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# COVERINGS BY $n$-CUBES AND THE GAUSS–BONNET THEOREM

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**Abstract.** Instead of standard $n$-simplexes we deal with $n$-dimensional cubes with coordinates on real manifolds. The transition matrices for any two cubes having $(n-1)$-dimensional common side form a group $H_n$ of orthogonal matrices composed of zeros and exactly one non-zero value 1 or $-1$ in each row (column). Considering the coverings, a theorem of Gauss–Bonnet type which holds also for odd-dimensional or non-orientable manifolds is proved. We conjecture that a real manifold admits a restriction of the transition matrices to a Lie subgroup $G$ of $GL(n,\mathbf{R})$ of dimension $\geq 1$, or the unit element in $GL(n,\mathbf{R})$ if and only if the manifold can be covered by $n$-cubes such that the transition matrices take values in the intersection of $H_n$ and $G$ or the unit matrix. The complex case uses $GL(2n,\mathbf{R})$ and transition matrices of even dimension. The conjecture is supported with 5 examples. We give methods for calculation of the smallest admissible subgroup of $H_n$ and finally, some conclusions and open questions are presented.

## 1. Introduction

The Gauss–Bonnet theorem, in its classical form, shows that an essential geometric information of 2-dimensional closed manifold (or a manifold with boundary), the total curvature, is associated with the important topological invariant, the Euler characteristic. This theorem became standard in the lectures of differential geometry at universities in the previous century, for example [8], or [9] which also includes historical comments. However, the research continued in direction of its generalizations. Chern popularized it by giving a short and conceptual proof for the case of even-dimensional closed orientable manifolds [6] on sphere bundle of the manifold with Levi–Civita connection. Admittedly, a year before the Chern's proof, the paper [1] was published with a proof of an analogue of the classical Gauss–Bonnet theorem for all Riemannian manifolds. It is a complicated work which realizes the proof by using embedding of the manifolds locally isometrically into suitable Eucledian spaces. There have been generalizations in different directions of the Gauss–Bonnet–Chern theorem also, most notably in [4] and [7], but also for applications in relativity, for example [3] and [2] or physics, for example, [5].

The main result in this work is a proof of a theorem of Gauss–Bonnet type which holds for odd-dimensional manifolds and for non-orientable manifolds, also. We introduce coverings of real and complex manifolds consisting of $n$-dimensional cubes

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instead of standard $n$-simplexes, such that coordinates can be straightforwardly assigned. The transition matrices between any two such oriented cubes are determined so that they belong to a group denoted by $H_n$ of orthogonal matrices composed of mostly zeros and exactly one non-zero entry 1 or $-1$ in each row (column). Then, some methods for calculation of the smallest admissible subgroup of $H_n$ are presented. We give a conjecture that a manifold admits a restriction of the transition matrices to a Lie subgroup $G$ of $GL(n, \mathbf{R})$ of dimension $\geq 1$, or the unit element in $GL(n, \mathbf{R})$ if and only if the manifold can be covered by $n$-cubes such that the transition matrices take values in the intersection of $H_n$ and $G$ or the unit matrix. The conjecture is supported by several examples of the appearance of the transition matrices and the procedure of their obtaining. One of the examples addresses complex manifolds, where instead of the group $H_n$, the group $H'_{2n} = H_{2n} \cap U(n)$ is employed in the conjecture, where $U(n)$ is considered as a subset of $GL(2n, \mathbf{R})$. In the end, conclusions and open questions are issued.

