus

$$G_{z,\epsilon}(w) = -\int_{-\epsilon}^{\epsilon}\frac{d}{dt}\log(\gamma(x+t) - w)\,dt = -\int_{-\epsilon}^{\epsilon}\frac{\gamma'(x+t)}{\gamma(x+t) - w}\,dt. \tag{2.3}$$

Since $w = \gamma(y) \in \Gamma\setminus\Gamma_{z,N\epsilon}$, we have $y \notin (x - N\epsilon, x + N\epsilon)$ and so

$$|w-z| = |\gamma(y) - \gamma(x)| \ge \frac{|y-x|}{L} \ge \frac{N\epsilon}{L},$$

which gives, taking $N \ge 2L^2$,

$$\begin{aligned} |w - \gamma(x+t)| &\ge |w-z| - |\gamma(x) - \gamma(x+t)| \\ &\ge \frac{|w-z|}{2} + \frac{N\epsilon}{2L} - L\epsilon \ge \frac{|w-z|}{2}. \end{aligned}$$

Hence, by (2.3),

$$|G_{z,\epsilon}(w)| \le \int_{-\epsilon}^{\epsilon} \frac{|\gamma'(x+t)|}{|\gamma(x+t)-w|}\, dt \le \frac{4L\epsilon}{|w-z|}. \qquad \square$$

Lemma 2. *Let Γ be a chord-arc curve and γ a bilipschitz parametrization of Γ. Then the inequality*

$$T_*(f)(z) \le C\, M^2(Tf)(z), \quad z \in \Gamma, \quad f \in L^2(\Gamma,\mu), \tag{2.4}$$

is equivalent to

$$|F(x,\epsilon)||\log(\epsilon)| \le C, \quad 0<\epsilon<T, \quad x \in \mathbf{R}. \tag{2.5}$$

Proof. Assume that (2.5) holds. Then by Lemma 1

$$\begin{aligned} III_\epsilon &= \int_{\Gamma\setminus\Gamma_{z,N\epsilon}} Tf(w)\, T(K_{z,\epsilon})(w)\, dw \\ &= \frac{F(x,\epsilon)}{\pi^2} \int_{\Gamma\setminus\Gamma_{z,N\epsilon}} \frac{Tf(w)}{z-w}\, dw + \frac{1}{\pi^2}\int_{\Gamma\setminus\Gamma_{z,N\epsilon}} Tf(w)\, \frac{G_{z,\epsilon}(w)}{z-w}\, dw \\ &= F(x,\epsilon)\, IV_\epsilon + V_\epsilon, \end{aligned}$$

where the last identity is a definition of the terms IV_ϵ and V_ϵ. One can break the domain of integration in the integrals in IV_ϵ and V_ϵ into a union of dyadic annuli

$$A_j = \gamma\big\{y \in \mathbf{R} \colon N\epsilon\, 2^j < |y-x| \le N\epsilon\, 2^{j+1}\big\}, \quad j=0,1,\dots,$$

then perform standard estimates and apply (2.2) to get, thanks to the quadratic decay of the integrand,

$$|V_\epsilon| \le C\, M\big(T(f)\big)(z). \tag{2.6}$$

For IV_ϵ one only has a first order decay, which gives

$$|IV_\epsilon| \le C \left|\log\left(\frac{NL}{\epsilon}\right)\right| M(Tf)(z),$$

thus completing the proof of the sufficient condition.

Assume now (2.4). Recalling that $III_\epsilon = F(x,\epsilon)\, IV_\epsilon + V_\epsilon$ and (2.6), we obtain

$$\big|F(x,\epsilon)\, IV_\epsilon\big| \le C\, M^2\big(T(f)\big)(z), \quad z\in\Gamma, \quad f \in L^2(\Gamma,\mu). \tag{2.7}$$

The Cauchy Singular Integral operator T is an isomorphism of $L^2(\Gamma,\mu)$ onto itself. This is proved in Lemma 1 of [2, p. 661] for Lipschitz graphs, and the same proof works in our context. Thus (2.7) can be rewritten as

$$\left|F(x,\epsilon) \int_{\Gamma\setminus\Gamma_{z,N\epsilon}} \frac{g(w)}{z-w}\, dw\right| \le C\, M^2(g)(z), \quad z\in\Gamma, \quad g \in L^2(\Gamma,\mu). \tag{2.8}$$

To simplify the notation take $x = 0 = \gamma(x)$. Assume first that $0<\epsilon<1$. Apply (2.8) with g the characteristic function of $\gamma((\epsilon^n,\epsilon))$, where n is a large integer to be chosen. Then

$$|F(0,\epsilon)| \left| \int_{\epsilon^n}^{\epsilon} \frac{\gamma'(t)}{\gamma(t)}\, dt \right| \le C$$

and

$$\begin{aligned}\left|\int_{\epsilon^n}^{\epsilon} \frac{\gamma'(t)}{\gamma(t)}\,dt\right| &= |\log(\gamma(\epsilon)) - \log(\gamma(\epsilon^n))|\\ &\geq |\log(|\gamma(\epsilon)|) - \log(|\gamma(\epsilon^n)|)| \geq \log\left(\frac{1}{L^2\ \epsilon^{n-1}}\right)\\ &\geq -2\log(L) + (n-2)\log\left(\frac{1}{\epsilon}\right) + \log\left(\frac{1}{\epsilon}\right) \geq |\log(\epsilon)|\end{aligned}$$

provided $n = n(\epsilon)$ is large enough so that $-2\log(L) + (n-2)\log(1/\epsilon) \geq 0$. Therefore (2.5) follows in this case.

If $1 \leq \epsilon < T$ then we take as g the characteristic function of $\gamma((\epsilon^{-n}, \epsilon))$. In this case we get

$$\left|\int_{\epsilon^{-n}}^{\epsilon} \frac{\gamma'(t)}{\gamma(t)}\,dt\right| \geq -2\log(L) + n\log(\epsilon) + \log(\epsilon) \geq |\log(\epsilon)|$$

provided n is chosen so that $-2\log(L) + n\log(\epsilon) \geq 0$. □

3. Reduction to estimating truncations at small levels

In this section we reduce the proof of (1.6) to estimating the truncations $T_\epsilon f$ for small ϵ. In the previous section we showed that the estimate of $T_\epsilon f$ can be reduced to that of the term III_ϵ in (2.1).

Lemma 3. *If ϵ_0 is a given positive number, then there exists a large positive number $N = N(L)$ so that*

$$\left|\int_{\Gamma\setminus\Gamma_{z,N\epsilon}} Tf(w)\, g_{z,\epsilon}(w)\, dw\right| \leq C\, M(Tf)(z), \quad z \in \Gamma, \quad \epsilon_0 < \epsilon,$$

for a positive constant $C = C(\epsilon_0, L)$.

The small number ϵ_0 will be chosen in the next section.

