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A COMPACTNESS RESULT FOR BV FUNCTIONS IN METRIC SPACES

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Abstract. We prove a compactness result for bounded sequences $(u_j)_j$ of functions with bounded variation in metric spaces (X, d_j) where the space X is fixed but the metric may vary with j. We also provide an application to Carnot–Carathéodory spaces.

1. Introduction

One of the milestones in the theory of functions with bounded variation (BV) is the following Rellich–Kondrachov-type theorem: given a bounded open set $\Omega \subseteq \mathbf{R}^n$ with Lipschitz regular boundary, the space $BV(\Omega)$ of functions with bounded variation in Ω compactly embeds in $L^q(\Omega)$ for any $q \in [1, \frac{n}{n-1}[$. One notable consequence is the following property: if $(u_j)_j$ is a sequence of functions in $BV_{\mathrm{loc}}(\mathbf{R}^n)$ that are locally uniformly bounded in BV, then for any $q \in [1, \frac{n}{n-1}[$ a subsequence $(u_{j_h})_h$ converges in $L^q_{\mathrm{loc}}(\mathbf{R}^n)$.

Sobolev and BV functions in metric measure spaces have recently received a great deal of attention; to this regard we only mention the celebrated paper [7], where the authors show how the validity of Poincaré-type inequalities and a doubling property of the reference measure are enough to prove fundamental properties like Sobolev inequalities, Sobolev embeddings, Trudinger inequality, etc. We also point out a Rellich–Kondrachov-type result [7, Theorem 8.1]: if a sequence $(u_j)_j$ is bounded in some $W^{1,p}$, then a subsequence converges in some L^q.

In this paper we study similar compactness properties for sequences $(u_j)_j$ of locally uniformly bounded BV functions in metric measure spaces (X, λ, d_j) where the underlying measure space (X, λ) is fixed but the metric d_j varies with j. In our main result we prove that, if d_j converges locally uniformly to some distance d on X such that (X, λ, d) is a (locally) doubling separable metric measure space, and if the functions $u_j \colon X \to \mathbf{R}$ are locally uniformly (in j) bounded with respect to a BV-type norm in (X, d_j) and satisfy some local Poincaré inequality (with constant independent of j), then a subsequence of u_j converges in some $L^q_{\mathrm{loc}}(X, \lambda)$. See Theorem 2.1 for a precise statement. We prove Theorem 2.1 by the combined use of the Poincaré inequality and of an approximation scheme for functions by their averages on balls: these are of course very well-known ideas but, to our knowledge, this precise

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combination is novel even when the metric on X is not varying (i.e., when $d_j = d$ for any j). In particular, our strategy seems to provide a different proof of the case $p = 1$ in [7, Theorem 8.1] for separable metric spaces.

The motivation that led us to Theorem 2.1 comes from an application to the study of BV functions in *Carnot–Carathéodory* (CC) spaces. In Theorem 3.6 we indeed prove that, if $X^j = (X^j_1, \ldots, X^j_m)$ are families of smooth vector fields in $\mathbf{R}^n$ that, as $j \to \infty$, converge in $C^\infty_{\rm loc}(\mathbf{R}^n)$ to a family $X = (X_1, \ldots, X_m)$ satisfying the Chow-Hörmander condition, and if $u_j\colon \mathbf{R}^n \to \mathbf{R}$ are locally uniformly bounded in $BV_{X^j,loc}$, then a subsequence u_{j_h} converges in $L^1_{\rm loc}(\mathbf{R}^n)$ to some $u \in BV_{X,loc}(\mathbf{R}^n)$. Theorem 3.6 directly follows from Theorem 2.1 once we show that the CC distances induced by X^j converge locally uniformly to the one induced by X, and that (locally) a Poincaré inequality holds for BV_{X^j} functions with constant independent of j; these two results (Theorems 3.4 and 3.5, respectively) use in a crucial way some outcomes of the papers [1, 11].

Our interest in Theorem 3.6, in turn, was originally motivated by the study of fine properties of BV_X functions in CC spaces and, in particular, of their local properties. Here, one often needs to perform a blow-up procedure around a fixed point p: it is well-known that this produces a sequence of CC metric spaces $(\mathbf{R}^n, X^j)$ that converges to (a quotient of) a *Carnot group* structure $\mathbf{G}$. In this blow-up, the original BV_X function u_0 gives rise to a sequence $(u_j)_j$ of functions in BV_{X^j} which, up to a subsequence, will converge in $L^1_{\rm loc}$ to a $BV_{\mathbf{G},{\rm loc}}$ function u in $\mathbf{G}$. The function u (typically: a linear map, or a *jump map* taking two different values on complementary halfspaces of $\mathbf{G}$) will then provide some information on u_0 around p. We refer to [3] for more details.

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2. The main result

This section is devoted to the statement and the proof of our main result. See e.g. [10] for a definition of BV functions in metric spaces.

Theorem 2.1. *Let X be a set, $q \geq 1$, $\delta > 0$ and let d, d_j ($j \in \mathbf{N}$) be metrics on X such that (X, d) is locally compact and separable. Let λ, μ_j ($j \in \mathbf{N}$) be Radon measures on X and consider a sequence $(u_j)_j$ in $L^q_{\rm loc}(X; \lambda)$. Suppose that the following assumptions hold.*

(i) *The sequence $(d_j)_j$ converges to d in $L^\infty_{\rm loc}(X \times X)$.*

(ii) *(X, d, λ) is a locally doubling metric measure space, i.e., for any compact set $K \subseteq X$ there exist $C_D \geq 1$ and $R_D > 0$ such that*

$$\forall\, x \in K,\ \forall r \in (0, R_D) \quad \lambda(B(x, 2r)) \leq C_D \lambda(B(x, r)).$$

(iii) *For every compact set $K \subseteq X$ there exist $C_P, R_P > 0$ and $\alpha \geq 1$ such that*

$$\forall x \in K,\ \forall j \in \mathbf{N},\ \forall r \in (0, R_P) \quad \|u_j - u_j(B^j)\|_{L^q(B^j)} \leq C_P\, r^\delta \mu_j(\alpha B^j),$$

where $B^j := B^j(x, r)$ denotes a ball in (X, d_j), $\alpha B^j := B^j(x, \alpha r)$ and $u_j(B^j) := \fint_{B^j} u_j \, d\lambda$.

