Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 44, 2019, 459–484

WEIGHTED COMPLETE CONTINUITY FOR THE COMMUTATOR OF MARCINKIEWICZ INTEGRAL

Guoen Hu

Beijing Normal University Zhuhai, School of Applied Mathematics
Zhuhai 519087, P. R. China; guoenxx@163.com

Abstract. Let Ω be homogeneous of degree zero, have mean value zero and integrable on the unit sphere, and $\mathcal{M}_\Omega$ be the higher-dimensional Marcinkiewicz integral associated with Ω. In this paper, the author considers the complete continuity on weighted $L^p(\mathbf{R}^n)$ spaces with $A_p(\mathbf{R}^n)$ weights, weighted Morrey spaces with $A_p(\mathbf{R}^n)$ weights, for the commutator generated by $\mathrm{CMO}(\mathbf{R}^n)$ functions and $\mathcal{M}_\Omega$ when Ω satisfies certain size conditions.

1. Introduction

As an analogy of the classicial Littlewood–Paley g-function, Marcinkiewicz [30] introduced the operator

$$\mathcal{M}(f)(x) = \left(\int_0^\pi \frac{|F(x+t)-F(x-t)-2F(x)|^2}{t^3}\,\mathrm{d}t\right)^{\frac{1}{2}},$$

where $F(x)=\int_0^x f(t)\mathrm{d}t$. This operator is now called Marcinkiewicz integral. Zygmund [39] proved that $\mathcal{M}$ is bounded on $L^p([0,2\pi])$ for $p\in(1,\infty)$. Stein [33] generalized the Marcinkiewicz operator to the case of higher dimension. Let Ω be homogeneous of degree zero, integrable and have mean value zero on the unit sphere S^{n-1}. Define the Marcinkiewicz integral operator $\mathcal{M}_\Omega$ by

$$\mathcal{M}_\Omega(f)(x) = \left(\int_0^\infty |F_{\Omega,t}f(x)|^2\frac{\mathrm{d}t}{t^3}\right)^{\frac{1}{2}}, \tag{1.1}$$

where

$$F_{\Omega,t}f(x) = \int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}f(y)\,\mathrm{d}y$$

for $f\in\mathcal{S}(\mathbf{R}^n)$. Stein [33] proved that if $\Omega\in\mathrm{Lip}_\alpha(S^{n-1})$ with $\alpha\in(0,1]$, then $\mathcal{M}_\Omega$ is bounded on $L^p(\mathbf{R}^n)$ for $p\in(1,2]$. Benedek, Calderón and Panzon [6] showed that the $L^p(\mathbf{R}^n)$ boundedness ($p\in(1,\infty)$) of $\mathcal{M}_\Omega$ holds true under the condition that $\Omega\in C^1(S^{n-1})$. Using the one-dimensional result and Riesz transforms similarly as in the case of singular integrals (see [8]) and interpolation, Walsh [37] proved that for each $p\in(1,\infty)$, $\Omega\in L(\ln L)^{1/r}(\ln\ln L)^{2(1-2/r')}(S^{n-1})$ is a sufficient condition such that $\mathcal{M}_\Omega$ is bounded on $L^p(\mathbf{R}^n)$, where $r=\min\{p,p'\}$ and $p'=p/(p-1)$. Ding, Fan and Pan [18] proved that if $\Omega\in H^1(S^{n-1})$ (the Hardy space on S^{n-1}), then $\mathcal{M}_\Omega$ is bounded on $L^p(\mathbf{R}^n)$ for all $p\in(1,\infty)$; Al-Salmam, Al-Qassem, Cheng and Pan [3] proved that $\Omega\in L(\ln L)^{1/2}(S^{n-1})$ is a sufficient condition such that $\mathcal{M}_\Omega$ is bounded on $L^p(\mathbf{R}^n)$ for all $p\in(1,\infty)$. Ding, Fan and Pan [17] considered the boundedness

https://doi.org/10.5186/aasfm.2019.4428
2010 Mathematics Subject Classification: Primary 42B25; Secondary 47B07.
Key words: commutator, Marcinkiewicz integral, completely continuous operator, Fourier transform, weighted Morrey space.
The research was supported by the NNSF of China under grant #11871108.

on weighted $L^p(\mathbf{R}^n)$ with $A_p(\mathbf{R}^n)$ when $\Omega \in L^q(S^{n-1})$ for some $q \in (1,\infty]$, where and in the following, for $p \in [1,\infty)$, $A_p(\mathbf{R}^n)$ denotes the weight function class of Muckenhoupt, see [24] for the definitions and properties of $A_p(\mathbf{R}^n)$. For other works about the operator defined by (1.1), see [2, 3, 10, 18, 19, 21] and the related references therein.

The commutator of $\mathcal{M}_\Omega$ is also of interest and has been considered by many authors (see [35, 26, 20, 9, 25]). Let $b \in \mathrm{BMO}(\mathbf{R}^n)$, the commutator generated by $\mathcal{M}_\Omega$ and b is defined by

$$\mathcal{M}_{\Omega,b}f(x) = \bigg(\int_0^\infty \Big|\int_{|x-y|\le t}\big(b(x)-b(y)\big)\frac{\Omega(x-y)}{|x-y|^{n-1}}f(y)\,\mathrm{d}y\Big|^2\frac{dt}{t^3}\bigg)^{\frac{1}{2}}. \tag{1.2}$$

Torchinsky and Wang [35] showed that if $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$ ($\alpha \in (0,1]$), then $\mathcal{M}_{\Omega,b}$ is bounded on $L^p(\mathbf{R}^n)$ with bound $C\|b\|_{\mathrm{BMO}(\mathbf{R}^n)}$ for all $p \in (1,\infty)$. Hu and Yan [26] proved the $\Omega \in L(\ln L)^{3/2}(S^{n-1})$ is a sufficient condition such that $\mathcal{M}_{\Omega,b}$ is bounded on L^2. Ding, Lu and Yabuta [20] considered the weighted estimates for $\mathcal{M}_{\Omega,b}$, and proved that if $\Omega \in L^q(S^{n-1})$ for some $q \in (1,\infty]$, then for $p \in (q',\infty)$ and $w \in A_{p/q'}(\mathbf{R}^n)$, or $p \in (1,q)$ and $w^{-1/(p-1)} \in A_{p'/q'}(\mathbf{R}^n)$, $\mathcal{M}_{\Omega,b}$ is bounded on $L^p(\mathbf{R}^n,w)$. Chen and Lu [9] improved the result in [26] and showed that if $\Omega \in L(\ln L)^{3/2}(S^{n-1})$, then $\mathcal{M}_{\Omega,b}$ is bounded on $L^p(\mathbf{R}^n)$ with bound $C\|b\|_{\mathrm{BMO}(\mathbf{R}^n)}$ for all $p \in (1,\infty)$.

Let $\mathrm{CMO}(\mathbf{R}^n)$ be the closure of $C_0^\infty(\mathbf{R}^n)$ in the $\mathrm{BMO}(\mathbf{R}^n)$ topology, which coincide with $\mathrm{VMO}(\mathbf{R}^n)$, the space of functions of vanishing mean oscillation introduced by Coifman and Weiss [16], see also [7]. Uchiyama [36] proved that if T is a Calderón–Zygmund operator, and $b \in \mathrm{BMO}(\mathbf{R}^n)$, then the commutator of T defined by

$$[b,T]f(x) = b(x)Tf(x) - T(bf)(x),$$

is a compact operator on $L^p(\mathbf{R}^n)$ ($p \in (1,\infty)$) if and only if $b \in \mathrm{CMO}(\mathbf{R}^n)$. Chen and Ding [12] considered the compactness of $\mathcal{M}_{\Omega,b}$ on $L^p(\mathbf{R}^n)$, and proved that if Ω satisfies certain regularity condition of Dini type, then for $p \in (1,\infty)$, $\mathcal{M}_{\Omega,b}$ is compact on $L^p(\mathbf{R}^n)$ if and only if $b \in \mathrm{CMO}(\mathbf{R}^n)$. Using the ideas from [11], Mao, Sawano and Wu [29] considered the compactness of $\mathcal{M}_{\Omega,b}$ when Ω satisfies the size condition that

$$\sup_{\zeta\in S^{n-1}}\int_{S^{n-1}}|\Omega(\eta)|\Big(\ln\frac{1}{|\eta\cdot\zeta|}\Big)^\theta\,\mathrm{d}\eta < \infty, \tag{1.3}$$

and proved that if Ω satisfies (1.3) for some $\theta \in (3/2,\infty)$, then for $b \in \mathrm{CMO}(\mathbf{R}^n)$ and $p \in \big(4\theta/(4\theta-3), 4\theta/3\big)$, $\mathcal{M}_{\Omega,b}$ is compact on $L^p(\mathbf{R}^n)$. Our first purpose of this paper is to consider the complete continuity on weighted $L^p(\mathbf{R}^n)$ for $\mathcal{M}_{\Omega,b}$ when $\Omega \in L^q(S^{n-1})$ for some $q \in (1,\infty]$. To formulate our main result, we first recall some definitions.

Definition 1.1. Let $\mathcal{X}$ be a normed linear spaces and $\mathcal{X}^*$ be its dual space, $\{x_k\} \subset \mathcal{X}$ and $x \in \mathcal{X}$, If for all $f \in \mathcal{X}^*$,

$$\lim_{k\to\infty}|f(x_k) - f(x)| = 0,$$

then $\{x_k\}$ is said to converge to x weakly, or $x_k \rightharpoonup x$.

Definition 1.2. Let $\mathcal{X}$, $\mathcal{Y}$ be two Banach spaces and S be a bounded operator from $\mathcal{X}$ to $\mathcal{Y}$.

(i) If for each bounded set $\mathcal{G} \subset \mathcal{X}$, $S\mathcal{G} = \{Sx\colon x \in \mathcal{G}\}$ is a strongly pre-compact set in $\mathcal{Y}$, then S is called a compact operator from $\mathcal{X}$ to $\mathcal{Y}$;

(ii) if for $\{x_k\} \subset \mathcal{X}$ and $x \in \mathcal{X}$,

$$x_k \rightharpoonup x \text{ in } \mathcal{X} \implies \|Sx_k - Sx\|_{\mathcal{Y}} \to 0,$$

then S is said to be a completely continuous operator.

It is well known that, if $\mathcal{X}$ is a reflexive space, and S is completely continuous from $\mathcal{X}$ to $\mathcal{Y}$, then S is also compact from $\mathcal{X}$ to $\mathcal{Y}$. On the other hand, if S is a linear compact operator from $\mathcal{X}$ to $\mathcal{Y}$, then S is also a completely continuous operator. However, if S is not linear, then S is compact do not imply that S is completely continuous. For example, the operator

$$Sx = \|x\|_{l^2}$$

is compact from l^2 to $\mathbf{R}$, but not completely continuous.

Our first result in this paper can be stated as follows.

Theorem 1.3. *Let Ω be homogeneous of degree zero, have mean value zero on S^{n-1} and $\Omega \in L^q(S^{n-1})$ for some $q \in (1, \infty]$. Suppose that p and w satisfy one of the following conditions*

(i) $p \in (q', \infty)$ *and* $w \in A_{p/q'}(\mathbf{R}^n)$;

(ii) $p \in (1, q)$ *and* $w^{-1/(p-1)} \in A_{p'/q'}(\mathbf{R}^n)$;

(iii) $p \in (1, \infty)$ *and* $w^{q'} \in A_p(\mathbf{R}^n)$.

Then for $b \in \mathrm{CMO}(\mathbf{R}^n)$, $\mathcal{M}_{\Omega,b}$ is completely continuous on $L^p(\mathbf{R}^n, w)$.

Our argument used in the proof of Theorem 1.3 also leads to the complete continuity of $\mathcal{M}_{\Omega,b}$ on weighted Morrey spaces.

Definition 1.4. Let $p \in (0, \infty)$, w be a weight and $\lambda \in (0, 1)$. The weighted Morrey space $L^{p,\lambda}(\mathbf{R}^n, w)$ is defined as

$$L^{p,\lambda}(\mathbf{R}^n, w) = \{f \in L^p_{\mathrm{loc}}(\mathbf{R}^n)\colon \|f\|_{L^{p,\lambda}(\mathbf{R}^n,w)} < \infty\},$$

with

$$\|f\|_{L^{p,\lambda}(\mathbf{R}^n,w)} = \sup_{y\in\mathbf{R}^n, r>0} \Big(\frac{1}{\{w(B(y,r))\}^\lambda} \int_{B(y,r)} |f(x)|^p w(x)\,\mathrm{d}x\Big)^{1/p},$$

here $B(y, r)$ denotes the ball in $\mathbf{R}^n$ centered at y and having radius r, and $w(B(y, r)) = \int_{B(y,r)} w(z)\mathrm{d}z$. For simplicity, we use $L^{p,\lambda}(\mathbf{R}^n)$ to denote $L^{p,\lambda}(\mathbf{R}^n, 1)$.

The Morrey space $L^{p,\lambda}(\mathbf{R}^n)$ was introduced by Morrey [17]. It is well-known that this space is closely related to some problems in PED (see [31, 32]), and has interest in harmonic analysis (see [1] and the references therein). Komori and Shiral [27] introduced the weighted Morrey spaces and considered the properties on weighted Morrey spaces for some classical operators. Chen, Ding and Wang [13] considered the compactness of $\mathcal{M}_{\Omega,b}$ on Morrey spaces. They proved that if $\lambda \in (0, 1)$, $\Omega \in L^q(S^{n-1})$ for $q \in (1/(1-\lambda), \infty]$ and satisfies a regularity condition of L^q-Dini type, then $\mathcal{M}_{\Omega,b}$ is compact on $L^{p,\lambda}(\mathbf{R}^n)$. Our second purpose of this paper is to prove the complete continuity of $\mathcal{M}_{\Omega,b}$ on weighted Morrey spaces with $A_p(\mathbf{R}^n)$ weights.

Theorem 1.5. *Let Ω be homogeneous of degree zero, have mean value zero on S^{n-1} and $\Omega \in L^q(S^{n-1})$ for some $q \in (1, \infty]$. Suppose that $p \in (q', \infty)$, $\lambda \in (0, 1)$ and $w \in A_{p/q'}(\mathbf{R}^n)$; or $p \in (1, q')$, $w^r \in A_1(\mathbf{R}^n)$ for some $r \in (q', \infty)$ and $\lambda \in (0, 1 - r'/q)$. Then for $b \in \mathrm{CMO}(\mathbf{R}^n)$, $\mathcal{M}_{\Omega,b}$ is completely continuous on $L^{p,\lambda}(\mathbf{R}^n, w)$.*

Remark 1.6. The proof of Theorems 1.3 involves some ideas used in [11] and a sufficient condition of strongly pre-compact set in $L^p(L^2([1,2]),l^2;\mathbf{R}^n,w)$ with $w\in A_p(\mathbf{R}^n)$. To prove Theorem 1.5, we will establish a lemma which clarify the relationship of the bounds on $L^p(\mathbf{R}^n,w)$ and the bounds on $L^{p,\lambda}(\mathbf{R}^n,w)$ for a class of sublinear operators, see Lemma 4.1 below.

