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BOUNDARY GROWTH OF GENERALIZED RIESZ POTENTIALS ON THE UNIT BALL IN THE VARIABLE SETTINGS

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Abstract. Riesz decomposition theorem says that a superharmonic function is locally represented as the sum of a potential and a harmonic function. In this paper we introduce a generalized Riesz kernel and study the boundary growth for its potential as an extension of Gardiner [3] in the variable settings.

1. Introduction

In the N-dimensional Euclidean space $\mathbf{R}^N$, we use the notation $B(x,r)$ to denote the open ball centered at x of radius r, whose boundary is written as $S(x,r)$. Set $\mathbf{B} = B(0,1)$. The spherical mean of u over $S(0,r)$ is defined by

$$M(u,r) = \frac{1}{|S(0,r)|} \int_{S(0,r)} u(x)\, dS(x),$$

where $|S(0,r)| = \omega_{N-1} r^{N-1}$ with ω_{N-1} denoting the area of the unit sphere and dS denotes the surface area measure on $S(0,1)$. It is known that if u is superharmonic in $\mathbf{B}$, then $M(u,r)$ is nonincreasing. If u is superharmonic in $\mathbf{B}$ and $\lim_{r\to 1} M(u,r) > -\infty$, then u is represented as the sum of the Green potential and a harmonic function :

$$u(x) = \int_{\mathbf{B}} G(x,y)\, d\mu(y) + \text{a harmonic function};$$

see Theorem 2.3 and Remark 2.4 below. For this, let us consider a generalized Riesz kernel

$$K_{\alpha,m}(x,y) = \frac{1}{(N-\alpha)\omega_{N-1}} \times \begin{cases} I_\alpha(x-y) & \text{when } y \in B(0,1/2), \\ I_\alpha(x-y) - \sum_{\ell=0}^{m} (1-|y|^2)^\ell \phi_{\alpha,\ell}(x,y^*) & \\ & \text{when } y \in \mathbf{B} \setminus B(0,1/2), \end{cases}$$

where $I_\alpha(x-y) = |x-y|^{\alpha-N}$ $(0<\alpha<N)$ and $m \ge 0$ (see Section 2 for the definition of $\phi_{\alpha,\ell}(x,y^*)$).

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The L^q ($1 \le q < \infty$) mean over the spherical surface $S(0,r)$ for a function u is defined by

$$S_q(u,r) = \left(\frac{1}{|S(0,r)|} \int_{S(0,r)} |u(x)|^q \, dS(x) \right)^{1/q} = \left(\frac{1}{\omega_{N-1}} \int_{S(0,1)} |u(r\sigma)|^q \, dS(\sigma) \right)^{1/q}.$$

Our starting point is a result by Gardiner [3, Theorem 2] which states that when $(N-3)/(N-1) < 1/q \le (N-2)/(N-1)$ and $q \ge 1$,

$$\liminf_{r \to 1-} (1-r)^{N-1-(N-1)/q} S_q(G\mu, r) = 0$$

for a Green potential $G\mu$ on the unit ball $\mathbf{B}$. We refer the reader to [10] for the plane case and [4, Sect. 5] for versions of Gardiner's result for Riesz potentials. Moreover, in [8], the first and the third authors studied the existence of boundary limits for BLD (Beppo Levi and Deny) functions u on the unit ball $\mathbf{B}$ of $\mathbf{R}^N$ satisfying

$$\int_{\mathbf{B}} |\nabla u(x)|^p (1-|x|)^\gamma \, dx < \infty,$$

where ∇ denotes the gradient, $1 < p < \infty$ and $-1 < \gamma < p-1$. More precisely, it was shown that

$$\liminf_{r \to 1-} (1-r)^{(N-p+\gamma)/p-(N-1)/q} S_q(u,r) = 0$$

when $q > 0$ and $(N-p-1)/(p(N-1)) < 1/q < (N-p+\gamma)/(p(N-1))$.

Set

$$C(0,r) = \mathbf{B} \setminus B(0,r)$$

for $0 < r < 1$. For $m \ge 0$, denote by $M^{p(\cdot),m,\omega}(\mathbf{B})$ the family of all functions $f \in L^1_{\mathrm{loc}}(\mathbf{B})$ such that

$$\|f\|_{M^{p(\cdot),m,\omega}(\mathbf{B})} = \sup_{0<r<1} \omega(1-r) \|f\|_{L^{p(\cdot),m}(C(0,r))} < \infty$$

with a variable exponent $p(\cdot)$ (see Section 3). In connection with Gardiner's result [3] and [8, Theorem 1], our main aim in this paper is to discuss the weighted limit:

$$\liminf_{r \to 1-} (1-r)^d \omega(1-r)^p S_q\left(|K_{\alpha,m} f|^{p(r)}, r\right)$$

for $f \in M^{p(\cdot),m+1,\omega}(\mathbf{B})$, where the exponent d will be given later and

$$K_{\alpha,m} f(x) = \int_{\mathbf{B}} K_{\alpha,m}(x,y) f(y) \, dy$$

(see Theorem 4.6 below). For Riesz potentials $K_{\alpha,-1} f(x) = \int_{\mathbf{B}} I_\alpha(x-y) f(y) \, dy$, we refer to [6].

For further related results on spherical means, see e.g. [5], [7] and [9].

2. Generalized Riesz kernels

Throughout this paper, let C denote various positive constants independent of the variables in question. The symbol $g \sim h$ means that $C^{-1} h \le g \le Ch$ for some constant $C > 0$.

Write

$$|x-y|^2 = |x-y^*+ty^*|^2 = |x-y^*|^2 (1 + s/|x-y^*|^2),$$

where $t = 1-|y|^2$, $y^* = y/|y|^2$ and $s = t^2 |y^*|^2 + 2t(x-y^*) \cdot y^*$. Note that

$$(1+a+b)^\gamma = \sum_{j=0}^\infty \binom{\gamma}{j} (a+b)^j = \sum_{j=0}^\infty \sum_{k=0}^j \binom{\gamma}{j} \binom{j}{k} a^k b^{j-k}.$$

The double series converges absolutely when $|a|+|b|<1$. Hence we have the following lemma.

Lemma 2.1. [1, Lemma 2.1] *Let $x, y \in \mathbf{R}^N$ and $t \in \mathbf{R}$. If $|t||y^*| < (\sqrt{2}-1)|x-y^*|$, then*

$$|x-y^*+ty^*|^{\alpha-N} = \sum_{\ell=0}^{\infty}\left(\sum_{\ell/2\le j\le \ell} a_{\ell,j}|x-y^*|^{\alpha-N-2j}(x\cdot y^*-|y^*|^2)^{2j-\ell}|y^*|^{2(\ell-j)}\right)t^\ell$$
$$= \sum_{\ell=0}^{\infty}\phi_{\alpha,\ell}(x,y^*)t^\ell,$$

where

$$\phi_{\alpha,\ell}(x,y^*) = \sum_{\ell/2\le j\le\ell} a_{\ell,j}|x-y^*|^{\alpha-N-2j}(x\cdot y^*-|y^*|^2)^{2j-\ell}|y^*|^{2(\ell-j)}$$

and

$$a_{\ell,j} = \begin{pmatrix}(\alpha-N)/2\\ j\end{pmatrix}\begin{pmatrix} j \\ \ell-j\end{pmatrix}2^{2j-\ell}.$$

In what follows, let $m \ge 0$. Now let us define

$$K_{\alpha,m}(x,y) = c(\alpha,N)\times\begin{cases}|x-y|^{\alpha-N} & \text{when } y\in B(0,1/2),\\ |x-y|^{\alpha-N}-\displaystyle\sum_{\ell=0}^{m}(1-|y|^2)^\ell\phi_{\alpha,\ell}(x,y^*) & \\ & \text{when } y \in \mathbf{B}\setminus B(0,1/2),\end{cases}$$

where $c(\alpha,N) = 1/((N-\alpha)\omega_{N-1})$.