Proof. Recall that

$$\begin{aligned} g_{z,\epsilon}(w) = T(K_{z,\epsilon})(w) &= -\frac{1}{\pi^2}\,\text{p. v.} \int_{\Gamma\setminus\Gamma_{z,\epsilon}} \frac{1}{(\zeta - w)(\zeta - z)}\, d\zeta\\ &= -\frac{1}{\pi^2}\frac{1}{w-z}\ \text{p. v.} \int_{\Gamma\setminus\Gamma_{z,\epsilon}} \left(\frac{1}{\zeta - w} - \frac{1}{\zeta - z}\right) d\zeta\\ &= -\frac{1}{\pi^2}\frac{1}{w-z}\ \text{p. v.} \int_{\Gamma\setminus\Gamma_{z,\epsilon}} \frac{1}{\zeta - w}\, d\zeta + \frac{1}{\pi^2}\frac{1}{w-z}\ \text{p. v.} \int_{\Gamma\setminus\Gamma_{z,\epsilon}} \frac{1}{\zeta - z}\, d\zeta\\ &= h(w) + k(w),\end{aligned}$$

where in the last identity we defined $h(w)$ and $k(w)$.

Applying the bilipschitz character of γ we conclude that

$$|k(w)| \leq \frac{1}{\pi^2}\frac{L^2}{N\,\epsilon_0^2}\,\text{length}(\Gamma), \quad w \in \Gamma\setminus\Gamma_{z,N\epsilon}, \quad \epsilon_0 < \epsilon\,. \tag{3.1}$$

The estimate of $h(w)$ is a little trickier. We have

$$h(w) = -\frac{1}{\pi^2}\frac{1}{w-z}\ \text{p. v.} \int_{\Gamma} \frac{1}{\zeta - w}\, d\zeta + \frac{1}{\pi^2}\frac{1}{w-z}\ \text{p. v.} \int_{\Gamma_{z,\epsilon}} \frac{1}{\zeta - w}\, d\zeta$$

A simple application of Cauchy's Theorem gives that, if Γ has a tangent at w,

$$\text{p. v.} \int_\Gamma \frac{1}{\zeta - w}\, d\zeta = \pi i.$$

As before, the bilipschitz character of γ yields

$$|w - z| \geq \frac{N\,\epsilon}{L}, \quad w \in \Gamma \setminus \Gamma_{z,N\epsilon}$$

and

$$|w - \zeta| \geq |w - z| - |z - \zeta| \geq \epsilon\Big(\frac{N}{L} - L\Big), \quad w \in \Gamma \setminus \Gamma_{z,N\epsilon}, \quad \zeta \in \Gamma_{z,\epsilon}$$

Choose N so that $N/L - L \geq 1$. Then

$$|w - \zeta| \geq \epsilon, \quad w \in \Gamma \setminus \Gamma_{z,N\epsilon}, \quad \zeta \in \Gamma_{z,\epsilon}.$$

Gathering all the previous estimates we finally get

$$|h(w)| \leq \frac{1}{\pi}\frac{L}{N\,\epsilon_0} + \frac{1}{\pi^2}\frac{\text{length}(\Gamma)}{\epsilon_0}, \quad w \in \Gamma \setminus \Gamma_{z,N\epsilon}, \quad \epsilon_0 < \epsilon\,. \tag{3.2}$$

Hence (3.1) and (3.2) yield

$$|g_{z,\epsilon}(w)| \leq C, \quad w \in \Gamma \setminus \Gamma_{z,N\epsilon}, \quad \epsilon_0 < \epsilon,$$

where $C = C(\epsilon_0, N, L, \text{length}(\Gamma))$ is a constant depending on ϵ_0, N, L and $\text{length}(\Gamma)$.

Therefore

$$\left|\int_{\Gamma\setminus\Gamma_{z,N\epsilon}} Tf(w)\, g_{z,\epsilon}(w)\, dw\right| \leq C \int_\Gamma |Tf(w)|\, d\mu(w) \leq C\, \text{length}(\Gamma) M(Tf)(z),$$

which completes the proof of the lemma. □

4. The proof of the Theorem

For $z \neq 0$ let $\text{Arg}(z)$ denote the principal argument of z, so that $0 \leq \text{Arg}(z) < 2\pi$.

Lemma 4. *Given $\alpha > 0$ there exists a positive number $\epsilon_0 = \epsilon_0(L)$ with the following property. Assume that $0 < \epsilon_1 \leq \epsilon_0$, $\epsilon_1/2 < \epsilon \leq \epsilon_1$ and that for a fixed $x \in \mathbf{R}$ we have $\gamma(x) = 0$. If $\gamma(x - \tau)$, $\tau > 0$, satisfies*

$$\frac{\epsilon_1}{2L} < |\gamma(x - \tau)| < L\,\epsilon_1,$$

then, for some θ such that $\gamma(x - \tau) = |\gamma(x - \tau)|e^{i\theta}$, we have

$$\big|\theta - \big(\text{Arg}(\gamma(x + \epsilon)) + \pi\big)\big| < \alpha.$$

Proof. Consider the triangle with vertices $0, \gamma(x-\tau)$ and $\gamma(x+\epsilon)$ and side lengths $A = |\gamma(x-\tau)|$, $B = |\gamma(x+\epsilon)|$ and $C = |\gamma(x+\epsilon) - \gamma(x-\tau)|$. By the cosine Theorem

$$C^2 = A^2 + B^2 - 2AB\cos(\phi),$$

where ϕ is the angle opposite to the side C. In other terms

$$1 + \cos(\phi) = \frac{(A + B - C)(A + B + C)}{2AB}.$$

By asymptotic conformality, given $\delta > 0$ there exists $\eta_0 > 0$ such that $C = |\gamma(x+\epsilon) - \gamma(x-\tau)| < \eta_0$ implies $A + B \le (1+\delta)C$. The bilipschitz property of γ (1.5) yields $\epsilon_1 /2L^2 \le \tau \le L^2 \epsilon_1$. Hence

$$1 + \cos(\phi) \le \delta L^4 \frac{(\epsilon_1 + \tau)^2}{\epsilon_1 \tau} \le 2\delta L^6 (1+L^2)^2.$$

Taking $\theta = \operatorname{Arg}(\gamma(x+\epsilon)) + \phi$ we see that $|\theta - (\operatorname{Arg}(\gamma(x+\epsilon)) + \pi)| < \alpha$ provided δ is small enough. Since

$$|\gamma(x+\epsilon) - \gamma(x-\tau)| \le L(\epsilon + \tau) \le \epsilon_0 L(1+L^2),$$

one has to choose ϵ_0 so that $\epsilon_0 L(1+L^2) \le \eta_0$, which shows the correct dependence of ϵ_0 and completes the proof of the Lemma. □

Given a point $z \in \Gamma$ we want now to construct a special Jordan arc Δ_z connecting z to ∞ in the complement of Γ. Assume, without loss of generality, that $z = 0$. Take $x \in \mathbf{R}$ with $\gamma(x) = 0$. Let ϵ_0 be the number given in the preceding lemma and define, for $j = 0, 1, 2, \ldots$, a polar rectangle by

$$R_j = \Big\{ w = |w| e^{i\theta} \colon \frac{\epsilon_0}{2^{j+1} L} < |w| < \frac{\epsilon_0 L}{2^j} \quad \text{and} \quad \Big| \theta - \operatorname{Arg}\Big(\gamma\Big(x + \frac{\epsilon_0}{2^j}\Big)\Big) + \pi \Big| < \alpha \Big\}.$$

Applying Lemma 4 with $\epsilon = \epsilon_1 = \epsilon_0 / 2^j$ we conclude that

$$\{\gamma(x-\tau) \colon 0 < \tau\} \cap \Big\{ w \colon \frac{\epsilon_0}{2^{j+1} L} < |w| < \frac{\epsilon_0 L}{2^j} \Big\} \subset R_j.$$

We need to introduce another polar rectangle

$$S_j = R_j \cap \Big\{ w \colon \frac{\epsilon_0 L}{2^{j+1}} < |w| \Big\}, \quad j = 0, 1, 2, \ldots.$$

We define inductively $\Delta_z = \Delta_0$ on S_j by just requiring that the Jordan arc $\Delta_0 \cap \overline{S_j}$ lies in the unbounded component of the complement of Γ, $\overline{S_j}$ being the closure of S_j. We then connect $\Delta_0 \cap \overline{S_0}$ with ∞ by a Jordan arc in the complement of Γ, with the only precaution of not reentering the disc $D(0, \epsilon_0)$ once Δ_0 has left it.