We make some conventions. In what follows, C always denotes a positive constant that is independent of the main parameters involved but whose value may differ from line to line. We use the symbol $A\lesssim B$ to denote that there exists a positive constant C such that $A\le CB$. For a set $E\subset\mathbf{R}^n$, χ_E denotes its characteristic function. Let M be the Hardy–Littlewood maximal operator. For $r\in(0,\infty)$, we use M_r to denote the operator $M_rf(x)=\big(M(|f|^r)(x)\big)^{1/r}$. For a locally integrable function f, the sharp maximal function $M^\sharp f$ is defined by

$$M^\sharp f(x)=\sup_{Q\ni x}\inf_{c\in\mathbf{C}}\frac{1}{|Q|}\int_Q|f(y)-c|\,\mathrm{d}y.$$

2. Approximation

Let Ω be homogeneous of degree zero, integrable on S^{n-1}. For $t\in[1,2]$ and $j\in\mathbf{Z}$, set

$$K_t^j(x)=\frac{1}{2^j}\frac{\Omega(x)}{|x|^{n-1}}\chi_{\{2^{j-1}t<|x|\le 2^jt\}}(x). \tag{2.1}$$

As it was proved in [23], if $\Omega\in L^q(S^{n-1})$ for some $q\in(1,\infty]$, then there exists a constant $\alpha\in(0,1)$ such that for $t\in[1,2]$ and $\xi\in\mathbf{R}^n\backslash\{0\}$,

$$|\widehat{K_t^j}(\xi)|\lesssim\|\Omega\|_{L^q(S^{n-1})}\min\{1,|2^j\xi|^{-\alpha}\}. \tag{2.2}$$

Here and in the following, for $h\in\mathcal{S}'(\mathbf{R}^n)$, $\widehat{h}$ denotes the Fourier transform of h. Moreover, if $\int_{S^{n-1}}\Omega(x')\,\mathrm{d}x'=0$, then

$$|\widehat{K_t^j}(\xi)|\lesssim\|\Omega\|_{L^1(S^{n-1})}\min\{1,|2^j\xi|\}. \tag{2.3}$$

Let

$$\widetilde{\mathcal{M}}_\Omega f(x)=\left(\int_1^2\sum_{j\in\mathbf{Z}}\big|F_jf(x,t)\big|^2\,\mathrm{d}t\right)^{\frac12},$$

with

$$F_jf(x,t)=\int_{\mathbf{R}^n}K_t^j(x-y)f(y)\,\mathrm{d}y.$$

For $b\in\mathrm{BMO}(\mathbf{R}^n)$, let $\widetilde{\mathcal{M}}_{\Omega,b}$ be the commutator of $\widetilde{\mathcal{M}}_\Omega$ defined by

$$\widetilde{\mathcal{M}}_{\Omega,b}f(x)=\left(\int_1^2\sum_{j\in\mathbf{Z}}\big|F_{j,b}f(x,t)\big|^2\,\mathrm{d}t\right)^{1/2},$$

with

$$F_{j,b}f(x,t)=\int_{\mathbf{R}^n}\big(b(x)-b(y)\big)K_t^j(x-y)f(y)\,\mathrm{d}y.$$

A trivial computation leads to that

$$\mathcal{M}_\Omega f(x)\approx\widetilde{\mathcal{M}}_\Omega f(x),\ \mathcal{M}_{\Omega,b}f(x)\approx\widetilde{\mathcal{M}}_{\Omega,b}f(x). \tag{2.4}$$

Let $\phi \in C_0^\infty(\mathbf{R}^n)$ be a nonnegative function such that $\int_{\mathbf{R}^n} \phi(x)\,\mathrm{d}x = 1$, $\operatorname{supp}\phi \subset \{x\colon |x| \le 1/4\}$. For $l \in \mathbf{Z}$, let $\phi_l(y) = 2^{-nl}\phi(2^{-l}y)$. It is easy to verify that for any $\varsigma \in (0,1)$,

$$|\widehat{\phi_l}(\xi) - 1| \lesssim \min\{1, |2^l\xi|^\varsigma\}. \tag{2.5}$$

Let

$$F_j^l f(x,t) = \int_{\mathbf{R}^n} K_t^j * \phi_{j-l}(x-y) f(y)\,\mathrm{d}y.$$

Define the operator $\widetilde{\mathcal{M}}_\Omega^l$ by

$$\widetilde{\mathcal{M}}_\Omega^l f(x) = \left(\int_1^2 \sum_{j\in\mathbf{Z}} \left|F_j^l f(x,t)\right|^2 \mathrm{d}t\right)^{\frac{1}{2}}. \tag{2.6}$$

This section is devoted to the approximation of $\widetilde{\mathcal{M}}_\Omega$ by $\widetilde{\mathcal{M}}_\Omega^l$. We will prove following theorem.

Theorem 2.1. *Let Ω be homogeneous of degree zero and have mean value zero. Suppose that $\Omega \in L^q(S^{n-1})$ for some $q \in (1,\infty]$, p and w are the same as in Theorem 1.3, then for $l \in \mathbf{N}$,*

$$\|\widetilde{\mathcal{M}}_\Omega f - \widetilde{\mathcal{M}}_\Omega^l f\|_{L^p(\mathbf{R}^n,w)} \lesssim 2^{-\varrho_p l}\|f\|_{L^p(\mathbf{R}^n,w)},$$

with $\varrho_p \in (0,1)$ a constant depending only on p,n and w.

To prove Theorem 2.1, we will use some lemmas.

Lemma 2.2. *Let Ω be homogeneous of degree zero and belong to $L^q(S^{n-1})$ for some $q \in (1,\infty]$, K_t^j be defined as in (2.1). Then for $t \in [1,2]$, $l \in \mathbf{N}$, $R > 0$ and $y \in \mathbf{R}^n$ with $|y| < R/4$,*

$$\sum_{j\in\mathbf{Z}} \sum_{k=1}^{\infty} (2^k R)^{\frac{n}{q'}} \left(\int_{2^kR<|x|\le 2^{k+1}R} \left|K_t^j * \phi_{j-l}(x+y) - K_t^j * \phi_{j-l}(x)\right|^q \mathrm{d}x\right)^{\frac{1}{q}} \lesssim l.$$

For the proof of Lemma 2.2, see [38].

Lemma 2.3. *Let Ω be homogeneous of degree zero and $\Omega \in L^q(S^{n-1})$ for some $q \in (1,\infty]$, $p \in (1,q)$ and $w^{-1/(p-1)} \in A_{p'/q'}(\mathbf{R}^n)$. Then*

$$\Big\|\Big(\sum_{j\in\mathbf{Z}} |K_t^j * \phi_{j-l} * f_j|^2\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathbf{R}^n,w)} \lesssim \Big\|\Big(\sum_{j\in\mathbf{Z}} |f_j|^2\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathbf{R}^n,w)}. \tag{2.7}$$

Proof. Let M_Ω be the maximal operator defined by

$$M_\Omega h(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |\Omega(x-y)h(y)|\,\mathrm{d}y. \tag{2.8}$$

We know from the proof of Lemma 1 in [22] that for $p \in (1,2]$,

$$\Big\|\Big(\sum_{j\in\mathbf{Z}} |M_\Omega f_j|^2\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathbf{R}^n,w)} \lesssim \Big\|\Big(\sum_{j\in\mathbf{Z}} |f_j|^2\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathbf{R}^n,w)}, \tag{2.9}$$

provided that $p \in (q',\infty)$ and $w \in A_{p/q'}(\mathbf{R}^n)$, or $p \in (1,q)$ and $w^{-1/(p-1)} \in A_{p'/q'}(\mathbf{R}^n)$. On the other hand, it is easy to verify that

$$|K_t^j * \phi_{j-l} * f_j(x)| \lesssim M_\Omega M f_j(x).$$

The inequality (2.9), together with the weighted vector-valued inequality of M (see Theorem 3.1 in [5]), proves that (2.7) holds when $p \in (1,2]$, $p \in (q',\infty)$ and $w \in A_{p/q'}(\mathbf{R}^n)$, or $p \in (1,q)$ and $w^{-1/(p-1)} \in A_{p'/q'}(\mathbf{R}^n)$. This, via a standard duality argument, shows that (2.7) holds when $p \in (2,\infty)$, $p \in (1,q)$ and $w^{-1/(p-1)} \in A_{p'/q'}(\mathbf{R}^n)$. □

Proof of Theorem 2.1. We employ the ideas used in [38]. By Fourier transform estimates (2.2) and (2.5), and Plancherel's theorem, we know that

$$\begin{aligned}\|\widetilde{\mathcal{M}}_\Omega f - \widetilde{\mathcal{M}}_\Omega^l f\|_{L^2(\mathbf{R}^n)}^2 &= \int_1^2 \Big\| \Big(\sum_{j\in\mathbf{Z}} \big|F_l f(\cdot,t) - F_j^l f(\cdot,t)\big|^2\Big)^{\frac12}\Big\|_{L^2(\mathbf{R}^n)}^2 \,\mathrm{d}t \\ &= \int_1^2 \sum_{j\in\mathbf{Z}} \int_{\mathbf{R}^n} |\widehat{K_t^j}(\xi)|^2 |1-\widehat{\phi_{j-l}}(\xi)|^2 |\widehat{f}(\xi)|^2 \,\mathrm{d}\xi\,\mathrm{d}t \\ &\lesssim 2^{-\alpha l}\|f\|_{L^2(\mathbf{R}^n)}^2.\end{aligned}$$

Now let p and w be the same as in Theorem 1.3. Recall that $\mathcal{M}_\Omega$ is bounded on $L^p(\mathbf{R}^n,w)$ and so is $\widetilde{\mathcal{M}}_\Omega$. Thus, by interpolation with changes of measures of Stein and Weiss [34], it suffices to prove that

$$\|\widetilde{\mathcal{M}}_\Omega^l f\|_{L^p(\mathbf{R}^n,w)} \lesssim l\|f\|_{L^p(\mathbf{R}^n,w)}. \tag{2.10}$$

We now prove (2.10) for the case $p \in (1,q)$ and $w^{-1/(p-1)} \in A_{p'/q'}(\mathbf{R}^n)$. Let $\psi \in C_0^\infty(\mathbf{R}^n)$ be a radial function such that $\operatorname{supp}\psi \subset \{1/4 \le |\xi| \le 4\}$ and

$$\sum_{i\in\mathbf{Z}} \psi(2^{-i}\xi) = 1, \quad |\xi| \neq 0.$$

Define the multiplier operator S_i by

$$\widehat{S_i f}(\xi) = \psi(2^{-i}\xi)\widehat{f}(\xi).$$

Set

$$\mathrm{E}_1 f(x) = \sum_{m=-\infty}^{0} \Big(\int_1^2 \sum_j \Big| K_t^j * \phi_{j-l} * (S_{m-j}f)(x)\Big|^2 \,\mathrm{d}t\Big)^{\frac12},$$

$$\mathrm{E}_2 f(x) = \sum_{m=1}^{\infty} \Big(\int_1^2 \sum_j \Big| K_t^j * \phi_{j-l} * (S_{m-j}f)(x)\Big|^2 \,\mathrm{d}t\Big)^{\frac12}.$$

It then follows that for $f \in \mathcal{S}(\mathbf{R}^n)$,

$$\Big\| \Big(\int_1^2 \sum_j \big|K_t^j * \phi_{j-l} * f(x)\big|^2 \,\mathrm{d}t\Big)^{\frac12} \Big\|_{L^p(\mathbf{R}^n)} \le \sum_{i=1}^{2} \|\mathrm{E}_i f\|_{L^p(\mathbf{R}^n)}.$$

We now estimate the term E_1. By Fourier transform estimate (2.3), we know that

$$\begin{aligned}&\Big\| \Big(\int_1^2 \sum_j \big|K_t^j * \phi_{j-l} * (S_{m-j}f)(x)\big|^2 \,\mathrm{d}t\Big)^{\frac12} \Big\|_{L^2(\mathbf{R}^n)}^2 \qquad (2.11)\\ &= \int_1^2 \int_{\mathbf{R}^n} \sum_{j\in\mathbf{Z}} \big|K_t^j * \phi_{j-l} * (S_{m-j}f)(x)\big|^2 \,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

$$\lesssim \sum_{j\in\mathbf{Z}} \int_{\mathbf{R}^n} |2^j\xi||\psi(2^{-m+j}\xi)|^2|\widehat{f}(\xi)|^2\,\mathrm{d}\xi \le 2^{2m}\|f\|^2_{L^2(\mathbf{R}^n)}.$$

On the other hand, applying Minkowski's inequality, Lemma 2.3 and the weighted Littlewood–Paley theory, we have that

$$\begin{aligned}(2.12)\qquad &\left\|\left(\int_1^2 \sum_j \left|K_t^j * \phi_{j-l} * (S_{m-j}f)(x)\right|^2\,\mathrm{d}t\right)^{\frac12}\right\|^2_{L^p(\mathbf{R}^n,w)}\\ &\le \int_1^2 \left(\int_{\mathbf{R}^n}\left(\sum_{j\in\mathbf{Z}} \left|K_t^j * \phi_{j-l} * (S_{m-j}f)(x)\right|^2\right)^{p/2} w(x)\,\mathrm{d}x\right)^{2/p}\,\mathrm{d}t\\ &\le \|f\|^2_{L^p(\mathbf{R}^n,w)},\quad p\in[2,\infty).\end{aligned}$$

To estimate

$$\left\|\left(\int_1^2 \sum_j \left|K_t^j * \phi_{j-l} * (S_{m-j}f)(x)\right|^2\,\mathrm{d}t\right)^{\frac12}\right\|_{L^p(\mathbf{R}^n,w)}$$

for $p\in(1,2)$, we consider the mapping $\mathcal{F}$ defined by

$$\mathcal{F}\colon \{h_j(x)\}_{j\in\mathbf{Z}} \longrightarrow \{K_t^j * \phi_{j-l} * h_j(x)\}.$$