(iv) *For every compact set $K \subseteq X$ there exists $M_K > 0$ such that*

$$\forall j \in \mathbf{N} \quad \|u_j\|_{L^1(K;\lambda)} + \mu_j(K) \leq M_K.$$

Then there exist $u \in L^q_{\rm loc}(X; \lambda)$ and a subsequence $(u_{j_h})_h$ of $(u_j)_j$ such that $(u_{j_h})_h$ converges to u in $L^q_{\rm loc}(X; \lambda)$ as $h \to +\infty$.

Concerning the classical Euclidean case when

$$(X, d_j, \lambda) = (X, d, \lambda) = (\mathbf{R}^n, |\cdot|, \mathscr{L}^n),$$

we invite the reader to compare the assumption in (iii) with the well-known Poincaré inequality

$$\|u - u(B_r)\|_{L^q(B_r)} \le C r^\delta |Du|(B_r) \quad \forall\, q \in [1, \tfrac{n}{n-1}[\text{ with } \delta := \tfrac{n}{q} + 1 - n > 0$$

valid for for any BV function u on any ball $B_r \subseteq \mathbf{R}^n$ of radius r and where $u(B_r)$ denotes the mean value $\mathscr{L}^n(B_r)^{-1}\int_{B_r} u\, d\mathscr{L}^n$ of u in B_r, $C > 0$ is a geometric constant, and $|Du|$ denotes the total variation measure associated with u (i.e., the total variation of the distributional derivatives of u).

Proof. We recall the following result that is needed later in the proof: given a locally compact and separable metric space (X, d) and a Radon measure λ on (X, d), then there exists a sequence (K_j) of compact sets such that $K_j \subseteq \operatorname{int}(K_{j+1})$ and $\bigcup_{j\in\mathbf{N}} K_j = X$.

Let $K \subseteq X$ be a fixed compact set and let $\varepsilon > 0$. We first prove that there exists a subsequence $(u_{j_h})_h$ such that

$$\limsup_{h,k\to+\infty} \|u_{j_h} - u_{j_k}\|_{L^q(K;\lambda)} \le 2C_0\varepsilon, \tag{1}$$

for some $C_0 > 0$ depending on K only.

Consider an open set $U_1 \subseteq X$ such that $K \subseteq U_1$, $\overline{U}_1$ is compact and

$$\lambda(U_1 \setminus K) \le \frac{1}{4C_D^{\beta+3}}\lambda(K), \tag{2}$$

where β is an integer such that $2^\beta > 2\alpha$ and α is given by condition (iii). By the $5r$-covering Theorem (see e.g. [8, Theorem 1.2]) we can find a family $\{B(x_\ell, r_\ell) \colon \ell \in \mathbf{N}\}$ of pairwise disjoint balls such that $x_\ell \in K$, $0 < r_\ell < \min\{\varepsilon^{1/\delta}, R_D/4, 2\alpha R_P\}$, $\overline{B(x_\ell, 5r_\ell)} \subseteq U_1$ and

$$K \subseteq \bigcup_{\ell=0}^{\infty} \overline{B(x_\ell, 5r_\ell)}.$$

Denote for shortness $B_\ell := B(x_\ell, r_\ell)$; then

$$\lambda(K) \le \sum_{\ell=0}^{\infty} \lambda(5\overline{B}_\ell) \le \sum_{\ell=0}^{\infty} \lambda(8B_\ell) \le C_D^{\beta+3} \sum_{\ell=0}^{\infty} \lambda(\tfrac{1}{2^\beta} B_\ell) = C_D^{\beta+3} \lambda\left(\bigcup_{\ell=0}^{\infty} \tfrac{1}{2^\beta} B_\ell\right).$$

Hence we can choose $L \in \mathbf{N}$ such that

$$\lambda\left(\bigcup_{\ell=0}^{L} \tfrac{1}{2^\beta} B_\ell\right) \ge \frac{1}{2C_D^{\beta+3}}\lambda(K).$$

Taking into account (2) we easily get that $A_1 := K \cap \bigcup_{\ell=0}^{L} \frac{1}{2^\beta} B_\ell$ satisfies

$$\lambda(A_1) \ge \frac{1}{4C_D^{\beta+3}}\lambda(K).$$

For $j \in \mathbf{N}$ and $\ell = 0, \ldots, L$ set for shortness $B_\ell^j := B^j(x_\ell, r_\ell)$. By assumption (i), and since $\overline{B_\ell} \subseteq U_1$ are compact for $\ell = 0, \ldots, L$, there exists $J \in \mathbf{N}$ such that for every $j \ge J$, and for every $\ell = 0, \ldots, L$

$$\tfrac{1}{2^\beta} B_\ell \subseteq \tfrac{1}{2\alpha} B_\ell^j \quad \text{and} \quad \tfrac{1}{2} B_\ell^j \subseteq B_\ell. \tag{3}$$

Hence for every $j \geq J$ one has

$$\left|u_j\left(\tfrac{1}{2\alpha}B_\ell^j\right)\right| \leq \lambda\left(\tfrac{1}{2\alpha}B_\ell^j\right)^{-1}\|u_j\|_{L^1(U_1;\lambda)} \leq M_{\overline{U_1}}\max\{\lambda\left(\tfrac{1}{2^\beta}B_\ell\right)^{-1} : \ell=0,\dots,L\} < +\infty.$$

By Bolzano–Weierstrass Theorem we get an increasing function $\nu_1 \colon \mathbf{N} \to \mathbf{N}$ such that

(4) the sequence $\left(u_{\nu_1(j)}\left(\tfrac{1}{2\alpha}B_\ell^{\nu_1(j)}\right)\right)_j$ is convergent for every $\ell = 0,\dots,L$.