## 2. Coverings by $n$-cubes and methods for calculation of the smallest admissible subgroup of $\mathbf{H}_n$

We firstly consider real manifolds, particularly, coverings of $n$-dimensional real manifolds. Instead of standard $n$-simplexes we can choose a covering consisting of $n$-dimensional cubes $I^n$. It means that we can introduce coordinates $x^1, \dots, x^n$ such that a cube is given by $0 \leq x^i \leq 1$, $(1 \leq i \leq n)$, (see Figure 1). By arbitrarily choosing such a coordinate system on each cube, the transition matrices for any two cubes having $(n-1)$-dimensional common side are determined and they are linear mappings such that the corresponding Jacobi matrix is an orthogonal matrix containing only 1, $-1$ or 0. Indeed, in each row (column) there is exactly one non-zero value 1 or $-1$. We will denote this group by $H_n$. In order to realize this transformation, it is sufficient to rotate one of the cubes so that these two simplexes lie in the same affine subspaces (without loss of generality we can assume that the cubes are not curvilinear areas). These $n$-cubes with coordinates will be called *oriented cubes*.

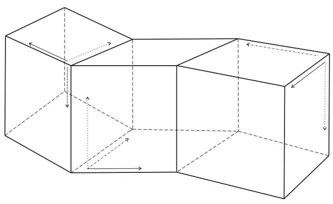

Figure 1.

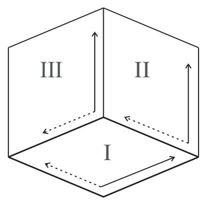


Figure 2.

Let $G$ denotes a Lie subgroup of $GL(n, \mathbf{R})$ of dimension $\geq 1$, or $G$ is the unit element in $GL(n, \mathbf{R})$, i.e., the unit matrix. We give the following lemma.

**Lemma**. *If the manifold $M$ admits oriented cubes with transition matrices in $G \leq H_n$ then it admits oriented cubes with transition matrices in each conjugate subgroup* $\mathrm{AGA}^{-1} \leq H_n$ *for $A \in H_n$.*

*Proof.* It is sufficient to consider the coordinates $y^i = \sum_{j=1}^{n} A^i_j x^j$ in each $n$-cube instead of the coordinates $x^i$. □

It is convenient to consider only the orientable manifolds and assume that $H_n$ consists of all unimodular matrices with elements 1,$-1$ and 0. We notice that *if the manifold is covered with $n$-cubes such that it admits the transition matrices in the group $G \leq H_n$, then each $n$-cube can change its coordinates with transition matrix* *in $G$.* After these changes, all transition matrices will be also in $G$. It follows from here that *if the manifold is covered with $n$-cubes such that it admits the transition matrices in the group $G \leq H_n$, then, in order to find the coordinates at each $n$-cube, it is sufficient to know the coordinates of only one arbitrary $n$-cube. Then the coordinates of the other $n$-cubes should be chosen one after one $n$-cube in such a way, that the transition matrix of each $n$-cube with respect to one of the previous neighboring $n$-cube belongs in $G$. Then the procedure must be successfully finished independently from the choice of the order of the $n$-cubes.* Thus, *it is sufficient to have such a covering with $n$-cubes, and if we did not have any given coordinate system, then we have to check no more than $|H_n \cap G|$ cases to see whether it admits restriction to the group $G$ or not.* Hence, the main problem is to know when a given covering with $n$-cubes has this property for the group $G$. If it has this property for the groups $G_1$ and $G_2$ starting with the same coordinates from the same $n$-cube, then it is easy to see that we can obtain restriction to the group $G_1 \cap G_2$. Moreover, if we can obtain the restriction to the group $G_1$ starting from one cube with coordinates $x^1, \dots, x^n$ and also, we can obtain the restriction to the group $G_2$ starting from the same $n$-cube but with another coordinates $y^1, \dots, y^n$, then we can obtain a restriction to the group $G_1 \cap AG_2A^{-1}$ where $A$ is the transition matrix $\partial y^i/\partial x^j$ in $H_n$.