It is worth pointing out that the axis of two consecutive polar rectangles R_j and R_{j+1} make an angle less than α. This follows by the defining property of ϵ_0 (see the proof of Lemma 4).

Lemma 5.

$$\log(\gamma(x-\epsilon)) - \pi i = \log(-\gamma(x-\epsilon)), \quad x \in \mathbf{R}, \quad 0 < \epsilon \le \epsilon_0 .$$

Proof. We know that

$$\log(\gamma(x-\epsilon)) - \pi i = \log(-\gamma(x-\epsilon)) + 2\pi m i \tag{4.1}$$

for some integer m. Our goal is to compute the difference

$$\log(\gamma(x-\epsilon)) - \log(-\gamma(x-\epsilon))$$

by the integral

$$\int_{\varsigma} \frac{1}{z}\, dz,$$

where ς is an appropriately chosen Jordan arc connecting $-\gamma(x-\epsilon)$ to $\gamma(x-\epsilon)$ in the complement of Δ_0.

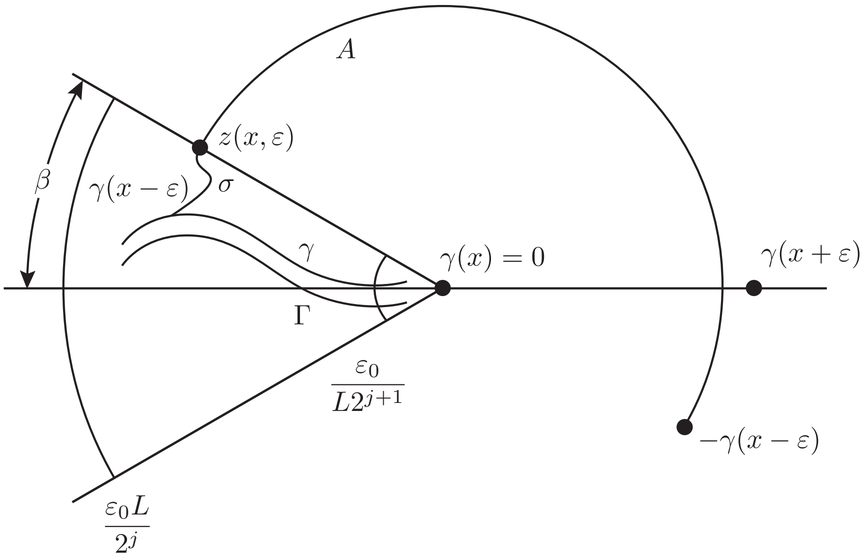

Figure 1. The curve ς.

Assume that $\epsilon_0/2^{j+1} < \epsilon \le \epsilon_0/2^j$, for some non-negative integer j. Define N as the smallest integer satisfying

$$\frac{L\,\epsilon_0}{2^{j+N}} \le \frac{\epsilon_0}{L2^{j+1}}.$$

This is equivalent to $L^2 \le 2^{N-1}$ and so N depends only on L. Hence $R_k \subset D(0, \epsilon_0/L\,2^{j+1})$, $k \ge j+N$, and, in particular, R_k, $k \ge j+N$, does not intersect the circumference $\partial D(0, |\gamma(x-\epsilon)|)$.

The angle between the axis of the polar rectangle R_{j+l} and that of R_j is not greater than $l\alpha \le N\alpha$, $l = 1, 2, \dots, N-1$. Set $\beta = N\alpha$, so that β can be as small as desired by taking $\alpha = \alpha(L)$ appropriately. We conclude that

$$R_{j+l} \subset \{w\colon w = |w|e^{i\theta} \text{ with } |\theta - \operatorname{Arg}(\gamma(x+\epsilon)+\pi)| < \beta\}, \quad l = 1, 2, \dots, N-1.$$

We are now ready to define the Jordan arc ς. Let $z(x,\epsilon)$ be the point at the intersection of the circumference $\partial D(0, |\gamma(x-\epsilon)|)$ and the ray

$$\{w\colon w = |w|e^{i\theta} \text{ with } \theta = \operatorname{Arg}(\gamma(x+\epsilon)+\pi) - \beta\}.$$

Let A stand for the arc in $\partial D(0, |\gamma(x-\epsilon)|)$ having $-\gamma(x-\epsilon)$ as initial point and $z(x,\epsilon)$ as end point (counterclockwise oriented).

There exists a rectifiable Jordan arc σ joining the points $z(x,\epsilon)$ and $\gamma(x-\epsilon)$ in the bounded component of the complement of Γ with the property that

$$\operatorname{length}(\sigma) \le C\,|z(x,\epsilon) - \gamma(x-\epsilon)|.$$

This can be seen readily as follows. Set $\tilde\gamma(e^{ix}) = \gamma(x)$, $x \in \mathbf{R}$. Then $\tilde\gamma$ is a bilipschitz homeomorphism between $\mathbf{T}$ and Γ and thus can be extended to a global bilipschitz homeomorphism of the plane onto itself (see [7, 8]). The existence of the arc σ is then easily proved by transferring the question via the extended bilipschitz homeomorphism.

Define $\varsigma = A \cup \sigma$, oriented as already specified. Note that ς lies in the complement of Δ_0, by the previous discussion, in particular, the definition of N and β. Therefore

$$\log(\gamma(x-\epsilon)) - \log(-\gamma(x-\epsilon)) = \int_\varsigma \frac{1}{z}\,dz.$$

On one hand we have

$$\int_A \frac{1}{z}\,dz = \pi i + O(\beta)$$

and on the other hand

$$\Big|\int_\sigma \frac{1}{z}\,dz\Big| \le \frac{C\,|z(x,\epsilon)-\gamma(x-\epsilon)|}{|\gamma(x-\epsilon)|} \le C\,\beta = O(\beta).$$

If β is small enough so that $O(\beta) < \pi$, then, by (4.1), we get that $m = 0$, and the lemma is proved. □

We need a final lemma, which concludes the proof of the Theorem.

Lemma 6. *Let Γ be an asymptotically conformal chord-arc curve and let γ be a bilipschitz parametrization of Γ (in the sense of (1.5)). Then there exists a constant $C > 1$ and a positive number ϵ_0 such that*

$$\begin{aligned} C^{-1}\,\frac{|\gamma(x+\epsilon)+\gamma(x-\epsilon)-2\gamma(x)|}{\epsilon} &\le |F(x,\epsilon)| \\ &\le C\,\frac{|\gamma(x+\epsilon)+\gamma(x-\epsilon)-2\gamma(x)|}{\epsilon}, \end{aligned} \tag{4.2}$$

for $x \in \mathbf{R}$ and $0 < \epsilon < \epsilon_0$.