Note that for $t\in[1,2]$,

$$\left|K_t^j * \phi_{j-l} * h_j(x)\right| \lesssim MM_\Omega h_j(x).$$

We choose $p_0\in(1,p)$ such that $w^{-1/(p_0-1)}\in A_{p_0'/q'}(\mathbf{R}^n)$. Then by the weighted estimates for M_Ω (see [22]), we have that

$$(2.13)\qquad \int_{\mathbf{R}^n}\int_1^2 \sum_{j\in\mathbf{Z}} \left|K_t^j * \phi_{j-l} * h_j(x)\right|^{p_0}\,\mathrm{d}t\,w(x)\,\mathrm{d}x \lesssim \int_{\mathbf{R}^n}\sum_{j\in\mathbf{Z}}|h_j(x)|^{p_0}w(x)\,\mathrm{d}x.$$

Also, we have that

$$\sup_{j\in\mathbf{Z}}\sup_{t\in[1,2]}\left|K_t^j * \phi_{j-l} * h_j(x)\right| \lesssim \sup_{j\in\mathbf{Z}}|h_j(x)|.$$

which implies that for $p_1\in(1,\infty)$,

$$(2.14)\qquad \left\|\sup_{j\in\mathbf{Z}}\sup_{t\in[1,2]}\left|K_t^j * \phi_{j-l} * h_j\right|\right\|_{L^{p_1}(\mathbf{R}^n,w)} \lesssim \left\|\sup_{j\in\mathbf{Z}}|h_j|\right\|_{L^{p_1}(\mathbf{R}^n,w)}.$$

By interpolation, we deduce from the inequalities (2.13) and (2.14) (with $p_0\in(1,2)$, $p_1\in(2,\infty)$ and $1/p=1/2+(2-p_0)/(2p_1)$) that

$$\left\|\left(\int_1^2 \sum_{j\in\mathbf{Z}} \left|K_t^j * \phi_{j-l} * h_j\right|^2\mathrm{d}t\right)^{\frac12}\right\|_{L^p(\mathbf{R}^n,w)} \lesssim \left\|\left(\sum_{j\in\mathbf{Z}}|h_j|^2\right)^{\frac12}\right\|_{L^p(\mathbf{R}^n,w)},$$

and so

$$\left\|\left(\int_1^2 \sum_j \left|K_t^j * \phi_{j-l} * (S_{m-j}f)\right|^2 \mathrm{d}t\right)^{\frac{1}{2}}\right\|_{L^p(\mathbf{R}^n,w)} \lesssim \left\|\left(\sum_{j\in\mathbf{Z}} |S_{m-j}f|^2\right)^{\frac{1}{2}}\right\|_{L^p(\mathbf{R}^n,w)}$$
$$\lesssim \|f\|_{L^p(\mathbf{R}^n,w)}, \quad p\in(1,2).$$

This, along with (2.12), states that for $p\in(1,q)$,

$$\left\|\left(\int_1^2 \sum_j \left|K_t^j * \phi_{j-l} * (S_{m-j}f)\right|^2 \mathrm{d}t\right)^{\frac{1}{2}}\right\|_{L^p(\mathbf{R}^n,w)} \lesssim \|f\|_{L^p(\mathbf{R}^n,w)}. \tag{2.15}$$

Again by interpolating, the inequalities (2.11) and (2.15) give us that for $p\in(1,q)$,

$$\left\|\left(\int_1^2 \sum_j \left|K_t^j * \phi_{j-l} * (S_{m-j}f)(x)\right|^2 \mathrm{d}t\right)^{\frac{1}{2}}\right\|_{L^p(\mathbf{R}^n,w)} \lesssim 2^{t_p m}\|f\|_{L^p(\mathbf{R}^n,w)}.$$

with $t_p\in(0,1)$ a constant depending only on p. Therefore,

$$\|\mathrm{E}_1 f\|_{L^p(\mathbf{R}^n,w)} \lesssim \|f\|_{L^p(\mathbf{R}^n,w)}.$$

We consider the term E_2. Again by Plancherel's theorem and the Fourier transform estimates (2.2) and (2.5), we have that

$$\left\|\left(\int_1^2 \sum_{j\in\mathbf{Z}} \left|K_t^j * \phi_{j-l} * (S_{m-j}f)(x)\right|^2 \mathrm{d}t\right)^{\frac{1}{2}}\right\|^2_{L^2(\mathbf{R}^n)} \tag{2.16}$$
$$= \int_1^2 \sum_{j\in\mathbf{Z}} \int_{\mathbf{R}^n} |\widehat{K_t^j}(\xi)|^2 |\psi(2^{-m+j}\xi)|^2 |\widehat{f}(\xi)|^2 \,\mathrm{d}\xi\,\mathrm{d}t$$
$$\lesssim \sum_{j\in\mathbf{Z}} \int_{\mathbf{R}^n} |2^j\xi|^{-2\alpha} |2^{j-l}\xi|^{\alpha} \psi(2^{-m+j}\xi)|^2 |\widehat{f}(\xi)|^2 \,\mathrm{d}\xi \lesssim 2^{-m\alpha}\|f\|^2_{L^2(\mathbf{R}^n)}.$$

As in the inequality (2.15), we have that

$$\left\|\left(\int_1^2 \sum_{j\in\mathbf{Z}} \left|K_t^j * \phi_{j-l} * (S_{m-j}f)(x)\right|^2 \mathrm{d}t\right)^{\frac{1}{2}}\right\|_{L^p(\mathbf{R}^n,w)} \lesssim \|f\|_{L^p(\mathbf{R}^n,w)}. \tag{2.17}$$

Interpolating the inequalities (2.16) and (2.17) then shows that

$$\left\|\left(\int_1^2 \sum_{j\in\mathbf{Z}} \left|K_t^j * \phi_{j-l} * (S_{m-j}f)(x)\right|^2 \mathrm{d}t\right)^{\frac{1}{2}}\right\|_{L^p(\mathbf{R}^n,w)} \lesssim 2^{-t_p m}\|f\|_{L^p(\mathbf{R}^n,w)}.$$

This gives the desired estimate for E_2. Combining the estimates for E_1 and E_2 then yields (2.10) for the case $p\in(1,q)$ and $w^{-1/(p-1)}\in A_{p'/q'}(\mathbf{R}^n)$.

We now prove (2.10) for the case of $p\in(q',\infty)$ and $w\in A_{p/q'}(\mathbf{R}^n)$. By a standard argument, it suffices to prove that

$$M^\sharp(\widetilde{\mathcal{M}}^l_\Omega f)(x) \lesssim l M_{q'} f(x), \tag{2.18}$$

To prove (2.18), let $x\in\mathbf{R}^n$ and Q be a cube containing x. Decompose f as

$$f(y) = f(y)\chi_{4nQ}(y) + f(y)\chi_{\mathbf{R}^n\setminus 4nQ}(y) =: f_1(y) + f_2(y).$$

It is obvious that $\widetilde{\mathcal{M}}^l_\Omega$ is bounded on $L^{q'}(\mathbf{R}^n)$. Thus,

$$\frac{1}{|Q|}\int_Q \widetilde{\mathcal{M}}^l_\Omega f_1(y)\,\mathrm{d}y \lesssim \left(\frac{1}{|Q|}\int_Q \{\widetilde{\mathcal{M}}^l_\Omega f_1(y)\}^{q'}\,\mathrm{d}y\right)^{1/q'} \lesssim M_{q'}f(x). \tag{2.19}$$

Let $x_0 \in Q$ such that $\widetilde{\mathcal{M}}^l_\Omega f_2(x_0) < \infty$. For $y \in Q$ and $t \in [1,2]$, it follows from Lemma 2.2 that

$$\begin{aligned}&\sum_{j\in\mathbf{Z}} \big|K^j_t * \phi_{j-l} * f_2(y) - K^j_t * \phi_{j-l} * f_2(x_0)\big| \\ &\lesssim \sum_{j\in\mathbf{Z}}\sum_{k=2}^{\infty}\left(\int_{2^{k+1}nQ\setminus 2^k nQ}\big|K^j_t * \phi_{j-l}(y-z) - K^j_t * \phi_{j-l}(x_0-z)\big|^q\,\mathrm{d}z\right)^{\frac1q} \\ &\quad\cdot\left(\int_{2^{k+1}nQ}|f(z)|^{q'}\,\mathrm{d}z\right)^{\frac{1}{q'}} \lesssim lM_{q'}f(x).\end{aligned}$$

Thus, for all $y \in Q$,

$$\begin{aligned}&\Big|\widetilde{\mathcal{M}}^l_\Omega f_2(y) - \widetilde{\mathcal{M}}^l_\Omega f_2(y_0)\Big| \\ &\lesssim \left(\int_1^2 \sum_{j\in\mathbf{Z}}\big|K^j_t * \phi_{j-l} * f_2(y) - K^j_t * \phi_{j-l} * f_2(x_0)\big|^2\,\mathrm{d}t\right)^{\frac12} \\ &\lesssim \left(\int_1^2\left(\sum_{j\in\mathbf{Z}}\big|K^j_t * \phi_{j-l} * f_2(y) - K^j_t * \phi_{j-l} * f_2(x_0)\big|\right)^2\,\mathrm{d}t\right)^{\frac12} \lesssim lM_{q'}f(x).\end{aligned} \tag{2.20}$$

Combining the estimates (2.19) and (2.20) leads to that

$$\inf_{c\in\mathbf{C}}\frac{1}{|Q|}\int_Q\big|\widetilde{\mathcal{M}}^l_\Omega f(y) - c\big|\,\mathrm{d}y \lesssim lM_{q'}f(x)$$

and then establishes (2.18).

Finally, we see that (2.10) holds for the case of $p \in (1,\infty)$ and $w^{q'} \in A_p(\mathbf{R}^n)$, if we invoke the interpolation argument used in the proof of Theorem 2 in [28]. This completes the proof of Theorem 2.1. □

3. Proof of Theorem 1.3

We begin with some preliminary lemmas.

Lemma 3.1. *Let Ω be homogeneous of degree zero and belong to $L^1(S^{n-1})$, K^j_t be defined as in (2.1). Then for $l \in \mathbf{N}$, $t \in [1,2]$, $s \in (1,\infty]$, $j_0 \in \mathbf{Z}_-$ and $y \in \mathbf{R}^n$ with $|y| < 2^{j_0-4}$,*

$$\sum_{j>j_0}\sum_{k\in\mathbf{Z}} 2^{kn/s}\left(\int_{2^k<|x|\le 2^{k+1}}\big|K^j_t * \phi_{j-l}(x+y) - K^j_t * \phi_{j-l}(x)\big|^{s'}\,\mathrm{d}x\right)^{\frac{1}{s'}} \lesssim 2^{l(n+1)}2^{-j_0}|y|.$$

Proof. We follow the argument used in [38] (see also [11]), with suitable modification. Observe that $\operatorname{supp} K^j_t * \phi_{j-l} \subset \{x \in \mathbf{R}^n : 2^{j-2} \le |x| \le 2^{j+2}\}$ and

$$\|\phi_{j-l}(\cdot+y) - \phi_{j-l}(\cdot)\|_{L^{s'}(\mathbf{R}^n)} \lesssim 2^{(l-j)n/s}2^{l-j}|y|.$$

Thus, for all $k \in \mathbf{N}$,

$$
\begin{aligned}
&2^{\frac{kn}{s}}\sum_{j\in\mathbf{Z}}\bigg(\int_{2^k<|x|\le 2k+1}\big|K_t^j*\phi_{j-l}(x+y)-K_t^j*\phi_{j-l}(x)\big|^{s'}\,\mathrm{d}x\bigg)^{\frac{1}{s'}}\\
&\lesssim 2^{\frac{kn}{s}}\sum_{j\in\mathbf{Z}\colon |j-k|\le 3}\|K_t^j\|_{L^1(\mathbf{R}^n)}\|\phi_{j-l}(\cdot+y)-\phi_{j-l}(\cdot)\|_{L^{s'}(\mathbf{R}^n)}\lesssim 2^{l(n+1)}\frac{|y|}{2^k}.
\end{aligned}
$$

This, in turn, leads to that

$$
\begin{aligned}
&\sum_{j>j_0}\sum_{k\in\mathbf{Z}}2^{\frac{kn}{s}}\bigg(\int_{2^k<|x|\le 2k+1}\big|K_t^j*\phi_{j-l}(x+y)-K_t^j*\phi_{j-l}(x)\big|^{s'}\,\mathrm{d}x\bigg)^{\frac{1}{s'}}\\
&\lesssim\sum_{k>j_0-3}2^{\frac{kn}{s}}\sum_{j\in\mathbf{Z}}\bigg(\int_{2^k<|x|\le 2k+1}\big|K_t^j*\phi_{j-l}(x+y)-K_t^j*\phi_{j-l}(x)\big|^{s'}\mathrm{d}x\bigg)^{\frac{1}{s'}}\\
&\lesssim 2^{l(n+1)}2^{-j_0}|y|,
\end{aligned}
$$

and completes the proof of Lemma 3.1. □

For $t\in[1,2]$ and $j\in\mathbf{Z}$, let K_t^j be defined as in (2.1), ϕ and ϕ_l (with $l\in\mathbf{N}$) be the same as in Section 2. For $b\in\mathrm{BMO}(\mathbf{R}^n)$, let $\widetilde{\mathcal{M}}_{\Omega,b}^l$ be the commutator of $\widetilde{\mathcal{M}}_{\Omega}^l$ defined by

$$
\widetilde{\mathcal{M}}_{\Omega,b}^l f(x)=\bigg(\int_1^2\sum_{j\in\mathbf{Z}}\big|F_{j,b}^l f(x,t)\big|^2\,\mathrm{d}t\bigg)^{\frac12},
$$

with

$$
F_{j,b}^l f(x,t)=\int_{\mathbf{R}^n}\big(b(x)-b(y)\big)K_t^j*\phi_{j-l}(x-y)f(y)\,\mathrm{d}y.
$$

For $j_0\in\mathbf{Z}$, define the operator $\widetilde{\mathcal{M}}_{\Omega}^{l,j_0}$ by

$$
\widetilde{\mathcal{M}}_{\Omega}^{l,j_0} f(x)=\bigg(\int_1^2\sum_{j\in\mathbf{Z}\colon j>j_0}\big|F_{j}^l f(x,t)\big|^2\,\mathrm{d}t\bigg)^{\frac12},
$$

and the commutator $\widetilde{\mathcal{M}}_{\Omega,b}^{l,j_0}$ by

$$
\widetilde{\mathcal{M}}_{\Omega,b}^{l,j_0} f(x)=\bigg(\int_1^2\sum_{j\in\mathbf{Z}\colon j>j_0}\big|F_{j,b}^l f(x,t)\big|^2\,\mathrm{d}t\bigg)^{\frac12},
$$

with $b\in\mathrm{BMO}(\mathbf{R}^n)$.