Note that if a manifold is given with a given covering of $n$-cubes, the procedure to check whether it admits restriction to the group $G$ starts from an arbitrary $n$-cube with arbitrary "coordinates". Then we find coordinates to $n$-cubes one by one such that the coordinates of each next one has unit Jacobi matrix with respect to at least one of the previously chosen coordinates of a neighboring $n$-cube. In the end we obtain a set $S$ of all possible Jacobi matrices. This is easy to do. Finally, *this covering of $n$-cubes admits restriction to a discrete group $G \cap H_n$ if and only if there exists a matrix $A \in H_n$ such that*

$$ASA^{-1} \subset G \cap H_n, \quad \text{i.e.} \quad S \subset A^{-1}(G \cap H_n)A.$$

Now, for a given covering with $n$-cubes and with fixed coordinates in chosen $n$-cube $C$, the smallest discrete group of "discrete transition matrices" is generated as follows: choose arbitrary sequence of neighboring $n$-cubes $C_0, C_1, \dots, C_r$, ($C_0 = C, C_r = C$ and $r$ is arbitrary) and choose coordinates in $C_1, \dots, C_r$ such that the Jacobians between $C_0$ and $C_1, C_1$ and $C_2, \dots, C_{r-1}$ and $C_r$ are unit matrices. Then, the Jacobi matrices between $C_0$ and $C_r$ generate the smallest discrete group of transition matrices. Changing the initial coordinates in $C$, we obtain its conjugate groups.

## 3. A theorem of Gauss–Bonnet type

Let us consider a manifold covered by $n$-cubes without orientation. Then, it defines a parallel transport of vectors (parallel to the edges). The induced connection is flat in the interior of each $n$-cube. Indeed, each manifold covered by $n$-cubes can be considered just as limit of Riemannian manifold such that the curvature is almost flat and the metric $g$ in the interior is $g = I$. This connection has singularity at $k$-cubes for $k < n-1$, similarly as the cone has singularity at the vertex. If the $n$-cubes are chosen to be unit $n$-cells in the Euclidean space, this connection is induced by the flat connection of the surrounding space.

Let us consider, for example, $S^3$. It admits 3 linearly independent vectors fields, but it does not mean that the transition matrices reduce to the unit matrix. Indeed, one can verify that if the vector fields are denoted by $X, Y$ and $Z$, then $[X, Y] \neq 0$ or $[X, Z] \neq 0$ or $[Y, Z] \neq 0$. If all of the transition matrices were the unit matrix, then it would follow that $S^3$ admits a flat Riemannian connection which is not true. The standard covering of $S^3$ with eight 3-cubes, does not admit transition matrices of the form $\{1\} \times H_2$. The reason is that in this case at each point 4 edges meet, which does not yield the result. In order to find the required covering with 3-cubes, we should consider one vector field on $S^3$, then find a Riemannian metric on $S^3$ such that the vector field is everywhere parallel. This Riemannian manifold should be isometrically embedded into Euclidean space, and then the metric should be "homotoped" up to almost everywhere flat metric. This is only an example, but it may give an idea how to prove the conjecture from section 4.

Now, let us consider 2-dimensional manifold $M$ covered by $n$-cubes and also, let us consider the characteristic class $w$ obtained from the almost flat connection. Then, the Gauss–Bonnet theorem

$$\int w = 2\pi\chi(M) \tag{1}$$

reduces to the following form

$$\sum K(A_i) = 2\pi\chi(M), \tag{2}$$

where $K(A_i)$ is the curvature at the vertex $A_i$. Our aim is to find these curvatures $K(A_i)$.

If $n = 2$, then $K(A_i) = \frac{\pi}{2}(4 - m_i)$ where $m_i$ is the number of 2-cubes which have $A_i$ as a vertex. Indeed, now we verify the formula (2). The number of vertices is

$$V = v_3 + v_4 + v_5 + v_6 + \ldots$$

where $v_i$ is the number of vertices where i edges meet. Then, the number of edges is