Then

$$\begin{aligned}
&\limsup_{h,k\to+\infty}\|u_{\nu_1(h)} - u_{\nu_1(k)}\|_{L^q(A_1;\lambda)} \\
&\leq \limsup_{h,k\to+\infty}\sum_{\ell=0}^{L}\Bigg(\left\|u_{\nu_1(h)} - u_{\nu_1(h)}\left(\tfrac{1}{2\alpha}B_\ell^{\nu_1(h)}\right)\right\|_{L^q\left(\frac{1}{2^\beta}B_\ell;\lambda\right)} \\
&\qquad + \left\|u_{\nu_1(k)} - u_{\nu_1(k)}\left(\tfrac{1}{2\alpha}B_\ell^{\nu_1(k)}\right)\right\|_{L^q\left(\frac{1}{2^\beta}B_\ell;\lambda\right)} \\
&\qquad + \left\|u_{\nu_1(h)}\left(\tfrac{1}{2\alpha}B_\ell^{\nu_1(h)}\right) - u_{\nu_1(k)}\left(\tfrac{1}{2\alpha}B_\ell^{\nu_1(k)}\right)\right\|_{L^q\left(\frac{1}{2^\beta}B_\ell;\lambda\right)}\Bigg)
\end{aligned}$$

and, using (3) and (4),

$$\begin{aligned}
&\leq \limsup_{h,k\to+\infty}\sum_{\ell=0}^{L}\Bigg(\left\|u_{\nu_1(h)} - u_{\nu_1(h)}\left(\tfrac{1}{2\alpha}B_\ell^{\nu_1(h)}\right)\right\|_{L^q\left(\frac{1}{2\alpha}B_\ell^{\nu_1(h)};\lambda\right)} \\
&\qquad + \left\|u_{\nu_1(k)} - u_{\nu_1(k)}\left(\tfrac{1}{2\alpha}B_\ell^{\nu_1(k)}\right)\right\|_{L^q\left(\frac{1}{2\alpha}B_\ell^{\nu_1(k)};\lambda\right)}\Bigg) \\
&\leq \limsup_{h,k\to+\infty}\sum_{\ell=0}^{L}\frac{C_P\, r_\ell^\delta}{(2\alpha)^\delta}\left(\mu_{\nu_1(h)}\left(\tfrac{1}{2}B_\ell^{\nu_1(h)}\right) + \mu_{\nu_1(k)}\left(\tfrac{1}{2}B_\ell^{\nu_1(k)}\right)\right) \\
&\leq \limsup_{h,k\to+\infty}\frac{C_P\,\varepsilon}{(2\alpha)^\delta}\left(\mu_{\nu_1(h)}\left(\overline{U}_1\right) + \mu_{\nu_1(k)}\left(\overline{U}_1\right)\right) \leq C_0\varepsilon,
\end{aligned}$$

where C_0 depends only on U_1 and thus only on K.

We proved that there exist $A_1 \subseteq K$ and a subsequence $(u_{\nu_1(h)})_h$ of $(u_j)_j$ such that

$$\lambda(K \setminus A_1) \leq \left(1 - \frac{1}{4C_D^{\beta+3}}\right)\lambda(K),$$
$$\limsup_{h,k\to+\infty}\|u_{\nu(h)} - u_{\nu(k)}\|_{L^q(A_1;\lambda)} \leq C_0\varepsilon.$$

Since the set $K_2 = K \setminus A_1$ is compact we can repeat the same argument on K_2, with $\frac{\varepsilon}{2}$ in place of ε, and paying attention to choose an open set $U_2 \subseteq U_1$ so that C_0 can be left unchanged. By a recursive argument, for every $j \in \mathbf{N}$ we get pairwise disjoint sets $A_j \subseteq K$ and subsequences $(u_{\nu_j(h)})_h$ such that for every $j \geq 1$

(a) $(u_{\nu_{j+1}(h)})_h$ is a subsequence of $(u_{\nu_j(h)})_h$;

(b) $\lambda\left(K \setminus \bigcup_{i=1}^j A_i\right) \leq \left(1 - \frac{1}{4C_D^{\beta+3}}\right)^j \lambda(K)$;

(c) $\limsup_{h,k\to+\infty}\|u_{\nu_j(h)} - u_{\nu_j(k)}\|_{L^q(A_j;\lambda)} \leq C_0 2^{1-j}\varepsilon$.

Inequality (b) immediately implies that $\lambda\,(K \setminus \bigcup_{i=1}^{\infty} A_i) = 0$. Working on the diagonal subsequence $(u_{\nu_h(h)})_h$ we can conclude that

$$
\begin{aligned}
\limsup_{h,k\to+\infty} \|u_{\nu_h(h)} - u_{\nu_k(k)}\|_{L^q(K;\lambda)} &= \limsup_{h,k\to+\infty} \|u_{\nu_h(h)} - u_{\nu_k(k)}\|_{L^q\left(\bigcup_{i=1}^{\infty} A_i;\lambda\right)} \\
&\le \sum_{i=1}^{\infty} \limsup_{h,k\to+\infty} \|u_{\nu_h(h)} - u_{\nu_k(k)}\|_{L^q(A_i;\lambda)} \le 2C_0\varepsilon.
\end{aligned} \tag{5}
$$

This proves (1).

