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ABSOLUTELY CONTINUOUS FUNCTIONS ON COMPACT AND CONNECTED 1-DIMENSIONAL METRIC SPACES

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Abstract. In this paper, we study the absolutely continuous characterization of Sobolev functions on compact and connected 1-dimensional metric spaces X. We generalize the definition of absolutely continuous functions to such spaces and prove the equivalence between the absolutely continuous functions and Newtonian Sobolev functions. We also show that a compact and 1-Ahlfors regular metric space X supports a p-Poincaré inequality for $1 \leq p \leq \infty$ if and only if X is quasiconvex. As a result, the absolutely continuous functions are equivalent to the Sobolev functions defined via several different approaches.

1. Introduction

On the real line, the absolutely continuous function u on a compact interval $[a, b]$ is defined in the following way: for all $\epsilon > 0$, there exists a $\delta > 0$ such that for any finite collection of pairwise disjoint subintervals $[a_i, b_i] \subset [a, b]$,

$$\sum_i |u(b_i) - u(a_i)| < \epsilon$$

provided that $\sum_i |b_i - a_i| < \delta$. We denote the absolutely continuous functions on $[a, b]$ by $AC([a, b])$. An absolutely continuous function $f \in AC([a, b])$ is differentiable almost everywhere. Furthermore, if the derivative f' is p-integrable, this function f belongs to the Sobolev space $W^{1,p}([a, b])$. On the other hand, every Sobolev function in $W^{1,p}([a, b])$ has an absolutely continuous representative. Upon choosing such a representative, we can identify a Sobolev function $u \in W^{1,p}([a, b])$ with an absolutely continuous function in $AC([a, b])$ with p-integrable derivative [5]. In this work, we extend the definition of absolutely continuous functions to some general one-dimensional metric spaces and study the connection between these absolutely continuous functions and Sobolev functions.

There are many extensions of the definitions of absolutely continuous functions to general settings and some results on the connections between the absolutely continuous functions and Sobolev functions. Malý [14] introduces a class of "n-absolutely continuous functions" in $\mathbf{R}^n$ and proves that absolute continuity implies continuity, weak differentiability with gradient in L^n and some other results. Kauhanen, Koskela and Malý [12, Theorem A] show that functions $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ whose weak partial derivatives belong to the Lorentz space $L^{n,1}(\Omega)$ have n-absolutely continuous representatives in a domain $\Omega \subset \mathbf{R}^n$. Romanov [15, Theorem 2] generalizes this result to a locally s-Ahlfors regular metric space X for $1 \leq p < s$. He shows if u is

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a Poincaré Sobolev function [8] and the corresponding “Poincaré gradient” g belongs to the Lorentz space $L^{s,1}(X)$, then u has an s-absolutely continuous representative.

In a compact and connected metric space X with finite 1-dimensional Hausdorff measure, a classical result [1, Theorem 4.4.7] states that every two points can be joined by the shortest curve. By curve, we mean a continuous mapping $\gamma\colon [a,b] \to X$. The arc-length parametrization of this shortest curve is injective. We call the curves without self intersections simple curves. The above results guarantees there are sufficient simple curves in a compact, connected metric space X with finite 1-dimensional Hausdorff measure. We define the absolutely continuous functions on such spaces as follows.

Definition 1.1. Let $(X, d, \mathcal{H}^1)$ be a compact and connected metric measure space with $\mathcal{H}^1(X) < \infty$. A function $u\colon X \to \mathbf{R}$ is absolutely continuous if for any $\epsilon > 0$, there is a positive number δ such that

$$\sum_i |u \circ \gamma_i(\ell_i) - u \circ \gamma_i(0)| < \epsilon,$$

for any countable collection of pairwise disjoint arc-length parametrized simple curves $\gamma_i\colon [0, \ell_i] \to X$ with total length $\sum_i \ell_i < \delta$.

Remark 1.1. We denote the above class of absolutely continuous functions on X by $AC(X)$. Let $X = [a, b]$, it is easy to verify this definition coincides with the classical definition. It is also clear that the absolutely continuous functions defined above are uniformly continuous when X is quasiconvex.

Remark 1.2. Let $u_i = u \circ \gamma_i$ and $\mathrm{Var}_0^{\ell_i}(u_i)$ denote the total variation of u_i on the interval $[0, \ell_i]$ defined as

$$\mathrm{Var}_0^{\ell_i}(u_i) = \sup\Big\{ \sum_{k=1}^{n-1} |u \circ \gamma_i(t_k) - u \circ \gamma_i(t_{k+1})| \mid 0 \le t_1 < \cdots < t_n \le \ell_i \Big\}.$$

For simplicity, we write $\mathrm{Var}(u_i) = \mathrm{Var}_0^{\ell_i}(u_i)$. We can divide each simple curve In Definition 1.1 into smaller pieces and add the oscillation on the sub-curves. In this way, we can replace $\sum_i |u \circ \gamma_i(\ell_i) - u \circ \gamma_i(0)|$ in the definition by $\sum_i \mathrm{Var}(u_i)$ and get an equivalent characterization.