$$E = \frac{1}{2}(3v_3 + 4v_4 + 5v_5 + \ldots)$$

and the number of faces is

$$F = \frac{1}{4}(3v_3 + 4v_4 + 5v_5 + \ldots).$$

Hence,

$$\chi(M) = V - E + F = \frac{1}{4}(v_3 + 0v_4 - v_5 - 2v_6 - \ldots),$$

and

$$\begin{aligned}
\sum K(A_i) &= \sum \frac{\pi}{2}(4 - m_i) = 2\pi V - \sum \frac{\pi}{2} m_i \\
&= 2\pi(v_3 + v_4 + v_5 + v_6 + \ldots) - \frac{\pi}{2} \cdot 4F \\
&= 2\pi\Big(v_3 + v_4 + v_5 + v_6 + \ldots - \frac{1}{4}(3v_3 + 4v_4 + 5v_5 + \ldots)\Big) \\
&= \frac{1}{2}\pi(v_3 - 0v_4 - v_5 - 2v_6 - \ldots) = 2\pi\chi(M).
\end{aligned}$$

Now, we give a generalization for an arbitrary dimension. Indeed, let the manifold be covered by $n$-cubes. At each vertex $V$ we determine index function as follows

$$\mathrm{ind}(V) = (2^n - v)/2^n, \tag{3}$$

where $v$ is the number of $n$-cubes which have $V$ as a vertex. Note that for the standard covering of $\mathbf{R}^n$ by unit $n$-cubes at each vertex $V$ it is $v = 2^n$. Thus ind(V) is the relative deviation from this standard case. The definition of index of any greater-dimensional cube is analogous. At each edge $E$

$$\operatorname{ind}(E) = (2^{n-1} - e)/2^{n-1}, \tag{4}$$

where $e$ is the number of $n$-cubes which have $E$ as an edge. At each 2-dimensional cube $C$

$$\operatorname{ind}(C) = (2^{n-2} - c)/2^{n-2} \tag{5}$$

where $c$ is the number of $n$-cubes which have $C$ as 2-dimensional cube. This procedure continues in the obvious way. Note that the index of each side $S$ (i.e. $(n-1)$-dimensional cube) is 0, because $s = 2$ and $(2^1 - 2)/2^1 = 0$. Also, the index of each $n$-cube is again 0 because $(2^0 - 1)/2^0 = 0$.

Now, we will prove the main theorem. Let us denote by $P_1, P_2, \dots$ the vertices, by $E_1, E_2, \dots$ the edges, by $C_1, C_2, \dots$ the 2-dimensional cubes and so on.

**Theorem.** *The following formula holds*

$$\sum_{p=0}^{n} (-1)^p \sum_i \operatorname{ind}(Q_i^{(p)}) = \chi(M) \tag{6}$$

*where $Q_i^{(p)}$ denotes a $p$-dimensional cube of the manifold $M$ and $\chi(M)$ denotes the Euler characteristic of $M$.*

*Proof.* The formula (6) means that

$$\left(1 - \frac{v_1}{2^n}\right) + \left(1 - \frac{v_2}{2^n}\right) + \left(1 - \frac{v_3}{2^n}\right) + \cdots - \left(1 - \frac{e_1}{2^{n-1}}\right) - \left(1 - \frac{e_2}{2^{n-1}}\right)$$
$$- \left(1 - \frac{e_3}{2^{n-1}}\right) - \cdots + \left(1 - \frac{c_1}{2^{n-2}}\right) + \left(1 - \frac{c_2}{2^{n-2}}\right) + \left(1 - \frac{c_3}{2^{n-2}}\right) + \cdots - \cdots = \chi(M).$$

This equality is equivalent to

$$\left(\sum v_i\right) - 2\left(\sum e_i\right) + 2^2\left(\sum c_i\right) - \cdots = 0. \tag{7}$$

Let the number of all $n$-cubes be $N$. Then,

$$\sum v_i = N \cdot 2^n \cdot \binom{n}{0},$$
$$\sum e_i = N \cdot 2^{n-1} \cdot \binom{n}{1},$$
$$\sum c_i = N \cdot 2^{n-2} \cdot \binom{n}{2},$$
$$\cdots\cdots\cdots$$

because each $n$-cube contains exactly $2^{n-j} \cdot \binom{n}{j}$ cubes of dimension $j$ $(0 \le j \le n)$. Now (7) is satisfied because