Lemma 2.2. (cf. [1, Lemma 2.2])

(1) *For $y \in \mathbf{B}$ and $N>2$, $\Delta K_{2,m}(\cdot,y) = \delta_y$ on $\mathbf{B}$;*

(2) *there exists a constant $C>0$ such that*

$$|K_{\alpha,m}(x,y)| \le C|x-y|^{\alpha-N}$$

for all $x,y\in\mathbf{B}$;

(3) *there exists a constant $C>0$ such that*

$$|K_{\alpha,m}(x,y)| \le C|x-y|^{\alpha-N-m-1}(1-|y|)^{m+1}$$

for all $x,y\in\mathbf{B}$.

Proof. First we show assertion (1). Consider $F_\alpha(t) = |x-y^*+ty^*|^{\alpha-N}$. Then

$$\phi_{2,\ell}(x,y^*) = F_2^{(\ell)}(0)/\ell!,$$

so that $\phi_{2,\ell}(\cdot,y^*)$ is harmonic in $\mathbf{B}$. Thus (1) follows.

Next we show assertion (2). We may assume $y\in\mathbf{B}\setminus B(0,1/2)$. Note that

$$|\phi_{\alpha,\ell}(x,y^*)| \le \sum_{\ell/2\le j\le\ell}|a_{\ell,j}||x-y^*|^{\alpha-N-2j}|x\cdot y^*-|y^*|^2|^{2j-\ell}|y^*|^{2(\ell-j)}$$
$$= \sum_{\ell/2\le j\le\ell}|a_{\ell,j}||x-y^*|^{\alpha-N-2j}|x\cdot y^*/|y^*|-|y^*||^{2j-\ell}|y^*|^\ell$$
$$= C|x-y^*|^{\alpha-N-\ell}|y^*|^\ell,$$

so that

$$|\phi_{\alpha,\ell}(x,y^*)|(1-|y|^2)^\ell \le C|x-y^*|^{\alpha-N-\ell}|y^*|^\ell(1-|y|^2)^\ell \le C|x-y^*|^{\alpha-N}$$

since

$$\frac{|x-y^*|}{1-|y|} \ge \frac{|y^*|-|x|}{1-|y|} \ge \frac{|y^*|-1}{1-|y|} = |y^*|.$$

Hence we obtain

$$|K_{\alpha,m}(x,y)| \le C\left(|x-y|^{\alpha-N} + |x-y^*|^{\alpha-N}\right) \le C|x-y|^{\alpha-N}$$

since

$$|x-y^*| > |y||x-y^*| = |x||x^*-y| = \left(|x-y|^2 + (1-|x|^2)(1-|y|^2)\right)^{1/2} > |x-y|.$$

Finally, we show assertion (3). If $1-|y|^2 \ge |x-y|/4$, then

$$|K_{\alpha,m}(x,y)| \le C|x-y|^{\alpha-N} \le C|x-y|^{\alpha-N-m-1}(1-|y|)^{m+1}.$$

Hence we show the case $1-|y|^2 < |x-y|/4$ and $1/2 \le |y| < 1$. By Taylor's theorem, one can find $0<\theta<1$ such that

$$K_{\alpha,m}(x,y) = \frac{1}{(N-\alpha)\sigma_N(m+1)!}F_\alpha^{(m+1)}(\theta(1-|y|^2))(1-|y|^2)^{m+1}.$$

Set

$$G(S) = (1+S)^{(\alpha-N)/2},$$
$$S = S(t) = \frac{2t(x-y^*)\cdot y^*}{|x-y^*|^2} + \frac{t^2|y^*|^2}{|x-y^*|^2}$$

and

$$H(t) = G(S(t)).$$

Then we see by induction on m that $H^{(m+1)}(t)$ is of the form

$$H^{(m+1)}(t) = \sum_{0\le\ell\le(m+1)/2} c_{m;\ell}G^{(m+1-\ell)}(S(t))\left(S^{(1)}(t)\right)^{m+1-2\ell}\left(S^{(2)}(t)\right)^\ell,$$

where $c_{m;\ell}$ are constants. Here note that in case $0\le t\le 1-|y|^2 \le |x-y|/4$,

$$|x-y| \le |x-y^*| \le |x-y| + |y-y^*| \le 3|x-y|/2,$$
$$|x-y^*+ty^*| \ge |x-y^*| - t|y^*| \ge |x-y|/2$$

and hence

$$-\frac{8}{9} \le \frac{(|x-y|/2)^2 - |x-y^*|^2}{|x-y^*|^2} \le S(t) \le 0.$$

Thus

$$|S^{(1)}(t)| \le \frac{2|y^*|}{|x-y^*|} + \frac{2(1-|y|^2)|y^*|^2}{|x-y^*|^2} \le C|x-y|^{-1}$$

and

$$|H^{(m+1)}(t)| \le C\sum_{0\le\ell\le(m+1)/2}|x-y|^{-(m+1-2\ell)}|x-y|^{-2\ell} \le C|x-y|^{-(m+1)}$$

when $1-|y|^2 < |x-y|/4$ and $1/2\le|y|<1$. Now we obtain

$$|K_{\alpha,m}(x,y)| \le C|x-y^*|^{\alpha-N}\left|H^{(m+1)}(\theta(1-|y|^2))\right|(1-|y|^2)^{m+1}$$
$$\le C|x-y|^{(\alpha-N)-(m+1)}(1-|y|^2)^{m+1},$$

which proves the result. □

For reader's convenience we show Riesz decomposition theorem in the following.

Theorem 2.3. [1, Theorem 5.5] *Let u be superharmonic in $\mathbf{B}$.*

(1) *If*
$$\lim_{r\to 1} M(u,r) > -\infty,$$
then
$$u(x) = \int_{\mathbf{B}} K_{2,0}(x,y)\,d\mu(y) + h(x),$$
where h is harmonic in $\mathbf{B}$.

(2) *If*
$$\liminf_{r\to 1} (1-r)^a M(u,r) > -\infty$$
for some $a > 0$, then
$$u(x) = \int_{\mathbf{B}} K_{2,m}(x,y)\,d\mu(y) + h_0(x),$$
where h_0 is harmonic in $\mathbf{B}$ and m is an integer greater than a.