If $u \in AC([a, b])$, the pointwise derivative u' exists almost everywhere and $u' \in L^1([a, b])$. Likewise, if $u \in AC(X)$, we prove that there exists an upper gradient associated to u and this upper gradient belongs to $L^1(X)$.

Theorem 1.1. *Let $(X, d, \mathcal{H}^1)$ be a compact and connected metric measure space with $\mathcal{H}^1(X) < \infty$. If $u \in AC(X)$, then there is an upper gradient $g \in L^1(X)$ for u, that is, for any rectifiable curve $\gamma\colon [a, b] \to X$, we have*

$$|u(\gamma(a)) - u(\gamma(b))| \le \int_\gamma g.$$

The construction of the upper gradient g can be roughly described as follows: the one-dimensional space X in the assumption can be decomposed as a countable collection of the images of pariwise disjoint simple curves $\gamma_i\colon [0, \ell_i] \to X$ and a null set. On each simple curve piece, the definition that $u \in AC(X)$ guarantees the absolute continuity of $u \circ \gamma_i$ and the existence of the pointwise derivative $(u \circ \gamma_i)'(t)$ for almost everywhere $t \in [0, \ell_i]$. Then the upper gradient $g(x)$ can be defined as $(u \circ \gamma_i)'(t)$ with $t = \gamma_i^{-1}(x)$ up to a set of measure zero. To verify that this function g

is a upper gradient for u requires a careful comparison between the integral of g along an arbitrary curve $\gamma\colon [a,b] \to X$ and a integral of the pointwise derivative $(u \circ \gamma_0)'$ over $[0,\ell_0]$, where $\gamma_0\colon [0,\ell_0] \to X$ is the shortest curve connecting $\gamma(a)$ and $\gamma(b)$.

There are different approaches to extend the classical theory of Sobolev functions to metric measure spaces [2, 6, 8, 17]. In this work, we mainly employ the definition introduced by Shanmugalingam [17] and denote it by $N^{1,p}(X)$. For an absolutely continuous function $u \in AC(X)$, if the upper gradient g associated to u (defined explicitly in the proof of Theorem 1.1) is p-integrable, we write $u \in AC^p(X)$. By definition, it follows that $AC^p(X) \cap L^p(X) \subset N^{1,p}(X)$. On the other hand, we can verify that all Sobolev functions in $N^{1,p}(X)$ belong to the class $AC^p(X)$ in our setting X. Thus, we get the following result.

Theorem 1.2. *Let $1 \le p < \infty$ and $(X, d, \mathcal{H}^1)$ be a compact, connected metric measure space with $\mathcal{H}^1(X) < \infty$. Then $u \in N^{1,p}(X)$ if and only if $u \in AC^p(X)$ and $u \in L^p(X)$. In other words,*

$$N^{1,p}(X) = AC^p(X) \cap L^p(X).$$

Remark 1.3. The upper gradient g we find in Theorem 1.1 is the least upper gradient of $u \in N^{1,p}(X)$, that is, if $\rho \in L^p(X)$ is an upper gradient of $u \in N^{1,p}(X)$, then $g(x) \le \rho(x)$ holds almost everywhere. This follows from the construction of g described above and Lemma 3.1. More details can be found in the proof of Theorem 1.2.

The notion of an abstract Poincaré inequality on metric measure spaces was introduced by Heinonen and Koskela [10]. Metric measure spaces that are doubling and support an abstract Poincaré inequality provide a good structure to study the first-order analysis. In some spaces, a pure geometric characterization of the Poincaré condition is possible. For example, Durand-Cartagena, Jaramillo and Shanmugalingam [3] show a connected complete doubling metric measure space supports a ∞-Poincaré inequality if and only if it is thick quasiconvex. In Q-Ahlfors regular spaces with $Q > 1$, they also obtain a characterization of p-Poincaré condition for $p > Q - 1$ in terms of a quantitative estimate of the p-modulus of the family of all quasiconvex curves [4]. In our settings, we prove that for each compact and 1-Ahlfors regular space, supporting a p-Poincaré inequality for $1 \le p \le \infty$ is equivalent to being quasiconvex.