Lemma 3.2. *Let Ω be homogeneous of degree zero and integrable on S^{n-1}. Then for $b\in C_0^\infty(\mathbf{R}^n)$, $l\in\mathbf{N}$, $j_0\in\mathbf{Z}_-$,*

$$
\big|\widetilde{\mathcal{M}}_{\Omega,b}^{l,j_0} f(x)-\widetilde{\mathcal{M}}_{\Omega,b}^{l} f(x)\big|\lesssim 2^{j_0}MM_\Omega f(x).
$$

Proof. Let $b \in C_0^\infty(\mathbf{R}^n)$ with $\|\nabla b\|_{L^\infty(\mathbf{R}^n)} = 1$. For $t \in [1, 2]$, by the fact that $\operatorname{supp} K_t^j * \phi_{j-l} \subset \{x\colon 2^{j-2} \le |x| \le 2^{j+2}\}$, it is easy to verify that

$$\begin{aligned}
&\sum_{j\le j_0}\int_{\mathbf{R}^n}\big|K_t^j*\phi_{j-l}(x-y)\big||x-y||f(y)|\mathrm{d}y\\
&\lesssim \sum_{j\le j_0}\sum_{k\in\mathbf{Z}}2^k\int_{2^k<|x-y|\le 2^{k+1}}\big|K_t^j*\phi_{j-l}(x-y)\big||f(y)|\,\mathrm{d}y\\
&\lesssim \sum_{j\le j_0}\sum_{|k-j|\le 3}2^k\int_{2^k<|x-y|\le 2^{k+1}}\big|K_t^j*\phi_{j-l}(x-y)\big||f(y)|\,\mathrm{d}y \lesssim 2^{j_0}M_\Omega Mf(x).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\Big|\widetilde{\mathcal{M}}_{\Omega,b}^{l,j_0}f(x)-\widetilde{\mathcal{M}}_{\Omega,b}^{l}f(x)\Big|^2\\
&\le \sum_{j\le j_0}\int_1^2\Big|\int_{\mathbf{R}^n}\big(b(x)-b(y)\big)K_t^j*\phi_{j-l}(x-y)f(y)\Big|^2\,\mathrm{d}t\\
&\lesssim \int_1^2\Big(\sum_{j\le j_0}\int_{\mathbf{R}^n}|x-y|\big|K_t^j*\phi_{j-l}(x-y)f(y)\big|\,\mathrm{d}y\Big)^2\,\mathrm{d}t \lesssim \{2^{j_0}M_\Omega Mf(x)\}^2.
\end{aligned}$$

The desired conclusion now follows immediately. □

Let $p, r \in [1, \infty)$, $q \in [1, \infty]$ and w be a weight, $L^p(L^q([1, 2]), l^r; \mathbf{R}^n, w)$ be the space of sequences of functions defined by

$$L^p(L^q([1,2]),l^r;\mathbf{R}^n,w)=\big\{\vec{f}=\{f_k\}_{k\in\mathbf{Z}}:\ \|\vec{f}\|_{L^p(L^q([1,2]),l^r;\mathbf{R}^n,w)}<\infty\big\},$$

with

$$\|\vec{f}\|_{L^p(L^q([1,2]),l^r;\mathbf{R}^n,w)}=\left\|\left(\int_1^2\Big(\sum_{k\in\mathbf{Z}}|f_k(x,t)|^r\Big)^{\frac{q}{r}}\mathrm{d}t\right)^{1/q}\right\|_{L^p(\mathbf{R}^n,w)}.$$

With usual addition and scalar multiplication, $L^p(L^q([1, 2]), l^r; \mathbf{R}^n, w)$ is a Banach space.

Lemma 3.3. *Let $p \in (1, \infty)$ and $w \in A_p(\mathbf{R}^n)$, $\mathcal{G} \subset L^p(L^2([1, 2]), l^2; \mathbf{R}^n, w)$. Suppose that $\mathcal{G}$ satisfies the following five conditions:*

(a) *$\mathcal{G}$ is bounded, that is, there exists a constant C such that for all $\vec{f} = \{f_k\}_{k\in\mathbf{Z}} \in \mathcal{G}$, $\|\vec{f}\|_{L^p(L^2([1,2]),l^2;\mathbf{R}^n,w)} \le C$;*

(b) *for each fixed $\epsilon > 0$, there exists a constant $A > 0$, such that for all $\vec{f} = \{f_k\}_{k\in\mathbf{Z}} \in \mathcal{G}$,*

$$\left\|\left(\int_1^2\sum_{k\in\mathbf{Z}}|f_k(\cdot,t)|^2\,\mathrm{d}t\right)^{\frac12}\chi_{\{|\cdot|>A\}}(\cdot)\right\|_{L^p(\mathbf{R}^n,w)}<\epsilon;$$

(c) *for each fixed $\epsilon > 0$ and $N \in \mathbf{N}$, there exists a constant $\varrho > 0$, such that for all $\vec{f} = \{f_k\}_{k\in\mathbf{Z}} \in \mathcal{G}$,*

$$\left\|\sup_{|h|\le\varrho}\left(\int_1^2\sum_{|k|\le N}|f_k(x,t)-f_k(x+h,t)|^2\,\mathrm{d}t\right)^{\frac12}\right\|_{L^p(\mathbf{R}^n,w)}<\epsilon;$$

(d) *for each fixed* $\epsilon > 0$ *and* $N \in \mathbf{N}$, *there exists a constant* $\sigma \in (0, 1/2)$ *such that for all* $\vec{f} = \{f_k\}_{k\in\mathbf{Z}} \in \mathcal{G}$,

$$\left\| \sup_{|s|\le\sigma} \left(\int_1^2 \sum_{|k|\le N} |f_k(\cdot, t+s) - f_k(\cdot, t)|^2 \,\mathrm{d}t \right)^{\frac{1}{2}} \right\|_{L^p(\mathbf{R}^n, w)} < \epsilon,$$

(e) *for each fixed* $D > 0$ *and* $\epsilon > 0$, *there exists* $N \in \mathbf{N}$ *such that for all* $\vec{f} = \{f_k\}_{k\in\mathbf{Z}} \in \mathcal{G}$,

$$\left\| \left(\int_1^2 \sum_{|k|>N} |f_k(\cdot, t)|^2 \,\mathrm{d}t \right)^{\frac{1}{2}} \chi_{B(0,D)} \right\|_{L^p(\mathbf{R}^n, w)} < \epsilon.$$

Then $\mathcal{G}$ *is a strongly pre-compact set in* $L^p(L^2([1, 2]), l^2; \mathbf{R}^n, w)$.

Proof. We employ the argument used in the proof of [14, Theorem 5], with some refined modifications. Our goal is to prove that, for each fixed $\epsilon > 0$, there exists a $\delta = \delta_\epsilon > 0$ and a mapping Φ_ϵ on $L^p(L^2([1, 2]), l^2; \mathbf{R}^n, w)$, such that $\Phi_\epsilon(\mathcal{G}) = \{\Phi_\epsilon(\vec{f}) : \vec{f} \in \mathcal{G}\}$ is a strong pre-compact set in the space $L^p(L^2([1, 2]), l^2; \mathbf{R}^n, w)$, and for any $\vec{f}$, $\vec{g} \in \mathcal{G}$,

$$\|\Phi_\epsilon(\vec{f}) - \Phi_\epsilon(\vec{g})\|_{L^p(L^2([1,2]),l^2;\mathbf{R}^n,w)} < \delta \implies \|\vec{f} - \vec{g}\|_{L^p(L^2([1,2]),l^2;\mathbf{R}^n,w)} < 8\epsilon. \tag{3.1}$$

If we can prove this, then by Lemma 6 in [14], we see that $\mathcal{G}$ is a strongly pre-compact set in $L^p(L^2([1, 2]), l^2; \mathbf{R}^n, w)$.

Now let $\epsilon > 0$. We choose $A > 1$ large enough as in assumption (b), $N \in \mathbf{N}$ such that for all $\{f_k\}_{k\in\mathbf{Z}} \in \mathcal{G}$,

$$\left\| \left(\int_1^2 \sum_{|k|>N} |f_k(\cdot, t)|^2 \,\mathrm{d}t \right)^{1/2} \chi_{B(0,2A)} \right\|_{L^p(\mathbf{R}^n, w)} < \epsilon.$$

Let $\varrho \in (0, 1/2)$ small enough as in assumption (c) and $\sigma \in (0, 1/2)$ small enough such that (d) holds true. Let Q be the largest cube centered at the origin such that $2Q \subset B(0, \varrho)$, $Q_1, \ldots, Q_J$ be J copies of Q such that they are non-overlapping, and $\overline{B(0, A)} \subset \overline{\bigcup_{j=1}^J Q_j} \subset B(0, 2A)$. Let $I_1, \ldots, I_L \subset [1, 2]$ be non-overlapping intervals with same length $|I|$, such that $|s - t| \le \sigma$ for all $s, t \in I_j$ $(j = 1, \ldots, L)$ and $\bigcup_{j=1}^L I_j = [1, 2]$. Define the mapping Φ_ϵ on $L^p(L^2([1, 2]), l^2; \mathbf{R}^n, w)$ by

$$\Phi_\epsilon(\vec{f})(x, t) = \Bigg\{ \ldots, 0, \ldots, 0, \sum_{i=1}^J \sum_{j=1}^L m_{Q_i\times I_j}(f_{-N})\chi_{Q_i\times I_j}(x, t),$$
$$\sum_{i=1}^J \sum_{j=1}^L m_{Q_i\times I_j}(f_{-N+1})\chi_{Q_i\times I_j}(x, t), \ldots, \sum_{i=1}^J \sum_{j=1}^L m_{Q_i\times I_j}(f_N)\chi_{Q_i\times I_j}(x, t), 0, \ldots \Bigg\},$$

where and in the following,

$$m_{Q_i\times I_j}(f_k) = \frac{1}{|Q_i|}\frac{1}{|I_j|}\int_{Q_i\times I_j} f_k(x, t)\,\mathrm{d}x\,\mathrm{d}t.$$

We claim that Φ_ϵ is bounded on $L^p(L^2([1,2]),l^2;\,\mathbf{R}^n,w)$. In fact, if $p\in[2,\infty)$, we have by Hölder's inequality that

$$|m_{Q_i\times I_j}(f_k)|\le\left(\frac{1}{|Q_i||I_j|}\int_{I_j\times Q_i}|f_k(y,t)|^pw(y)\,\mathrm{d}y\,\mathrm{d}t\right)^{\frac{1}{p}}\left(\frac{1}{|Q_i|}\int_{Q_i}w^{-\frac{1}{p-1}}(y)\,\mathrm{d}y\right)^{\frac{1}{p'}},$$

and

$$\begin{aligned}&\sum_{|k|\le N}\left(\frac{1}{|Q_i||I_j|}\int_{I_j}\int_{Q_i}|f_k(y,t)|^pw(y)\,\mathrm{d}y\,\mathrm{d}t\right)^{2/p}\\&\lesssim N^{1-2/p}\left(\sum_{|k|\le N}\frac{1}{|Q_i||I_j|}\int_{I_j\times Q_i}|f_k(y,t)|^pw(y)\,\mathrm{d}y\,\mathrm{d}t\right)^{2/p}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|\Phi_\epsilon(\vec{f})\|^p_{L^p(L^2([1,2]),l^2;\mathbf{R}^n,w)}&=\sum_{i=1}^J\sum_{j=1}^L\int_{I_j}\int_{Q_i}\left(\sum_{|k|\le N}|m_{Q_i\times I_j}(f_k)|^2\right)^{p/2}w(x)\,\mathrm{d}x\,\mathrm{d}t\\&\lesssim N^{p/2-1}\sum_{i=1}^J\sum_{j=1}^L\int_{I_j}\int_{Q_i}\sum_{|k|\le N}|f_k(y,t)|^pw(y)\,\mathrm{d}y\,\mathrm{d}t\\&\le N^{p/2}\sum_{i=1}^J\sum_{j=1}^L\int_{I_j}\int_{Q_i}\left\{\sum_{|k|\le N}|f_k(y,t)|^2\right\}^{\frac{p}{2}}w(y)\,\mathrm{d}y\,\mathrm{d}t\\&\le N^{p/2}\|\vec{f}\|^p_{L^p(L^2([1,2]),l^2;\,\mathbf{R}^n,w)}.\end{aligned}$$

On the other hand, for $p\in(1,2)$ and $w\in A_p(\mathbf{R}^n)$, we choose $\gamma\in(0,1)$ such that $w\in A_{p-\gamma}(\mathbf{R}^n)$. Note that

$$\sup_{-N\le k\le N}\sup_{t\in[1,2]}\left|\sum_{i=1}^J\sum_{j=1}^L m_{Q_i\times I_j}(f_k)\chi_{Q_i\times I_j}(x,t)\right|\lesssim\sup_{k\in\mathbf{Z}}\sup_{t\in[1,2]}|f_k(x,t)|,$$

which implies that for $p_1\in(1,\infty)$,

$$\|\Phi_\epsilon(\vec{f})\|_{L^{p_1}(L^\infty([1,2]),l^\infty;\,\mathbf{R}^n,w)}\lesssim\|\vec{f}\|_{L^{p_1}(L^\infty([1,2]),l^\infty;\,\mathbf{R}^n,w)}.\tag{3.2}$$

We also have that for $p_0=p-\gamma$,

$$|m_{Q_i\times I_j}(f_k)|\le\left(\frac{1}{|Q_i||I_j|}\int_{I_j}\int_{Q_i}|f_k(y,t)|^{p_0}w(y)\,\mathrm{d}y\,\mathrm{d}t\right)^{\frac{1}{p_0}}\left(\frac{1}{|Q_i|}\int_{Q_i}w^{-\frac{1}{p_0-1}}(y)\,\mathrm{d}y\right)^{\frac{1}{p_0'}},$$

and so

$$\|\Phi_\epsilon(\vec{f})\|_{L^{p_0}(L^{p_0}([1,2]),l^{p_0};\,\mathbf{R}^n,w)}\lesssim\|\vec{f}\|_{L^{p_0}(L^{p_0}([1,2]),l^{p_0};\,\mathbf{R}^n,w)}.\tag{3.3}$$

By interpolation, we can deduce from (3.2) and (3.3) that in this case

$$\|\Phi_\epsilon(\vec{f})\|_{L^p(L^2([1,2]),l^2;\,\mathbf{R}^n,w)}\lesssim\|\vec{f}\|_{L^p(L^2([0,1]),l^2,\mathbf{R}^n,w)}.$$

Our claim then follows directly, and so $\Phi_\epsilon(\mathcal{G})=\{\Phi_\epsilon(\vec{f})\colon\vec{f}\in\mathcal{G}\}$ is strongly precompact in $L^p(L^2([1,2]),l^2;\mathbf{R}^n,w)$.