Remark 2.4. Note that
$$K_{2,0}(x,y) = G(x,y) + (|y|^{2-N} - 1)|x-y^*|^{2-N}.$$
If u is superharmonic in $\mathbf{B}$ and
$$\lim_{r\to 1} M(u,r) > -\infty,$$
then
$$u(x) = \int_{\mathbf{B}} G(x,y)\,d\mu(y) + v(x) = \int_{\mathbf{B}} K_{2,0}(x,y)\,d\mu(y) + h(x),$$
where v and h are harmonic in $\mathbf{B}$.

3. Variable exponent on the unit ball

Let $p(\cdot)$ be a variable exponent on $\mathbf{B}$ such that

($p1$) $1 \le p^- \equiv \inf_{x\in\mathbf{B}} p(x) \le \sup_{x\in\mathbf{B}} p(x) \equiv p^+ < \infty$;

($p2$) $|p(x)-p(y)| \le \dfrac{c_{\mathbf{B}}}{\log(e/||x|-|y||)}$ for $x, y \in \mathbf{B}$ with a constant $c_{\mathbf{B}} > 0$.

By (p2), we see that $p(\cdot)$ is uniformly continuous on $\mathbf{B}$ and a radial function on $\mathbf{B}$. Thus we have

($p3$) there exists a constant $p \ge 1$ such that
$$|p(x) - p| \le \frac{c_{\mathbf{B}}}{\log(e/(1-|x|))} \quad \text{for } x \in \mathbf{B}.$$

For simplicity, we set $p(r) = p(x)$ with $r = |x|$. A typical example of $p(\cdot)$ is of the form
$$p(x) = p + \frac{c}{\log(e/(1-|x|))}$$
as in [2].

Let Ω be a measurable set in $\mathbf{B}$. For $m \ge 0$, the variable exponent Lebesgue spaces
$$L^{p(\cdot),m}(\Omega) = \left\{ f \in L^1_{\mathrm{loc}}(\Omega)\,;\ \int_\Omega ((1-|y|)^m |f(y)|)^{p(y)}\,dy < \infty \right\}$$

is a Banach space with respect to the norm

$$\|f\|_{L^{p(\cdot),m}(\Omega)} = \inf\left\{\lambda > 0\,;\, \int_{\Omega} \left(\frac{(1-|y|)^m|f(y)|}{\lambda}\right)^{p(y)} dy \le 1\right\}.$$

Further we consider a weight ω such that

(ω1) $\omega(r) > 0$ for $0 < r \le 1$;

(ω2) ω is almost decreasing in $(0, 1]$, that is, there is a constant $C > 0$ such that

$$\omega(t) \le C\omega(s) \quad \text{when } 0 < s < t \le 1;$$

(ω3) ω is doubling on $(0, 1]$.

Throughout this paper, we always assume that ω satisfies all of (ω1)–(ω3). We see that $\omega(r) = r^{-\nu}(\log(e + r^{-1}))^{\tau}$ is almost decreasing when $\nu > 0$ and $\tau \in \mathbf{R}$. Set

$$C(0, r) = \mathbf{B} \setminus B(0, r)$$

for $0 < r < 1$. For $m \ge 0$, denote by $M^{p(\cdot),m,\omega}(\mathbf{B})$ the family of all functions $f \in L^1_{\rm loc}(\mathbf{B})$ such that

$$\|f\|_{M^{p(\cdot),m,\omega}(\mathbf{B})} = \sup_{0<r<1} \omega(1-r)\|f\|_{L^{p(\cdot),m}(C(0,r))} < \infty.$$

Let us begin with the following elementary estimates for spherical means.

Lemma 3.1. [6, Lemma 2.1] *Let $0 < a < 1$ and c_1 be positive constants. If $y \in \mathbf{B}$ and $1/2 < t < \min\{1, c_1|y|\}$, then there exists a constant $C > 0$ such that*

$$\int_{S(0,1)} |t\sigma - y|^{a-N}\, dS(\sigma) \le C|t - |y||^{a-1}.$$

For later use, we need a version of Lemma 3.1 when $a > 1$.

Lemma 3.2. [6, Lemma 2.2] *Let $1 < a < N$ and c_1 be positive constants. If $y \in \mathbf{B}$ and $1/2 < t < \min\{1, c_1|y|\}$, then there exists a constant $C > 0$ such that*

$$\int_{\{\sigma\in S(0,1):\, |t\sigma-y|<1-t\}} |t\sigma - y|^{a-N}\, dS(\sigma) \le C(1-t)^{a-1}.$$

Set

$$I = \frac{1}{|B(x,t)|}\int_{B(x,t)\cap\mathbf{B}} |f(y)|\, dy$$

and

$$J = \left(\frac{1}{|B(x,t)|}\int_{B(x,t)\cap\mathbf{B}} |f(y)|^{p(y)}\, dy\right)^{1/p(x)},$$

where $|B(x, t)|$ denotes the volume of balls $B(x, t)$. Then I is estimated by J as follows.

Lemma 3.3. [6, Lemma 2.4] *Let $\gamma > 0$. If $J \le \beta_1 t^{-\beta_2}$ for some constants $\beta_1, \beta_2 > 0$, then there exists a constant $C > 0$ such that*

$$I \le C\,(t^{\gamma} + J)$$

for all $x \in \mathbf{B}$, $0 < t < 1$ and $f \in L^1_{\rm loc}(\mathbf{B})$, where a constant C depends only on β_1, β_2, γ and c_1.

Finally it is convenient to see the following estimates.

Lemma 3.4. [6, Lemma 2.5] *For $1/2 < r < 1$,*

$$\omega(1-r)^{p(r)} \sim \omega(1-r)^{p}, \quad (1-r)^{p(r)} \sim (1-r)^{p}.$$

4. Spherical means near the boundary

In what follows we prepare several estimates for Riesz potentials of functions in $M^{p(\cdot),m+1,\omega}(\mathbf{B})$. For this purpose, write

$$
\begin{aligned}
K_{\alpha,m}f(x) &= \int_{B(x,(1-|x|)/2)} K_{\alpha,m}(x,y)f(y)\,dy \\
&\quad + \int_{\{y\in\mathbf{B}\setminus B(x,(1-|x|)/2):\ 1-|y|\le 1-|x|\}} K_{\alpha,m}(x,y)f(y)\,dy \\
&\quad + \int_{\{y\in\mathbf{B}\setminus B(x,(1-|x|)/2):\ 1-|y|>1-|x|\}} K_{\alpha,m}(x,y)f(y)\,dy \\
&= K_1(x)+K_2(x)+K_3(x).
\end{aligned}
$$

We first give an estimate for $K_1(x)$. For this note by Lemma 2.2 (2)

$$|K_1(x)| \le C\int_{B(x,(1-|x|)/2)} |x-y|^{\alpha-N} f(y)\,dy.$$

Set

$$A(0,r) = B(0,r+(1-r)/2)\setminus B(0,r-(1-r)/2)$$

for $1/2<r<1$.