Let us denote for simplicity $(u_h)_h$ instead of $(u_{\nu_h(h)})_h$. We now prove that for every compact set $K \subseteq X$ there exists a subsequence $(u_{j_h})_h$ of $(u_h)_h$ such that

$$
\lim_{h,k\to+\infty} \|u_{j_h} - u_{j_k}\|_{L^q(K;\lambda)} = 0. \tag{6}
$$

By (5), for every $i \in \mathbf{N}$, we can recursively build a subsequence $(u_{\nu_{i+1}(h)})_h$ of $(u_{\nu_i(h)})_h$ such that

$$
\limsup_{h,k\to+\infty} \|u_{\nu_i(h)} - u_{\nu_i(k)}\|_{L^q(K;\lambda)} \le \tfrac{2}{i+1} C_0.
$$

Then the diagonal sequence $(u_{\nu_h(h)})$ satisfies (6).

Eventually, take a sequence (K_j) of compact sets such that $K_j \subseteq \operatorname{int}(K_{j+1})$ and $\bigcup_{j\in\mathbf{N}} K_j = X$. By (6), for every $i \in \mathbf{N}$ we can recursively build a subsequence $(u_{\nu_i(h)})_h$ such that $(u_{\nu_{i+1}(h)})_h$ is a subsequence of $(u_{\nu_i(h)})_h$ and

$$
\lim_{h,k\to+\infty} \|u_{\nu_i(h)} - u_{\nu_i(k)}\|_{L^q(K_i;\lambda)} = 0.
$$

The diagonal subsequence $(u_{\nu_h(h)})_h$ will then converge to some u in $L^q_{\mathrm{loc}}(X;\lambda)$. This concludes the proof. $\square$

Remark 2.2. The careful reader will easily notice that Theorem 2.1 holds also when assumption (iii) is replaced by the following weaker one:

(iii') For every compact set $K \subseteq X$ there exist $R_P > 0, \alpha \ge 1$ and $f\colon (0,+\infty) \to (0,+\infty)$ such that $\lim_{r\to 0^+} f(r) = 0$ and

$$
\forall x \in K,\ \forall j \in \mathbf{N},\ \forall r \in (0,R_P) \quad \|u_j - u_j(B^j)\|_{L^q(B^j)} \le f(r)\,\mu_j(\alpha B^j).
$$

3. An application to Carnot–Carathéodory spaces

Let Ω be an open set in $\mathbf{R}^n$ and let $X = (X_1,\dots,X_m)$ be an m-tuple of smooth and linearly independent vector fields on $\mathbf{R}^n$, with $2 \le m \le n$. We say that an absolutely continuous curve $\gamma\colon [0,T] \to \mathbf{R}^n$ (briefly denoted by $\gamma \in AC([0,T];\mathbf{R}^n)$) is an X-subunit path joining x and y in $\mathbf{R}^n$ if $\gamma(0) = x$, $\gamma(T) = y$ and there exist $h_1,\dots,h_m\colon [0,T] \to \mathbf{R}$ with $\sum_{j=1}^m h_j^2 \le 1$ such that

$$
\dot\gamma(t) = \sum_{j=1}^m h_j(t) X_j(\gamma(t)) \quad \text{for a.e. } t \in [0,T]. \tag{7}
$$

Moreover, for every $x, y \in \mathbf{R}^n$ we define the quantity

$$
d(x,y) := \inf \bigl\{ T \in (0,+\infty) \colon \exists \gamma \in AC([0,T];\mathbf{R}^n)\ X\text{-subunit joining } x \text{ and } y \bigr\}, \tag{8}
$$

where we agree that $\inf \emptyset = +\infty$.

We will suppose in the following that the Chow–Hörmander condition holds, i.e., that for every $x \in \mathbf{R}^n$ the vector space spanned by $X_1,\dots,X_m$ and their commutators of any order computed at x is the whole $\mathbf{R}^n$. By the Chow–Rashevsky Theorem, if the Chow–Hörmander condition holds, the function d defined above is a distance and the couple $(\mathbf{R}^n, X)$ (or equivalently $(\mathbf{R}^n, d)$) is called *Carnot–Carathéodory space* (CC space for short). It is well known that d and the Euclidean distance d_e induce on $\mathbf{R}^n$ the same topology (see [13]).

We denote balls induced by d by $B(x,r)$ and Euclidean balls by $B_e(x,r)$. As customary in the literature, in what follows we also suppose that the metric balls $B(x,r)$ are bounded with respect to the Euclidean metric. One consequence of this assumption is the existence of geodesics, i.e., for any $x, y \in \mathbf{R}^n$ the infimum in (8) (as well as the one in (9) below) is indeed a minimum; see e.g. [12, Theorem 1.4.4].