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots = (-1)^n = 0. \qquad \square$$

Note that this theorem is of the type of the Gauss–Bonnet theorem [8]. Indeed, by multiplication of index function in the definitions (3), (4), (5), by an appropriate constant depending on $n$, we obtain Gauss–Bonnet type theorem for the case where all $n$-cubes and all $(n-1)$-dimensional cubes have flat metric. Moreover, comparing this theorem with the Gauss–Bonnet theorem we see that this theorem holds also for odd-dimensional manifolds and also for non-orientable manifolds, while the Gauss–Bonnet theorem considers only orientable and even-dimensional manifolds.

## 4. A conjecture and supporting examples

The conjecture we give in this paper refers to real and complex manifolds. Regarding the real case, we give the following

**Conjecture**. *A manifold admits a restriction of the transition matrices to the group $G$ if and only if the manifold can be covered by $n$-cubes such that the transition matrices take values in the set $H'_n = H_n \cap G$.*

**Remark 1.** For a given manifold which admits restriction of the transition matrices to the group $G$, if there are cubes without orientation, then it is not always possible to orient the cubes so that the transition matrices belong in $G$ (see Example 3 below).

**Remark 2.** Note that the transition matrices for the $n$-cubes are not the same as those for manifolds. For example, if I, II and III are three squares ($n = 2$) (Figure 2) such that the transition matrix between I and II is the identity matrix and between II and III is also the identity matrix, then the transition matrix between I and III might not necessarily be the identity matrix.

In order to support the given conjecture, we give the following examples.

**Example 1**. Let us consider 1-dimensional manifold. It can be considered as a sequence of segments $\dots, A_iA_{i+1}, A_{i+1}A_{i+2}, A_{i+2}A_{i+3}, \dots$ and each segment $A_iA_{i+1}$ can acquire only one direction $\overrightarrow{A_iA_{i+1}}$ or $\overrightarrow{A_{i+1}A_i}$. If two neighboring vectors have the same direction, then transition $1\times 1$ matrix is 1, and it is $-1$ for opposite directions. It is obvious that the orientation of each 1-cube can be chosen so that the transition $1 \times 1$ matrix is 1, i.e., to be orientable.

**Example 2**. Now we will prove that the conjecture holds for the trivial subgroup $G = GL(n, \mathbf{R})$. Indeed, we should only prove that each manifold can be covered by $n$-cubes, neglecting their orientations. Firstly, note that each $n$-dimensional manifold can be covered by $n$-dimensional simplexes. We give a method on how each $n$-dimensional simplex can be divided into $n + 1$ $n$-cubes. This method is inductive. Without loss of generality we suppose that the $n$-dimensional simplex is regular. If $n = 2$ the dividing points are the middle points of its sides and the center of the equilateral triangle (Figure 3). Further, in order to divide the tetrahedron into four 3-cubes, we divide each side of the tetrahedron into three 2-cubes and the vertices of the required 3-cubes are the vertices of the considered tetrahedron, the centers of its sides and the center of the tetrahedron (Figure 4). These procedure can be continued for each simplex of arbitrary dimension.

**Example 3**. It is known that an orientable manifold admits a non-zero vector field if and only if the transition matrices can be reduced to the subgroup $GL^+(1, \mathbf{R})\times GL^+(n-1, \mathbf{R})$. Hence, in discrete case we should have reduction to the subgroup $\{1\} \times SO(n-1, \mathbf{R})$. Specially, if $n = 2$, we should have reduction to the identity matrix $I_2$. It is obvious that it can happen only if locally at each point, exactly 4 squares meet. It is easy to see from here that the Euler characteristic is 0, and hence this must be the torus $S^1\times S^1$ (assuming that the manifold is compact and orientable). This shows how easily a conclusion can be made using the given conjecture, although in this example an already well-known fact is derived.