Proof. Without loss of generality assume that $\gamma(x) = 0$. Let ϵ_0 be the small number provided by Lemma 4. By the construction of the arc Δ_0 described in the proof of Lemma 4 we have that the segment joining $-\gamma(x-\epsilon)$ and $\gamma(x+\epsilon)$ lies in the complement of Δ_0. We have, by Lemma 5,

$$\begin{aligned} F(x,\epsilon) &= \log\big(\gamma(x+\epsilon)\big) - \log\big(\gamma(x-\epsilon)\big) + \pi i \\ &= \log\big(\gamma(x+\epsilon)\big) - \log\big(-\gamma(x-\epsilon)\big) \end{aligned}$$

and so

$$\begin{aligned} F(x,\epsilon) &= \int_0^1 \frac{d}{dt}\log\big(-\gamma(x-\epsilon)+t(\gamma(x+\epsilon)+\gamma(x-\epsilon))\big)\,dt \\ &= \int_0^1 \frac{\gamma(x+\epsilon)+\gamma(x-\epsilon)}{-\gamma(x-\epsilon)+t(\gamma(x+\epsilon)+\gamma(x-\epsilon))}\,dt. \end{aligned}$$

Set, to simplify notation, $a = -\gamma(x-\epsilon)$, $b = \gamma(x+\epsilon)$ and let θ denote the angle between a and b. By Lemma 4 we know that θ is as small as we wish. In particular we can assume that $\cos(\theta) \ge 1/2$. Thus, using the cosine Theorem,

$$\begin{aligned} |a+t(b-a)|^2 &= (1-t)^2|a|^2 + t^2|b|^2 + 2(1-t)t|a||b|\cos(\theta) \\ &\ge \frac{1}{2}\,((1-t)|a|+t|b|)^2 \ge \frac{\epsilon^2}{2L^2}, \end{aligned}$$

and

$$|F(x,\epsilon)| \le \frac{\sqrt{2}L}{\epsilon}|\gamma(x+\epsilon)+\gamma(x-\epsilon)|,$$

which is the upper estimate in (4.2).

For the lower estimate we set $z_t = -\gamma(x-\epsilon)+t(\gamma(x+\epsilon)+\gamma(x-\epsilon))$. Since $\mathrm{Re}(z_t) \ge |z_t|/2$ and $|z_t| \le 2L\,\epsilon$

$$\Big|\int_0^1 \frac{1}{z_t}\,dt\Big| \ge \mathrm{Re}\int_0^1 \frac{1}{z_t}\,dt = \int_0^1 \frac{\mathrm{Re}(z_t)}{|z_t|^2}\,dt \ge \int_0^1 \frac{1}{2|z_t|}\,dt \ge \frac{1}{4L\,\epsilon}.$$

To complete the proof of the Theorem one only needs to combine Lemmas 2, 3 and 6. □

Remark. Let $a = \gamma(x) - \gamma(x-\epsilon)$, $b = \gamma(x+\epsilon) - \gamma(x)$ and let $\alpha(x,\epsilon)$ be the angle spanned by a and b. For a bilipschitz parametrization γ such that

$$c\,|x-y| \le |\gamma(x) - \gamma(y)| \le C\,|x-y|, \quad x, y \in \mathbf{R}, \quad |x-y| \le \frac{T}{2},$$

we have the estimate

$$|\gamma(x+\epsilon) + \gamma(x-\epsilon) - 2\gamma(x)|^2 \le 2C^2\epsilon^2 - 2c^2\epsilon^2 \cos\alpha(x,\epsilon).$$

So, in the general case, we can guarantee just a linear decay of the second finite difference $|\gamma(x+\epsilon) + \gamma(x-\epsilon) - 2\gamma(x)|$ and the logarithmic condition (1.7) gives informations about the local behavior of the best constants c and C around x and about the decay of $\alpha(x,\epsilon)$ for ϵ small. This remark will be useful in the next section.

5. An example

In this section we provide an example of curve γ which is not C^1 but for which the improved Cotlar's inequality (1.6) holds. The curve will be constructed in a recursive way and will be parametrized by arc-length. Without loss of generality, we will focus on defining a curve which is not closed. Indeed, possibly by connecting the ends of this curve in a smooth way, we can reduce to the same environment of the previous sections. The idea in the construction of the example is that the curve should resemble a suitable spiraling sequence of smoothened corners of decreasing aperture.

Let $0 < \alpha < \pi/2$. Let $F_\alpha \colon [0,1] \to \mathbf{R}$ be the function with support in $[1/4, 3/4]$ which is linear in $[1/4, 1/2]$ and $[1/2, 3/4]$ with slope $\tan\alpha$ in $[1/4, 1/2]$ and $-\tan\alpha$ in $[1/2, 3/4]$. In other words

$$F_\alpha(t) := \max\Big\{0, \Big(\frac{1}{4} - \Big|t - \frac{1}{2}\Big|\Big)\tan\alpha\Big\}.$$

Let $\xi > 0$. For $t \in \mathbf{R}$ we define the function

$$\eta_\xi(t) := \eta\Big(\frac{t}{\xi}\Big)\frac{1}{\xi},$$

where η is a smooth, even and positive function such that $\operatorname{supp}\eta \subset [-1,1]$ and $\int \eta(t)\,dt = 1$. For $0 < \xi < 1/100$ we define the regularized function

$$\lambda_\alpha := F_\alpha * \eta_\xi.$$

We will call the curve $\Lambda_\alpha := \big(t, \lambda_\alpha(t)\big)_{t\in[0,1]}$ *α-patch.*

An α-patch has the following properties:

- Λ_α is the graph of a function $\lambda_\alpha \colon [0,1] \to \mathbf{R}$ which is symmetric around $1/2$.
- if we denote by $[a,b]$ the segment joining the points $a, b \in \mathbf{R}^2$, then Λ_α contains the segments $I_\alpha := [(0,0), (1/4 - \xi, 0)]$, $II_\alpha := [(1/4+\xi, \xi\tan\alpha), (1/2 - \xi, (1/4-\xi)\tan\alpha)]$, $III_\alpha := [(1/2+\xi, (1/4-\xi)\tan\alpha), (3/4-\xi, \xi\tan\alpha)]$ and $IV_\alpha := [(3/4+\xi, 0), (1, 0)]$. We denote by C_α^i, $i = 1,2,3$ the remaining three non-affine parts of the graph. Precisely, C_α^1 joins the segments I_α and II_α, C_α^2 the segments II_α and III_α and C_α^3 the segments III_α and IV_α.
- the function λ_α is convex on the intervals below C_α^1 and C_α^3 and concave on the interval below C_α^2.

The idea is that the α-patch is a smoothened corner, as shown in Figure 2.

