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ON A POWERED BOHR INEQUALITY

Ilgiz R. Kayumov and Saminathan Ponnusamy

Kazan Federal University
Kremlevskaya 18, 420 008 Kazan, Russia; ikayumov@kpfu.ru

Indian Institute of Technology Madras, Department of Mathematics
Chennai–600 036, India; samy@isichennai.res.in, samy@iitm.ac.in

Abstract. The object of this paper is to study the powered Bohr radius ρ_p, $p \in (1,2)$, of analytic functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$ defined on the unit disk $|z| < 1$ and such that $|f(z)| < 1$ for $|z| < 1$. More precisely, if $M_p^f(r) = \sum_{k=0}^{\infty} |a_k|^p r^k$, then we show that $M_p^f(r) \leq 1$ for $r \leq r_p$ where r_ρ is the powered Bohr radius for conformal automorphisms of the unit disk. This answers the open problem posed by Djakov and Ramanujan in 2000. A couple of other consequences of our approach is also stated, including an asymptotically sharp form of one of the results of Djakov and Ramanujan. In addition, we consider a similar problem for sense-preserving harmonic mappings in $|z| < 1$. Finally, we conclude by stating the Bohr radius for the class of Bieberbach–Eilenberg functions.

1. Preliminaries and main results

Let $\mathcal{B}$ denote the class of analytic functions f defined on the unit disk $\mathbf{D} := \{z \in \mathbf{C} \colon |z| < 1\}$, with the power series expansion $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and such that $|f(z)| < 1$ for $z \in \mathbf{D}$. Then the classical Bohr's inequality states that there is a constant ρ such that

$$M^f(r) := \sum_{k=0}^{\infty} |a_k| r^k \leq 1 \quad \text{for all } r = |z| \leq \rho$$

and the value $\rho = 1/3$ is optimal. The number $\rho = 1/3$, known as Bohr's radius, was originally obtained in 1914 by Bohr [6] with $\rho = 1/6$, but subsequently later, Wiener, Riesz and Schur, independently established the sharp inequality for $r = |z| \leq 1/3$. This little article of Bohr generates intensive research activities even after a century of its appearance. We refer to the recent survey article on this topic [4] and the references therein. Multidimensional generalizations of this result were obtained by Boas and Khavinson [5] by establishing upper and lower bounds for the Bohr radius of the unit polydisk $\mathbf{D}^n$. Aizenberg [2, 3] extended the concept of Bohr radius in several different directions for further studies in this topic. In 2000, Djakov and Ramanujan [10] investigated the same phenomenon from different point of view. For $f \in \mathcal{B}$ and a fixed $p > 0$, we consider the powered Bohr sum $M_p^f(r)$ defined by

$$M_p^f(r) = \sum_{k=0}^{\infty} |a_k|^p r^k.$$

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Observe that for $p=1$, $M_p^f(r)$ reduces to the classical Bohr sum defined as above by $M^f(r)$. The best possible constant ρ_p for which

$$M_p^f(r) \leq 1 \quad \text{for all } r \leq \rho_p$$

is called the (powered) Bohr radius for the family $\mathcal{B}$.

We now introduce

$$M_p(r) := \sup_{f\in\mathcal{B}} M_p^f(r)$$

and

$$r_p := \sup\left\{r\colon a^p + \frac{r(1-a^2)^p}{1-ra^p} \leq 1,\ 0\leq a<1\right\} = \inf_{a\in[0,1)} \frac{1-a^p}{a^p(1-a^p)+(1-a^2)^p}.$$

Let us first proceed to recall the following results.

Theorem A. [10, Theorem 3] *For each $p\in(1,2]$ and $f(z)=\sum_{k=0}^\infty a_k z^k$ belongs to $\mathcal{B}$, we have $M_p^f(r)\leq 1$ for $r\leq T_p$, where*

$$m_p \leq T_p \leq r_p.$$

Here r_p is as above,

$$m_p := \frac{p}{\left(2^{1/(2-p)} + p^{1/(2-p)}\right)^{2-p}} \quad \text{for } 1<p<2,$$

and $m_2 := \lim_{p\to 2} m_p = 1$.