Theorem 1.3. *Let $(X, d, \mathcal{H}^1)$ be a compact 1-Ahlfors regular metric measure space. Then it supports p-Poincaré inequality for $1 \le p \le \infty$ if and only if X is quasiconvex.*

Let X be a complete metric space equipped with a doubling measure and $1 \le q < p$. If X supports a q-Poincaré inequality, the Sobolev spaces $N^{1,p}(X)$ defined by upper gradients, the Sobolev spaces $P^{1,p}(X)$ defined by Poincaré inequalities, and the Sobolev spaces $M^{1,p}(X)$ defined by a pointwise inequality using maximal functions all coincide [7, Theorem 11.3]. Consequently, we obtain the following corollary.

Corollary 1.4. *Let $(X, d, \mathcal{H}^1)$ be a compact, quasiconvex, 1-Ahlfors regular metric measure space and $1 < p < \infty$. Then*

$$AC^p(X) \cap L^p(X) = N^{1,p}(X) = P^{1,p}(X) = M^{1,p}(X).$$

This paper is organized in the following way. In Section 2, we review several basic definitions and some well-known results. In Section 3, we prove Theorem 1.1 and Theorem 1.2. In Section 4, we prove Theorem 1.3 and Corollary 1.4 follows.

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2. Definitions and notations

In this section, we will give some basic definitions and notations, as well as a list of several known results. Most of the results in this section can be found in [11].

2.1. s-Ahlfors regular space and doubling space.

Definition 2.1. We say that a space (X, d, μ) is s-Ahlfors regular if there is a fixed constant C, such that

$$C^{-1}r^s \le \mu(B(x,r)) \le Cr^s,$$

where $x \in X$ and $0 < r < \operatorname{diam}(X)$.

If X is s-Ahlfors regular with respect to μ, we can replace μ by the Hausdorff measure $\mathcal{H}^s$ without losing essential information [9, Exercise 8.11].

We say that μ is a doubling measure if there is a fixed constant $C > 0$ such that $\mu(B(x,2r)) \le C\mu(B(x,r))$ for each $x \in X$ and all $r > 0$. If X is equipped with a doubling measure μ, we call X is a doubling space.

2.2. Upper gradients and Newtonian Sobolev spaces $N^{1,p}(X)$. Rectifiable curve always admits an arc-length parametrization. Let $\gamma\colon [a,b] \to X$ be a rectifiable curve and $g\colon \gamma([a,b]) \to [0,\infty]$ be a Borel measure function, we define

$$\int_\gamma g := \int_0^\ell g(\tilde\gamma(t))\,dt,$$

where $\tilde\gamma\colon [0,\ell] \to X$ is the arc-length parametrization of γ.

Definition 2.2. Let $u\colon X \to \mathbf{R}$ be a Borel function. We say that a Borel function $g\colon X \to [0,\infty]$ is an upper gradient of u if

$$|u(\gamma(a)) - u(\gamma(b))| \le \int_\gamma g$$

for every rectifiable curve $\gamma\colon [a,b] \to X$.

Let $1 \le p < \infty$, $\widetilde{N}^{1,p}(X,d,\mu)$ is the class of all L^p-integrable functions on X for which there exists a p-integrable upper gradient. For each $u \in \widetilde{N}^{1,p}(X,d,\mu)$ we associate a seminorm

$$\|u\|_{\widetilde{N}^{1,p}(X,d,\mu)} = \|u\|_{L^p} + \inf_g \|g\|_{L^p},$$

where the infimum is taken over all upper gradients of u.

Definition 2.3. We define an equivalence relation in $\widetilde{N}^{1,p}(X,d,\mu)$ by $u \sim v$ if $\|u-v\|_{\widetilde{N}^{1,p}(X,d,\mu)} = 0$. Then the space $N^{1,p}(X,d,\mu)$ is defined as the quotient space $\widetilde{N}^{1,p}(X,d,\mu)/\sim$ and it is a Banach space equipped with the norm

$$\|u\|_{N^{1,p}} = \|u\|_{\widetilde{N}^{1,p}}.$$

2.3. Poincaré Sobolev spaces $P^{1,p}(X)$ and Hajłasz Sobolev spaces $M^{1,p}(X)$.

Definition 2.4. Fix $\sigma \geq 1$ and $0 < p < \infty$. We say that a pair (u, g), $u \in L^1(X)$, $0 \leq g \in L^p(X)$ satisfies the p-Poincaré inequality if the following inequality holds:

$$\fint_B |u - u_B| d\mu \leq r \left(\fint_{\sigma B} g^p d\mu \right)^{\frac{1}{p}}$$

on every ball B of radius r and $\sigma B \subset X$.