Lemma 4.1. *Let $1\le q<\infty$.*

(1) *Let $\beta>0$. Suppose*

$$(N-1)/q \le N-\alpha p.$$

Then, for $\varepsilon>0$, there exist constants $C>0$ and $1/2<r_1<1$ such that

$$
\begin{aligned}
&S_q\left(|K_1|^{p(r)},r\right) \le C\omega(1-r)^{-p}\Big\{(1-r)^{\beta}+(1-r)^{\varepsilon(2p-1)-(m+1)p} \\
&\times\int_{A(0,r)} |r-|y||^{\alpha p(r)-\varepsilon(2p(r)-1)-N+(N-1)/q}\left(\omega(1-|y|)(1-|y|)^{m+1}f(y)\right)^{p(y)}\,dy\Big\}
\end{aligned}
$$

for all $r_1<r<1$ and nonnegative measurable functions f on $\mathbf{B}$ with $\|f\|_{M^{p(\cdot),m+1,\omega}(\mathbf{B})}\le 1$.

(2) *Suppose*

$$(N-1)/q > N-\alpha p.$$

Then there exist constants $C>0$ and $1/2<r_1<1$ such that

$$S_q\left(|K_1|^{p(r)},r\right) \le C(1-r)^{(\alpha-m-1)p-N+(N-1)/q}\omega(1-r)^{-p}$$

for all $r_1<r<1$ and nonnegative measurable functions f on $\mathbf{B}$ with $\|f\|_{M^{p(\cdot),m+1,\omega}(\mathbf{B})}\le 1$.

Proof. Let f be a nonnegative measurable function on $\mathbf{B}$ with $\|f\|_{M^{p(\cdot),m+1,\omega}(\mathbf{B})}\le 1$ and let $1/2<r=|x|<1$. First we show the assertion (1). Let $\beta>0$ and let $\varepsilon>0$ such that

$$(N-1)/q < N-\alpha p+\varepsilon(2p-1).$$

We have

$$|K_1(x)| \le C \int_{B(x,(1-r)/2)} |x-y|^{\alpha-N} f(y)\, dy$$
$$\le C \int_0^{1-r} \left(\frac{1}{|B(x,t)|} \int_{B(x,t)\cap A(0,r)} f(y)\, dy \right) t^{\alpha-1}\, dt$$
$$\le C(1-r)^{\varepsilon} \int_0^{1-r} \left(\frac{1}{|B(x,t)|} \int_{B(x,t)\cap A(0,r)} t^{\alpha-2\varepsilon} f(y)\, dy \right) t^{\varepsilon-1}\, dt$$

since $B(x,(1-r)/2) \subset A(0,r)$. Take $1/2 < r_1 < 1$ such that

$$C_{1,p} = \sup_{r_1<r<1} \{\alpha p(r) - \varepsilon(2p(r)-1) - N + (N-1)/q\} < 0.$$

Letting $s = r-(1-r)/2$, we see that

$$\int_{A(0,r)} (\omega(1-r)(1-|y|)^{m+1} f(y))^{p(y)}\, dy$$
$$\le \int_{C(0,s)} \left(\omega(2(1-s)/3)(1-|y|)^{m+1} f(y)\right)^{p(y)}\, dy$$
$$\le C \int_{C(0,s)} \left(\omega(1-s)(1-|y|)^{m+1} f(y)\right)^{p(y)}\, dy \le C,$$

so that

$$\int_{A(0,r)} (\omega(1-r)(1-|y|)^{m+1} f(y))^{p(y)}\, dy \le C. \tag{4.1}$$

Note here from (4.1) that

$$\left(\frac{1}{|B(x,t)|} \int_{B(x,t)\cap A(0,r)} (\omega(1-r)(1-|y|)^{m+1} f(y))^{p(y)}\, dy \right)^{1/p(r)} \le C t^{-N/p^-}$$

for $0 < t < 1$.

Take $\gamma > 0$ such that

$$\gamma > \max\left\{\varepsilon(2-1/p^+) - \alpha, \beta/p - \alpha + m + 1\right\}.$$

By Jensen's inequality and Lemmas 3.3 and 3.4, we have

$$\left(\omega(1-r)(1-r)^{-2\varepsilon}|K_1(x)|\right)^{p(r)}$$
$$\le C\Bigg(\omega(1-r)(1-r)^{-\varepsilon-m-1}$$
$$\times \int_0^{1-r} \left(\frac{1}{|B(x,t)|} \int_{B(x,t)\cap A(0,r)} t^{\alpha-2\varepsilon}(1-|y|)^{m+1} f(y)\, dy \right) t^{\varepsilon-1}\, dt\Bigg)^{p(r)}$$
$$\le C(1-r)^{-\varepsilon-(m+1)p}$$
$$\times \int_0^{1-r} \left(\frac{1}{|B(x,t)|} \int_{B(x,t)\cap A(0,r)} t^{\alpha-2\varepsilon}\omega(1-r)(1-|y|)^{m+1} f(y)\, dy \right)^{p(r)} t^{\varepsilon-1}\, dt$$
$$= C(1-r)^{-\varepsilon-(m+1)p}$$
$$\times \int_0^{1-r} t^{(\alpha-2\varepsilon)p(r)} \left(\frac{1}{|B(x,t)|} \int_{B(x,t)\cap A(0,r)} \omega(1-r)(1-|y|)^{m+1} f(y)\, dy \right)^{p(r)} t^{\varepsilon-1}\, dt$$

$$\le C(1-r)^{-\varepsilon-(m+1)p}\Big\{\int_0^{1-r} t^{\alpha p(r)-\varepsilon(2p(r)-1)+\gamma p(r)-1}\,dt$$

$$+\int_0^{1-r} t^{(\alpha-2\varepsilon)p(r)-N}\left(\int_{B(x,t)\cap A(0,r)}\big(\omega(1-r)(1-|y|)^{m+1}f(y)\big)^{p(y)}\,dy\right)t^{\varepsilon-1}\,dt\Big\}$$

$$\le C\Big\{(1-r)^{(\alpha-m-1+\gamma)p-2\varepsilon p}+(1-r)^{-\varepsilon-(m+1)p}$$

$$\times\int_0^{1-r} t^{(\alpha-2\varepsilon)p(r)-N}\left(\int_{B(x,t)\cap A(0,r)}\big(\omega(1-r)(1-|y|)^{m+1}f(y)\big)^{p(y)}\,dy\right)t^{\varepsilon-1}\,dt\Big\}$$

$$\le C\Big\{(1-r)^{\beta-2\varepsilon p}$$

$$+(1-r)^{-\varepsilon-(m+1)p}\int_{A(0,r)}|x-y|^{\alpha p(r)-\varepsilon(2p(r)-1)-N}\big(\omega(1-|y|)(1-|y|)^{m+1}f(y)\big)^{p(y)}\,dy\Big\}$$