For $j \in \mathbf{N}$ let $X^j = (X_1^j, \dots, X_m^j)$ be a family of linearly independent vector fields such that, for every fixed $i = 1, \dots, m$, X_i^j converges to X_i in $C^\infty_{\mathrm{loc}}(\mathbf{R}^n)$ as $j \to \infty$. We denote by d_j, $j \in \mathbf{N}$, the CC distance associated with X^j. If $h \in L^\infty([0,T]; \mathbf{R}^m)$ with $\|h\| \le 1$, $T > 0$ and $x \in \mathbf{R}^n$, it is convenient to define $\gamma_{h,x}, \gamma_{h,x}^j \colon [0,T] \to \mathbf{R}^n$ as the AC curves such that $\gamma_{h,x}(0) = \gamma_{h,x}^j(0) = x$ and for almost every $t \in [0,T]$

$$\dot\gamma_{h,x}(t) = \sum_{i=1}^m h_i(t) X_i(\gamma_{h,x}(t)), \quad \dot\gamma_{h,x}^j(t) = \sum_{i=1}^m h_i(t) X_i^j(\gamma_{h,x}^j(t)).$$

With this notation, an equivalent definition of the CC distance is

$$d(x,y) = \inf\{\|h\|_{L^\infty(0,1)} \colon h \in L^\infty([0,1]; \mathbf{R}^m) \text{ and } \gamma_{h,x}(1) = y\}. \tag{9}$$

The boundedness of metric balls implies that, for every $T > 0$ and $h \in L^\infty([0,T]; \mathbf{R}^m)$, the curve $\gamma_{h,x}$ is well-defined on $[0,T]$.

It can be easily seen that, if the Chow–Hörmander condition holds, then for every compact set $K \subseteq \mathbf{R}^n$ there exists an integer $s(K)$ such that the following holds: for any $x \in K$, $X_1, \dots, X_m$ and their commutators up to order $s(K)$ computed at x span the whole $\mathbf{R}^n$. The following theorem gives a sort of quantitative version of some of the celebrated results of [13]. The proof of Theorem 3.1 follows fairly easily from [1, 11] (see in particular [1, Proposition 5.8 and Claim 3.3]) and from the following observation: for any compact set $K \subseteq \mathbf{R}^n$ there exists $J \in \mathbf{N}$ such that, for any $x \in K$ and $j \ge J$, the vector fields $X_1^j, \dots, X_m^j$ and their commutators up to order $s(K)$ computed at x span the whole $\mathbf{R}^n$.

Theorem 3.1. *For every compact set $K \subseteq \mathbf{R}^n$ there exist $J_0 \in \mathbf{N}$ and $C_K > 0$ such that for every $x, y \in K$ and $j \ge J_0$*

$$\frac{1}{C_K}|x-y| \le d(x,y) \le C_K |x-y|^{1/s(K)}$$
$$\frac{1}{C_K}|x-y| \le d_j(x,y) \le C_K |x-y|^{1/s(K)}.$$

We aim at proving that the sequence of distances d_j converges to d locally uniformly; we need some preparatory lemmata.

Lemma 3.2. *Let K be a compact set in $\mathbf{R}^n$. Then for every $T > 0$, there exist $J_1 = J_1(K,T) \in \mathbf{N}$ and $R = R(K,T) > 0$ such that for every $x \in K$, $h \in L^\infty([0,T]; \mathbf{R}^m)$ with $\|h\| \le 1$ and any $j \ge J_1$ the following hold:*

(a) *the curve $\gamma_{h,x}^j$ is well defined on $[0,T]$;*

(b) *$\gamma_{h,x}^j([0,T]) \subseteq B_e(0,R)$.*

Proof. Define first

$$K' := \{\gamma_{h,x}(T) \colon x \in K, h \in L^\infty([0,T]; \mathbf{R}^m), \|h\| \le 1\} = \bigcup_{x \in K} \overline{B(x,T)}.$$

Since metric balls are bounded, also K' is bounded. We can therefore find $R > 0$ such that $K' \subseteq B_e(0,R)$ and $d_e(K', \mathbf{R}^n \setminus B_e(0,R)) > 1$. Choose $J_1 \in \mathbf{N}$ such that for every $j \geq J_1$

$$T\left(\sum_{i=1}^m \sup_{B_e(0,R)} |X_i^j - X_i|\right) e^{mCT} \leq \frac{1}{2},$$

where $C > 0$ will be determined later. Let $h \in L^\infty([0,T];\mathbf{R}^m)$ and $j \geq J_1$ be fixed; define

$$t_j := \sup\{t > 0\colon \gamma^j_{h,x} \text{ is well-defined on } [0,t] \text{ and } \gamma^j_{h,x}([0,t]) \subseteq B_e(0,R)\}$$

and suppose by contradiction that $t_j < T$. Then $\gamma^j_{h,x}(t_j) \in \partial B_e(0,R)$ and for every $\tau < t_j$ one has

$$\begin{aligned}\left|\gamma^j_{h,x}(\tau) - \gamma_{h,x}(\tau)\right| &\leq \int_0^\tau \sum_{i=1}^m \left|h_i(s) X_i^j(\gamma^j_{h,x}(s)) - h_i(s) X_i(\gamma_{h,x}(s))\right| ds \\ &\leq \int_0^\tau \sum_{i=1}^m \left|X_i^j(\gamma^j_{h,x}(s)) - X_i^j(\gamma_{h,x}(s))\right| ds \\ &\quad + \int_0^\tau \sum_{i=1}^m \left|X_i^j(\gamma_{h,x}(s)) - X_i(\gamma_{h,x}(s))\right| ds.\end{aligned}$$

Notice that, since X_i^j is converging to X_i locally in C^1, and since $\gamma^j_{h,x}(s), \gamma_{h,x}(s) \in B_e(0,R)$, the Lipschitz constants

$$c_i^j := \sup_{x,y \in B_e(0,R)} \frac{|X_i^j(x) - X_i^j(y)|}{|x-y|}$$

are converging to the Lipschitz constant $c_i := \sup_{x,y\in B_e(0,R)} \frac{|X_i(x)-X_i(y)|}{|x-y|}$. Therefore there exists $C > 0$ such that $c_i^j, c_i \leq C$ for any $j \in \mathbf{N}$ and $i = 1, \dots, m$, which gives

$$\left|\gamma^j_{h,x}(\tau) - \gamma_{h,x}(\tau)\right| \leq \int_0^\tau \left(mC\left|\gamma^j_{h,x}(s) - \gamma_{h,x}(s)\right| + \sum_{i=1}^m \sup_{B_e(0,R)} |X_i^j - X_i| \right) ds.$$