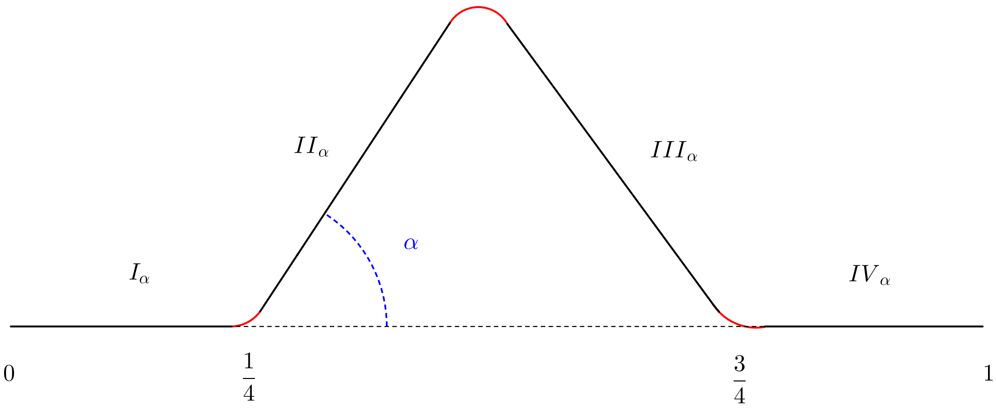

Figure 2. An α-patch.

Remark 1. Let us denote by $\tau(\alpha)$ the difference between the length of the (non-smoothened) graph of F_α and the length of Λ_α. For what follows, we need to estimate its behavior for small values of α. It suffices to observe that

$$\begin{aligned}\tau(\alpha) &:= \text{length}(F_\alpha) - \text{length}(\Lambda_\alpha)\\ &= \int_0^1 \Big(\sqrt{1+|f'_\alpha * \eta_\xi|^2(t)}\Big) - \Big(\sqrt{1+|f'_\alpha|^2(t)}\Big)dt \\ &= \int_0^1 \frac{|f'_\alpha * \eta_\xi|^2(t) - |f'_\alpha|^2(t)}{\Big(\sqrt{1+|f'_\alpha * \eta_\xi|^2(t)}\Big) + \Big(\sqrt{1+|f'_\alpha|^2(t)}\Big)}\,dt \le 2\|f'_\alpha\|_\infty = 2\tan\alpha.\end{aligned} \tag{5.1}$$

Definition of the curve Γ. Let $\alpha_j := 1/j$ for $j = 1, 2, \ldots$ positive integer. For the sake of notational convenience we replace the subscript α_j by j; for instance, we write Λ_j for Λ_{α_j}, I_j for $I_{\alpha_j}, \ldots, IV_j$ for IV_{α_j} and C_j^i for $C_{\alpha_j}^i$. Moreover, $\tau_j := \tau(\alpha_j)$. Now we can define Γ according to the following recursive steps:

- $\Gamma_1 := \Lambda_1$.
- We would like to glue on II_1 an appropriate rescaled, translated and rotated copy $\tilde\Lambda_2$ of Λ_2. The angle of rotation is α_1. The scaling factor and the translation are chosen so that the origin of $\tilde\Lambda_2$ is $(1/4, 0)$ and the end is $\big(1/2, (\tan\alpha)/4\big)$. Denote by $\widetilde{II}_2$ the image of II_2 via the same affinity which maps Λ_2 to $\tilde\Lambda_2$; let us use the tilde to denote the images of the other parts of the patch via the same map, too. Delete the segment II_1 from Λ_1 and add $\tilde\Lambda_2$. Now the endings of $\tilde\Lambda_2$ should be deleted in order to make a connection with Λ_1. The precise expression for the second step curve is
$$\Gamma_2 := \big((\Lambda_1 \setminus II_1) \cup \tilde\Lambda_2\big) \setminus \big((\tilde I_2 \cup \widetilde{IV}_2) \setminus II_1\big).$$
- given Γ_n, which is a "gluing" of affine copies $\tilde\Lambda_j$ of Λ_j for $j \in \{1, \ldots, n\}$, where $\widetilde{II}_n$ is the image of II_j under the same affinity which maps Λ_j to $\tilde\Lambda_j$, we define
$$\Gamma_{n+1} := ((\tilde\Lambda_n \setminus \widetilde{II}_n) \cup \tilde\Lambda_{n+1}) \setminus ((\tilde I_{n+1} \cup \widetilde{IV}_{n+1}) \setminus \widetilde{II}_n),$$
where $\tilde\Lambda_{n+1}$ is an re-scaled copy of Λ_{n+1} rotated by an angle $\sum_{j=1}^{n+1}\alpha_j$ whose vertices coincide with the images of $(1/4, 0)$ and $\big(1/2, \tan\alpha/4\big)$ via the transformation of the plain that sends Λ_n to $\tilde\Lambda_n$.

Then, $\{\Gamma_n\}_n$ converges in the Hausdorff distance (a similar case is presented, for example, in [1]) and we can simply define $\Gamma := \lim_n \Gamma_n$.

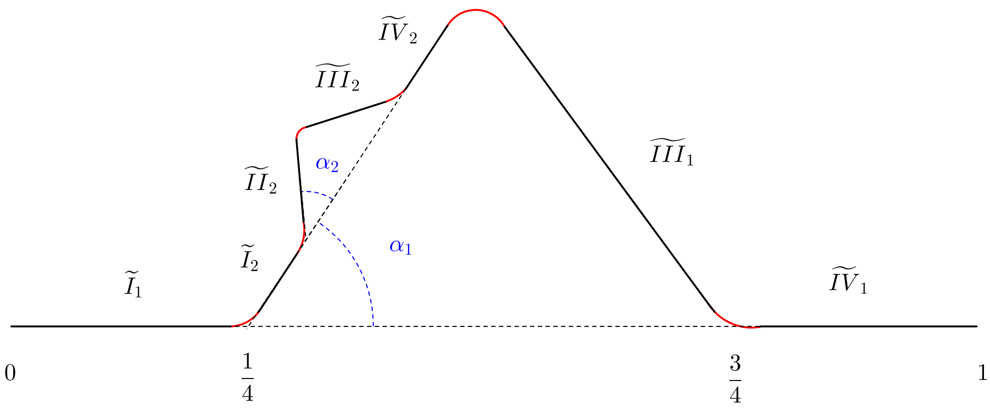

Figure 3. The second step in the construction of the curve Γ.

Let us now state an estimate that we will use in what follows.

Lemma 7. *Given $0 < \alpha < \pi/2$ and $z_1, z_2 \in \Lambda_\alpha$, we have*

$$l(z_1, z_2) \leq \frac{|z_1 - z_2|}{\cos\alpha}, \tag{5.2}$$

where $l(z_1, z_2)$ denotes the length of the arc of Λ_α joining z_1 and z_2.

Proof. Let $t_1 := \lambda_\alpha^{-1}(z_1)$ and $t_2 := \lambda_\alpha^{-1}(z_2)$. We have $|t_1 - t_2| \leq |z_1 - z_2|$. Moreover, because of the way we constructed Λ_α, we have that $|\lambda'_\alpha(t)| \leq \tan\alpha$ for every $t \in [0, 1]$. Collecting all these observations,

$$\begin{aligned} l(z_1, z_2) &= \int_{t_1}^{t_2} \sqrt{1 + |\lambda'_\alpha(t)|^2}\, dt \leq \int_{t_1}^{t_2} \sqrt{1 + |\tan\alpha|^2}\, dt \\ &= |t_2 - t_1|\sqrt{1 + |\tan\alpha|^2} = \frac{|t_2 - t_1|}{\cos\alpha} \leq \frac{|z_2 - z_1|}{\cos\alpha}. \end{aligned}$$ □

Remark 2. Notice that the inequality (5.2) keeps holding for a scaling of Λ_α, in particular for the $\tilde{\Lambda}_j$, $j \in \mathbf{N}$.