In this paper, we call g in the above inequality the "Poincaré gradient" for u. The class of $u \in L^p(X)$ for which there exists $0 \leq g \in L^p(X)$ so that the pair (u, g) satisfies the p-Poincaré inequality will be denoted by $P^{1,p}_\sigma(X)$ and

$$P^{1,p}(X) = \bigcup_{\sigma \geq 1} P^{1,p}_\sigma(X).$$

Definition 2.5. For $0 < p < \infty$ we define $M^{1,p}(X)$ to be the set of all functions $u \in L^p(X)$ for which there exists $g \geq 0$ such that

$$|u(x) - u(y)| \leq d(x, y)(g(x) + g(y)) \quad \mu\text{-a.e.}$$

Denote by $D(u)$ the class of all nonnegative Borel functions g that satisfy the above inequality. Thus $u \in M^{1,p}(X)$ if and only if $u \in L^p(X)$ and $D(u) \cap L^p(X) \neq \emptyset$. The space $M^{1,p}(X)$ is linear and we equip it with the norm

$$||u||_{M^{1,p}} = ||u||_{L^p} + \inf_{g \in D(u)} ||g||_{L^p},$$

for $1 \leq p < \infty$.

2.4. Spaces supporting a Poincaré inequality. Recall that u_B stands for an integral average, that is, $u_B = \frac{1}{\mu(B)} \int_B u \, d\mu$. A metric measure space (X, d, μ) supporting Poincaré inequality is defined in the following way.

Definition 2.6. Let $p \geq 1$. A metric measure space (X, d, μ) is said to support a p-Poincaré inequality if there exists constants $C > 0$ and $\lambda \geq 1$ such that for all measurable functions, the following holds for every pair of functions $u \colon X \to \mathbf{R}$ and $g \to [0, \infty]$ where u is measurable, and g is an upper gradient for u:

$$\fint_B |u - u_B| \, d\mu \leq C \operatorname{diam} B \left(\fint_{\lambda B} g^p \, d\mu \right)^{\frac{1}{p}},$$

on every ball B.

A metric space X is quasiconvex if every two points can be joined by a curve with length comparable to the distance of these two points. If X is complete, doubling and supports a p-Poincaré inequality for $p \geq 1$, then X is quasiconvex [2, Theorem 17.1.][8, Proposition 4.4].

The following theorem can be found in [7, Theorem 11.3].

Theorem 2.1. *Let (X, d, μ) be a complete metric space equipped with a doubling measure. If $1 < p < \infty$ and the space supports the q-Poincaré inequality for some $1 \leq q < p$, then*

$$N^{1,p}(X) = P^{1,p}(X) = M^{1,p}(X).$$

2.5. Area Formula. We need the following generalization of the Euclidean Area Formula to the case of Lipschitz maps f from the Euclidean space $\mathbf{R}^n$ into a metric space X. The proof can be found in [13, Corollary 8].

Theorem 2.2. *Let $f\colon \mathbf{R}^n \to X$ be Lipschitz. Then*

$$\int_{\mathbf{R}^n} g(x) J_n(mdf_x)\, dx = \int_X \sum_{x \in f^{-1}(y)} g(x)\, d\mathcal{H}^n(y)$$

for any Borel function $g\colon \mathbf{R}^n \to [0,\infty]$ and

$$\int_A g(f(x)) J_n(mdf_x)\, dx = \int_X g(y) \mathcal{H}^0(A \cap f^{-1}(y))\, d\mathcal{H}^n(y)$$

for $A \subset \mathbf{R}^n$ is measurable and any Borel function $g\colon X \to [0,\infty]$.

We apply the above theorem to an injective, arc-length parametrized curve. Let $f = \gamma$ and $\gamma\colon [0,\ell] \to X$. In this case, $J_1(mdf_x)$ equals to the metric derivative defined as

$$|\dot\gamma|(t) = \lim_{h \to 0} \frac{d(\gamma(t+h), \gamma(t))}{|h|},$$

and $|\dot\gamma|(t) = 1$ almost everywhere for $t \in [0,\ell]$. Let $\Gamma = \gamma([0,\ell])$ and $g : X \to [0,\infty]$ be a Borel function. It follows from Theorem 2.2 that

$$\int_0^\ell g(\gamma(s))\, ds = \int_\Gamma g(y)\, d\mathcal{H}^1(y).$$

This implies that $\mathcal{H}^1(\Gamma) = \ell$ for an injective, arc-length parametrized curve. If $N \subset X$ and $\mathcal{H}^1(N) = 0$, the above Area Formula also shows that $\mathcal{H}^1(\gamma^{-1}(N)) = 0$.