Theorem B. [10, Theorem 2] *For each $p\in(0,2)$,*

$$M_p(r) \asymp \left(\frac{1}{1-r}\right)^{1-p/2}.$$

Our first aim is to investigate the problem posed by Djakov and Ramanujan [10] about the Bohr radius for $M_p^f(r)$. Their question is the following.

Problem 1. [10, Question 1, p. 71] What is the exact value of the (powered) Bohr radius ρ_p, $p\in(1,2)$? Is it true that $\rho_p=r_p$?

Using the method of proofs of our recent approach from [12, 13], we solve this problem affirmatively in the following form.

Theorem 1. *If $f(z)=\sum_{k=0}^\infty a_k z^k$ belongs to $\mathcal{B}$ and $0<p\leq 2$, then*

$$M_p(r) = \max_{a\in[0,1]}\left\{a^p + \frac{r(1-a^2)^p}{1-ra^p}\right\}, \quad 0\leq r\leq 2^{p/2-1},$$

and

$$M_p(r) < \left(\frac{1}{1-r^{2/(2-p)}}\right)^{1-p/2}, \quad 2^{p/2-1} < r < 1.$$

Proofs of Theorem 1 and a couple of its corollaries will be given in Section 2.

Let us remark that $M_p(r)=1$ for $p\geq 2$ and $r<1$. So, the interesting case is to consider the problem only for $p\in(1,2)$.

One may ask about the second inequality of Theorem 1: how close it to be sharp? To get an answer to this question we will use a Bombieri–Bourgain estimate [8] which reads as follows: for a given $\varepsilon>0$, there exists a positive constant $C(\varepsilon)>0$, such that

$$M_1(\rho) \geq \frac{1}{\sqrt{1-\rho^2}} - C(\varepsilon)\left(\log\frac{1}{1-\rho}\right)^{(3/2)+\varepsilon}, \quad \rho\geq 1/\sqrt{2}.$$

The Hölder inequality implies that

$$\begin{aligned}M_1^f(r^{1/(2-p)}) &= \sum_{k=0}^{\infty} |a_k| r^{k/p} r^{(2k(p-1))/(p(2-p))} \\ &\leq \left(\sum_{k=0}^{\infty} |a_k|^p r^k\right)^{1/p} \left(\sum_{k=0}^{\infty} r^{2k/(2-p)}\right)^{1-1/p} \\ &= \left(M_p^f(r)\right)^{1/p} \frac{1}{(1-r^{2/(2-p)})^{(p-1)/p}}\end{aligned}$$

so that

$$M_p^f(r) \geq \left(\frac{1}{\sqrt{1-r^{2/(2-p)}}} - C(\varepsilon)\left(\log\frac{1}{1-r^{1/(2-p)}}\right)^{3/2+\varepsilon}\right)^p (1-r^{2/(2-p)})^{p-1}$$

for $2^{p/2-1} < r < 1$; or equivalently

$$M_p^f(r) \geq \left(\frac{1}{1-r^{2/(2-p)}}\right)^{1-p/2} - C_1(\varepsilon)(1-r^{2/(2-p)})^{(p-1)/2}\left(\log\frac{1}{1-r^{1/(2-p)}}\right)^{3/2+\varepsilon}.$$

This estimate together with the second estimate of Theorem 1 implies that

$$M_p(r) - \left(\frac{1}{1-r^{2/(2-p)}}\right)^{1-p/2} \to 0 \quad \text{as} \quad r \to 1^-$$

for $1 < p < 2$ while we do not know whether this fact is true for $p = 1$. Also the last estimate can be considered as an asymptotically sharp form of Theorem B in the case $p > 1$.