We can therefore apply Grönwall's Lemma (see [6]) to get

$$\left|\gamma^j_{h,x}(t_j) - \gamma_{h,x}(t_j)\right| \leq t_j \left(\sum_{i=1}^m \sup_{B_e(0,R)} |X_i^j - X_i|\right) e^{mCt_j} \leq \frac{1}{2}.$$

Notice that $\gamma_{h,x}(t_j) \in K'$ and $\gamma^j_{h,x}(t_j) \in \partial B_e(0,R)$: this contradicts the definition of R, giving $t_j = T$. The lemma is proved. □

Lemma 3.3. *Fix $\varepsilon \in (0,1)$ and a compact set K in $\mathbf{R}^n$. Then, for every $T > 0$ there exists $J_2 = J_2(K,T,\varepsilon) \in \mathbf{N}$ such that for every $x \in K$, $j \geq J_2$, $h \in L^\infty([0,T];\mathbf{R}^m)$ with $\|h\| \leq 1$ and $t \in [0,T]$ one has*

$$|\gamma^j_{h,x}(t) - \gamma_{h,x}(t)| \leq \varepsilon$$

Proof. Let $J_1 = J_1(K,T)$ and $R = R(K,T)$ be given by Lemma 3.2 and let $C > 0$ be the constant appearing in its proof. We can reason as in Lemma 3.2 above and use Grönwall's Lemma to get, for any x, j, h, t as in the statement, that

$$\left|\gamma^j_{h,x}(t) - \gamma_{h,x}(t)\right| \leq t\left(\sum_{i=1}^m \sup_{B_e(0,R)} |X_i^j - X_i|\right) e^{mCt}.$$

The proof is then accomplished by choosing $J_2 \geq J_1$ sufficiently large to have

$$T\left(\sum_{i=1}^{m} \sup_{B_e(0,R)} |X_i^j - X_i|\right) e^{mCT} < \varepsilon. \qquad \square$$

Clearly, J_2 can be chosen with the additional property that $J_2(K, T_1, \varepsilon) \leq J_2(K, T_2, \varepsilon)$ whenever $0 < T_1 \leq T_2$.

Theorem 3.4. *Let $X = (X_1, \ldots, X_m)$ and $X^j = (X_1^j, \ldots, X_m^j)$, $j \in \mathbf{N}$, be m-tuples of linearly independent smooth vector fields on $\mathbf{R}^n$ such that X satisfies the Chow–Hörmander condition and its CC balls are bounded in $\mathbf{R}^n$; assume that, for every $i = 1, \ldots, m$, $X_i^j \to X_i$ in $C^\infty_{\mathrm{loc}}(\mathbf{R}^n)$ as $j \to \infty$. Then the sequence $(d_j)_j$ converges to d in $L^\infty_{\mathrm{loc}}(\mathbf{R}^n \times \mathbf{R}^n)$ as $j \to +\infty$.*

Proof. Let $K \subseteq \mathbf{R}^n$ be a fixed compact set. We first prove that for every $\varepsilon \in (0,1)$ there exists $J_3 = J_3(K, \varepsilon) \in \mathbf{N}$ such that for every $x, y \in K$ and $j \geq J_3$ one has

$$d_j(x,y) \leq d(x,y) + \varepsilon.$$

Consider $x, y \in K$; by the existence of geodesics, there exists $h \in L^\infty([0,1]; \mathbf{R}^m)$ such that $\|h\|_{L^\infty} = d(x,y)$ and $\gamma_{h,x}(1) = y$. We set $y_j := \gamma^j_{h,x}(1)$ and consider J_0 and $C_K > 0$ as given by Theorem 3.1. By Lemma 3.3, for $j \geq J_3 := \max\{J_0, J_2(K, \mathrm{diam}_d K, (\varepsilon/C_K)^{s(K)})\}$ we have

$$|y_j - y| = |\gamma^j_{h,x}(1) - \gamma_{h,x}(1)| \leq \left(\frac{\varepsilon}{C_K}\right)^{s(K)}.$$

By Theorem 3.1 we deduce that $d_j(y_j, y) \leq \varepsilon$; in particular, for any $j \geq J_3$ one has

$$d_j(x,y) \leq d_j(x, y_j) + d_j(y_j, y) \leq d(x,y) + \varepsilon, \tag{10}$$

as claimed. Notice also that $\sup_{j \geq J_3} \mathrm{diam}_{d_j} K \leq \mathrm{diam}_d K + 1 =: L$ is finite.

We now prove that for any $x, y \in K$ and $\varepsilon \in (0,1)$ there exists $J_4 = J_4(K, x, y, \varepsilon) \in \mathbf{N}$ such that for every $j \geq J_4$

$$d(x,y) \leq d_j(x,y) + \varepsilon. \tag{11}$$

For every $j \geq J_3$ let $h^j \in L^\infty([0,1]; \mathbf{R}^m)$ be such that

$$\gamma^j_{h^j,x}(1) = y \quad \text{and} \quad \|h^j\|_{L^\infty} = d_j(x,y) \leq L.$$

The sequence $(h^j)_j$ is bounded in L^∞ and therefore there exists a subsequence $(h^{j_\ell})_\ell$ and $h \in L^\infty([0,1]; \mathbf{R}^m)$ such that

$$h^{j_\ell} \overset{*}{\rightharpoonup} h \text{ in } L^\infty \quad \text{and} \quad \lim_{\ell \to \infty} \|h^{j_\ell}\|_{L^\infty} = \liminf_{j \to \infty} \|h^j\|_{L^\infty} = \liminf_{j \to \infty} d_j(x,y).$$