2.6. Rectifiability of 1-dimensional connected spaces. We list several important results about the parametrization of compact and connected 1-dimensional metric spaces. The first result is proved by Schul [16, Lemma 2.3] and gives a global Lipschitz parametrization of finite 1-dimensional compact, connected metric spaces.

Lemma 2.3. *Let $K \subset X$ be a compact connected set of finite $\mathcal{H}^1$ measure. Then there is a Lipschitz function $\gamma\colon [0,1] \to K$ such that* Image$(\gamma) = K$ *and $\|\gamma\|_{\mathrm{Lip}} \le 32\mathcal{H}^1(K)$. Moreover, if K is 1-Ahlfors-regular, then*

$$\frac{R}{C} \le \mathcal{H}^1(\gamma^{-1}(B(x,R))) \le CR \quad \forall x \in K,\ 0 < R \le \operatorname{diam}(K),$$

where C is a constant depending only on the 1-Ahlfors-regularity constant of the set K.

The proofs of the following two classical results can be found in [1, Theorem 4.4.7, Theorem 4.4.8].

Theorem 2.4. (First Rectifiability Theorem) *If E is complete and $C \subset E$ is a closed connected set such that $\mathcal{H}^1(C) < \infty$, then C is compact and connected by rectifiable curves.*

Actually, we can replace the rectifiable curve joining any two points x, y by geodesic, the shortest curve connecting x, y in C in the above theorem [7, Theorem 3.9].

Theorem 2.5. (Second Rectifiability Theorem) *If E is complete, $C \subset E$ is closed and connected, and $\mathcal{H}^1(C) < \infty$, then there exist countably many arc-length parametrized simple curves $\gamma_i\colon [0,\ell_i] \to C$ such that*

$$\mathcal{H}^1\Big(C \setminus \bigcup_{i=1}^\infty \gamma_i([0,\ell_i])\Big) = 0.$$

We briefly describe the construction of this parametrization. Since C is compact, we can choose $x, y \in C$ such that

$$d(x, y) = \operatorname{diam}(C).$$

By First Rectifiability Theorem (Theorem 2.4), we can join x, y by an arc-length parametrized geodesic $\gamma_0\colon [0, \ell_0] \to C$ and we denote the range of this curve as Γ_0. Suppose that we have already constructed $\Gamma_0, \cdots, \Gamma_k$ with the following properties:

(1) $\Gamma_i \subset C$, $i = 0, \cdots, k$;
(3) Each curve $\gamma_i\colon [0, \ell_i] \to C$ with $\Gamma_i = \gamma_i([0, \ell_i])$ is an arc-length parametrized geodesic;
(2) Each intersection $\Gamma_i \cap \bigcup_{j<i} \Gamma_j$ consists of a single point, for $i = 1, \cdots, k$.

Let

$$d_k = \sup_{x \in C} d\big(x, \bigcup_{i=0}^{k} \Gamma_i\big).$$

If $d_k = 0$, then $C = \bigcup_{i=0}^{k} \Gamma_i$ and we are done. If $d_k > 0$ for all k, by compactness we can choose $x_k \in C$ and $y_k \in \cup_{i=0}^{k} \Gamma_i$ such that $d(x_k, y_k) = d_k$. Connect x_k and y_k with an arc-length parametrized geodesic γ_{k+1} such that $\gamma_{k+1}(0) = x_k$ and $\gamma_{k+1}(\ell_{k+1}) = y_{k+1}$. Let

$$\tilde{t} = \inf \left\{ t \in [0, \ell_{k+1}] \mid \gamma_{k+1}(t) \in \bigcup_{i=0}^{k} \Gamma_i \right\}$$

and define $\Gamma_{k+1} = \gamma_{k+1}([0, \tilde{t}])$. We get that $\Gamma_{k+1} \subset X$ is an arc-length parametrized geodesic and the intersection of Γ_{k+1} and $\bigcup_{i=0}^{k} \Gamma_i$ consists of one single point $\gamma_{k+1}(\tilde{t})$. We can continue this construction. Since $\bigcup_{i=0}^{\infty} \Gamma_i$ may not be closed, we may have

$$X \setminus \bigcup_{i=0}^{\infty} \Gamma_i \neq \emptyset.$$

We omit the proof for $\mathcal{H}^1(X \setminus \bigcup_{i=0}^{\infty} \Gamma_i) = 0$.

Remark 2.1. From the above construction, if two curves intersect with each other, the intersection point must be the endpoint of one of them. Thus, if we remove the endpoints of these curves, they are pairwise disjoint.

Remark 2.2. Since we choose γ_i to be geodesic in each step, for any two points $x, y \in \Gamma_i$, γ_i is the shortest curve joining x and y in C.