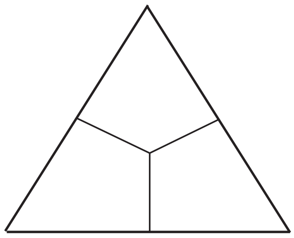

Figure 3.

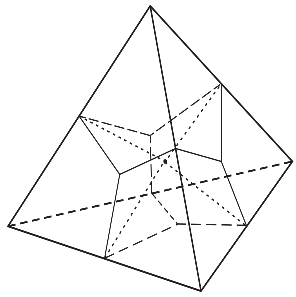

Figure 4.

The vector field has a very good geometrical interpretation using the $n$-cubes (Figure 5). Note that if we have a point $P$ on the torus which is a vertex of three squares (Figure 6), then in I, II and III coordinates can not be defined in such way that the transition matrices are the unit matrix, although the torus admits a vector field.

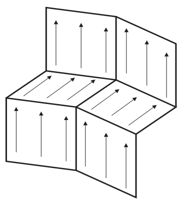

Figure 5.

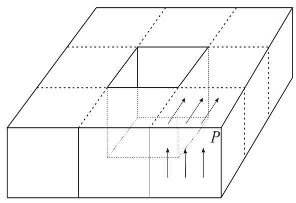


Figure 6.

**Example 4**. (Orientable manifolds) According to the conjecture above, one manifold is orientable if it admits $n$-cubes such that the transition matrices are in the finite group $SO(n, \mathbf{R}) \cap H_n$. The conjecture holds for the orientable manifolds, i.e. for $G = GL^+(n, \mathbf{R})$.

**Example 5**. (Complex manifolds) In this case the conjecture is:

*A manifold admits a complex structure if and only if the manifold admits covering by $n$-cubes such that the transition matrices take values in the group*

$$H'_{2n} = H_{2n} \cap U(n)$$

*where $U(n)$ is considered as a subset of $GL(2n, \mathbf{R})$.*

Firstly, let us see why each 2-dimensional orientable manifold admits complex structure. Indeed, the covering with 2-cubes for orientable manifolds requires that the transition matrices should be

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \quad \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}, \quad \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}, \quad \text{and} \quad \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}. \tag{8}$$

The covering with 2-cubes for complex manifold requires that the transition matrices should also be given by (8). Thus, each 2-dimensional orientable manifold admits a complex structure. Indeed, this well known fact confirms our conjecture for complex manifolds for $n = 2$.

Further, we will consider one special case. Let us consider the sphere of dimension 6 embedded in $\mathbf{R}^7$ and covered with 14 6-cubes: $C_i^+$ and $C_i^-$, $(1 \leq i \leq 7)$. The cubes $C_i^+$ and $C_i^-$ intersect the $x^i$-axis, and they do not have common points. All other pairs $C_i^*$ and $C_j^*(i \neq j)$ have common 5-cube. The group $H_6'$ consists of the following block matrices

$$\begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}, \quad \begin{bmatrix} 0 & A & 0 \\ 0 & 0 & B \\ C & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & A \\ B & 0 & 0 \\ 0 & C & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & A & 0 \\ B & 0 & 0 \\ 0 & 0 & C \end{bmatrix}, \quad \begin{bmatrix} A & 0 & 0 \\ 0 & 0 & B \\ 0 & C & 0 \end{bmatrix},$$

where the $2 \times 2$ blocks $A, B$ and $C$ are given by (8). This matrix group contains $6 \cdot 4^3 = 384$ elements. It is convenient to consider all these matrices as $7 \times 7$ matrices by setting 1 at (7,7) and 0 at the additional remaining entries. For each $i$ and $j$ $(i \neq j)$ we define matrices $S_{ij}^+$ and $S_{ij}^-$ as follows:

$$(S_{ij}^+)_{rs} = \begin{cases} -1 & \text{if } r = i \text{ and } s = j, \\ 1 & \text{if } r = j \text{ and } s = i, \\ 1 & \text{if } i \neq r = s \neq j, \\ 0 & \text{otherwise,} \end{cases}$$

$$(S_{ij}^-)_{rs} = \begin{cases} 1 & \text{if } r = i \text{ and } s = j, \\ -1 & \text{if } r = j \text{ and } s = i, \\ 1 & \text{if } i \neq r = s \neq j, \\ 0 & \text{otherwise.} \end{cases}$$