We now verify (3.1). Denote $\mathcal{D}=\bigcup_{i=1}^{J} Q_i$ and write

$$\begin{aligned}
&\big\|\vec{f}\chi_{\mathcal{D}}-\Phi_\epsilon(\vec{f})\big\|_{L^p(L^2([1,2]),l^2;\,\mathbf{R}^n,w)}\\
&\le\bigg\|\bigg(\int_1^2\sum_{|k|\le N}\Big|f_k(\cdot,t)\chi_{\mathcal{D}}-\sum_{i=1}^{J}\sum_{j=1}^{L}m_{Q_i\times I_j}(f_k)\chi_{Q_i\times I_j}(x,t)\Big|^2\,\mathrm{d}t\bigg)^{1/2}\bigg\|_{L^p(\mathbf{R}^n,w)}\\
&\quad+\bigg\|\bigg(\int_1^2\sum_{|k|>N}\big|f_k(\cdot,t)\big|^2\bigg)^{\frac12}\chi_{B(0,2A)}\bigg\|_{L^p(\mathbf{R}^n,w)}.
\end{aligned}$$

Noting that for $x\in Q_i$ with $1\le i\le J$,

$$\begin{aligned}
&\bigg\{\int_1^2\sum_{|k|\le N}\Big|f_k(x,t)\chi_{\mathcal{D}}(x)-\sum_{u=1}^{J}\sum_{v=1}^{L}m_{Q_u\times I_v}(f_k)\chi_{Q_u\times I_v}(x,t)\Big|^2\,\mathrm{d}t\bigg\}^{\frac12}\\
&\lesssim|Q|^{-1/2}|I|^{-1/2}\bigg\{\sum_{j=1}^{L}\int_{I_j}\int_{Q_i}\int_{I_j}\sum_{|k|\le N}\big|f_k(x,t)-f_k(y,s)\big|^2\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}t\bigg\}^{\frac12}\\
&\lesssim|Q|^{-1/2}\bigg\{\int_{2Q}\int_1^2\sum_{|k|\le N}|f_k(x,s)-f_k(x+h,s)|^2\,\mathrm{d}s\,\mathrm{d}h\bigg\}^{\frac12}\\
&\quad+|I|^{-1/2}\bigg\{\sum_{j=1}^{L}\int_{I_j}\int_{I_j}\sum_{|k|\le N}|f_k(x,t)-f_k(x,s)|^2\,\mathrm{d}t\,\mathrm{d}s\bigg\}^{\frac12}\\
&\lesssim\sup_{|h|\le\varrho}\bigg(\int_1^2\sum_{|k|\le N}|f_k(x,t)-f_k(x+h,t)|^2\,\mathrm{d}t\bigg)^{\frac12}\\
&\quad+\sup_{|s|\le\sigma}\bigg(\int_1^2\sum_{|k|\le N}|f_k(x,t+s)-f_k(x,t)|^2\,\mathrm{d}t\bigg)^{\frac12},
\end{aligned}$$

we then get that

$$\sum_{i=1}^{J}\int_{Q_i}\bigg\{\int_1^2\sum_{|k|\le N}\Big|f_k(x,t)-\sum_{l=1}^{J}m_{Q_l}(f_k)\chi_{Q_l}(x)\Big|^2\,\mathrm{d}t\bigg\}^{p/2}w(x)\,\mathrm{d}x\lesssim 2\epsilon.$$

It then follows from the assumption (b) that for all $\vec{f}\in\mathcal{G}$,

$$\begin{aligned}
\|\vec{f}-\Phi_\epsilon(\vec{f})\|_{L^p(L^2([1,2]),l^2;\,\mathbf{R}^n,w)}&\le\big\|\vec{f}\chi_{\mathcal{D}}-\Phi_\epsilon(\vec{f})\big\|_{L^p(L^2([1,2]),l^2;\,\mathbf{R}^n,w)}\\
&\quad+\bigg\|\bigg(\int_1^2\sum_{k\in\mathbf{Z}}|f_k(\cdot,t)|^2\,\mathrm{d}t\bigg)^{\frac12}\chi_{\{|\cdot|>A\}}(\cdot)\bigg\|_{L^p(\mathbf{R}^n,w)}\\
&<3\epsilon.
\end{aligned}$$

Noting that

$$\begin{aligned}\|\vec{f}-\vec{g}\|_{L^p(L^2([1,2]),l^2;\mathbf{R}^n,w)} \le \|\vec{f}-\Phi_\epsilon(\vec{f})\|_{L^p(L^2([1,2]),l^2;\mathbf{R}^n,w)}\\ +\|\Phi_\epsilon(\vec{f})-\Phi_\epsilon(\vec{g})\|_{L^p(L^2([1,2]),l^2;\mathbf{R}^n,w)}\\ +\|\vec{g}-\Phi_\epsilon(\vec{g})\|_{L^p(L^2([1,2]),l^2;\mathbf{R}^n,w)},\end{aligned}$$

we then get (3.1) and finish the proof of Lemma 3.3. □

Proof of Theorem 1.3. Let $j_0\in\mathbf{Z}_-$, $b\in C_0^\infty(\mathbf{R}^n)$ with $\operatorname{supp} b\subset B(0,R)$, p and w be the same as in Theorem 1.3. Without loss of generality, we may assume that $\|b\|_{L^\infty(\mathbf{R}^n)}+\|\nabla b\|_{L^\infty(\mathbf{R}^n)}=1$. We claim that

(i) for each fixed $\epsilon>0$, there exists a constant $A>0$ such that

$$\left\|\left(\int_1^2\sum_{j\in\mathbf{Z}}|F_{j,b}^l f(x,t)|^2\,\mathrm{d}t\right)^{1/2}\chi_{\{|\cdot|>A\}}(\cdot)\right\|_{L^p(\mathbf{R}^n,w)}<\epsilon\|f\|_{L^p(\mathbf{R}^n,w)};$$

(ii) for $s\in(1,\infty)$,

$$\begin{aligned}&\left(\int_1^2\sum_{j>j_0}|F_{j,b}^l f(x,t)-F_{j,b}^l f(x+h,t)|^2\,\mathrm{d}t\right)^{1/2}\\ &\lesssim 2^{-j_0}|h|\left(\widetilde{\mathcal{M}}_\Omega^{l,j_0}f(x)+2^{l(n+1)}M_s f(x)\right);\end{aligned}\tag{3.4}$$

(iii) for each $\epsilon>0$ and $N\in\mathbf{N}$, there exists a constant $\sigma\in(0,1/2)$ such that

$$\left\|\sup_{|s|\le\sigma}\left(\int_1^2\sum_{|j|\le N}|F_{j,b}^l f(x,s+t)-F_{j,b}^l f(x,t)|^2\,\mathrm{d}t\right)^{\frac12}\right\|_{L^p(\mathbf{R}^n,w)}<\epsilon\|f\|_{L^p(\mathbf{R}^n,w)};\tag{3.5}$$

(iv) for each fixed $D>0$ and $\epsilon>0$, there exists $N\in\mathbf{N}$ such that

$$\left\|\left(\int_1^2\sum_{j>N}|F_{j,b}^l f(\cdot,t)|^2\,\mathrm{d}t\right)^{1/2}\chi_{B(0,D)}\right\|_{L^p(\mathbf{R}^n,w)}<\epsilon\|f\|_{L^p(\mathbf{R}^n,w)}.\tag{3.6}$$

We now prove claim (i). Let $t\in[1,2]$. For each fixed $x\in\mathbf{R}^n$ with $|x|>4R$, observe that $\operatorname{supp} K_t^j*\phi_{j-l}\subset\{2^{j-2}\le|y|\le 2^{j+2}\}$, and $\int_{|z|<R}|K_t^j*\phi_{j-l}(x-z)|\,\mathrm{d}z\ne 0$ only if $2^j\approx|x|$. A trivial computation shows that

$$\begin{aligned}\int_{|z|<R}\left|K_t^j*\phi_{j-l}(x-z)\right|\mathrm{d}z&\lesssim\left(\int_{|z|<R}\left|K_t^j*\phi_{j-l}(x-z)\right|^2\mathrm{d}z\right)^{\frac12}R^{\frac n2}\\ &\lesssim\left(\int_{\frac{|x|}{2}\le|z|<2|x|}\left|K_t^j*\phi_{j-l}(z)\right|^2\mathrm{d}z\right)^{\frac12}R^{\frac n2}\\ &\lesssim\|K_t^j\|_{L^1(S^{n-1})}\|\phi_{j-l}\|_{L^2(\mathbf{R}^n)}R^{\frac n2}\\ &\lesssim 2^{nl/2}|x|^{-\frac n2}R^{\frac n2}.\end{aligned}$$

On the other hand, we have that for $s \in (1,p)$,

$$\begin{aligned}
&\sum_{j\in\mathbf{Z}}\Big(\int_{|y|<R}|K_t^j*\phi_{j-l}(x-y)||f(y)|^s\,\mathrm{d}y\Big)^{\frac{1}{s}}\\
&=\sum_{j\in\mathbf{Z}:\,2^j\approx|x|}\Big(\int_{|x|/2\le|y-x|\le2|x|}|K_t^j*\phi_{j-l}(x-y)||f(y)|^s\,\mathrm{d}y\Big)^{\frac{1}{s}}\\
&\lesssim\big(M_\Omega M(|f|^s)(x)\big)^{1/s}.
\end{aligned}$$

Another application of Hölder's inequality then yields

$$\begin{aligned}
\sum_{j\in\mathbf{Z}}\big|F_{j,b}^lf(x,t)\big|^2&\lesssim\sum_{j\in\mathbf{Z}}\Big(\int_{|y|<R}|K_t^j*\phi_{j-l}(x-y)||f(y)|^s\,\mathrm{d}y\Big)^{2/s}\\
&\quad\cdot\Big(\int_{|y|<R}|K_t^j*\phi_{j-l}(x-y)|\,\mathrm{d}y\Big)^{2/s'}\\
&\lesssim 2^{\frac{nl}{s'}}|x|^{-\frac{n}{s'}}R^{\frac{n}{s'}}\big(M_\Omega M(|f|^s)(x)\big)^{2/s}.
\end{aligned}\tag{3.7}$$

This, in turn leads to our claim (i).

We turn our attention to claim (ii). Write

$$|F_{j,b}^lf(x,t)-F_{j,b}^lf(x+h,t)|\le|b(x)-b(x+h)||F_j^lf(x,t)|+\mathrm{J}_j^lf(x,t),$$

with

$$\mathrm{J}_j^lf(x,t)=\Big|\int_{\mathbf{R}^n}\big(K_t^j*\phi_{j-l}(x-y)-K_t^j*\phi_{j-l}(x+h-y)\big)\big(b(x+h)-b(y)\big)f(y)\,\mathrm{d}y\Big|.$$

It follows from Hölder's inequality and Lemma 3.1 that

$$\begin{aligned}
\Big(\sum_{j>j_0}|\mathrm{J}_j^lf(x,t)|^2\Big)^{\frac12}&\lesssim\sum_{j>j_0}\int_{\mathbf{R}^n}\big|K_t^j*\phi_{j-l}(x-y)-K_t^j*\phi_{j-l}(x+h-y)\big||f(y)|\,\mathrm{d}y\\
&\lesssim\sum_{j>j_0}\sum_{k\in\mathbf{Z}}\Big(\int_{2^k<|x-y|\le2^{k+1}}\big|K_t^j*\phi_{j-l}(x-y)\\
&\quad-K_t^j*\phi_{j-l}(x+h-y)\big|^{s'}rmdy\Big)^{\frac{1}{s'}}\Big(\int_{|x-y|\le2^{k+1}}|f(y)|^s\,\mathrm{d}y\Big)^{\frac1s}\\
&\lesssim 2^{l(n+1)}|h|2^{-j_0}M_sf(x).
\end{aligned}$$

Therefore,

$$\Big(\int_1^2\sum_{j>j_0}|F_{j,b}^lf(x,t)-F_{j,b}^lf(x+h,t)|^2\,\mathrm{d}t\Big)^{\frac12}\lesssim|h|\widetilde{\mathcal{M}}_\Omega^{l,j_0}f(x)+2^{l(n+1)}2^{-j_0}|h|M_sf(x).$$

We now verify claim (iii). For each fixed $\sigma\in(0,1/2)$ and $t\in[1,2]$, let

$$U_{t,\sigma}^j(z)=\frac{1}{2^j}\frac{|\Omega(z)|}{|z|^{n-1}}\chi_{\{2^j(t-\sigma)\le|z|\le2^jt\}}+\frac{1}{2^j}\frac{|\Omega(z)|}{|z|^{n-1}}\chi_{\{2^{j+1}t\le|z|\le2^{j+1}(t+\sigma)\}},$$

and

$$G_{l,t,\sigma}^jf(x)=\int_{\mathbf{R}^n}\big(U_{t,\sigma}^j*|\phi_{j-l}|\big)(x-y)|f(y)|\,\mathrm{d}y.$$

Note that for $t \in [1,2]$,

$$\|U^j_{t,\sigma} * |\phi_{j-l}|\|_{L^1(\mathbf{R}^n)} \lesssim \sigma, \quad \sup_{|j|\le N}\sup_{t\in[1,2]} |G^j_{l,t,\sigma} f(x)| \lesssim MM_\Omega f(x).$$

Thus,

$$\left\| \sup_{|j|\le N}\sup_{t\in[1,2]} |G^j_{l,t,\sigma} f| \right\|_{L^\infty(\mathbf{R}^n)} \lesssim \sigma \|f\|_{L^\infty(\mathbf{R}^n)}, \tag{3.8}$$

and

$$\left\| \sup_{|j|\le N}\sup_{t\in[1,2]} |G^j_{l,t,\sigma} f| \right\|_{L^p(\mathbf{R}^n,w)} \lesssim \|MM_\Omega f\|_{L^p(\mathbf{R}^n,w)} \lesssim \|f\|_{L^p(\mathbf{R}^n,w)}. \tag{3.9}$$

Interpolating the estimates (3.8) and (3.9) shows that if $p_1 \in (p,\infty)$,

$$\left\| \sup_{|j|\le N}\sup_{t\in[1,2]} |G^j_{l,t,\sigma} f| \right\|_{L^{p_1}(\mathbf{R}^n,w)} \lesssim \sigma^{1-p/p_1} \|f\|_{L^{p_1}(\mathbf{R}^n,w)}. \tag{3.10}$$