Corollary 1. *Let $p \in (1, 2)$. Then $M_p(r) = 1$ for $r \leq r_p$.*

In [16, Corollary 2.8], Paulsen et al. showed that if $f \in \mathcal{B}$, then for $r \in [0, 1)$,

$$M_1^f(r) \leq m(r) = \inf\{M(r), 1/\sqrt{1-r^2}\} \tag{1}$$

where

$$M(r) = \sup\left\{t + (1-t^2)\frac{r}{1-r} : 0 \leq t \leq 1\right\} = \begin{cases} 1 & \text{for } 0 \leq r \leq 1/3, \\ \dfrac{4r^2 + (1-r)^2}{4r(1-r)} & \text{for } 1/3 < r < 1. \end{cases}$$

In 2002, Paulsen et al. [16] raised a question whether the inequality (1) is sharp for any r with $1/3 < r < 1$. However, in 1962 this has been answered by Bombieri [7] who determined the exact value of this constant for r in the range $1/3 \leq r \leq 1/\sqrt{2}$. This constant is

$$m(r) = \frac{3 - \sqrt{8(1-r^2)}}{r}.$$

Further results on this and related topics can be found in [10, 16]. On the other hand, it is worth mentioning that the answer to the above question is indeed a consequence of Theorem 1 and so, we state it as a corollary.

Corollary 2. *We have the following sharp estimate:*

$$M_1(r) = \frac{1}{r}(3 - \sqrt{8(1-r^2)}) \quad \textit{for } r \in \left[\frac{1}{3}, \frac{1}{\sqrt{2}}\right].$$

Finally, we recall the following corollary which was proved in [13] and so we omit the proof.

Corollary 3. *Let $p \in \mathbf{N}$ and $0 \le m \le p$, $f(z) = \sum_{k=0}^{\infty} a_{pk+m} z^{pk+m}$ be analytic in $\mathbf{D}$ and $|f(z)| < 1$ in $\mathbf{D}$. Then*

$$\sum_{k=0}^{\infty} |a_{pk+m}| r^{pk+m} \le 1 \quad \text{for } r \le r_{p,m},$$

where $r_{p,m}$ is the maximal positive root of the equation

$$-6r^{p-m} + r^{2(p-m)} + 8r^{2p} + 1 = 0.$$

The extremal function has the form $z^m(z^p - a)/(1 - az^p)$, where

$$a = \left(1 - \frac{\sqrt{1 - {r_{p,m}}^{2p}}}{\sqrt{2}}\right) \frac{1}{{r_{p,m}}^{p}}.$$

Our next result concerns sense-preserving harmonic mappings defined on the unit disk. Recall that the family $\mathcal{H}$ of complex-valued harmonic functions $f = h + \overline{g}$ defined on the unit disk $\mathbf{D}$ and its univalent subfamilies are investigated in details. Here h and g are analytic on $\mathbf{D}$ with the form

$$h(z) = \sum_{k=0}^{\infty} a_k z^k \quad \text{and} \quad g(z) = \sum_{k=1}^{\infty} b_k z^k$$

so that the Jacobian of f is given by $J_f = |f_z|^2 - |f_{\overline{z}}|^2 = |h'|^2 - |g'|^2$. We say that the locally univalent harmonic mapping f is sense-preserving if $J_f(z) > 0$ in $\mathbf{D}$. We call $\omega(z) = g'(z)/h'(z)$ the complex dilatation of $f = h + \overline{g}$. Lewy's theorem implies that every harmonic function f on $\mathbf{D}$ is locally one-to-one and sense-preserving on $\mathbf{D}$ if and only if $|\omega(z)| < 1$ for $z \in \mathbf{D}$. See [9, 11] for detailed discussion on the class of univalent harmonic mappings and its geometric subclasses.