Each of these 14 cubes should be oriented, i.e. for each cube there should be 6 ordered unit vectors in $\mathbf{R}^7$ parallel to the edges of the corresponding cube. Moreover, we also determine the 7-th vectors $\mathbf{a}_{17}^-, \dots, \mathbf{a}_{77}^-$ and $\mathbf{a}_{17}^+, \dots, \mathbf{a}_{77}^+$ such that $(a_{i7}^-)_j = \delta_{ij}$ and $(a_{i7}^+)_j = -\delta_{ij}$. For each cube, the corresponding 7 vectors uniquely determine a matrix in $H_7$, considered as a set of $7 \times 7$ matrices, using these 7 vectors as vector columns. Thus, we should find the following 14 matrices $A_i^+$ and $A_i^-$ in $H_6'$ such that:

1. These matrices belong in $H_6'$ and moreover
$$(A_i^+)_{i7} = -1 \quad \text{and} \quad (A_i^-)_{i7} = 1 \qquad (1 \leq i \leq 7).$$
2. They should also satisfy the following property for the transition matrices:
   - For the neighbor cubes $C_i^-$ and $C_j^+$ $(i \neq j)$ the transition matrix $P = (A_i^-)^{-1} \cdot S_{ij}^+ \cdot A_j^+$ belongs to $H_6'$;
   - For the neighbor cubes $C_i^-$ and $C_j^-$ $(i \neq j)$ the transition matrix $P = (A_i^-)^{-1} \cdot S_{ij}^- \cdot A_j^-$ belongs to $H_6'$;
   - For the neighbor cubes $C_i^+$ and $C_j^+$ $(i \neq j)$ the transition matrix $P = (A_i^+)^{-1} \cdot S_{ij}^- \cdot A_j^+$ belongs to $H_6'$;
   - For the neighbor cubes $C_i^+$ and $C_j^-$ $(i \neq j)$ the transition matrix $P = (A_i^+)^{-1} \cdot S_{ij}^+ \cdot A_j^-$ belongs to $H_6'$.

We will prove that these 14 cubes can not be oriented so that the sphere admits a complex structure. This is only a special case, and it does not say anything for another covering of $S^6$ by cubes. Without loss of generality we can suppose that $A_7^- = I_7$. Then according to the property 2 above, we obtain

$$(A_7^-)^{-1} \cdot S_{71}^+ \cdot A_1^+ = S_{71}^+ \cdot A_1^+ \in H_6', \quad (A_7^-)^{-1} \cdot S_{73}^+ \cdot A_3^+ = S_{73}^+ \cdot A_3^+ \in H_6',$$
$$(A_7^+)^{-1} \cdot S_{71}^- \cdot A_1^+ \in H_6', \quad (A_7^+)^{-1} \cdot S_{73}^- \cdot A_3^+ \in H_6',$$

and hence

$$(A_7^+)^{-1} \cdot S_{71}^- \cdot A_1^+ \cdot (A_1^+)^{-1} \cdot (S_{71}^+)^{-1} = (A_7^+)^{-1} \cdot S_{71}^- \cdot S_{71}^- \in H_6',$$
$$(A_7^+)^{-1} \cdot S_{73}^- \cdot A_3^+ \cdot (A_3^+)^{-1} \cdot (S_{73}^+)^{-1} = (A_7^+)^{-1} \cdot S_{73}^- \cdot S_{73}^- \in H_6'.$$

Thus, $(S_{73}^- \cdot S_{73}^-)^{-1} \cdot S_{71}^- \cdot S_{71}^- \in H_6'$. But this is not true because

$$(S_{73}^- \cdot S_{73}^-)^{-1} \cdot S_{71}^- \cdot S_{71}^- = \operatorname{diag}(-1, 1, -1, 1, 1, 1, 1) \notin H_6'.$$

Note that this example can be generalized also for arbitrary $2n$-dimensional sphere ($n > 1$) embedded in $\mathbf{R}^{2n+1}$ and covered by the standard $4n + 2$-cubes.