for $r_1<r<1$, since

$$\alpha p(r)-\varepsilon(2p(r)-1)-N<-(N-1)/q+C_{1,p}<0$$

for $r_1<r<1$. Then Minkowski's inequality and Lemma 3.1 yield

$$S_q\left(|K_1|^{p(r)},r\right)$$

$$\le C\omega(1-r)^{-p}\Big\{(1-r)^{\beta}+(1-r)^{\varepsilon(2p-1)-(m+1)p}$$

$$\times\int_{A(0,r)}S_q(|\cdot-y|^{\alpha p(r)-\varepsilon(2p(r)-1)-N},r)\,\big(\omega(1-|y|)(1-|y|)^{m+1}f(y)\big)^{p(y)}\,dy\Big\}$$

$$\le C\omega(1-r)^{-p}\Big\{(1-r)^{\beta}+(1-r)^{\varepsilon(2p-1)-(m+1)p}$$

$$\times\int_{A(0,r)}|r-|y||^{\alpha p(r)-\varepsilon(2p(r)-1)-N+(N-1)/q}\,\big(\omega(1-|y|)(1-|y|)^{m+1}f(y)\big)^{p(y)}\,dy\Big\}$$

for $r_1<r<1$, since $r\sim|y|$ on $A(0,r)$ and

$$\alpha p(r)-\varepsilon(2p(r)-1)-N+(N-1)/q\le C_{1,p}<0$$

for $r_1<r<1$. Thus assertion (1) is proved.

Next we shall show assertion (2). Let $\varepsilon>0$ such that

$$(N-1)/q>N-\alpha p+\varepsilon(p-1)>0.$$

Take $1/2<r_1<1$ such that

$$\inf_{r_1<r<1}\{\alpha p(r)-\varepsilon(p(r)-1)-N+(N-1)/q\}>0,$$
$$\sup_{r_1<r<1}\{\alpha p(r)-\varepsilon(p(r)-1)-N\}<0$$

and $\gamma>0$ such that

$$\gamma>\varepsilon(1-1/p^+)-\alpha.$$

As in the above considerations, we obtain by Lemma 3.2

$$
\begin{aligned}
&S_q\left(\left(\omega(1-r)(1-r)^{-\varepsilon}|K_1|\right)^{p(r)}, r\right) \\
&\le C(1-r)^{-\varepsilon-(m+1)p}\Big\{(1-r)^{\alpha p-\varepsilon(p-1)+\gamma p} \\
&\quad + \int_{A(0,r)} S_q(|\cdot-y|^{\alpha p(r)-\varepsilon(p(r)-1)-N}\chi_{B(y,(1-r)/2)}, r)\left(\omega(1-r)(1-|y|)^{m+1}f(y)\right)^{p(y)}\,dy\Big\} \\
&\le C(1-r)^{-\varepsilon-(m+1)p}\Big\{(1-r)^{\alpha p-\varepsilon(p-1)+\gamma p} \\
&\quad + (1-r)^{\alpha p-\varepsilon(p-1)-N+(N-1)/q}\int_{C(0,r-(1-r)/2)}\left(\omega(1-r)(1-|y|)^{m+1}f(y)\right)^{p(y)}\,dy\Big\} \\
&\le C(1-r)^{(\alpha-m-1-\varepsilon)p-N+(N-1)/q}
\end{aligned}
$$

for $r_1 < r < 1$. Thus assertion (2) is proved. □

Let $d(\cdot)$ be a valuable exponent on $[0,1)$ such that

(d1) $0 < \inf_{t\in[0,1)} d(t) \le \sup_{t\in[0,1)} d(t) < 1$;

(d2) there exists a positive constant $0 < d < 1$ such that

$$|d(t) - d| \le \frac{c_d}{\log(e/(1-t))} \quad \text{for } 0 < t < 1$$

with a constant $c_d > 0$.

Set

$$G(t) = (1-t)^d \int_{A(0,t)} |t - |y||^{-d(t)} g(y)\,dy$$

for a nonnegative measurable function g.

To complete the estimate for K_1, we use the following result.

Lemma 4.2. [7, Lemma 2.7] *Let $M > 0$. If $\sup_{0<t<1} \int_{A(0,t)} g(y)\,dy \le M$, then there exists a constant $C > 0$ such that*

$$\inf_{1-2^{-j+1}<t<1-2^{-j}} G(t) < CM \quad \textit{for each positive integer } j.$$

Next we treat $K_2(x)$. For this note from Lemma 2.2 (3) that

$$|K_2(x)| \le C \int_{\{y\in\mathbf{B}\setminus B(x,(1-|x|)/2):1-|y|\le 1-|x|\}} |x-y|^{\alpha-N-m-1}(1-|y|)^{m+1} f(y)\,dy.$$

Lemma 4.3. *Let $1 \le q < \infty$, and suppose*

$$(N-1)/q < N - (\alpha - m - 1)p.$$

Then there exists a constant $C > 0$ such that

$$S_q\left(|K_2|^{p(r)}, r\right) \le C(1-r)^{(\alpha-m-1)p-N+(N-1)/q}\omega(1-r)^{-p}$$

for all $1/2 < r < 1$ and nonnegative measurable functions f on $\mathbf{B}$ with $\|f\|_{M^{p(\cdot),m+1,\omega}(\mathbf{B})} \le 1$.

Proof. Let f be a nonnegative measurable function on $\mathbf{B}$ with $\|f\|_{M^{p(\cdot),m+1,\omega}(\mathbf{B})} \le 1$ and let $1/2 < r = |x| < 1$. Let $\varepsilon > 0$ such that

$$(N-1)/q < N - (\alpha - m - 1)p - \varepsilon(p-1).$$

We have by Lemma 2.2 (3)

$$|K_2(x)| \le C \int_{\{y\in\mathbf{B}\setminus B(x,(1-|x|)/2):\ 1-|y|\le 1-|x|\}} |x-y|^{\alpha-N-m-1}(1-|y|)^{m+1} f(y)\,dy$$
$$\le C\int_{(1-r)/2}^{2} \left(\frac{1}{|B(x,t)|}\int_{B(x,t)} f_{2,x}(y)\,dy\right) t^{\alpha-m-2}\,dt$$
$$\le C\int_{(1-r)/2}^{2} \left(\frac{1}{|B(x,t)|}\int_{B(x,t)} t^{\alpha-m-1+\varepsilon} f_{2,x}(y)\,dy\right) t^{-\varepsilon-1}\,dt,$$

where $f_{2,x}(y) = (1-|y|)^{m+1} f(y)\chi_{E_{2,x}}(y)$ with $E_{2,x} = \{y\in\mathbf{B}\setminus B(x,(1-r)/2)\colon 1-|y|\le 1-r\}$ and χ_E is the characteristic function of E.