Let us define $L_1 = 1/2$ and, for $n > 1$,

$$L_n := 2^{-2n+1}\Big(\prod_{j=1}^{n-1} \cos\alpha_j\Big)^{-1},$$

which is half of the diameter of the rescaled patch $\tilde{\Lambda}_n$ in the construction of the curve Γ. Indeed, some trigonometry gives

$$L_1 = \frac{1}{2}, L_2 = \frac{1}{2}\Big(\frac{1}{2}L_1\frac{1}{\cos\alpha_1}\Big), L_3 = \frac{1}{2}\Big(\frac{1}{2}L_2\frac{1}{\cos\alpha_2}\Big), \cdots, L_n = \frac{1}{2}\Big(\frac{1}{2}L_{n-1}\frac{1}{\cos\alpha_{n-1}}\Big).$$

Observe that the definition of L_n does not depend on α_n because the scaling of $\tilde{\Lambda}_n$ is determined just by the previous $(n-1)$ angles. We will use L_n as a quantifier of the scale.

Lemma 8. *For every $\delta > 0$ there exists $k \in \mathbf{N}$ big enough such that for $z_1, z_2 \in \Gamma \cap (\bigcup_{j=k}^\infty \tilde{\Lambda}_j)$ we have*

$$l(z_1, z_2) \leq (1 + \delta)|z_1 - z_2|. \tag{5.3}$$

Proof. Let us start with some geometrical observation. Let $k \in \mathbf{N}$ and $\zeta_1, \zeta_2 \in \Gamma$. Suppose, moreover, that $\zeta_1 \in \tilde{I}_k$ and $\zeta_2 \in \widetilde{IV}_k$. It is useful to define

$$R_k := l(\zeta_1, \zeta_2) - |\zeta_1 - \zeta_2|.$$

Observe that the definition of R_k does not depend on the choice of ζ_1 and ζ_2 in the respective segments. In particular, by the construction of the curve Γ and by the definition of the error term τ_j in (5.1), it is not difficult to check that we have

$$R_k = \left(3 \sum_{j=k+1}^{\infty} L_j - L_k\right) - \sum_{j=k+1}^{\infty} 2L_j\tau_j. \tag{5.4}$$

The term between parentheses in the right hand side is the length of the gluing of the 'non-regularized' α-patches in the construction and the second sum is an error term due to the smoothing in the definition of α-patch.

Because of how we chose L_j and τ_j, the quantity R_k represents the error we make in estimating the length of the arch of the curve between $\zeta_1 \in \tilde{I}_k$ and $\zeta_2 \in \widetilde{IV}_k$ compared to $|\zeta_1 - \zeta_2|$. The presence of factor $2L_j$ in the last sum in the right hand side of (5.4) is due to the fact that the diameter of $\tilde{\Lambda}_j$ is equal to $2L_j$ and, thus, the error term τ_j has to be rescaled by that value. It turns out that

$$\frac{R_k}{L_k} \to 0 \quad \text{as} \quad k \to \infty, \tag{5.5}$$

which justifies the interpretation of R_k as an error term. Indeed, recalling that $\cos\alpha_l \geq \cos\alpha_k$ for $l \geq k$, we have

$$\begin{aligned}\frac{3}{L_k} \sum_{j=k+1}^{\infty} L_j &= \sum_{j=k+1}^{\infty} \frac{3}{4^{j-k}} \left(\prod_{l=k}^{j-1} \cos\alpha_l\right)^{-1} \\ &\leq 3 \sum_{j=k+1}^{\infty} \left(\frac{1}{4\cos\alpha_k}\right)^{j-k} = \frac{3}{4\cos\alpha_k - 1}\end{aligned}$$

and the last term tends to 1 as $k \to \infty$. Moreover, using (5.1) and since $L_j \leq 2^{k-j} L_k$ for $j > k$, we have that

$$\frac{1}{L_k} \sum_{j=k+1}^{\infty} 2L_j\tau_j \lesssim \tau_{k+1} \sum_{j=k+1}^{\infty} 2^{k-j} \to 0 \quad \text{as} \quad k \to \infty,$$

so that (5.5) follows.

Let us combine this observation with (5.2) to prove (5.3). Let $z_1, z_2 \in \Gamma$. Observe that each point of Γ belongs to $\widetilde{\Lambda}_j$ for at most two different j. Let k_1 be the maximum index such that $z_1 \in \tilde{\Lambda}_{k_1}$ and let k_2 be the maximum index such that $z_2 \in \tilde{\Lambda}_{k_2}$. The rest of the proof works with minor changes if we take the minimum instead of the maximum in the definitions of k_1 and k_2. The use of this indices helps to make the calculations more systematic.

Without loss of generality, suppose $k_1 \leq k_2$. If $k_1 = k_2$, the points belong to the image of the same patch. We have two possible scenarios depending on the relative position of these points. The definition of R_k and the estimate (5.2) allow us to write

$$l(z_1, z_2) \leq \frac{|z_1 - z_2|}{\cos\alpha_{k_1}}, \tag{5.6}$$

if the point are at a distance $|z_1 - z_2| \leq L_{k_1+1}$. For $|z_1 - z_2| \geq L_{k_1+1}/4$, we have to consider the additional error term R_{k_1+1}, which comes from the 'spiraling' part of the curve. In particular

$$l(z_1, z_2) \leq \frac{|z_1 - z_2|}{\cos \alpha_{k_1}} + R_{k_1+1} \leq \frac{|z_1 - z_2|}{\cos \alpha_{k_1}} + \frac{R_{k_1+1}}{4L_{k_1+1}}|z_1 - z_2|,$$

so that, invoking (5.5), the lemma is proven in the case $k_1 = k_2$.

Let us consider the other case, $k_1 < k_2$. If $z_2 \in \widetilde{\Lambda}_{k_1}$, (5.6) easily applies because the two points belong to the image of the same patch. So we can suppose $z_2 \notin \widetilde{\Lambda}_{k_1}$. In this case

$$|z_1 - z_2| \geq \frac{L_{k_1+1}}{4}. \tag{5.7}$$

Let $z_2' \in \widetilde{II}_{k_1}$ be the orthogonal projection of z_2 on the segment $\widetilde{II}_{k_1}$. The idea now is, by means of projections, to reduce to the case in which the points belong to the image of the same patch. For this purpose it is also useful to use the length of the arcs of the m-th step curve Γ_m that we used to define Γ. By the triangular inequality and denoting by

$$h_{k_1+1} := \min\{h\colon \widetilde{\Lambda}_{k_1+1} \subset [0,h]n_V + V \text{ for some affine line } V \text{ with normal } n_V\} \tag{5.8}$$

the width of $\tilde{\Lambda}_{k_1+1}$, we have

$$|z_1 - z_2'| \leq |z_1 - z_2| + h_{k_1+1}. \tag{5.9}$$

Let us remark that, by construction of Γ,

$$\frac{h_{k_1+1}}{L_{k_1+1}} \to 0 \quad \text{as} \quad k \to \infty. \tag{5.10}$$