## 5. Conclusions and open questions

Let us assume that the manifold admits restriction of the following subgroup $H_{p_1} \oplus H_{p_2} \oplus \cdots \oplus H_{p_r}$ of $H_n$ ($n = p_1 + \cdots + p_r$). Then it admits restriction to each subgroup $G$ such that

$$A \cdot H_{p_1} \oplus H_{p_2} \oplus \cdots \oplus H_{p_r} \cdot A^{-1} \leq G \leq H_n,$$

where $A$ is an arbitrary element of $H_n$. Now, the following question appears: *Considering all such subgroups $G$ which are obtained by any matrix $A \in H_n$ and any admissible subgroup $H_{p_1} \oplus H_{p_2} \oplus \cdots \oplus H_{p_r}$, do we obtain all possible subgroups $H$ of $H_n$ such that the manifold admits restriction to $H$?*

Note that the manifold admits restriction to the subgroup $H_{p_1} \oplus H_{p_2} \oplus \cdots \oplus H_{p_r}$ if and only if the tangent bundle is a Whitney sum of $\mathbf{R}^{p_1} \oplus \mathbf{R}^{p_2} \oplus \cdots \oplus \mathbf{R}^{p_r}$. Thus, if the answer to the question is positive, it is easy to obtain all required subgroups $H$ of $H_n$ for each manifold. Specially, $S^6$ does not admit any Whitney decomposition of the tangent bundle and so, $S^6$ does not admit restriction to a subgroup of $H_6$. Hence $S^6$ does not admit any complex structure, assuming that the answer of the previous question is positive.

The answer of the previous question would be negative if we consider the Lie subgroups of $GL(n; \mathbf{R})$, instead of subgroups of the discrete group $H_n$. For example, let us consider the sphere $S^2$. Because it does not admit any Whitney decomposition of the tangent bundle, it follows now that it would not admit any restriction to subgroup of $SL(2, \mathbf{R})$. Hence, $S^2$ would not admit restriction to the subgroup $U(1) \leq GL(2, \mathbf{R})$, because $\dim U(1) = 2 < 4 = \dim GL(2, \mathbf{R})$. Hence, $S^2$ would not admit complex structure, which is not true.

Note that not all coverings by $n$-cubes are satisfactory. So, let us give the following definition. The covering of a manifold $M$ by $n$-cubes is *good*, if for each subgroup $G \leq GL(n; \mathbf{R})$ such that the manifold admits transition matrices in $G$, the covering on $n$-cubes admits orientations of $n$-cubes such that transition matrices belong to $H_n' = H_n \cap G$. The conjecture asserts that for each given subgroup $G$ there exists an orientation of the $n$-cubes. So, the existence of a "good covering" is stronger conjecture of the previous one. The following question arises: *In order to obtain a good covering, is it sufficient to consider only the subgroups of the form* $A \cdot H_{p_1} \oplus H_{p_2} \oplus \cdots \oplus H_{p_r} \cdot A^{-1}$? If the answer of this question is positive, then the Example 5 shows that the sphere $S^6$ does not admit complex structure, because $S^6$ does not admit any restriction of subgroups of the form $A \cdot H_{p_1} \oplus H_{p_2} \oplus \cdots \oplus H_{p_r} \cdot A^{-1}$, and thus each covering is good. Specially, the covering in the Example 5 is also good.

Although we are not sure that "good covering(s)" exists for each manifold, the following question appears very naturally. *How can we recognize a good covering?*

Indeed, if we have a manifold given by $n$-cubes, *which are the necessary and sufficient conditions for that covering with $n$-cubes to be a good covering*?

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