Note from (p3) that

$$t^{p(r)} = t^p t^{p(r)-p} \le Ct^p t^{-c_{\mathbf{B}}/\log(e/(1-r))} \le Ct^p(1-r)^{-c_{\mathbf{B}}/\log(e/(1-r))} \le Ct^p$$

and

$$t^{p(r)} \ge Ct^p t^{c_{\mathbf{B}}/\log(e/(1-r))} \ge Ct^p(1-r)^{c_{\mathbf{B}}/\log(e/(1-r))} \ge Ct^p$$

for $(1-r)/2 < t < 2$. Since

$$\int_{B(x,t)} (\omega(1-r)f_{2,x}(y))^{p(y)}\,dy \le \int_{C(0,r)} \left(\omega(1-r)(1-|y|)^{m+1}f(y)\right)^{p(y)}\,dy \le C$$

by the fact that $E_{2,x}\subset C(0,r)$, we have

$$\left(\frac{1}{|B(x,t)|}\int_{B(x,t)} (\omega(1-r)f_{2,x}(y))^{p(y)}\ dy\right)^{1/p(r)} \le Ct^{-N/p^-}$$

for $(1-r)/2 < t < 2$. We have by Jensen's inequality and Lemma 3.3 with $\gamma > -\varepsilon(1-1/p)-\alpha+m+1$

$$(\omega(1-r)(1-r)^{\varepsilon}|K_2(x)|)^{p(r)}$$
$$\le C(1-r)^{\varepsilon}\int_{(1-r)/2}^{2}\left(\frac{1}{|B(x,t)|}\int_{B(x,t)} t^{\alpha-m-1+\varepsilon}\omega(1-r)f_{2,x}(y)\,dy\right)^{p(r)} t^{-\varepsilon-1}\,dt$$
$$\le C(1-r)^{\varepsilon}\left\{1+\int_{(1-r)/2}^{2} t^{(\alpha-m-1+\varepsilon)p}\left(\frac{1}{|B(x,t)|}\int_{B(x,t)}(\omega(1-r)f_{2,x}(y))^{p(y)}\,dy\right)t^{-\varepsilon-1}dt\right\}$$
$$\le C(1-r)^{\varepsilon}\left\{1+\int_{\mathbf{B}} |x-y|^{(\alpha-m-1)p+\varepsilon(p-1)-N}\left(\omega(1-r)f_{2,x}(y)\right)^{p(y)}\ dy\right\}$$

for $1/2 < r < 1$, since

$$\int_{(1-r)/2}^{2} t^{(\alpha-m-1)p+\varepsilon(p-1)+\gamma p-1}dt \le C$$

and

$$(\alpha-m-1)p+\varepsilon(p-1)-N < -(N-1)/q < 0.$$

By Lemma 3.1, we see that

$$\int_{\{\sigma\in S(0,1):\ |t\sigma-y|>(1-t)/2\}} |t\sigma-y|^{a-N}\,dS(\sigma)$$
$$\le \int_{\{\sigma\in S(0,1):\ |t\sigma-y|>(1-t)/2\}} \left(C|(1+(1-t))\sigma-y|\right)^{a-N}\ dS(\sigma)$$
$$\le C|(1+(1-t))-|y||^{a-1} \le C|1-t|^{a-1}$$

for $1/2 < t < 1$ and $1/2 < |y| < 1$, when $a < 1$. Hence Minkowski's inequality yields

$$\begin{aligned}
&S_q\left((\omega(1-r)(1-r)^{\varepsilon}|K_2|)^{p(r)}, r\right)\\
&\le C(1-r)^{\varepsilon}\\
&\quad\times\left\{1+\int_{\mathbf{B}} S_q(|\cdot-y|^{(\alpha-m-1)p+\varepsilon(p-1)-N}\chi_{E_{2,x}}(y), r)\left(\omega(1-r)(1-|y|)^{m+1}f(y)\right)^{p(y)}dy\right\}\\
&\le C(1-r)^{\varepsilon}\\
&\quad\times\left\{1+(1-r)^{(\alpha-m-1)p+\varepsilon(p-1)-N+(N-1)/q}\int_{C(0,r)}\left(\omega(1-r)(1-|y|)^{m+1}f(y)\right)^{p(y)}dy\right\}\\
&\le C(1-r)^{(\alpha-m-1+\varepsilon)p-N+(N-1)/q}
\end{aligned}$$

for $1/2 < r < 1$, since

$$(\alpha-m-1)p+\varepsilon(p-1)-N+(N-1)/q < 0.$$

Thus the assertion is proved. □

Finally we treat $K_3(x)$. Note from Lemma 2.2 (3) that

$$|K_3(x)| \le C\int_{\{y\in\mathbf{B}\setminus B(x,(1-|x|)/2):1-|y|>1-|x|\}}|x-y|^{\alpha-N-m-1}(1-|y|)^{m+1}f(y)\,dy.$$

Lemma 4.4. *Let $1 \le q < \infty$, and suppose*

(ω4) $t^{(\alpha-m-1)p+\varepsilon_0-N+(N-1)/q}\omega(t)^{-p}$ *is almost decreasing on $(0,1]$ for some $\varepsilon_0 > 0$.*

Then there exists a constant $C > 0$ such that

$$S_q\left(|K_3|^{p(r)}, r\right) \le C(1-r)^{(\alpha-m-1)p-N+(N-1)/q}\omega(1-r)^{-p}$$

for all $1/2 < r < 1$ and nonnegative measurable functions f on $\mathbf{B}$ with $\|f\|_{M^{p(\cdot),m+1,\omega}(\mathbf{B})} \le 1$.

Remark 4.5. If (ω4) holds, then

$$(\alpha-m-1)p-N+(N-1)/q < 0.$$

Proof of Lemma 4.4. Let f be a nonnegative measurable function on $\mathbf{B}$ with $\|f\|_{M^{p(\cdot),m+1,\omega}(\mathbf{B})} \le 1$ and let $1/2 < r = |x| < 1$. Note that $t^{p(r)} \sim t^p$ for $c(1-r) < t < 2$.

Let $\varepsilon > 0$ and $\varepsilon(p-1) < \varepsilon_0$. Note from ($\omega$4) that $t^{(\alpha-m-1)p+\varepsilon(p-1)-N+(N-1)/q}\omega(t)^{-p}$ is almost decreasing on $(0,1]$ and

$$(N-1)/q < N-(\alpha-m-1)p-\varepsilon(p-1).$$

We see that

$$\int_{B(0,1/4)}|x-y|^{\alpha-N-m-1}(1-|y|)^{m+1}f(y)\,dy \le C\int_{B(0,1/4)}(1-|y|)^{m+1}f(y)\,dy \le C$$

since $\|f\|_{L^{p(\cdot),m+1}(\mathbf{B})} \le \omega(1)^{-1} \le C$. As in the proof of Lemma 4.3, we have

$$\begin{aligned}
|K_3(x)| &\le C\left\{1+\int_{\mathbf{B}}|x-y|^{\alpha-N-m-1}f_{3,x}(y)\,dy\right\}\\
&\le C\left\{1+\int_{(1-r)/2}^{2}\left(\frac{1}{|B(x,t)|}\int_{B(x,t)}f_{3,x}(y)\,dy\right)t^{\alpha-m-2}\,dt\right\}\\
&\le C\left\{1+\int_{(1-r)/2}^{2}\left(\frac{1}{|B(x,t)|}\int_{B(x,t)}t^{\alpha-m-1+\varepsilon}f_{3,x}(y)\,dy\right)t^{-\varepsilon-1}\,dt\right\},
\end{aligned}$$

where $f_{3,x}(y) = (1-|y|)^{m+1} f(y)\chi_{E_{3,x}}(y)$ with

$$E_{3,x} = \{y \in \mathbf{B} \setminus (B(0,1/4) \cup B(x,(1-r)/2)) \colon 1-|y| > 1-r\}.$$

