Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 43, 2018, 1023–1026

OUTER FUNCTIONS AND UNIFORM INTEGRABILITY

Javad Mashreghi and Thomas Ransford

Université Laval, Département de mathématiques et de statistique
Québec City (Québec), Canada G1V 0A6; javad.mashreghi@mat.ulaval.ca

Université Laval, Département de mathématiques et de statistique
Québec City (Québec), Canada G1V 0A6; thomas.ransford@mat.ulaval.ca

Abstract. We show that, if f is an outer function and $a \in [0,1)$, then the set of functions

$$\{\log|(f\circ\psi)^*|\colon \psi\colon \mathbf{D}\to\mathbf{D} \text{ holomorphic}, |\psi(0)|\le a\}$$

is uniformly integrable on the unit circle. As an application, we derive a simple proof of the fact that, if f is outer and $\phi\colon \mathbf{D}\to\mathbf{D}$ is holomorphic, then $f\circ\phi$ is outer.

1. Introduction

Let $\mathbf{D}$ be the open unit disk and $\mathbf{T}$ be the unit circle. We write $\mathcal{S}$ for the set of holomorphic functions $\phi\colon \mathbf{D}\to\mathbf{D}$ (essentially the *Schur class*, except that we exclude constant unimodular functions).

A holomorphic function $f\colon \mathbf{D}\to\mathbf{C}$ is called *outer* if it has the form

$$f(z) = c\exp\left(\int_{\mathbf{T}} \frac{e^{i\theta}+z}{e^{i\theta}-z}\log\rho(e^{i\theta})\,\frac{d\theta}{2\pi}\right) \quad (z\in\mathbf{D}), \tag{1}$$

where c is a unimodular constant and $\rho\colon \mathbf{T}\to\mathbf{R}^+$ is a function such that $\log\rho\in L^1(\mathbf{T})$. Outer functions are a key tool in the theory of Hardy spaces. Among their many nice properties is the following folklore fact: if f is outer and $\phi\in\mathcal{S}$, then $f\circ\phi$ is also outer. This note arose as an attempt to better understand why this fact is true.

We shall study two classes of functions. The *Nevanlinna class* N consists of those functions of the form $f = f_1/f_2$, where f_1, f_2 are bounded and holomorphic on $\mathbf{D}$ and f_2 has no zeros. The *Smirnov class* N^+ is the subclass of N consisting of those $f = f_1/f_2$, where f_1, f_2 are bounded and holomorphic on $\mathbf{D}$ and f_2 is outer.

If $f\in N$, then its radial boundary limits

$$f^*(e^{i\theta}) := \lim_{r\to 1^-} f(re^{i\theta})$$

exist a.e. on $\mathbf{T}$. This is a simple consequence of the corresponding result for bounded holomorphic functions, due to Fatou. Also, it is clear that, if $f\in N$ and $\phi\in\mathcal{S}$, then $f\circ\phi\in N$. The corresponding result for N^+ is also true, but rather less obvious. As we shall see, it is more or less equivalent to the analogous result for outer functions.

The following theorem lists a number of well-known characterizations of N^+. We write $f_r(z) := f(rz)$. Also, we recall that f is called *inner* if it is a bounded holomorphic function on $\mathbf{D}$ satisfying $|f^*| = 1$ a.e. on $\mathbf{T}$.

https://doi.org/10.5186/aasfm.2018.4360
2010 Mathematics Subject Classification: Primary 30H15; Secondary 28A20.
Key words: Outer function, Smirnov class, uniformly integrable.
JM supported by an NSERC grant.
TR supported by grants from NSERC and the Canada Research Chairs program.

Theorem A. *Let $f \in N$. The following statements are equivalent:*

(i) $f \in N^+$,

(ii) $f = f_i f_o$, *where f_i is inner and f_o is outer,*

(iii) $\lim_{r\to 1^-} \int_0^{2\pi} \log^+ |f(re^{i\theta})|\,d\theta = \int_0^{2\pi} \log^+ |f^*(e^{i\theta})|\,d\theta$,

(iv) *the set* $\{\log^+ |f_r^*|\colon 0 < r < 1\}$ *is uniformly integrable on* $\mathbf{T}$.