Theorem 2. *Suppose that $f(z) = h(z) + \overline{g(z)} = \sum_{k=0}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k}$ is a harmonic mapping of the disk $\mathbf{D}$, where h is a bounded function in $\mathbf{D}$ and $|g'(z)| \le |h'(z)|$ for $z \in \mathbf{D}$ (the later condition obviously holds if f is sense-preserving). If $p \in [0, 2]$ then the following sharp inequality holds*

$$|a_0|^p + \sum_{k=1}^{\infty} (|a_k|^p + |b_k|^p) r^k \le ||h||_\infty \max_{a \in [0,1]} \left\{ a^p + \frac{2r(1 - a^2)^p}{1 - ra^p} \right\}$$

for $r \le (2^{1/(p-2)} + 1)^{p/2-1}$. In the case $p > 2$ we have

$$|a_0|^p + \sum_{k=1}^{\infty} (|a_k|^p + |b_k|^p) r^k \le ||h||_\infty \max\{1, 2r\}.$$

Corollary 4. *Suppose that $f(z) = h(z) + \overline{g(z)} = \sum_{k=0}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k}$ is a sense-preserving harmonic mapping of the disk $\mathbf{D}$, where h is a bounded function in $\mathbf{D}$. Then the following sharp inequalities holds:*

$$|a_0| + \sum_{k=1}^{\infty} (|a_k| + |b_k|) r^k \le \frac{||h||_\infty}{r} (5 - 2\sqrt{6}\sqrt{1 - r^2}) \quad \text{for } \frac{1}{5} \le r \le \sqrt{\frac{2}{3}},$$

and

$$|a_0| + \sum_{k=1}^{\infty} (|a_k| + |b_k|) r^k \le ||h||_\infty \quad \text{for } r \le \frac{1}{5}.$$

Proofs of Theorem 2 and Corollary 4 will be given in Section 2. In Section 3, we discuss Bohr radius for the class of Bieberbach–Eilenberg functions.

2. Proofs of Theorems 1 and 2 and their corollaries

The proofs of the theorems rely on a couple of lemmas established by the present authors in [12] (see also [13]).

Lemma 1. [12] *Let* $|a|<1$ *and* $0<R\leq 1$. *If* $g(z)=\sum_{k=0}^{\infty} b_k z^k$ *belongs to* $\mathcal{B}$, *then the following sharp inequality holds:*

$$\sum_{k=1}^{\infty} |b_k|^2 R^k \leq R\frac{(1-|b_0|^2)^2}{1-|b_0|^2 R}.$$

Lemma 2. *For all* $p\in(0,2)$, *we have* $r_p<(1/2)^{1-p/2}=R_p$, *where* r_p *is defined as in the beginning.*

Proof. Let $r=R_p$ and set $a=(1/2)^{1-p/2}$. Then we conclude that

$$a^p+r\frac{(1-a^2)^p}{1-ra^p}=2\left(\frac{1}{2}\right)^{p/2}>1$$

which contradicts to the definition of r_p. □

Proof of Theorem 1. Let $|a_0|=a>0$ and $r\leq 2^{p/2-1}$. At first we suppose that $a>r^{1/(2-p)}$. In this case we have

$$\begin{aligned}
M_p^f(r) &= a^p+\sum_{k=1}^{\infty}\rho^k|a_k|^p\left(\frac{r}{\rho}\right)^k\\
&\leq a^p+\left(\sum_{k=1}^{\infty}\left(\rho^k|a_k|^p\right)^{2/p}\right)^{p/2}\left(\sum_{k=1}^{\infty}\left(\frac{r}{\rho}\right)^{2k/(2-p)}\right)^{1-p/2}\\
&= a^p+\left(\sum_{k=1}^{\infty}(\rho^{2/p})^k|a_k|^2\right)^{p/2}\left(\sum_{k=1}^{\infty}\left(\left(\frac{r}{\rho}\right)^{2/(2-p)}\right)^k\right)^{(2-p)/2}\\
&\leq a^p+\left(\frac{\rho^{2/p}(1-a^2)^2}{1-a^2\rho^{2/p}}\right)^{p/2}\left(\frac{(r/\rho)^{2/(2-p)}}{1-(r/\rho)^{2/(2-p)}}\right)^{(2-p)/2} \quad \text{(by Lemma 1)}\\
&= a^p+r\left(\frac{(1-a^2)^2}{1-a^2\rho^{2/p}}\right)^{p/2}\left(\frac{1}{1-(r/\rho)^{2/(2-p)}}\right)^{(2-p)/2}.
\end{aligned}$$