Since $\|f\|_{L^{p(\cdot),m+1}(\mathbf{B})} \le C$, we have

$$\left(\frac{1}{|B(x,t)|} \int_{B(x,t)} f_{3,x}(y)^{p(y)}\, dy \right)^{1/p(r)} \le C t^{-N/p^-}$$

for $(1-r)/2 < t < 2$. Since $r \sim |y|$ for $y \in \mathbf{B} \setminus B(0,1/4)$, in the same way as in the proof of Lemma 4.3, we see that

$$\begin{aligned}
&((1-r)^{\varepsilon}|K_3(x)|)^{p(r)} \\
&\le C\left\{ (1-r)^{\varepsilon} \left(1 + \int_{(1-r)/2}^{2} \left(\frac{1}{|B(x,t)|} \int_{B(x,t)} t^{\alpha-m-1+\varepsilon} f_{3,x}(y)\, dy \right) t^{-\varepsilon-1}\, dt \right) \right\}^{p(r)} \\
&\le C(1-r)^{\varepsilon} \left\{ 1 + \int_{\mathbf{B}} |x-y|^{(\alpha-m-1)p+\varepsilon(p-1)-N} f_{3,x}(y)^{p(y)}\, dy \right\}
\end{aligned}$$

for $1/2 < r < 1$, so that

$$\begin{aligned}
&S_q\left(((1-r)^{\varepsilon}|K_3|)^{p(r)}, r \right) \\
&\le C(1-r)^{\varepsilon} \left(1 + \int_{\mathbf{B}} S_q(|\cdot - y|^{(\alpha-m-1)p+\varepsilon(p-1)-N} \chi_{E_{3,x}}(y), r) f_{3,x}(y)^{p(y)}\, dy \right) \\
&\le C(1-r)^{\varepsilon} \left(1 + \int_{B(0,r)} (1-|y|)^{(\alpha-m-1)p+\varepsilon(p-1)-N+(N-1)/q} f_{3,x}(y)^{p(y)}\, dy \right)
\end{aligned}$$

for $1/2 < r < 1$. Let j_0 be the smallest integer such that $r \le 1 - 2^{-j_0-1}$. Note here that

$$\begin{aligned}
&\int_{B(0,r)} (1-|y|)^{(\alpha-m-1)p+\varepsilon(p-1)-N+(N-1)/q} f_{3,x}(y)^{p(y)}\, dy \\
&\le \sum_{j=0}^{j_0} \int_{B(0,1-2^{-j-1}) \setminus B(0,1-2^{-j})} (1-|y|)^{(\alpha-m-1)p+\varepsilon(p-1)-N+(N-1)/q} f_{3,x}(y)^{p(y)}\, dy \\
&\le C \sum_{j=0}^{j_0} 2^{-j((\alpha-m-1)p+\varepsilon(p-1)-N+(N-1)/q)} \int_{B(0,1-2^{-j-1}) \setminus B(0,1-2^{-j})} f_{3,x}(y)^{p(y)}\, dy \\
&\le C \sum_{j=0}^{j_0} 2^{-j((\alpha-m-1)p+\varepsilon(p-1)-N+(N-1)/q)} \omega(2^{-j})^{-p} \\
&\le C(1-r)^{(\alpha-m-1)p+\varepsilon(p-1)-N+(N-1)/q} \omega(1-r)^{-p}
\end{aligned}$$

for $1/2 < r < 1$ by $(\omega 4)$, which gives the assertion. □

We are now ready to show our main result.

Theorem 4.6. *Let $1 \le q < \infty$. Suppose $(\omega 4)$ holds for some $\varepsilon_0 > 0$.*

(1) *If*

$$N - \alpha p - 1 < (N-1)/q \le N - \alpha p,$$

then there exists a constant $C > 0$ such that

$$\liminf_{r \to 1-} (1-r)^{N-(\alpha-m-1)p-(N-1)/q} \omega(1-r)^p S_q\left(|K_{\alpha,m} f|^{p(r)}, r \right) \le C$$

for all nonnegative measurable functions f with $\|f\|_{M^{p(\cdot),m+1,\omega}(\mathbf{B})} \le 1$.

(2) *If*

$$N - \alpha p < (N-1)/q < N - (\alpha - m - 1)p,$$

then there exist constants $C > 0$ *and* $1/2 < r_0 < 1$ *such that*

$$S_q\left(|K_{\alpha,m}f|^{p(r)}, r\right) \le C(1-r)^{(\alpha-m-1)p-N+(N-1)/q}\omega(1-r)^{-p}$$

for all $r_0 < r < 1$ *and nonnegative measurable functions* f *with* $\|f\|_{M^{p(\cdot),m+1,\omega}(\mathbf{B})} \le 1$.

Proof. Let f be a nonnegative measurable function with $\|f\|_{M^{p(\cdot),m+1,\omega}(\mathbf{B})} \le 1$. For $x \in \mathbf{B}$, write

$$K_{\alpha,m}f(x) = K_1(x) + K_2(x) + K_3(x)$$

as before.

We first show assertion (1). Let $\varepsilon > 0$ such that

$$N - \alpha p - 1 + \varepsilon(2p-1) < (N-1)/q < N - \alpha p + \varepsilon(2p-1) < N - \alpha p + (m+1)p.$$

Set

$$d = -\alpha p + \varepsilon(2p-1) + N - (N-1)/q$$

and

$$d(r) = -\alpha p(r) + \varepsilon(2p(r)-1) + N - (N-1)/q.$$

Take $1/2 < r_0 < 1$ such that $r_0 \ge r_1$, $\inf_{r_0<r<1} d(r) > 0$ and $\sup_{r_0<r<1} d(r) < 1$, where r_1 is a constant appeared in Lemma 4.1. Let $r_0 < r < 1$ and $\beta > 0$. First note by Lemma 4.3 that

$$(1-r)^{N-(\alpha-m-1)p-(N-1)/q}\omega(1-r)^p S_q\left(|K_2|^{p(r)}, r\right) \le C.$$

By Lemma 4.4, we have

$$(1-r)^{N-(\alpha-m-1)p-(N-1)/q}\omega(1-r)^p S_q\left(|K_3|^{p(r)}, r\right) \le C.$$

Finally, we obtain by Lemma 4.1 (1)

$$\begin{aligned} &S_q\left(|K_1|^{p(r)}, r\right) \\ &\le C\omega(1-r)^{-p}\left\{(1-r)^{\beta} + (1-r)^{\varepsilon(2p-1)-(m+1)p}\int_{A(0,r)} |r-|y||^{-d(r)}g(y)\,dy\right\}, \end{aligned}$$

where $g(y) = (\omega(1-|y|)(1-|y|)^{m+1}f(y))^{p(y)}$.