Denoting $\gamma^{j_\ell} := \gamma^{j_\ell}_{h^{j_\ell},x}$ and considering $R = R(K, L) > 0$ as given by Lemma 3.2, one has $\gamma^{j_\ell}([0,1]) \subseteq B_e(0,R)$. Since X_i^j are converging to X_i uniformly in C^∞ ($i = 1, \ldots, m$), such vector fields are equibounded on $B_e(0,R)$. By Ascoli–Arzelà Theorem, up to a further subsequence, there exists a curve $\gamma \in AC([0,1], \mathbf{R}^n)$ such that γ^{j_ℓ} uniformly converges to γ in $[0,1]$ as $\ell \to \infty$. For every $t \in [0,1]$ one has

$$\gamma^{j_\ell}(t) = x + \int_0^t \sum_{i=1}^{m} h_i^{j_\ell}(s) X_i^{j_\ell}(\gamma^{j_\ell}(s))\, ds$$

and, taking into account that $X_i^{j_\ell} \circ \gamma^{j_\ell} \to X_i \circ \gamma$ uniformly in $[0,1]$ and that $h^j \overset{*}{\rightharpoonup} h$ in L^∞, by letting $\ell \to \infty$ one gets

$$\gamma(t) = x + \int_0^t \sum_{i=1}^m h_i(s) X_i(\gamma(s))\, ds.$$

In particular $\gamma = \gamma_{h,x}$, $\gamma(1) = y$ and

$$d(x,y) \le \|h\|_{L^\infty} \le \liminf_{\ell\to\infty} \|h_{j_\ell}\|_{L^\infty} = \liminf_{j\to\infty} d_j(x,y),$$

which proves (11).

By the compactness of K we can find $x_1, \dots, x_k \in K$ such that $K \subseteq \bigcup_{\ell=1}^k B(x_\ell, \varepsilon)$. Using Theorem 3.1 and (11) we can find $\widetilde{C} = \widetilde{C}(K) > 0$ and $J_5 = J_5(K,\varepsilon) \in \mathbf{N}$ such that for $j \ge J_5$

$$\begin{aligned} B(x_\ell, \varepsilon) &\subseteq B^j(x_\ell, \widetilde{C}\varepsilon^{1/s(K)}) \qquad && \forall\, \ell = 1, \dots, k, \\ d(x_{\ell_1}, x_{\ell_2}) &\le d_j(x_{\ell_1}, x_{\ell_2}) + \varepsilon && \forall\, \ell_1, \ell_2 = 1, \dots, k. \end{aligned}$$

For every $x, y \in K$ we can find $x_{\ell_1}, x_{\ell_2} \in K$ (with $1 \le \ell_1, \ell_2 \le k$) such that $x \in B(x_{\ell_1}, \varepsilon)$ and $y \in B(x_{\ell_2}, \varepsilon)$, hence for $j \ge J_5$ we have

$$\begin{aligned} d(x,y) &\le d(x, x_{\ell_1}) + d(x_{\ell_1}, x_{\ell_2}) + d(y, x_{\ell_2}) \\ &\le \varepsilon + d_j(x_{\ell_1}, x_{\ell_2}) + \varepsilon + \varepsilon \\ &\le d_j(x_{\ell_1}, x) + d_j(x,y) + d_j(y, x_{\ell_2}) + 3\varepsilon \\ &= d_j(x,y) + 3\varepsilon + 2\widetilde{C}\varepsilon^{1/s(K)}, \end{aligned}$$

which, combined with (10), concludes the proof. □

Let us recall that, given a CC space $(\mathbf{R}^n, X)$, a function $u \in L^1_{\mathrm{loc}}(\Omega)$ is said to have *locally bounded X-variation* if the distributional derivatives $X_1 u, \dots, X_m u$ are represented by Radon measures. See e.g. [2, 4]. We denote by $BV_{X,\mathrm{loc}}(\mathbf{R}^n)$ the set of functions of locally bounded X-variation in $\mathbf{R}^n$ and by $|D_X u|$ the total variation of the vector-valued measure $D_X u := (X_1 u, \dots, X_m u)$.

Sobolev- and Poincaré-type inequalities in CC spaces have been largely investigated; among the vast literature we mention only [9, 5, 7]. The following result is an easy consequence of [1, Theorem 7.2] or [11, Theorem 1.1]. Notice that the latter results are proved only when u is a smooth function on $\mathbf{R}^n$; in order to prove Theorem 3.5 as stated here one has to approximate functions in $BV_{X,\mathrm{loc}}$ by smooth ones (see [4, 5]).

Theorem 3.5. *Let $X = (X_1, \dots, X_m)$ and $X^j = (X_1^j, \dots, X_m^j)$, $j \in \mathbf{N}$, be m-tuples of linearly independent smooth vector fields on $\mathbf{R}^n$ such that X satisfies the Chow–Hörmander condition and its CC balls are bounded in $\mathbf{R}^n$; assume that, for every $i = 1, \dots, m$, $X_i^j \to X_i$ in $C^\infty_{\mathrm{loc}}(\mathbf{R}^n)$ as $j \to \infty$. Then, for every compact set $K \subseteq \mathbf{R}^n$ there exist $C_P > 1$, $\alpha \ge 1$, $R_P > 0$ and $J \in \mathbf{N}$ such that for every $j \ge J$, $u \in BV_{X^j,\mathrm{loc}}(\mathbf{R}^n)$, $x \in K$ and $r \in (0, R_P)$ one has*

$$\int_{B^j} \left| u - u(B^j) \right| d\mathscr{L}^n \le C_P\, r\, |D_{X^j} u|(\alpha B^j), \tag{12}$$

where $B^j := B^j(x,r)$ and $u(B^j) = \fint_{B^j} u \, d\mathscr{L}^n$.