For the equivalence of the first three, see for example [3, §2.5]. A proof of the equivalence of (iii) and (iv) can be found in [4, Theorem A.3.7].

Our contribution is the following theorem. Given $a \in [0,1)$, we write

$$\mathcal{S}_a := \{\psi \in \mathcal{S}\colon |\psi(0)| \le a\}.$$

Theorem 1. *Let $f \in N$ and let $a \in [0,1)$. Then $f \in N^+$ if and only if the set*

$$\{\log^+ |(f\circ\psi)^*|\colon \psi \in \mathcal{S}_a\}$$

is uniformly integrable on $\mathbf{T}$.

As observed above, if $f \in N$, then $f\circ\psi \in N$ for all $\psi \in \mathcal{S}$, and so $(f\circ\psi)^*$ exists a.e. on $\mathbf{T}$. Thus the statement of the theorem makes sense. We shall prove this theorem in §2.

Clearly, if $\phi \in \mathcal{S}$ and $a \in [0,1)$, then

$$\{\phi\circ\psi\colon \psi \in \mathcal{S}_a\} \subset \mathcal{S}_b,$$

where $b := \sup_{|z|\le a} |\phi(z)| \in [0,1)$. Theorem 1 therefore immediately implies the following result, previously obtained by other methods in [5] and [2].

Corollary 2. *If $f \in N^+$ and $\phi \in \mathcal{S}$, then $f\circ\phi \in N^+$.*

We now return to the subject of outer functions. The link with N^+ is furnished by the observation that a nowhere-vanishing holomorphic function f on $\mathbf{D}$ is outer if and only if both $f \in N^+$ and $1/f \in N^+$. Indeed, the 'only if' is obvious, and the 'if' is an easy consequence of the characterization (ii) of N^+ in Theorem A.

Combining this remark with Theorem 1, we obtain the following theorem, which we believe to be new.

Theorem 3. *Let $f \in N$ with no zeros and let $a \in [0,1)$. Then f is outer if and only if the set*

$$\{\log |(f\circ\psi)^*|\colon \psi \in \mathcal{S}_a\}$$

is uniformly integrable on $\mathbf{T}$.

From this, we deduce the result mentioned at the beginning of the section.

Corollary 4. *If f is outer and $\phi \in \mathcal{S}$, then $f\circ\phi$ is outer.*

2. Proof of Theorem 1

The main idea of the proof is to exploit a criterion for uniform integrability due to de la Vallée Poussin. For convenience, we include a quick proof.

Let (X,μ) be a measure space and let $\mathcal{G}$ be a family of measurable complex-valued functions on X. We recall that $\mathcal{G}$ is *uniformly integrable* if

$$\sup_{g\in\mathcal{G}} \int_{\{|g|\ge t\}} |g|\,d\mu \to 0 \quad (t\to\infty).$$

Lemma B. *The family $\mathcal{G}$ is uniformly integrable on (X,μ) if and only if there exists a function $\omega\colon \mathbf{R}\to\mathbf{R}^+$ with $\lim_{t\to\infty}\omega(t)/t=\infty$ such that*

$$\sup_{g\in\mathcal{G}}\int_X \omega(|g|)\,d\mu<\infty. \tag{2}$$

The function ω may be chosen to be convex and increasing.

Proof. Suppose ω exists. Given $\epsilon>0$, choose t such that $\omega(s)/s\ge 1/\epsilon$ for all $s\ge t$. Then, for each $g\in\mathcal{G}$, we have

$$\int_{\{|g|\ge t\}}|g|\,d\mu\le\int_{\{|g|\ge t\}}\epsilon\omega(|g|)\,d\mu\le\epsilon\int_X\omega(|g|)\,d\mu\le\epsilon M,$$

where M is the supremum in (2).