3. Characterization of Sobolev functions by absolute continuity

3.1. Absolutely continuous functions belong to $N^{1,1}(X)$.

Proof of Theorem 1.1. By Second Rectifiability Theorem (Theorem 2.5), we know that there is a countable collection of simple curves in X and a set N_1 with $\mathcal{H}^1(N_1) = 0$ such that

$$X = \bigcup_{i=1}^{\infty} \gamma_i((0, \ell_i)) \cup N_1.$$

We denote the range of γ_i by $\Gamma_i = \gamma_i(0, \ell_i)$ and denote $u \circ \gamma_i$ by u_i. The intersection of Γ_i and Γ_j is empty for $i \neq j$. If $u \in AC(X)$, then $u_i \in AC((0, \ell_i))$. Thus, $u_i'(t)$

exists almost everywhere for $t \in (0, \ell_i)$. Moreover, we have

$$u_i(\ell_i) - u_i(0) = \int_0^{\ell_i} u_i'(s)\, ds,$$

and

$$\mathrm{Var}(u_i) = \int_0^{\ell_i} |u_i'(s)|\, ds.$$

We denote the collection of points $x \in \bigcup_{i=1}^{\infty} \Gamma_i$ such that $u_i'(\gamma_i^{-1}(x))$ does not exist as N_2, and let $N_0 = N_1 \cup N_2$. It is clear that $\mathcal{H}^1(N_0) = 0$. Then we define a function $g\colon X \to \mathbf{R}$ as follows,

$$g(x) = \begin{cases} |u_i'(\gamma_i^{-1}(x))| & \text{if } x \in X \setminus N_0, \\ \infty & \text{if } x \in N_0. \end{cases} \tag{1}$$

This function g is integrable on X. In fact, by the decomposition of X, we have

$$\int_X g\; d\mathcal{H}^1 = \int_{X\setminus N_1} g\, d\mathcal{H}^1 = \sum_i \int_{\Gamma_i} g\, d\mathcal{H}^1.$$

Apply Area Formula to each arc-length parametrized simple curve Γ_i, it follows

$$\sum_i \int_{\Gamma_i} g\, d\mathcal{H}^1 = \sum_i \int_0^{\ell_i} g(\gamma_i(s))\, ds.$$

Since the $\mathcal{L}^1(\gamma_i^{-1}(N_2)) = 0$ and by definition of the function g, we get

$$\int_X g\, d\mathcal{H}^1 = \sum_i \int_0^{\ell_i} g(\gamma_i(s))\, ds = \sum_i \int_0^{\ell_i} |u_i'(s)|\, ds = \sum_i \mathrm{Var}(u_i).$$

Since γ_i is injective and $\mathcal{H}^1(X) < \infty$, it implies that

$$\sum_i \ell_{\gamma_i} = \sum_i \mathcal{H}^1(\Gamma_i) \le \mathcal{H}^1(X) < \infty.$$

For any $\epsilon > 0$, there exists a natural number n_0 such that $\sum_{i=n_0+1}^{\infty} \ell_{\gamma_i} < \epsilon$. By Definition 1.1, it implies that $\sum_{i=n_0+1}^{\infty} \mathrm{Var}(u_i)$ can be sufficiently small. Thus,

$$\int_X g\, d\mathcal{H}^1 = \sum_i \mathrm{Var}(u_i) = \sum_{i=1}^{n_0} \mathrm{Var}(u_i) + \sum_{i=n_0+1}^{\infty} \mathrm{Var}(u_i) < \infty.$$

We next prove that g is an upper gradient for the function $u \in AC(X)$, that is, for any rectifiable curve $\gamma\colon [a,b] \to X$, we have

$$|u(\gamma(a)) - u(\gamma(b))| \le \int_\gamma g\; ds.$$

Let $\gamma\colon [a,b] \to X$ be a rectifiable curve and $\Gamma = \gamma([a,b])$. Since Γ is compact, connected and $\mathcal{H}^1(\Gamma) < \infty$, there exists a shortest curve joining $\gamma(a)$ and $\gamma(b)$ in Γ. We denote the arc-length parametrization of this injective curve by $\gamma_0\colon [0,\ell] \to \Gamma$ with

$$\gamma_0(0) = \gamma(a) \quad \textit{and} \quad \gamma_0(\ell) = \gamma(b). \tag{2}$$

Let $\Gamma_0 = \gamma_0([0,\ell])$. It follows from Area Formula that

$$\int_{\gamma_0} g\, ds = \int_0^{\ell} g(\gamma_0(s))\; ds = \int_{\Gamma_0} g(x)\, d\mathcal{H}^1 \le \int_\Gamma g(x)\, d\mathcal{H}^1 \le \int_\gamma g\, ds. \tag{3}$$

Since $u \in AC(X)$, it follows that $u_0 = u \circ \gamma_0 \in AC([0, \ell])$ and u_0' exists almost everywhere. Let $I = \{i \in \mathbf{N} \colon \mathcal{H}^1(\Gamma_0 \cap \Gamma_i) \neq 0\}$. Then Γ_0 is the union of $\bigcup_{i \in I}(\Gamma_0 \cap \Gamma_i)$ and a null set.