Given $m \in \mathbf{N}$ and $u, v \in \Gamma_m$, it is useful to denote by $l_m(u, v)$ the length of the arc of Γ_m joining u and v. Now we want to prove that

$$l(z_1, z_2) \leq l_{k_1}(z_1, z_2') + R_{k_1+1}. \tag{5.11}$$

Let us just consider the case $z_1 \in \tilde{I}_{k_1}$, since the other cases are analogous. If $z_{k_2} \in \tilde{I}_{k_1+1}$ or $z_{k_2} \in \widetilde{IV}_{k_1+1}$, (5.11) holds trivially because $z_2 = z_2'$. Otherwise, let ζ be a point on $\widetilde{IV}_{k_1+1}$ and let us consider the quantities $l(z_2, \zeta)$ and $|z_2' - \zeta|$. Observe that the consideration below does not depend on the auxiliary point ζ of $\widetilde{IV}_{k_1+1}$ we choose. Clearly $l(z_2, \zeta) \geq |z_2' - \zeta|$ and, because of the definition of R_{k_1+1}, the equality

$$l(z_1, z_2) + l(z_2, \zeta) = R_{k_1+1} + l_{k_1}(z_1, z_2') + |z_2' - \zeta|,$$

holds. So

$$l(z_1, z_2) = l_{k_1}(z_1, z_2') + R_{k_1+1} + (|z_2' - \zeta| - l(z_2, \zeta)) \leq l_{k_1}(z_1, z_2') + R_{k_1+1}.$$

The proof of the lemma is now over: indeed using (5.2), (5.5), (5.7), (5.9) and (5.10) we get

$$\begin{aligned} \frac{l(z_1, z_2)}{|z_1 - z_2|} &\leq \frac{l_{k_1}(z_1, z_2')}{|z_1 - z_2|} + \frac{R_{k_1+1}}{|z_1 - z_2|} \leq \frac{|z_1 - z_2'|}{|z_1 - z_2| \cos \alpha_{k_1}} + \frac{R_{k_1+1}}{|z_1 - z_2|} \\ &\leq \frac{1}{\cos \alpha_{k_1}} + \frac{4h_{k_1+1}}{\cos \alpha_{k_1} L_{k_1+1}} + \frac{4R_{k_1+1}}{L_{k_1+1}} \to 1 \quad \text{as} \quad k_1 \to \infty. \end{aligned}$$

□

A rectifiable curve Γ is said *asymptotically smooth* if, denoting by $l(w_1,w_2)$ the length of the shortest arc of Γ between $w_1,w_2\in\Gamma$,

$$\frac{l(w_1,w_2)}{|w_1-w_2|}\to 1 \quad \text{as} \quad |w_1-w_2|\to 0, \quad w_1,w_2\in\Gamma.$$

As shown in [6], an asymptotically smooth curve is also asymptotically conformal.

Proposition 1. *Γ is asymptotically smooth but not C^1.*

Proof. Let $\tilde{z}'_j\in\Gamma$ be the image of the point z'_{α_j} via the map which sends Λ_j to $\widetilde{\Lambda}_j$. We have that the curve Γ is not C^1 at the point $z_0:=\lim_j z_j$, where z_j is an arbitrary point of $\tilde{\Lambda}_j$. Indeed, by our choice of the angles in the construction, $\sum_j \alpha_j=+\infty$ and the curve spirals close to z_0.

Let us now turn prove that the curve is asymptotically smooth. Notice that we may write $\Gamma=\Gamma_1\cup\Gamma_2\cup\{z_0\}$, where Γ_1 and Γ_2 are smooth curves. Then, for every couple of points $\{z_1,z_2\}$ in one of those two smooth components we can exploit the smoothness to state that for every δ there exists $\bar\epsilon$ such that for $\epsilon<\bar\epsilon$ and $|z_1-z_2|=\epsilon$ we have

$$l(z_1,z_2)\le(1+\delta)\epsilon.$$

This, together with the result of Lemma 8 concludes the proof. □

Let us consider the arc-length parametrization γ of Γ. Being Γ asymptotically smooth, γ is bilipschitz. In particular,

$$\frac{1}{C}|x-y|\le|\gamma(x)-\gamma(y)|\le|x-y|$$

for a constant $C>1$ and $x,y\in[0,L(\Gamma)]$. As in Remark 4 we denote by $\alpha(x,\epsilon)$ the angle between the vectors $\gamma(x)-\gamma(x-\epsilon)$ and $\gamma(x+\epsilon)-\gamma(x)$. Because of the geometrical considerations in Remark 4, we have that

$$|\gamma(x+\epsilon)+\gamma(x-\epsilon)-2\gamma(x)|^2\le\epsilon^2\left(2-\frac{2}{C^2}\cos\alpha(x,\epsilon)\right) \tag{5.12}$$

for $\epsilon>0$ and $x\in[0,L(\Gamma)]$. Now we want to prove the estimate

$$|\gamma(x+\epsilon)+\gamma(x-\epsilon)-2\gamma(x)|\lesssim\frac{\epsilon}{|\log\epsilon|}.$$

Being Γ smooth off the point z_0 and arguing as in [2], the logarithmic condition (1.4) and the estimate (1.6) are satisfied off that point. Hence it suffices to prove (1.6) for $\gamma(x)\in\bigcup_{k\ge k_0}\tilde{\Lambda}_k\cap\Gamma$ and k_0 big enough. To do that, we will study the behavior of the angle $\alpha(x,\epsilon)$ and of the local value of the bilipschitz constant of γ close to the point z_0.

Being the curve asymptotically smooth, as a corollary of Lemma 4 we know that $\alpha(x,\epsilon)\to 0$ for ϵ small. Then, the second factor in the right hand side of (5.12) behaves as

$$2-\frac{2}{C^2}\cos\alpha(x,\epsilon)=\left[2-\frac{2}{C^2}\right]+\frac{2}{C^2}\alpha(x,\epsilon)^2+O\left(\alpha(x,\epsilon)^4\right)$$

for $\epsilon\to 0$.

Let $x_0:=\gamma^{-1}(z_0)$. For $\epsilon>0$, we denote by C_ϵ the smallest constant such that

$$\frac{1}{C_\epsilon}|x-y|\le|\gamma(x)-\gamma(y)|\le|x-y|$$

holds for $x, y \in [x_0 - \epsilon, x_0 + \epsilon]$, i.e. the local value of the lower bilipschitz constant close to x_0.

Using this notation, to our purposes it suffices to prove that

$$|\alpha(x,\epsilon)| \lesssim |\log\epsilon|^{-1}$$

and

$$\left[1 - \frac{1}{C_\epsilon}\right] \lesssim |\log\epsilon|^{-1} \tag{5.13}$$

for ϵ small and $\gamma(x)$ close enough to z_0.

The following two lemmas respectively prove the estimate for the angle and the estimate for C_ϵ.

Lemma 9. *For every ϵ_0 there exists an integer k_0 such that*

$$|\alpha(x,\epsilon)| \lesssim |\log\epsilon|^{-1}$$

for $\epsilon < \epsilon_0$, $|x - x_0| < \epsilon_0$ and $\gamma(x-\epsilon) \in \bigcup_{k=k_0}^{\infty} \widetilde{\Lambda}_k \cap \Gamma$.

Proof. Let $\epsilon > 0$ and $z = \gamma(x) \in \Gamma$. Moreover, let us define $z_\pm := \gamma(x \pm \epsilon)$. Let k be the maximum index such that $z \in \tilde{\Lambda}_k$ and let $k_\pm$ be the maximum index such that $z_\pm \in \tilde{\Lambda}_{k_\pm}$. Without loss of generality, we will prove the lemma for $x < x_0$. Let us proceed with some geometrical consideration.