Conversely, suppose that $\mathcal{G}$ is uniformly integrable. Choose a positive increasing sequence $t_n\to\infty$ such that, for each n,

$$\sup_{g\in\mathcal{G}}\int_{\{|g|\ge t_n\}}|g|\,d\mu\le 2^{-n}.$$

Define $\omega(t):=\sum_{n\ge1}(t-t_n)^+$. Clearly $\lim_{t\to\infty}\omega(t)/t=\infty$ and, for each $g\in\mathcal{G}$, we have

$$\int_X\omega(|g|)\,d\mu=\sum_{n\ge1}\int_X(|g|-t_n)^+\,d\mu\le\sum_{n\ge1}\int_{\{|g|\ge t_n\}}|g|\,d\mu\le\sum_{n\ge1}2^{-n}\le1.$$

Finally, we note that ω, as constructed above, is convex and increasing. □

Proof of Theorem 1. By considering ψ of the form $\psi(z):=rz$ $(0<r<1)$, we see that the 'if' part of the theorem follows from the characterization of N^+ given in Theorem A (iv).

We now turn to the 'only if' part. Let $f\in N^+$. By Theorem A (iv), the family $\{\log^+|f_r^*|\colon 0<r<1\}$ is uniformly integrable on $\mathbf{T}$. Therefore, by Lemma B, there exists a convex increasing function $\omega\colon\mathbf{R}\to\mathbf{R}^+$ with $\lim_{t\to\infty}\omega(t)/t=\infty$ such that

$$\sup_{0<r<1}\int_{\mathbf{T}}\omega\Big(\log^+|f(re^{i\theta})|\Big)\frac{d\theta}{2\pi}<\infty. \tag{3}$$

Now $\omega(\log^+|f|)$ is subharmonic on $\mathbf{D}$, because ω is a convex increasing function and $\log^+|f|$ a subharmonic function on $\mathbf{D}$ (see [1, Theorem 3.4.3(ii)]). By [1, Theorem 3.6.6], the condition (3) implies that $\omega(\log^+|f|)$ has a harmonic majorant on $\mathbf{D}$, let us call it h. Thus, if $\psi\in\mathcal{S}_a$, then for all $r\in(0,1)$ we have

$$\int_{\mathbf{T}}\omega\Big(\log^+|(f\circ\psi)(re^{i\theta})|\Big)\frac{d\theta}{2\pi}\le\int_{\mathbf{T}}(h\circ\psi)(re^{i\theta})\frac{d\theta}{2\pi}=h(\psi(0))\le M,$$

where $M:=\sup_{|z|\le a}h(z)$. Letting $r\to1^-$ and using Fatou's lemma, we deduce that

$$\int_{\mathbf{T}}\omega\Big(\log^+|(f\circ\psi)^*(e^{i\theta})|\Big)\frac{d\theta}{2\pi}\le M.$$

Thus

$$\sup_{\psi\in\mathcal{S}_a}\int_{\mathbf{T}}\omega\Big(\log^+|(f\circ\psi)^*(e^{i\theta})|\Big)\frac{d\theta}{2\pi}<\infty,$$

and the result now follows by applying Lemma B in the other direction. □

References

[1] Armitage, D., and S. Gardiner: Classical potential theory. - Springer, London, 2001.

[2] Choa, J., H. Kim, and J. Shapiro: Compact composition operators on the Smirnov class. - Proc. Amer. Math. Soc. 128, 2000, 2297–2308.

[3] Duren, P.: Theory of H^p spaces. - Academic Press, New York, 1970.

[4] El-Fallah, O., K. Kellay, J. Mashreghi, and T. Ransford: A primer on the Dirichlet space. - Cambridge Tracts in Math. 203, Cambridge Univ. Press, Cambridge, 2014.

[5] Yanagihara, N., and Y. Nakamura: Composition operators on N^+. - TRU Math. 14, 1978, 9–16.

Received 12 March 2018 • Accepted 18 May 2018