Let $i \in I$ and $x \in \Gamma_i \cap \Gamma_0$ such that $u_0'(\gamma_0^{-1}(x))$, $u_i'(\gamma_i^{-1}(x))$ both exist and $t = \gamma_0^{-1}(x)$ is a density point of $\gamma_0^{-1}(\Gamma_i \cap \Gamma_0)$. By definition,

$$|u_0'(t)| = \lim_{h \to 0} \frac{|u_0(t+h) - u_0(t)|}{|h|} = \lim_{h \to 0} \frac{|u(\gamma_0(t+h)) - u(\gamma_0(t))|}{|h|}.$$

Since t is a density point in the measurable set $\gamma_0^{-1}(\Gamma_i \cap \Gamma_0)$, we can take a sequence of nonzero number $\{h_k\}_{k=1}^{\infty}$ such that $t + h_k \in \gamma_0^{-1}(\Gamma_i \cap \Gamma_0)$ and $\lim_{k \to \infty} h_k = 0$. We denote $\gamma_i(s) = \gamma_0(t)$ and $\gamma_i(s_k) = \gamma_0(t + h_k)$. By Remark 2.2, the curve γ_i is the shortest curve joining $\gamma_i(s)$ and $\gamma_i(s_k)$. Since γ_0 and γ_i are both parametrized by arc-length, it follows that

$$|s - s_k| = \ell_{\gamma_i(s, s_k)} \leq \ell_{\gamma_0(t, t+h_k)} = |h_k|.$$

This implies that

$$\begin{aligned} |u_0'(t)| &= \lim_{k \to \infty} \frac{|u(\gamma_0(t+h_k)) - u(\gamma_0(t))|}{|h_k|} \\ &\leq \lim_{k \to \infty} \frac{|u(\gamma_i(s_k)) - u(\gamma_i(s))|}{|s_k - s|} = |u_i'(s)|. \end{aligned}$$

Thus,

$$|u_0'(t)| \leq |u_i'(s)| = |u_i'(\gamma_i^{-1}(\gamma_0(t)))| = g(\gamma_0(t))$$

holds almost everywhere for $t \in [0, \ell]$. In fact, Area Formula guarantees $\mathcal{H}^1(\gamma_0^{-1}(N)) = \mathcal{H}^1(N) = 0$ for any null set $N \subset X$. It follows that

$$|u_0(\ell) - u_0(0)| \leq \int_0^{\ell} |u_0'(t)|\, dt \leq \int_0^{\ell} g(\gamma_0(t))\, dt = \int_{\gamma_0} g\, ds. \tag{4}$$

Combining inequalities (2), (3), (4), we get

$$|u(\gamma(a)) - u(\gamma(b))| = |u(\gamma_0(0)) - u(\gamma_0(\ell))| = |u_0(0) - u_0(\ell)| \leq \int_{\gamma_0} g\, ds \leq \int_{\gamma} g\, ds.$$

Thus, we verify that $g \in L^1(X)$ is an upper gradient of u and the proof is complete. □

3.2. Sobolev functions in $N^{1,p}$ are absolutely continuous. To conclude the proof of Theorem 1.2, we need the following lemma [11, Proposition 6.3.3].

Lemma 3.1. *Let $u \colon X \to \mathbf{R}$ be a function and $\gamma \colon [0, \ell] \to X$ be an arc-length parametrized rectifiable curve in X. Assume that $\rho \colon X \to [0, \infty]$ is a Borel function such that ρ is integrable on γ and the pair (u, ρ) satisfies the upper gradient inequality on γ and each of its compact subcurves. Then $u \circ \gamma$ is absolutely continuous and the inequality*

$$|(u \circ \gamma)'(t)| \leq (\rho \circ \gamma)(t),$$

holds for almost every $t \in [0, \ell]$.