On the other hand, if $p_0 \in (1,p)$, it then follows from the weighted estimae M and M_Ω that

$$\int_{\mathbf{R}^n}\int_1^2 \sum_{|j|\le N} \left|G^j_{l,t,\sigma} f(x)\right|^{p_0} \mathrm{d}t w(x)\,\mathrm{d}x \lesssim N\|f\|^{p_0}_{L^{p_0}(\mathbf{R}^n,w)}. \tag{3.11}$$

Choosing $p_1 \in (2,\infty)$ such that $1/p = 1/2 + (2-p_0)/(2p_1)$ in (3.10), we get from (3.10) and (3.11) that for $p \in (1,2)$,

$$\left\| \left(\int_1^2 \sum_{|j|\le N} \left|G^j_{l,t,\sigma} f(x)\right|^2 \mathrm{d}t \right)^{\frac12} \right\|_{L^p(\mathbf{R}^n,w)} \lesssim N\sigma^{\tau_1} \|f\|_{L^p(\mathbf{R}^n,w)}. \tag{3.12}$$

with $\tau_1 \in (0,1)$ a constant. If $p \in [2,\infty)$, we obtain from Minkowski's inequality and Young's inequality that

$$\begin{aligned} &\left\| \left(\int_1^2 \sum_{|j|\le N} |G^j_{l,t,\sigma} f(x)|^2 \mathrm{d}t \right)^{\frac12} \right\|^2_{L^p(\mathbf{R}^n,w)} \\ &\lesssim \left\{ \int_{\mathbf{R}^n} \left(\int_1^2 \left(\sum_{|j|\le N} \int_{\mathbf{R}^n} \left(U^j_{l,t,\sigma} * |\phi_{j-l}|\right)(x-y)|f(y)|\,\mathrm{d}y \right)^2 \mathrm{d}t \right)^{\frac{p}{2}} w(x)\,\mathrm{d}x \right\}^{\frac{2}{p}} \\ &\lesssim \int_1^2 \left\{ \sum_{|j|\le N} \left(\int_{\mathbf{R}^n} \left(\int_{\mathbf{R}^n} \left(U^j_{l,t,\sigma} * |\phi_{j-l}|\right)(x-y)|f(y)|\,\mathrm{d}y \right)^p w(x)\,\mathrm{d}x \right)^{\frac1p} \right\}^2 \mathrm{d}t \\ &\lesssim N^2 \|f\|^2_{L^p(\mathbf{R}^n,w)}. \end{aligned} \tag{3.13}$$

Also, we have that

$$\left\{\int_{\mathbf{R}^n}\left(\int_1^2\sum_{|j|\le N}\left(\int_{\mathbf{R}^n}\left(U^j_{l,t,\sigma}*|\phi_{j-l}|\right)(x-y)|f(y)|\,\mathrm{d}y\right)^2\mathrm{d}t\right)^{\frac{p}{2}}\mathrm{d}x\right\}^{\frac{2}{p}} \tag{3.14}$$

$$\begin{aligned}&\lesssim\int_1^2\left\{\sum_{|j|\le N}\left\|U^j_{l,t,\sigma}*|\phi_{j-l}|*|f|\right\|_{L^p(\mathbf{R}^n)}\right\}^2\mathrm{d}t\\&\lesssim(2N\sigma)^2\|f\|^2_{L^p(\mathbf{R}^n)},\quad p\in[2,\infty).\end{aligned}$$

The inequalities (3.13) and (3.14), via interpolation with changes of measures, give us that for $p\in[2,\infty)$,

$$\left\|\left(\int_1^2\sum_{|j|\le N}|G^j_{l,t,\sigma}f(x)|^2\,\mathrm{d}t\right)^{\frac12}\right\|_{L^p(\mathbf{R}^n,w)}\lesssim N\sigma^{\tau_2}\|f\|_{L^p(\mathbf{R}^n,w)}, \tag{3.15}$$

with $\tau_2\in(0,1)$ a constant. Since

$$\sup_{|s|\le\sigma}|F^l_{j,b}f(x,t)-F^l_{j,b}f(x,t+s)|\le G^j_{l,t,\sigma}f(x),$$

our claim (iii) now follow from (3.12) and (3.15) immediately if we choose $\sigma=\epsilon/(2N)$.

It remains to prove (iv). Let $D>0$ and $N\in\mathbf{N}$ such that $2^{N-2}>D$. Then for $j>N$ and $x\in\mathbf{R}^n$ with $|x|\le D$,

$$\begin{aligned}\int_{\mathbf{R}^n}\left|K^j_t*\phi_{j-l}(x-y)f(y)\right|\mathrm{d}y&\le\int_{\mathbf{R}^n}\left|K^j_t*\phi_{j-l}(x-y)f(y)\right|\chi_{\{|y|\le2^{j+3}\}}(y)\,\mathrm{d}y\\&\lesssim\int_{|y|\le2^{j+3}}|f(y)|\,\mathrm{d}y\,\|K^j_t\|_{L^1(\mathbf{R}^n)}\|\phi_{j-l}\|_{L^\infty(\mathbf{R}^n)}\\&\lesssim2^{nl}2^{-nj/p}\|f\|_{L^p(\mathbf{R}^n)}.\end{aligned}$$

Therefore,

$$\left\|\left(\int_1^2\sum_{j>N}\left|F^l_{j,b}f(\cdot,t)\right|^2\mathrm{d}t\right)^{\frac12}\chi_{B(0,D)}\right\|_{L^p(\mathbf{R}^n)}\lesssim2^{nl}\left(\frac{D}{2^N}\right)^{n/p}\|f\|_{L^p(\mathbf{R}^n)}. \tag{3.16}$$

It is obvious that

$$\left\|\left(\int_1^2\sum_{j>N}\left|F^l_{j,b}f(\cdot,t)\right|^2\mathrm{d}t\right)^{\frac12}\chi_{B(0,D)}\right\|_{L^p(\mathbf{R}^n,w)}\lesssim l\|f\|_{L^p(\mathbf{R}^n,w)}. \tag{3.17}$$

Interpolating the inequalities (3.16) and (3.17) yields

$$\left\|\left(\int_1^2\sum_{j>N}\left|F^l_{j,b}f(\cdot,t)\right|^2\mathrm{d}t\right)^{\frac12}\chi_{B(0,D)}\right\|_{L^p(\mathbf{R}^n,w)}\lesssim2^{\tau_3nl}\left(\frac{D}{2^N}\right)^{\frac{\tau_3n}{p}}\|f\|_{L^p(\mathbf{R}^n,w)}.$$

with $\tau_3\in(0,1)$ a constant depending only on w. The claim (iv) now follows directly.

We can now conclude the proof of Theorem 1.3. Let $p\in(1,\infty)$. Note that

$$\widetilde{\mathcal{M}}^{l,j_0}_{\Omega,b}f(x)\le\widetilde{\mathcal{M}}^l_{\Omega,b}f(x),$$

and so $\widetilde{\mathcal{M}}^{l,j_0}_{\Omega,b}$ is bounded on $L^p(\mathbf{R}^n,w)$. Our claims (i)–(iv), via Lemma 3.3, prove that for $b\in C_0^\infty(\mathbf{R}^n)$, $l\in\mathbf{N}$ and $j_0\in\mathbf{Z}_-$, the operator $\mathcal{F}^{l,}_{j_0}$ defined by

$$\mathcal{F}^l_{j_0}\colon f(x)\to\{\dots,0,\dots,F^l_{j_0,b}f(x,t),F^l_{j_0+1,b}f(x,t),\dots\} \tag{3.18}$$

is compact from $L^p(\mathbf{R}^n,w)$ to $L^p(L^2([1,2]),l^2;\mathbf{R}^n,w)$. Thus, $\widetilde{\mathcal{M}}^{l,j_0}_{\Omega,b}$ is completely continuous on $L^p(\mathbf{R}^n,w)$. This, via Lemma 3.2 and Theorem 2.1, shows that for $b\in C_0^\infty(\mathbf{R}^n)$, $\widetilde{\mathcal{M}}_{\Omega,b}$ is completely continuous on $L^p(\mathbf{R}^n,w)$. Note that

$$\big|\mathcal{M}_{\Omega,b}f_k(x)-\mathcal{M}_{\Omega,b}f(x)\big|\lesssim\mathcal{M}_{\Omega,b}(f_k-f)(x)\lesssim\widetilde{\mathcal{M}}_{\Omega,b}(f_k-f)(x).$$

Thus, for $b\in C_0^\infty(\mathbf{R}^n)$, $\mathcal{M}_{\Omega,b}$ is completely continuous on $L^p(\mathbf{R}^n,w)$. Recalling that $\mathcal{M}_{\Omega,b}$ is bounded on $L^p(\mathbf{R}^n,w)$ with bound $C\|b\|_{\mathrm{BMO}(\mathbf{R}^n)}$, we obtain that for $b\in\mathrm{CMO}(\mathbf{R}^n)$, $\mathcal{M}_{\Omega,b}$ is completely continuous on $L^p(\mathbf{R}^n,w)$. □

4. Proof of Theorem 1.5

The following lemma will be useful in the proof of Theorem 1.5, and is of independent interest.

Lemma 4.1. *Let $u\in(1,\infty)$, $m\in\mathbf{N}\cup\{0\}$, S be a sublinear operator which satisfies that*

$$|Sf(x)|\le\int_{\mathbf{R}^n}|b(x)-b(y)|^m|W(x-y)f(y)|\mathrm{d}y,$$

with $b\in\mathrm{BMO}(\mathbf{R}^n)$, and

$$\sup_{R>0}R^{n/u}\bigg(\int_{R\le|x|\le2R}|W(x)|^{u'}\,\mathrm{d}x\bigg)^{1/u'}\lesssim1. \tag{4.1}$$

(a) *Let $p\in(u,\infty)$, $\lambda\in(0,1)$ and $w\in A_{p/u}(\mathbf{R}^n)$. If S is bounded on $L^p(\mathbf{R}^n,w)$ with bound $D\|b\|^m_{\mathrm{BMO}(\mathbf{R}^n)}$, then for some $\varepsilon\in(0,1)$,*

$$\big\|Sf\big\|_{L^{p,\lambda}(\mathbf{R}^n,w)}\lesssim(D+D^\varepsilon)\|b\|^m_{\mathrm{BMO}(\mathbf{R}^n)}\|f\|_{L^{p,\lambda}(\mathbf{R}^n,w)}.$$

(b) *Let $p\in(1,u)$, $w^r\in A_1(\mathbf{R}^n)$ for some $r\in(u,\infty)$ and $\lambda\in(0,1-r'/u')$. If S is bounded on $L^p(\mathbf{R}^n,w)$ with bound D, then for some $\varepsilon\in(0,1)$,*

$$\big\|Sf\big\|_{L^{p,\lambda}(\mathbf{R}^n)}\lesssim(D+D^\varepsilon)\|b\|^m_{\mathrm{BMO}(\mathbf{R}^n)}\|f\|_{L^{p,\lambda}(\mathbf{R}^n)}.$$

Proof. For simplicity, we only consider the case of $m=1$ and $\|b\|_{\mathrm{BMO}(\mathbf{R}^n)}=1$. For fixed ball B and $f\in L^{p,\lambda}(\mathbf{R}^n,w)$, decompose f as

$$f(y)=f(y)\chi_{2B}(y)+\sum_{k=1}^\infty f(y)\chi_{2^{k+1}B\backslash2^kB}(y)=\sum_{k=0}^\infty f_k(y).$$

It is obvious that

$$\int_B|Sf_0(y)|^pw(y)\,\mathrm{d}y\lesssim D^p\int_{2B}|f(y)|^pw(y)\,\mathrm{d}y\lesssim D^p\|f\|^p_{L^{p,\lambda}(\mathbf{R}^n,w)}\{w(B)\}^\lambda.$$

Let $\theta\in(1,p/u)$ such that $w\in A_{p/(\theta u)}(\mathbf{R}^n)$. For each $k\in\mathbf{N}$, let $S_kf(x)=S\big(f\chi_{2^{k+1}B\backslash2^kB}\big)(x)$. Then S_k is also sublinear. We have by Hölder's inequality that for each $x \in B$,

$$|S_k f(x)| \lesssim |b(x) - m_B(b)| \|f_k\|_{L^u(\mathbf{R}^n)} \Big(\int_{2^k B} |W(x-y)|^{u'} \,\mathrm{d}y \Big)^{1/u'}$$
$$+ \big\| \big(b - m_B(b)\big) f_k \big\|_{L^u(\mathbf{R}^n)} \Big(\int_{2^k B} |W(x-y)|^{u'} \,\mathrm{d}y \Big)^{1/u'}$$
$$\lesssim |b(x) - m_B(b)| \|f_k\|_{L^p(\mathbf{R}^n,w)} \Big(\int_{2^k B} w^{-\frac{1}{p/u-1}}(y) \,\mathrm{d}y \Big)^{\frac{1}{u(p/u)'}} |2^k B|^{-\frac{1}{u}}$$
$$+ \Big(\int_{2^{k+1} B} |b(y) - m_B(b)|^{p\theta'} \,\mathrm{d}y \Big)^{1/(p\theta')} \|f_k\|_{L^p(\mathbf{R}^n,w)}$$
$$\cdot \Big(\int_{2^k B} w^{-\frac{1}{p/(\theta u)-1}}(y) \,\mathrm{d}y \Big)^{\frac{1}{u\left(p/(\theta u)\right)'}} |2^k B|^{-\frac{1}{u}},$$

here, $m_B(b)$ denotes the mean value of b on B. It follows from the John–Nirenberg inequality that

$$\Big(\int_{2^{k+1} B} |b(y) - m_B(b)|^{p\theta'} \,\mathrm{d}y \Big)^{\frac{1}{p\theta'}} \lesssim k |2^k B|^{\frac{1}{p\theta'}}.$$

Therefore, for $q \in (1, \infty)$ and $k \in \mathbf{N}$, we have

$$\big\| S_k f \big\|_{L^q(B,w)} \lesssim k \{w(B)\}^{\frac{1}{q}-\frac{1}{p}} \Big(\frac{w(B)}{w(2^k B)} \Big)^{1/p} \|f_k\|_{L^p(\mathbf{R}^n,w)}. \tag{4.2}$$

On the other hand, we deduce from the $L^p(\mathbf{R}^n, w)$ boundedness of S that

$$\int_B |S_k f(y)|^p w(y) \,\mathrm{d}y \lesssim D^p \int_{2^k B} |f(x)|^p w(x) \,\mathrm{d}x \tag{4.3}$$

We then get from (4.2) (with $q = p$) and (4.3) that for $\sigma \in (0, 1)$,

$$\int_B |S_k f(y)|^p w(y) \,\mathrm{d}y \lesssim k^p D^{p(1-\sigma)} \big(\frac{w(B)}{w(2^k B)} \big)^{\sigma} \int_{2^k B} |f(x)|^p w(x) \,\mathrm{d}x \tag{4.4}$$

Recall that $w \in A_{p/u}(\mathbf{R}^n)$. Thus, there exists a constant $\tau \in (0, 1)$,

$$\frac{w(B)}{w(2^k B)} \lesssim \big(\frac{|B|}{|2^k B|} \big)^{\tau},$$

see [24]. For fixed $\lambda \in (0, 1)$, we choose σ sufficiently close to 1 such that $0 < \lambda < \sigma$. It then follows from (4.4) that

$$\sum_{k=1}^{\infty} \Big(\int_B |S f_k(y)|^p w(y) \,\mathrm{d}y \Big)^{\frac{1}{p}} \lesssim D^{1-\sigma} \{w(B)\}^{\frac{\lambda}{p}} \sum_{k=1}^{\infty} k 2^{-kn\tau(\sigma-\lambda)/p} \|f\|_{L^{p,\lambda}(\mathbf{R}^n,w)}$$
$$\lesssim D^{1-\sigma} \{w(B)\}^{\lambda/p} \|f\|_{L^{p,\lambda}(\mathbf{R}^n,w)}.$$

This leads to the conclusion (a).