Setting $\rho=r^{p/2}a^{(p-2)p/2}$ we obtain the inequality

$$M_p^f(r)\leq a^p+r\frac{(1-a^2)^p}{1-ra^p},$$

which proves the theorem in the case $a>r^{1/(2-p)}$.

In the case $a\leq r^{1/(2-p)}$, we set $\rho=1$ and obtain

$$M_p^f(r)=\sum_{k=0}^{\infty}|a_k|^p r^k\leq a^p+r\frac{(1-a^2)^{p/2}}{(1-r^{2/(2-p)})^{1-p/2}}.$$

Let us remark that the inequality $M_p^f(r)\leq 1$ is valid in the cases $a=0$ and $a=r^{1/(2-p)}$. This fact can be established as a limiting case of the previous case. Finally, we let $t=a^2$. We have then to maximize the expression

$$A(t)=t^{p/2}+r\frac{(1-t)^{p/2}}{(1-r^{2/(2-p)})^{1-p/2}},\quad t\leq r^{2/(2-p)}.$$

Using differentiation we obtain the stationary point

$$t = 1 - r^{2/(2-p)}$$

which must satisfy under the restriction $t \leq r^{2/(2-p)}$ which is impossible because $r \leq 2^{p/2-1}$.

However, in the case $r > 2^{p/2-1}$ the critical point t is admissible so that

$$A(t) = t^{p/2} + r\frac{(1-t)^{p/2}}{(1-r^{2/(2-p)})^{1-p/2}} = \left(\frac{1}{1-r^{2/(2-p)}}\right)^{1-p/2}.$$

This observation shows that

$$M_p^f(r) \leq \left(\frac{1}{1-r^{2/(2-p)}}\right)^{1-p/2}, \quad 2^{p/2-1} < r < 1.$$

Now let us show that this inequality cannot be sharp. To do this we will use the method presented by Bombieri and Bourgain [8].

Suppose that the estimate sharp in this case. Then by analyzing Hölder's inequality we immediately conclude that

$$|a_k| = \sqrt{1-r^{2/(2-p)}}\, r^{k/(2-p)}, \quad k \geq 0.$$

Also it is easy to show that the extremal function must be a Blashke product with a finite degree $d \geq 1$. Computing the area, one obtains that

$$\pi d = \text{Area}\, f(\mathbf{D}) = \pi \sum_{k=1}^{\infty} k|a_k|^2 = \pi \frac{\lambda^2}{1-\lambda^2}, \quad \lambda = r^{1/(2-p)}.$$

From here we easily deduce that $d = \lambda^2/(1-\lambda^2)$ and thus, $\lambda = \sqrt{d/(d+1))}$, which gives

$$\sqrt{\frac{d}{d+1}} = r^{1/(2-p)}, \quad \text{i.e.} \quad r = \left(\frac{d}{d+1}\right)^{1-(p/2)}. \tag{2}$$

Therefore our inequality could be sharp for these values only. Now let us show that this is possible for $d = 1$ only. Using the same reasoning as in [8] (in fact we apply their considerations in which r is replaced by $r^{1/(2-p)}$) we arrive at the identity

$$\sqrt{1-r^{2/(2-p)}} = r^{d/(2-p)}$$

which together with (2) implies that

$$\sqrt{1-\frac{d}{d+1}} = \left(\frac{d}{d+1}\right)^{d/2}$$

which is equivalent to the equality

$$\frac{1}{d+1} = \left(\frac{d}{d+1}\right)^{d}.$$

From classical analysis we know that the right hand side of this equality is greater than $1/e$ for $d \geq 1$ so that $d+1 \leq e$ and from here we easily deduce that $d = 1$ which concludes the proof of Theorem 1. □