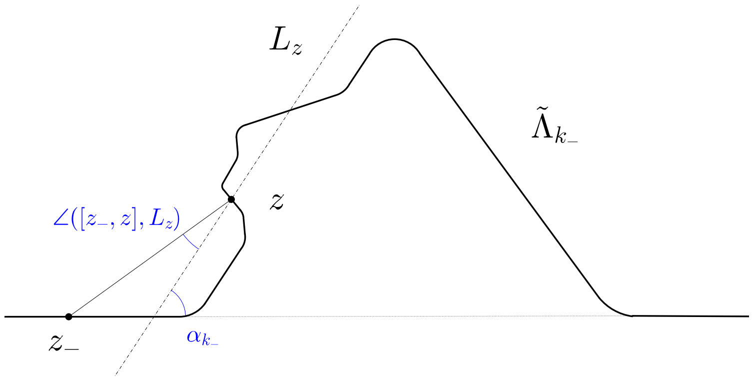

Figure 4. A schematic representation of the setting of the proof of Lemma 9.

Let L_z denote the line passing through z and parallel to the segment $\widetilde{II}_{k_-}$. Due to the definition of the angle $\alpha(x,\epsilon)$, we can fix the line L_z and bound $|\alpha(x,\epsilon)|$ by the absolute value of the smallest angle $\angle([z_-, z], L_z)$ that L_z forms with the segment $[z_-, z]$ plus the absolute value of the smallest angle $\angle([z, z_+], L_z)$ that L_z forms with the segment $[z, z_+]$.

If z belongs to $\widetilde{\Lambda}_{k_-}$, due to the properties of the $\alpha_{k_-}-$patch, the arc $\gamma([x-\epsilon, x])$ is entirely contained in a cone of vertex z and aperture $\angle([z_-, z], L_z)$. By elementary geometric considerations, we can write

$$|\angle([z_-, z], L_z)| \le \alpha_{k_-}. \tag{5.14}$$

Again, due to few geometric observations (that are not substantial for the sequel and we decide to omit in order to make the proof more concise) and to the way Γ is defined, it is not difficult to see that

$$|\angle([z_+, z], L_z)| \le 2\alpha_{k_-}. \tag{5.15}$$

We are left to consider the case $z \notin \widetilde{\Lambda}_{k_-}$. As we observed in Lemma 8, in this case we have $|z_- - z| \ge L_{k_-+1}/4$. Moreover, $\bigcup_{j=k_-+1}^{\infty} \widetilde{\Lambda}_j \cap \Gamma$ is contained in a rectangle whose base lays on $\widetilde{II}_{k_-}$, whose length is smaller than, say, $5L_{k_-+1}/3$ and with height h_{k_-+1} (for its definition we refer to (5.8) in Lemma 8). We recall that

$$\frac{h_j}{L_j} \to 0 \quad \text{for} \quad j \to \infty.$$

Now observe that $z_+ \in \bigcup_{j=k_-}^{\infty} \widetilde{\Lambda}_j \cap \Gamma$. For every point z in this rectangle, using that $|z - z_+| \gtrsim L_{k_-+1}$, it holds that

$$|\angle([z_-, z], L_z)| \lesssim \alpha_{k_-} \tag{5.16}$$

and

$$|\angle([z, z_+], L_z)| \lesssim \alpha_{k_-}. \tag{5.17}$$

Joining (5.14),(5.15),(5.16) and (5.17), we get

$$|\alpha(z, \epsilon)| \lesssim \alpha_{k_-}.$$

Then, by the construction of Γ and the definition of L_m, $L_{m+1}/L_m \leq 1/2$ for every m, that by iteration leads to

$$L_m \leq 2^{-m}.$$

Now, if $\gamma(x - \epsilon) \in \widetilde{\Lambda}_{k_-}$ for k_- big enough, we have that $\epsilon \lesssim L_{k_-}$ so that

$$k_- \gtrsim |\log \epsilon|$$

for ϵ small enough. So, gathering all the considerations and recalling that $\alpha_{k_-} = 1/k_-$, we get the desired result. ☐

Lemma 10. *There exists* $\epsilon_1 > 0$ *such that the inequality* (5.13) *holds for* $\epsilon < \epsilon_1$.

Proof. Let us consider $z_1, z_2 \in \Gamma$. Let k_1 be the maximum index such that $z_1 \in \tilde{\Lambda}_{k_1}$ and k_2 the maximum index such that $z_2 \in \tilde{\Lambda}_{k_2}$. Without loss of generality, $k_1 \leq k_2$ and $\gamma^{-1}(z_1) \leq \gamma^{-1}(\tilde{z})$. The idea is to prove that C_ϵ^{-1} is greater than a quantity which approximates $\cos \alpha_{k_1}$. It is convenient to split the study into different cases.

If $k_1 = k_2$ and $\gamma^{-1}(z_2) < \bar{x}$ or $k_2 = k_1 + 1$ and $z_2 \in \widetilde{I}_{k_1+1}$, then (5.2) gives

$$|z_1 - z_2| \geq \cos \alpha_{k_1} l(z_1, z_2).$$

If $k_1 = k_2$ and $\gamma^{-1}(z_2) > \bar{x}$ or $k_2 = k_1 + 1$ and $z_2 \in \widetilde{IV}_{k_1+1}$, then we can write

$$|z_1 - z_2| \geq \cos \alpha_{k_1} \big(l(z_1, z_2) - R_{k_1+1}\big) = \left(\cos \alpha_{k_1} - \cos \alpha_{k_1} \frac{R_{k_1+1}}{l(z_1, z_2)}\right) l(z_1, z_2)$$

and we recall that

$$\frac{R_{k_1+1}}{l(z_1, z_2)} \lesssim \frac{R_{k_1+1}}{L_{k_1+1}} \to 0 \quad \text{for} \quad k_1 \to \infty.$$

In the remaining cases, we know from the proof of Lemma 8 that

$$|z_1 - z_2| \geq \Big(\cos \alpha_{k_1} - \cos \alpha_{k_1} \frac{h_{k_1+1}}{l(z_1, z_2)} - \cos \alpha_{k_1} \frac{R_{k_1+1}}{l(z_1, z_2)}\Big) l(z_1, z_2),$$

so that, using the same argument as at the end of the proof of Lemma 9 together with the Taylor expansion for the cosine, the proof is completed. ☐

The two previous lemmas show that the arc-length parametrization γ of Γ is such that the estimate

$$T_*(f)(z) \lesssim M^2(Tf)(z)$$

holds for every $z \in \Gamma$.

Final remarks on the curve Γ. The curve Γ that we studied in this section can be considered as an example of a critical curve for which the main theorem holds. Indeed, another look at the estimates we got tells that most of those concerning the geometry of the curve are close to being sharp. Moreover, the finite second difference $|\gamma(x+\epsilon)+\gamma(x-\epsilon)-2\gamma(x)|$ has the right decay we need; the choice of a slower decay for the angles α_j causes worse estimates for $|\alpha(x,\epsilon)|$ and, hence, the finite second difference estimate to fail. Let us notice that the spiraling of Γ close to the point z_0 also gives an idea of how the critical curves may look like.

Asymptotically smooth curves that are not C^1 may also be defined by means of complex analysis (exploiting, for example, the results in [6]) but we found a constructive approach more convenient to our purposes.

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