Now we turn our attention to conclusion (b). From (4.1), it is obvious that for $y \in 2^{k+1} B \backslash 2^k B$,

$$\int_B |W(x-y)| |b(x) - m_B(b)| w(x) \,\mathrm{d}x \lesssim |2^k B|^{-1/u} |B|^{\frac{1}{u\vartheta'}} \Big(\int_B w^{u\vartheta}(x) \,\mathrm{d}x \Big)^{\frac{1}{u\vartheta}},$$

with $\vartheta \in (1,\infty)$ small enough such that $w^{u\vartheta} \in A_1(\mathbf{R}^n)$. This, in turn implies that

$$\begin{aligned}&\int_B \int_{2^{k+1}B\backslash 2^kB} |W(x-y)h(y)|\,\mathrm{d}y\,|b(x)-m_B(b)|w(x)\,\mathrm{d}x\\&\lesssim 2^{kn/u'}\frac{w(B)}{w(2^kB)}\int_{2^kB} h(y)w(y)\,\mathrm{d}y.\end{aligned}$$

Therefore, for $s \in (1,\infty)$,

$$\begin{aligned}\int_B |S_kf(x)|w(x)\,\mathrm{d}x &\lesssim 2^{kn/u'}\frac{w(B)}{w(2^kB)}\int_{2^kB}|f(x)|w(x)\,\mathrm{d}x\\&\quad+2^{kn/u'}\frac{w(B)}{w(2^kB)}\int_{2^kB}|f(x)||b(x)-m_B(b)|w(x)\,\mathrm{d}x\\&\lesssim k2^{\frac{kn}{u'}}\frac{w(B)}{w(2^{k+1}B)}\left(\int_{2^kB}|f(x)|^sw(x)\,\mathrm{d}x\right)^{\frac{1}{s}}\left\{w(2^kB)\right\}^{\frac{1}{s'}}.\end{aligned}\tag{4.5}$$

Also, we get by (4.2) that for $q \in (u,\infty)$ and $\theta \in (0,1)$ with $\theta q \in (u,\infty)$,

$$\big\|S_kf\big\|_{L^q(B,w)} \lesssim k\{w(B)\}^{\frac{1}{q}-\frac{1}{\theta q}}\left(\frac{w(B)}{w(2^kB)}\right)^{\frac{1}{\theta q}}\|f\|_{L^{\theta q}(2^{k+1}B,w)}.\tag{4.6}$$

For $p \in (1,\infty)$, we choose $q \in (u,\infty)$ and $\theta \in (0,1)$, $s \in (1,\infty)$ which is close to 1 sufficiently such that $1/p = t+(1-t)/q$ and $1/p = t/s+(1-t)/(\theta q)$, with $t \in (0,1/p)$. By interpolating, we obtain from the inequalities (4.5) and (4.6) that

$$\|S_kf\|_{L^p(\mathbf{R}^n,w)} \lesssim k2^{\frac{kn}{pu'}}\left(\frac{w(B)}{w(2^kB)}\right)^{1/p}\|f\|_{L^p(2^kB,w)}.$$

The fact that $w^r \in A_1(\mathbf{R}^n)$ tells us that

$$\frac{w(B)}{w(2^kB)} \lesssim 2^{-kn(r-1)/r},$$

see [24, p. 306]. This, together with the fact that S is bounded on $L^p(\mathbf{R}^n,w)$ with bound D, gives us that for any $\omega \in (0,1)$,

$$\begin{aligned}\left(\int_B |S_kf(x)|^pw(x)\,\mathrm{d}x\right)^{1/p} &\lesssim D^{1-\omega}k2^{\frac{\omega kn}{pu'}}\left(\frac{w(B)}{w(2^kB)}\right)^{\omega/p}\|f\|_{L^p(2^kB,w)}\\&\lesssim \{w(B)\}^{\lambda/p}D^{1-\omega}k2^{\frac{kn}{p}\left(\frac{\omega}{u'}-\frac{\omega-\lambda}{r'}\right)}\|f\|_{L^{p,\lambda}(\mathbf{R}^n,w)}.\end{aligned}$$

For fixed $\lambda \in (0,1-r'/u')$, we choose $\omega \in (\lambda,1)$ sufficiently close to 1 such that $\omega/u'-(\omega-\lambda)/r' < 0$. Summing over the last inequality yields conclusion (b). □

Let $p,r \in [1,\infty)$, $\lambda \in (0,1)$, $q \in [1,\infty]$ and w be a weight. Define the space $L^{p,\lambda}(L^q([1,2]),l^r;\mathbf{R}^n,w)$ by

$$L^{p,\lambda}(L^q([1,2]),l^r;\mathbf{R}^n,w) = \big\{\vec{f} = \{f_k\}_{k\in\mathbf{Z}}\colon \|\vec{f}\|_{L^{p,\lambda}(L^q([1,2]),l^r;\mathbf{R}^n,w)} < \infty\big\},$$

with

$$\|\vec{f}\|_{L^{p,\lambda}(L^q([1,2]),l^r;\mathbf{R}^n,w)} = \left\|\left(\int_1^2\left(\sum_{k\in\mathbf{Z}}|f_k(x,t)|^r\right)^{\frac{q}{r}}\mathrm{d}t\right)^{\frac{1}{q}}\right\|_{L^{p,\lambda}(\mathbf{R}^n,w)}.$$

With usual addition and scalar multiplication, $L^{p,\lambda}(L^q([1,2]),l^r;\mathbf{R}^n,w)$ is a Banach space.

Lemma 4.2. *Let $p \in (1,\infty)$, $\lambda \in (0,1)$ and $w \in A_p(\mathbf{R}^n)$, $\mathcal{G}$ be a subset in $L^{p,\lambda}(L^2([1,2]), l^2;\, \mathbf{R}^n, w)$. Suppose that $\mathcal{G}$ satisfies the following five conditions:*

(a) *$\mathcal{G}$ is a bounded set in $L^{p,\lambda}(L^2([1,2]), l^2;\, \mathbf{R}^n, w)$;*

(b) *for each fixed $\epsilon > 0$, there exists a constant $A > 0$, such that for all $\{f_k\}_{k\in\mathbf{Z}} \in \mathcal{G}$,*

$$\left\| \left(\int_1^2 \sum_{k\in\mathbf{Z}} |f_k(\cdot,t)|^2\,\mathrm{d}t \right)^{\frac12} \chi_{\{|\cdot|>A\}}(\cdot) \right\|_{L^{p,\lambda}(\mathbf{R}^n,w)} < \epsilon;$$

(c) *for each fixed $\epsilon > 0$ and $N \in \mathbf{N}$, there exists a constant $\varrho > 0$, such that for all $\vec{f} = \{f_k\}_{k\in\mathbf{Z}} \in \mathcal{G}$,*

$$\left\| \sup_{|h|\le\varrho} \left(\int_1^2 \sum_{|k|\le N} |f_k(\cdot,t) - f_k(\cdot+h,t)|^2\,\mathrm{d}t \right)^{\frac12} \right\|_{L^{p,\lambda}(\mathbf{R}^n,w)} < \epsilon;$$

(d) *for each fixed $\epsilon > 0$ and $N \in \mathbf{N}$, there exists a constant $\sigma \in (0,1/2)$ such that for all $\vec{f} = \{f_k\}_{k\in\mathbf{Z}} \in \mathcal{G}$,*

$$\left\| \sup_{|s|\le\sigma} \left(\int_1^2 \sum_{|k|\le N} |f_k(\cdot,t+s) - f_k(\cdot,t)|^2\,\mathrm{d}t \right)^{\frac12} \right\|_{L^{p,\lambda}(\mathbf{R}^n,w)} < \epsilon,$$

(e) *for each fixed $D > 0$ and $\epsilon > 0$, there exists $N \in \mathbf{N}$ such that for all $\vec{f} = \{f_k\}_{k\in\mathbf{Z}} \in \mathcal{G}$,*

$$\left\| \left(\int_1^2 \sum_{|k|>N} |f_k(\cdot,t)|^2\,\mathrm{d}t \right)^{\frac12} \chi_{B(0,D)} \right\|_{L^{p,\lambda}(\mathbf{R}^n,w)} < \epsilon.$$

Then $\mathcal{G}$ is strongly pre-compact in $L^{p,\lambda}(L^2([1,2]), l^2;\, \mathbf{R}^n, w)$.

Proof. The proof is similar to the proof of Lemma 3.3, and so we only give the outline here. It suffices to prove that, for each fixed $\epsilon > 0$, there exists a $\delta = \delta_\epsilon > 0$ and a mapping Φ_ϵ on $L^{p,\lambda}(L^2([1,2]), l^2;\, \mathbf{R}^n, w)$, such that $\Phi_\epsilon(\mathcal{G}) = \{\Phi_\epsilon(\vec{f})\colon \vec{f} \in \mathcal{G}\}$ is a strongly pre-compact set in $L^{p,\lambda}(L^2([1,2]), l^2;\, \mathbf{R}^n, w)$, and for $\vec{f}$, $\vec{g} \in \mathcal{G}$,

$$\|\Phi_\epsilon(\vec{f}) - \Phi_\epsilon(\vec{g})\|_{L^{p,\lambda}(L^2([1,2]),l^2;\,\mathbf{R}^n,w)} < \delta \implies \|\vec{f} - \vec{g}\|_{L^{p,\lambda}(L^2([1,2]),l^2;\,\mathbf{R}^n,w))} < 8\epsilon.$$

For fixed $\epsilon > 0$, we choose $A > 1$ large enough as in assumption (b), and $N \in \mathbf{N}$ such that for all $\{f_k\}_{k\in\mathbf{Z}} \in \mathcal{G}$,

$$\left\| \left(\int_1^2 \sum_{|k|>N} |f_k(\cdot,t)|^2\,\mathrm{d}t \right)^{\frac12} \chi_{B(0,2A)} \right\|_{L^{p,\lambda}(\mathbf{R}^n,w)} < \epsilon.$$

Let Q, $Q_1, \ldots, Q_J$, $\mathcal{D}$, $I_1, \ldots, I_L \subset [1,2]$, and Φ_ϵ be the same as in the proof of Lemma 3.2. For such fixed N, let ϱ and $\sigma \in (0,1/2)$ small enough such that for all

$\vec{f}=\{f_k\}_{k\in\mathbf{Z}}\in\mathcal{G}$,

$$\left\|\sup_{|h|\le\varrho}\left(\int_1^2\sum_{|k|\le N}|f_k(\cdot,t)-f_k(\cdot+h,t)|^2\,\mathrm{d}t\right)^{\frac12}\right\|_{L^{p,\lambda}(\mathbf{R}^n,w)}<\frac{\epsilon}{2J}; \tag{4.7}$$

$$\left\|\sup_{|s|\le\sigma}\left(\int_1^2\sum_{|k|\le N}|f_k(\cdot,t+s)-f_k(\cdot,t)|^2\,\mathrm{d}t\right)^{\frac12}\right\|_{L^{p,\lambda}(\mathbf{R}^n,w)}<\frac{\epsilon}{2J}, \tag{4.8}$$

We can verify that Φ_ϵ is bounded on $L^{p,\lambda}(L^2([1,2]),l^2;\,\mathbf{R}^n,w)$, and consequently, $\Phi_\epsilon(\mathcal{G})=\{\Phi_\epsilon(\vec{f})\colon \vec{f}\in\mathcal{G}\}$ is a strongly pre-compact set in $L^{p,\lambda}(L^2([1,2]),l^2;\,\mathbf{R}^n,w)$. Recall that for $x\in Q_i$ with $1\le i\le J$,

$$\begin{aligned}&\left\{\int_1^2\sum_{|k|\le N}\Big|f_k(x,t)-\sum_{v=1}^{L}m_{Q_i\times I_v}(f_k)\chi_{I_v}(t)\Big|^2\,\mathrm{d}t\right\}^{\frac12}\\&\lesssim\sup_{|h|\le\varrho}\left(\int_1^2\sum_{|k|\le N}|f_k(x,t)-f_k(x+h,t)|^2\,\mathrm{d}t\right)^{\frac12}\\&\quad+\sup_{|s|\le\sigma}\left(\int_1^2\sum_{|k|\le N}|f_k(x,t+s)-f_k(x,t)|^2\,\mathrm{d}t\right)^{\frac12}.\end{aligned}$$

For a ball $B(y,r)$, a trivial computation involving (4.7) and (4.8), leads to that

$$\begin{aligned}&\int_{B(y,r)}\left(\int_1^2\sum_{|k|\le N}\left|f_k(x,t)\chi_{\mathcal{D}}-\sum_{i=1}^{J}\sum_{j=1}^{L}m_{Q_i\times I_j}(f_k)\chi_{Q_i\times I_j}(x,t)\right|^2\,\mathrm{d}t\right)^{\frac p2}w(x)\,\mathrm{d}x\\&=\sum_{i=1}^{J}\int_{B(y,r)\cap Q_i}\left(\int_1^2\sum_{|k|\le N}\left|f_k(x,t)-\sum_{j=1}^{L}m_{Q_i\times I_j}(f_k)\chi_{I_j}(t)\right|^2\,\mathrm{d}t\right)^{\frac p2}w(x)\,\mathrm{d}x\\&\lesssim\epsilon\{w(B(y,r))\}^{\lambda}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_{B(y,r)}\big\|\vec{f}\chi_{\mathcal{D}}-\Phi_\epsilon(\vec{f})\big\|^p_{L^2([1,2]),l^2)}w(x)\,\mathrm{d}x\\&\lesssim\int_{B(y,r)}\left(\int_1^2\sum_{|k|\le N}\left|f_k(x,t)\chi_{\mathcal{D}}-\sum_{i=1}^{J}\sum_{j=1}^{L}m_{Q_i\times I_j}(f_k)\chi_{Q_i\times I_j}(x,t)\right|^2\,\mathrm{d}t\right)^{\frac p2}w(x)\,\mathrm{d}x\\&\quad+\int_{B(y,r)}\left(\int_1^2\sum_{|k|>N}\big|f_k(x,t)\big|^2\right)^{p/2}\chi_{B(0,2A)}(x)w(x)\,\mathrm{d}x\\&\lesssim 2\epsilon\{w(B(y,r))\}^{\lambda}.\end{aligned}$$