Proof of Corollary 1. Easily follows from Theorem 1 and Lemma 2. □

Proof of Corollary 2. Theorem 1 for $p = 1$ gives that

$$M_p(r) = \max_{a\in[0,1]} \left\{a + \frac{r(1-a^2)}{1-ra}\right\}.$$

By using differentiation it is easy to show that in the case $1/3 \le r \le 1/\sqrt{2}$ the maximum of the last expression is achieved at the point

$$a = \left(1 - \frac{\sqrt{1-r^2}}{\sqrt{2}}\right)\frac{1}{r}$$

and consequently, we obtain that

$$M_1(r) = \frac{1}{r}(3 - 2\sqrt{2}\sqrt{1-r^2}).$$

The proof is complete. $\square$

Proof of Theorem 2. Without lost of generality we may assume that $||h||_\infty = 1$. As in [14], the condition $|g'(z)| \le |h'(z)|$ gives that for each $r \in [0,1)$,

$$\sum_{k=1}^{\infty} |b_k|^2 r^k \le \sum_{k=1}^{\infty} |a_k|^2 r^k. \tag{3}$$

Let $|a_0| = a > 0$. Then, by using the same method as in the previous theorem in the case $a > r^{1/(2-p)}$, we obtain

$$|a_0|^p + \sum_{k=1}^{\infty}(|a_k|^p + |b_k|^p)r^k \le a^p + 2r\frac{(1-a^2)^p}{1-ra^p}.$$

In the case $a \le r^{1/(2-p)}$, we let $\rho = 1$ and obtain

$$\sum_{k=0}^{\infty} |a_k|^p r^k \le a^p + 2r\frac{(1-a^2)^{p/2}}{(1-r^{2/(2-p)})^{1-p/2}}.$$

We set $t = a^2$. We have to maximize the expression

$$B(t) = t^{p/2} + 2r\frac{(1-t)^{p/2}}{(1-r^{2/(2-p)})^{1-p/2}}, \quad t \le r^{2/(2-p)}.$$

Using differentiation we see that the function $B(t)$ is increasing on the interval

$$0 \le t \le \frac{1-r^{2/(2-p)}}{1+(2r)^{2/(2-p)} - r^{2/(2-p)}}.$$

The upper bound of this interval is greater than or equal to $2^{p/2-1}$ in the case $r \le (2^{1/(p-2)}+1)^{p/2-1}$. It means that the function $B(t)$ has maximum at the point $t = r^{2/(2-p)}$ which corresponds to the case $a = r^{1/(2-p)}$ so that we can apply our previous case. This completes the proof of Theorem 2. $\square$

Let $p = 1$ and then we apply the previous theorem. As a result, we obtain the inequality

$$|a_0| + \sum_{k=1}^{\infty}(|a_k| + |b_k|)r^k \le \max_{a\in[0,1]}\left\{a + \frac{2r(1-a^2)}{1-ra}\right\} \quad \text{for } r \le \sqrt{2/3}.$$

Straightforward calculations confirm the proof of Corollary 4.