Note here that

$$\int_{A(r)} g(y)\,dy \le C\omega(1-r)^p \int_{A(r)} ((1-|y|)^{m+1}f(y))^{p(y)}\,dy \le C.$$

Therefore

$$\begin{aligned} &(1-r)^{N-(\alpha-m-1)p-(N-1)/q}\omega(1-r)^p S_q\left(|K_1|^{p(r)}, r\right) \\ &\le C\left\{(1-r)^{N-(\alpha-m-1)p-(N-1)/q+\beta} + (1-r)^d\int_{A(0,r)} |r-|y||^{-d(r)}g(y)\,dy\right\}. \end{aligned}$$

In view of Lemma 4.2, we can find a sequence $\{r_j\}$ of positive numbers and a positive integer j_0 such that $r_{j_0} \ge r_0$, $1 - 2^{-j+1} < r_j < 1 - 2^{-j}$ and

$$\sup_{j \ge j_0} (1-r_j)^{N-(\alpha-m-1)p-(N-1)/q}\omega(1-r_j)^p S_q\left(|K_1|^{p(r_j)}, r_j\right) \le C,$$

which proves assertion (1).

Assertion (2) is obtained by Lemmas 4.1 (2), 4.3 and 4.4 . □

Setting $M^{p(\cdot),m+1,\omega}(\mathbf{B}) = M^{p,m+1,\nu}(\mathbf{B})$ when $p(x) = p$ and $\omega(r) = r^{-\nu}$ with $\nu \geq 0$, we obtain the following corollary.

Corollary 4.7. *Let $1 \leq p \leq q < \infty$.*

(1) *If*

$$\frac{1}{q} < \frac{N-(\alpha+\nu-m-1)p}{p(N-1)}$$

and

$$\frac{N-\alpha p-1}{p(N-1)} < \frac{1}{q} \leq \frac{N-\alpha p}{p(N-1)},$$

then there exists a constant $C > 0$ such that

$$\liminf_{r\to 1-} (1-r)^{N/p-\alpha-\nu+m+1-(N-1)/q} S_q(|K_{\alpha,m}f|, r) \leq C$$

for all nonnegative measurable functions f with $\|f\|_{M^{p,m+1,\nu}(\mathbf{B})} \leq 1$.

(2) *If*

$$\frac{N-\alpha p}{p(N-1)} < \frac{1}{q} < \frac{N-(\alpha+\nu-m-1)p}{p(N-1)},$$

then there exists a constant $C > 0$ such that

$$\limsup_{r\to 1-} (1-r)^{N/p-\alpha-\nu+m+1-(N-1)/q} S_q(|K_{\alpha,m}f|, r) \leq C$$

for all nonnegative measurable functions f with $\|f\|_{M^{p,m+1,\nu}(\mathbf{B})} \leq 1$.

Remark 4.8. In Theorem 4.6 (1), "lim inf" can not be replaced by "lim sup". For this purpose, we first note from the proof of Lemma 2.2 (2) that

$$K_{\alpha,m}(x,y) \geq C|x-y|^{\alpha-N} - C|x-y^*|^{\alpha-N}.$$

Hence, if $0 < \varepsilon < 1$ is small enough, then

$$K_{\alpha,m}(x,y) \geq 2^{-1}|x-y|^{\alpha-N} \quad \text{when } |x-y| < \varepsilon(1-|x|)$$

since $1-|x| \leq |x-y^*|$.

Let $p > 1$ and $1 \leq q < \infty$ satisfy

$$\frac{1}{q} < \frac{N-\alpha p}{p(N-1)}$$

and take $a \in \mathbf{R}$ such that

$$\alpha + \frac{N-1}{q} < a \leq \frac{N}{p}.$$

Let $r_j = 2^{-j}$ for each positive integer j and $\gamma > 0$. Consider the function

$$f(y) = \sum_{j=1}^{\infty} (j^{-\gamma} r_j)^{-a} \chi_{B_j},$$

where $a < m+1+N/p$, $B_j = B((1-r_j)\mathbf{e}, j^{-\gamma} r_{j+1})$ and $\mathbf{e} = (1,0,\ldots,0)$, and set

$$u(x) = \int K_{\alpha,m}(x,y) f(y)\, dy.$$

Then, for $0 < r < 1$, we have by $a \leq N/p$

$$\int_{C(0,r)} \left\{(1-|y|)^{m+1} f(y)\right\}^p \, dy \leq C \sum_{j=j_0}^{\infty} r_j^{(m+1)p} (j^{-\gamma} r_j)^{-ap+N}$$

$$\leq C(1-r)^{(m+1-a)p+N} (\log(e/(1-r)))^{-\gamma(-ap+N)},$$

where $1-r < 2^{-j_0} \le 2(1-r)$, so that $f \in M^{p,m+1,\nu}(\mathbf{B})$ with $\nu = (m+1-a)+N/p > 0$. For $x \in B_j$ with $|x| = 1 - r_j$, we have by Lemma 2.2 (2) and $\alpha < a < N$

$$\begin{aligned} u(x) &\ge C \int_{B_j} |x-y|^{\alpha-N} f(y)\,dy - C \int_{\mathbf{B}\setminus B_j} |x-y|^{\alpha-N} f(y)\,dy \\ &\ge C(j^{-\gamma} r_j)^{-a+\alpha} - C \sum_{k\ne j} |r_j - r_k|^{\alpha-N} (k^{-\gamma} r_k)^{N-a} \\ &\ge C(j^{-\gamma} r_j)^{-a+\alpha} - C \sum_{k<j} r_k^{\alpha-N} (k^{-\gamma} r_k)^{N-a} - C \sum_{k>j} r_j^{\alpha-N} (k^{-\gamma} r_k)^{N-a} \\ &\ge C(j^{-\gamma} r_j)^{-a+\alpha} - C r_j^{\alpha-N} (j^{-\gamma} r_j)^{N-a} - C r_j^{\alpha-N} (j^{-\gamma} r_j)^{N-a} \ge C(j^{-\gamma} r_j)^{-a+\alpha} \end{aligned}$$

when j is large enough, so that

$$\begin{aligned} S_q(u, 1-r_j) &\ge C \left(\frac{1}{|S(0,1-r_j)|} \int_{S(0,1-r_j)\cap B_j} (j^{-\gamma} r_j)^{(-a+\alpha)q}\, dS(x) \right)^{1/q} \\ &\ge C(j^{-\gamma} r_j)^{-a+\alpha+(N-1)/q} \end{aligned}$$

for large j. This gives

$$\begin{aligned} r_j^{N/p-(\alpha+\nu-m-1)-(N-1)/q} S_q(u, 1-r_j) &= r_j^{-(\alpha-a)-(N-1)/q} S_q(u, 1-r_j) \\ &\ge C j^{\gamma(a-\alpha-(N-1)/q)} \end{aligned}$$

for large j. Hence if $\alpha + (N-1)/q < a \le N/p$, then

$$\limsup_{r\to 1}\, (1-r)^{N/p-(\alpha+\nu-m-1)-(N-1)/q} S_q(u, r) = \infty,$$

as required.

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