It then follows from the assumption (b) that for all $\vec{f}\in\mathcal{G}$,

$$\|\vec{f}-\Phi_\epsilon(\vec{f})\|_{L^{p,\lambda}(L^2([1,2]),l^2;\mathbf{R}^n,w)}\le\big\|\vec{f}\chi_{\mathcal{D}}-\Phi_\epsilon(\vec{f})\big\|_{L^p(L^2([1,2]),l^2;\mathbf{R}^n,w)}+\epsilon<3\epsilon,$$

and

$$\|\vec{f}-\vec{g}\|_{L^{p,\lambda}(\mathbf{R}^n)}<6\epsilon+\|\Phi_\epsilon(f)-\Phi_\epsilon(\vec{g})\|_{L^{p,\lambda}(\mathbf{R}^n)}.$$

This completes the proof of Lemma 4.2. □

Proof of Theorem 1.5. We only consider the case of $p\in(q',\infty)$, $w\in A_{p/q'}(\mathbf{R}^n)$ and $\lambda\in(0,1)$. Recall that $\mathcal{M}_{\Omega,b}$ is bounded on $L^p(\mathbf{R}^n,w)$. By Lemma 4.2, we know that $\mathcal{M}_{\Omega,b}$ is bounded on $L^{p,\lambda}(\mathbf{R}^n,w)$. Thus, it suffices to prove that for $b\in C_0^\infty(\mathbf{R}^n)$, $\mathcal{M}_{\Omega,b}$ is completely continuous on $L^{p,\lambda}(\mathbf{R}^n,w)$.

Let $j_0\in\mathbf{Z}_-$, $b\in C_0^\infty(\mathbf{R}^n)$ with $\operatorname{supp}b\subset B(0,R)$ and $\|b\|_{L^\infty(\mathbf{R}^n)}+\|\nabla b\|_{L^\infty(\mathbf{R}^n)}=1$. Let $\widetilde{K^j}(z)=\frac{|\Omega(z)|}{|z|^n}\chi_{\{2^{j-1}\le|z|\le2^{j+2}\}}(z)$. By Minkowski's inequality,

$$\begin{aligned}\left(\int_1^2\sum_{j\in\mathbf{Z}}|F_{j,b}^lf(x,t)|^2\,\mathrm{d}t\right)^{\frac12}&\le\left(\sum_{j\in\mathbf{Z}}\int_1^2|F_{j,b}^lf(x,t)|^2\,\mathrm{d}t\right)^{\frac12}\\&\lesssim\sum_{j\in\mathbf{Z}}\int_{\mathbf{R}^n}\widetilde{K^j}*\phi_{j-l}(x-y)|f(y)|\,\mathrm{d}y.\end{aligned}$$

It is obvious that $\operatorname{supp}\widetilde{K^j}*\phi_{j-l}\subset\{x\colon 2^{j-3}\le|x|\le2^{j+3}\}$, and for any $R>0$,

$$\int_{R\le|x|\le2R}\left|\sum_{j\in\mathbf{Z}}\widetilde{K^j}*\phi_{j-l}(x)\right|^q\mathrm{d}x\le\sum_{j\colon2^j\approx R}\left\|\widetilde{K^j}*\phi_{j-l}\right\|_{L^q(\mathbf{R}^n)}^q\lesssim R^{-nq+n}.$$

Let $\epsilon>0$. We deduce from Lemma 4.1 and the inequality (3.7) that, there exists a constant $A>0$, such that

$$\left\|\left(\int_1^2\sum_{j\in\mathbf{Z}}|F_{j,b}^lf(x,t)|^2\,\mathrm{d}t\right)^{\frac12}\chi_{\{|\cdot|>A\}}(\cdot)\right\|_{L^{p,\lambda}(\mathbf{R}^n,w)}<\epsilon\|f\|_{L^{p,\lambda}(\mathbf{R}^n,w)}.\tag{4.9}$$

Recall that $\widetilde{\mathcal{M}}_{\Omega}^{l,j_0}$ is bounded on $L^{p,\lambda}(\mathbf{R}^n,w)$. For $r>1$ small enough, M_r is also bounded on $L^{p,\lambda}(\mathbf{R}^n,w)$ (see [27]). Thus by (3.4), we know that there exists a constant $\varrho>0$, such that

$$\left\|\sup_{|h|\le\varrho}\left(\int_1^2\sum_{j>j_0}|F_{j,b}^lf(\cdot,t)-F_{j,b}^lf(\cdot+h,t)|^2\,\mathrm{d}t\right)^{\frac12}\right\|_{L^{p,\lambda}(\mathbf{R}^n,w)}\lesssim\epsilon\|f\|_{L^{p,\lambda}(\mathbf{R}^n,w)}.\tag{4.10}$$

It follows from Lemma 4.1, estimate (3.5) that for each $N\in\mathbf{N}$, there exists a constant $\sigma\in(0,1/2)$ such that

$$\left\|\sup_{|s|\le\sigma}\left(\int_1^2\sum_{|j|\le N}|F_{j,b}^lf(\cdot,s+t)-F_{j,b}^lf(\cdot,t)|^2\,\mathrm{d}t\right)^{\frac12}\right\|_{L^{p,\lambda}(\mathbf{R}^n,w)}<\epsilon\|f\|_{L^{p,\lambda}(\mathbf{R}^n,w)}.\tag{4.11}$$

We also obtain by Lemma 4.1 and (3.6) that for each fixed $D>0$, there exists $N\in\mathbf{N}$ such that

$$\left\|\left(\int_1^2\sum_{j>N}|F_{j,b}^lf(\cdot,t)|^2\,\mathrm{d}t\right)^{\frac12}\chi_{B(0,D)}\right\|_{L^{p,\lambda}(\mathbf{R}^n,w)}<\epsilon\|f\|_{L^{p,\lambda}(\mathbf{R}^n,w)}.\tag{4.12}$$

The inequalities (4.9)–(4.12), via Lemma 4.2, tell us for any $j_0 \in \mathbf{Z}_-$, the operator $\mathcal{F}^l_{j_0}$ defined by (3.18) is compact from $L^{p,\lambda}(\mathbf{R}^n, w)$ to $L^{p,\lambda}(L^2([1,2]), l^2; \mathbf{R}^n, w)$. On the other hand, by Lemma 4.1, Theorem 2.1 and Lemma 3.2, we know that

$$\|\widetilde{\mathcal{M}}_\Omega f - \widetilde{\mathcal{M}}^l_\Omega f\|_{L^{p,\lambda}(\mathbf{R}^n,w)} \lesssim 2^{-\varepsilon\varrho_p l}\|f\|_{L^{p,\lambda}(\mathbf{R}^n,w)},$$

and

$$\left\|\widetilde{\mathcal{M}}^{l,j_0}_{\Omega,b} f - \widetilde{\mathcal{M}}^l_{\Omega,b} f\right\|_{L^{p,\lambda}(\mathbf{R}^n,w)} \lesssim 2^{\varepsilon j_0}\|f\|_{L^{p,\lambda}(\mathbf{R}^n,w)}.$$

As it was shown in the proof of Theorem 1.3, we can deduce from the last facts that $\mathcal{M}_{\Omega,b}$ is completely continuous on $L^{p,\lambda}(\mathbf{R}^n, w)$ when $b \in C^\infty_0(\mathbf{R}^n)$. This completes the proof of Theorem 1.5. □

Acknowledgement. The author would like to thank the referee for his/her valuable suggestions and helpful comments.

References

[1] Adams, D. R., and J. Xiao: Morrey potential and harmonic analysis. - Ark. Mat. 50, 2012, 201–230.

[2] Al-Salman, A.: On the L^2 boundedness of parametric Marcinkiewicz integral operator. - J. Math. Anal. Appl. 375, 2011, 745–752.

[3] Al-Salman, A., H. Al-Qassem, L. Cheng, and Y. Pan: L^p bounds for the function of Marcinkiewicz. - Math. Res. Lett. 9, 2002, 697–700.

[4] Alvarez, J., R. Bagby, D. Kurtz, and C. Pérez: Weighted estimates for commutators of linear operators. - Studia Math. 104, 1993, 195–209.

[5] Anderson, K. F., and R. T. John: Weighted inequality for vector-valued maximal functions and singular integrals. - Studia Math. 69, 1980, 19–31.

[6] Banedek, A., A. P. Calderón, and R. Panzon: Convolution operators on Banach space valued functions. - Proc. Nat. Acad. Sci. U.S.A. 48, 1962, 356–365.

[7] Bourdaud, G., M. Lanza de Cristoforis, and W. Sickel: Functional calculus on BMO and related spaces. - J. Funct. Anal. 189, 2002, 515–538.

[8] Calderón, A. P., and A. Zygmund: On singular integrals. - Amer. J. Math. 78, 1956, 289–309.

[9] Chen, D., and S. Lu: L^p boundedness for commutators of parabolic Littlewood–Paley operators with rough kernels. - Math. Nachr. 284, 2011, 973–986.

[10] Chen, J., D. Fan, and Y. Pan: A note on a Marcinkiewicz integral operator. - Math. Nachr. 227, 2001, 33–42.

[11] Chen, J., and G. Hu: Compact commutators of rough singular integral operators. - Canad. Math. Bull. 64, 2012, 257–281.

[12] Chen, Y., and Y. Ding: Compactness characterization of commutators for Littlewood–Paley operators. - Kodai Math. J. 32, 2009, 256–323.

[13] Chen, Y., Y. Ding, and X. Wang: Compactness for commutators of Marcinkiewicz integral on Morrey spaces. - Taiwanese J. Math. 15, 2011, 633–658.

[14] Clop, A., and V. Cruz: Weighted estimates for Beltrami equations. - Ann. Acad. Sci. Fenn. Math. 38, 2013, 91–113.

[15] Coifman, R., R. Rochberg, and G. Weiss: Factorizaton theorems for Hardy spaces in several variables. - Ann. of Math. (2) 103, 1976, 611–635.

[16] Coifman, R., and G. Weiss: Extension of Hardy spaces and their use in analysis. - Bull. Amer. Math. Soc. 83, 1977, 569–645.

[17] Ding, Y., D. Fan, and Y. Pan: Weighted boundedness for a class of rough Marcinkiewicz integral. - Indiana Univ. Math. J. 48, 1999, 1037–1055.

[18] Ding, Y., D. Fan, and Y. Pan: L^p-boundedness of Marcinkiewicz integrals with Hardy space function kernel. - Acta Math. Sin. (Engl. Ser.) 16, 2000, 593–600.

[19] Ding, Y., S. Lu, and K. Yabuta: A problem on rough Marcinkiewicz functions. - J. Austral. Math. Soc. 71, 2001, 1–9.

[20] Ding, Y., S. Lu, and K. Yabuta: On commutators of Marcinkiewicz integrals with rough kernel. - J. Math. Anal. Appl. 275, 2002, 60–68.

[21] Ding, Y., Q. Xue, and K. Yabuta: A remark to the L^2 boundedness of parametric Marcinkiewicz integral. - J. Math. Anal. Appl. 387, 2012, 691–697.

[22] Duoandikoetxea, J.: Weighted norm inequalities for homogeneous singular integrals. - Trans. Amer. Math. Soc. 336, 1993, 869–880.

[23] Duoandikoetxea, J., and J. L. Rubio de Francia: Maximal and singular integrals via Fourier transform estimates. - Invent. Math. 84, 1986, 541–561.

[24] Grafakos, L.: Modern Fourier analysis. - Grad. Texts in Math. 250, 2nd Edition, Springer, New York, 2008.

[25] Hu, G.: $L^p(\mathbf{R}^n)$ boundedness for a class of g-functions and its applications. - J. Hokkaido Math. 32, 2003, 497–521.

[26] Hu, G., and D. Yan: On commutator of the Marcinkiewicz integral. - J. Math. Anal. Appl. 283, 2003, 351–361.

[27] Komori, Y., and S. Shirai: Weighted Morrey spaces and a singular integral operator. - Math. Nachr. 282, 2009, 219–232.

[28] Kurtz, D. S., and R. Wheeden: Results on weighted norm inequalities for multipliers. - Trans. Amer. Math. Soc. 255, 1979, 343–362.

[29] Mao, S., Y. Sawano, and H. Wu: On the compactness of commutators for rough Marcinkiewicz integral operators. - Taiwanese J. Math. 19, 2015, 1777–1793.

[30] Marcinkiewicz, J.: Sur quelques intégrales du type de Dini. - Ann. Soc. Polon. Math. 17, 1938, 42–50.

[31] Ruiz, A., and L. Vega: Unique continuation for Schrödinger operators with potential in Morrey space. - Publ. Mat. 35, 1991, 291–298.

[32] Shen, Z.: Boundary value problems in Morrey spaces for elliptic systems on Lipschitz domains. - Amer. J. Math. 125, 2003, 1079–1115.

[33] Stein, E. M.: On the function of Littlewood–Paley, Lusin and Marcinkiewicz. - Trans. Amer. Math. Soc. 88, 1958, 430–466.

[34] Stein, E. M., and G. Weiss: Interpolation of operators with changes of measures. - Trans. Amer. Math. Soc. 87, 1958, 159–172.

[35] Torchinsky, A., and S. Wang: A note on the Marcinkiewicz integral. - Colloq. Math. 47, 1990, 235–243.

[36] Uchiyama, A.: On the compactness of operators of Hankel type. - Tohoku Math. J. 30, 1978, 163–171.

[37] Walsh, T.: On the function of Marcinkiewicz. - Studia Math. 44, 1972, 203–217.

[38] Watson, D. K.: Weighted estimates for singular integrals via Fourier transform estimates. - Duke Math. J. 60, 1990, 389–399.

[39] Zygmund, A.: Trigonomatric series. - Cambridge Univ. Press, Cambridge, 3rd edition, 2002.

Received 30 June 2016 • Accepted 6 November 2018