3. Concluding remarks

Let $\mathcal{BE}$ denote the class of all functions $f(z) = \sum_{k=1}^{\infty} a_k z^k$ analytic in $\mathbf{D}$ such that $f(z_1)f(z_2) \neq 1$ for all pairs of points z_1, z_2 in $\mathbf{D}$. Each $f \in \mathcal{BE}$ is called a Bieberbach–Eilenberg function. Clearly, $\mathcal{BE}$ contains the class $\mathcal{B}_0$, where $\mathcal{B}_0 = \{f \in \mathcal{B}\colon f(0) = 0\}$. In 1970, Aharonov [1] and Nehari [15] independently showed that

$$\sum_{k=1}^{\infty} |a_k|^2 \leq 1 \quad \text{and} \quad |f(z)| \leq \frac{|z|}{\sqrt{1-|z|^2}} \tag{4}$$

hold for every $f \in \mathcal{BE}$. Equality holds only for the functions

$$f(z) = \frac{\eta z}{R \pm (\sqrt{R^2-1}) i\eta z}, \quad R > 1,\ |\eta| = 1.$$

Since $\mathcal{B}_0 \subset \mathcal{BE}$, it is natural to ask for the Bohr radius for the family $\mathcal{BE}$. Indeed, we see blow that the Bohr radius for $\mathcal{BE}$ and the class $\mathcal{B}_0$ remains the same.

Theorem 3. *Assume that $f(z) = \sum_{k=1}^{\infty} a_k z^k$ belongs to $\mathcal{BE}$. Then*

$$\sum_{k=1}^{\infty} |a_k| r^k \leq 1 \quad \text{for } |z| = r \leq 1/\sqrt{2}.$$

The number $1/\sqrt{2}$ is sharp.

Proof. Because $f \in \mathcal{BE}$ satisfies the coefficient inequality (4), it follows that

$$\sum_{k=1}^{\infty} |a_k| r^k \leq \sqrt{\sum_{k=1}^{\infty} |a_k|^2} \sqrt{\sum_{k=1}^{\infty} r^{2k}} \leq \frac{r}{\sqrt{1-r^2}}$$

which is less than or equal to 1 if $0 \leq r \leq 1/\sqrt{2}$. The number $1/\sqrt{2}$ is sharp as the function $f(z) = z(a-z)/(1-az)$ shows, where $a = 1/\sqrt{2}$. The proof is complete. □

Theorem 4. *Suppose that $f(z) = h(z) + \overline{g(z)} = \sum_{k=1}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k}$ is a harmonic mapping of the disk $\mathbf{D}$, where $h \in \mathcal{BE}$ and $|g'(z)| \leq |h'(z)|$ for $z \in \mathbf{D}$. Then for any $p \geq 1$ and $r < 1$, the following inequality holds:*

$$\sum_{k=1}^{\infty} (|a_k|^p + |b_k|^p)^{1/p} r^k \leq \max\{2^{(1/p)-1/2}, 1\} \frac{\sqrt{2}r}{\sqrt{1-r^2}}.$$

Proof. By hypothesis, (3) holds and thus, letting r approach 1, we get

$$\sum_{k=1}^{\infty} |b_k|^2 \leq \sum_{k=1}^{\infty} |a_k|^2 \leq 1.$$

Consequently, we obtain

$$\begin{aligned}
\sum_{k=1}^{\infty} (|a_k|^p + |b_k|^p)^{1/p} r^k &\leq \sqrt{\sum_{k=1}^{\infty} (|a_k|^p + |b_k|^p)^{2/p}} \sqrt{\sum_{k=1}^{\infty} r^{2k}} \\
&\leq \sqrt{\max\{2^{(2/p)-1}, 1\} \sum_{k=1}^{\infty} (|a_k|^2 + |b_k|^2)\, \frac{r}{\sqrt{1-r^2}}} \\
&\leq \max\{2^{(1/p)-1/2}, 1\} \frac{\sqrt{2}\, r}{\sqrt{1-r^2}}
\end{aligned}$$

and the proof is complete. □

Theorem 4 for $p = 1$ shows that for $r \le 1/\sqrt{5}$,

$$\sum_{k=1}^{\infty}(|a_k| + |b_k|)r^k \le 1.$$

Similarly, for $p = 2$, we see that for $r \le 1/\sqrt{3}$,

$$\sum_{k=1}^{\infty}(|a_k|^2 + |b_k|^2)^{1/2}r^k \le 1.$$

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