We can then state our main application. See [7, Section 8] for more references about compactness results for Sobolev or BV functions in CC spaces.

Theorem 3.6. *Let $X = (X_1, \dots, X_m)$ and $X^j = (X_1^j, \dots, X_m^j)$, $j \in \mathbf{N}$, be m-tuples of linearly independent smooth vector fields on $\mathbf{R}^n$ such that X satisfies the Chow–Hörmander condition and its CC balls are bounded in $\mathbf{R}^n$; assume that, for every $i = 1, \dots, m$, $X_i^j \to X_i$ in $C^\infty_{\mathrm{loc}}(\mathbf{R}^n)$ as $j \to \infty$. Let $u_j \in BV_{X^j,\mathrm{loc}}(\mathbf{R}^n)$ be a sequence of functions that is locally uniformly bounded in BV, i.e., such that for any compact set $K \subseteq \mathbf{R}^n$ there exists $M > 0$ such that*

$$\forall j \in \mathbf{N} \quad \|u_j\|_{L^1(K)} + |D_{X^j} u_j|(K) \le M < \infty.$$

Then, there exist $u \in BV_{X,\mathrm{loc}}(\mathbf{R}^n)$ and a subsequence $(u_{j_h})_h$ of $(u_j)_j$ such that $u_{j_h} \to u$ in $L^1_{\mathrm{loc}}(\mathbf{R}^n)$ as $h \to \infty$. Moreover, for any bounded open set $\Omega \subseteq \mathbf{R}^n$ the semicontinuity of the total variation

$$|D_X u|(\Omega) \le \liminf_{j\to\infty} |D_{X^j} u_j|(\Omega)$$

holds.

Proof. We use Theorem 2.1 with $X = \mathbf{R}^n$, $\lambda = \mathscr{L}^n$, $\delta = q = 1$, $\mu_j := |D_{X^j} u|$ and d, d_j the CC distances associated with X, X^j respectively. Assumption (i) follows from Theorem 3.4, while the local doubling property (ii) of d is a well-known fact (see e.g. [13]). The validity of (iii) (with $\delta = q = 1$) follows from Theorem 3.5, while (iv) is satisfied by assumption.

Theorem 2.1 ensures that, up to subsequences, u_j converges to some u in $L^1_{\mathrm{loc}}(\mathbf{R}^n)$; we need to show that $u \in BV_{X,\mathrm{loc}}(\mathbf{R}^n)$. To this aim, for any $i = 1, \dots, m$ we denote by X_i^* the formal adjoint to X_i and write

$$X_i(x) = \sum_{k=1}^n a_{i,k}(x)\partial_k \quad \text{and} \quad X_i^j(x) = \sum_{k=1}^n a_{i,k}^j(x)\partial_k$$

for suitable smooth functions $a_{i,k}, a_{i,k}^j$. Then, for any bounded open set $\Omega \subseteq \mathbf{R}^n$ and any test function $\varphi = (\varphi_1, \dots, \varphi_m) \in C^1_c(\Omega; \mathbf{R}^m)$ with $|\varphi| \le 1$ we have

$$\begin{aligned}
\int_\Omega u \sum_{i=1}^m X_i^* \varphi_i \, d\mathscr{L}^n &= \int_\Omega u \sum_{i=1}^m \sum_{k=1}^n \partial_k(a_{i,k}\varphi_i)\, d\mathscr{L}^n = \lim_{j\to\infty} \int_\Omega u_j \sum_{i=1}^m \sum_{k=1}^n \partial_k(a_{i,k}^j \varphi_i)\, d\mathscr{L}^n \\
&= \lim_{j\to\infty} \int_\Omega u_j \sum_{i=1}^m X_i^{j*} \varphi_i \, d\mathscr{L}^n = -\lim_{j\to\infty} \int_\Omega \sum_{i=1}^m \varphi_i \, dX_i^j u_j \\
&\le \liminf_{j\to\infty} |D_X^j u_j|(\Omega) < \infty.
\end{aligned}$$

This proves that $u \in BV_{X,\mathrm{loc}}(\mathbf{R}^n)$ as well as the semicontinuity of the total variation. The proof is accomplished. □

Remark 3.7. We conjecture that, when the CC space $(\mathbf{R}^n, X)$ is *equiregular*, the convergence $u_{j_h} \to u$ in Theorem 3.6 holds in L^q_{loc} for any $q \in [1, \frac{Q}{Q-1}[$, where Q is the Hausdorff dimension of $(\mathbf{R}^n, X)$. This would easily follow in case the Poincaré inequality (12) could be strengthened to

$$\|u - u(B^j)\|_{L^q(B^j)} \le C_P \, r^\delta \, |D_{X^j} u|(\alpha B^j)$$

for some $\delta > 0$ (arguably, $\delta = \frac{Q}{q} + 1 - Q$). The key point would be proving that the constant C_P can be chosen independent of j but, as far as we know, no investigation in this direction has been attempted in the literature.

Remark 3.8. Theorems 3.4, 3.5 and 3.6 hold also under a slightly weaker assumption: it is indeed enough that, for any compact set $K \subseteq \mathbf{R}^n$, the convergence $X_i^j \to X_i$ holds in $C^k(K)$ for a suitable $k = k(K)$ (actually, k depends only on $s(K)$) that one could explicitly compute. See [1, 11] for more details.

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