Proof of Theorem 1.2. If $u \in AC(X) \cap L^1(X)$, then Theorem 1.1 implies that $u \in N^{1,1}(X)$. If we further assume that $u \in AC^p(X)$, that is, the upper gradient g defined in (1) belongs to $L^p(X)$. It follows that $u \in N^{1,p}(X)$ and

$$AC^p(X) \cap L^p(X) \subset N^{1,p}(X).$$

On the other hand, if $u \in N^{1,p}(X)$, and ρ is a p-integrable upper gradient of u. For any arc-length parametrized simple rectifiable curves $\gamma\colon [0,\ell] \to X$, ρ is integrable along γ and $\mathcal{H}^1(\Gamma) = \ell$ by Area Formula. Since (u, ρ) satisfies the following upper gradient inequality

$$|u(\gamma(\ell)) - u(\gamma(0))| \le \int_0^\ell \rho(\gamma(s))\, ds = \int_\Gamma \rho(y)\, d\mathcal{H}^1(y),$$

the absolute continuity of integral implies immediately that $u \in AC(X)$.

Moreover, the upper gradient g defined in (1) is the least upper gradient for $u \in N^{1,p}(X)$. Let γ_i be the simple curves constructed in the Second Rectifiability Theorem. Lemma 3.1 implies that

$$|u_i'(t)| = |(u \circ \gamma_i)'(t)| \le (\rho \circ \gamma_i)(t).$$

For almost every point $x \in X$, the above inequality implies that

$$g(x) = |u_i'(\gamma_i^{-1}(x))| \le \rho(x).$$

The upper gradient g defined in (1) is bounded by a p-integrable function ρ almost everywhere in X. It implies that $g \in L^p(X)$ and $u \in AC^p(X)$. Thus, we get

$$N^{1,p}(X) \subset AC^p(X) \cap L^p(X).$$

The proof of Theorem 1.2 is complete. □

Note the proofs of the above two theorems also work in the case $p = \infty$.

4. Space supporting Poincaré inequality

Proof of Theorem 1.3. Since every complete and doubling metric measure space that supports a p-Poincaré inequality for $p \ge 1$ is quasiconvex [8, Proposition 4.4] [11, Theorem 8.3.2], the necessity part of the theorem is clear.

On the other hand, by Hölder's inequality, when the compact 1-Ahlfors regular metric space is quasiconvex, it suffices to show that X supports a 1-Poincaré inequality. Let $B(O, r) \subset X$ be an arbitrary ball in X and $x, y \in B$. There exists a shortest rectifiable curve connecting x and y [7, Theorem 3.9]. We denote the arc-length parametrization of this shortest curve by $\gamma\colon [0, \ell] \to X$ such that $\gamma(0) = x$ and $\gamma(\ell) = y$. At the same time, X being quasiconvex guarantees that there is a curve $\tilde\gamma(x, y)$ joining x and y with length $\ell_{\tilde\gamma(x,y)}$ such that

$$\ell \le \ell_{\tilde\gamma(x,y)} \le C d(x, y).$$

Let u be a Borel function and g be an upper gradient of u. Then

$$|u(x) - u(y)| \le \int_0^\ell g(\gamma(s))\, ds. \tag{5}$$

Denote $\Gamma = \gamma([0, \ell])$ and $\lambda = 3C$, where C is the quasiconvexity constant of X. We next verify that the whole curve $\Gamma \subset B(O, \lambda r)$. By triangle inequality, $d(z, O) \le d(z, x) + d(x, O)$. Let $\ell_{\gamma(x,z)}$ denote the length of the shortest curve joining x and z. We have $d(z, x) \le \ell_{\gamma(x,z)} \le \ell$. Thus,

$$d(z, O) \le \ell + d(x, O) \le \ell_{\tilde\gamma(x,y)} + r \le \lambda r.$$

The fact that $\Gamma \subset B(O, \lambda r)$ implies that

$$\int_0^\ell g(\gamma(s))\, ds = \int_\Gamma g(y)\, d\mathcal{H}^1 \le \int_{\lambda B} g(y)\, d\mathcal{H}^1. \tag{6}$$

Finally, combining (5) and (6), we get

$$
\begin{aligned}
|u(x)-u_B| &\le \fint_B |u(x)-u(y)|\,d\mathcal{H}^1 \le \fint_B \int_0^\ell g(\gamma(s))\;ds\,d\mathcal{H}^1 \\
&\le \fint_B \int_{\lambda B} g(x)\,d\mathcal{H}^1 d\mathcal{H}^1 = \mathcal{H}^1(\lambda B) \fint_{\lambda B} g\,d\mathcal{H}^1.
\end{aligned} \tag{7}
$$

Since X is 1-Ahlfors regular, it follows that

$$\fint_B |u(x)-u_B|\,d\mathcal{H}^1 \le C\operatorname{diam}(B) \fint_{\lambda B} g\,d\mathcal{H}^1. \qquad \square$$

Finally, combining Theorem 1.2, Theorem 1.3 and Theorem 2.1, we get Corollary 1.4.

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