Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 44, 2019, 569–580

REMARKS ON ONE-COMPONENT INNER FUNCTIONS

Atte Reijonen

Tohoku University, Graduate School of Information Sciences
Aoba-ku, Sendai 980-8579, Japan; atte.reijonen@uef.fi

Abstract. A one-component inner function Θ is an inner function whose level set

$$\Omega_\Theta(\varepsilon) = \{z \in \mathbf{D} \colon |\Theta(z)| < \varepsilon\}$$

is connected for some $\varepsilon \in (0,1)$. We give a sufficient condition for a Blaschke product with zeros in a Stolz domain to be a one-component inner function. Moreover, a sufficient condition is obtained in the case of atomic singular inner functions. We study also derivatives of one-component inner functions in the Hardy and Bergman spaces. For instance, it is shown that, for $0 < p < \infty$, the derivative of a one-component inner function Θ is a member of the Hardy space H^p if and only if Θ'' belongs to the Bergman space A^p_{p-1}, or equivalently $\Theta' \in A^{2p}_{p-1}$.

1. Examples of one-component inner functions

Let $\mathbf{D}$ be the open unit disc of the complex plane $\mathbf{C}$. A bounded and analytic function in $\mathbf{D}$ is an inner function if it has unimodular radial limits almost everywhere on the boundary $\mathbf{T}$ of $\mathbf{D}$. In this note, we study so-called one-component inner functions [14], which are inner functions Θ whose level set

$$\Omega_\Theta(\varepsilon) = \{z \in \mathbf{D} \colon |\Theta(z)| < \varepsilon\}$$

is connected for some $\varepsilon \in (0,1)$. In particular, Blaschke products in this class are of interest. For a given sequence $\{z_n\} \subset \mathbf{D} \setminus \{0\}$ satisfying $\sum_n (1-|z_n|) < \infty$, the Blaschke product with zeros $\{z_n\}$ is defined by

$$B(z) = \prod_n \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \overline{z}_n z}, \quad z \in \mathbf{D}.$$

Here each zero z_n is repeated according to its multiplicity. In addition, we assume that $\{z_n\}$ is ordered by non-decreasing moduli.

Recently several authors have studied one-component inner functions in the context of model spaces and operator theory; see for instance [6, 8, 9, 10]. In addition, Aleksandrov's paper [5], which contains several characterizations for one-component inner functions, is worth mentioning. These references do not offer any concrete examples of infinite one-component Blaschke products; even though, reference [5] offers tools for this purpose. In recent paper [13] by Cima and Mortini, one can find some examples. However, all one-component Blaschke products constructed in [13] have some heavy restrictions. Roughly speaking, zeros of all of them are at least uniformly separated. Recall that $\{z_n\} \subset \mathbf{D}$ is called uniformly separated if

$$\inf_{n \in \mathbf{N}} \prod_{k \neq n} \left| \frac{z_k - z_n}{1 - \overline{z}_k z_n} \right| > 0.$$

https://doi.org/10.5186/aasfm.2019.4434
2010 Mathematics Subject Classification: Primary 30J05; Secondary 30H10.
Key words: Bergman space, Blaschke product, Hardy space, one-component inner function, singular inner function.
This research was supported by Finnish Cultural Foundation.

As a concrete example, we mention that the Blaschke product with zeros $z_n = 1-2^{-n}$ for $n \in \mathbf{N}$ is a one-component inner function [13]. In addition, it is a well-known fact that every finite Blaschke product is a one-component inner function.

For $\gamma \geq 1$, $\xi \in \mathbf{T}$ and $C > 0$, we define

$$R(\gamma, \xi, C) = \{z \in \mathbf{D} \colon |1 - \overline{\xi} z|^\gamma \leq C(1-|z|)\}.$$

The region $R(1, \xi, C)$ is a Stolz domain with vertex at ξ. Note that in the case $\gamma = 1$ we have to assume $C > 1$. For $\gamma > 1$, $R(\gamma, \xi, C)$ is a tangential approaching region in $\mathbf{D}$, which touches $\mathbf{T}$ at ξ. Denote by $\mathcal{R}_\gamma$ the family of all Blaschke products whose zeros lie in some $R(\gamma, \xi, C)$ with a fixed γ. References related to $\mathcal{R}_\gamma$ are for instance [4, 11, 21]. With these preparations we are ready to state our first main result.

Theorem 1. *Let B be a member of $\mathcal{R}_1$ with zeros $\{z_n\}_{n=1}^\infty$. If*

$$\liminf_{n\to\infty} \frac{\sum_{|z_j|>|z_n|}(1-|z_j|)}{1-|z_n|} > 0, \tag{1.1}$$

then B is a one-component inner function.

As a consequence of Theorem 1, we obtain the affirmative answer to the following question posed in [13]: Is the Blaschke product B with zeros $z_n = 1 - n^{-2}$ for $n \in \mathbf{N}$ a one-component inner function? Some other examples of one-component inner functions are listed below. All of these examples can be verified by using the fact that condition (1.1) is valid if $\{z_n\}$ is ordered by strictly increasing moduli and

$$\liminf_{n\to\infty} \frac{1-|z_{n+1}|}{1-|z_n|} > 0.$$

Example 2. Let $1 < \alpha < \infty$ and B be a Blaschke product with zeros

(a) $z_n = 1 - n^{-\alpha}$ for $n \in \mathbf{N}$, or
(b) $z_n = 1 - \frac{1}{n(\log n)^\alpha}$ for $n \in \mathbf{N} \setminus \{1\}$, or
(c) $z_n = 1 - \alpha^{-n}$ for $n \in \mathbf{N}$.

Then B is a one-component inner function.

A Blaschke product B is said to be thin if its zeros $\{z_n\}_{n=1}^\infty$ satisfy

$$\lim_{n\to\infty} (1-|z_n|^2)|B'(z_n)| = 1.$$

We interpret that finite Blaschke products are not thin. By [13, Corollary 21], any thin Blaschke product is not a one-component inner function. Using this fact and [12, Proposition 4.3(i)], we can give an example which shows that condition (1.1) in Theorem 1 is essential.

Example 3. Let B be the Blaschke product with zeros $\{w_n\}_{n=1}^\infty$ ordered by strictly increasing moduli and satisfying

$$\frac{1-|w_{n+1}|}{1-|w_n|} \longrightarrow 0, \quad n \to \infty.$$

Then, by [12, Proposition 4.3(i)], B is a thin Blaschke product (with uniformly separated zeros). Consequently, for instance, the Blaschke product with zeros $z_n = 1 - 2^{-2^n}$ for $n \in \mathbf{N}$ is not a one-component inner function. Note that zeros $\{z_n\}$ lie in $R(1,1,C)$ for every $C > 1$ but they do not satisfy (1.1).

Let us recall a classical result of Frostman [17]: The Blaschke product B with zeros $\{z_n\}$ has a unimodular radial limit at $\xi \in \mathbf{T}$ if and only if

$$\sum_n \frac{1-|z_n|}{|\xi - z_n|} < \infty. \tag{1.2}$$

A Blaschke product is called a Frostman Blaschke product if it has a unimodular radial limit at every point on $\mathbf{T}$. It is a well-known fact that an infinite Frostman Blaschke product cannot be a one-component inner function; see for instance [5, Theorem 1.11] or Theorem A in Section 2. Using this fact, we show that any $\mathcal{R}_\gamma$ with $\gamma > 1$ contains a member which is not a one-component inner function but its zeros $\{z_n\}$ satisfy (1.1). This means that the hypothesis $B \in \mathcal{R}_1$ in Theorem 1 is essential.

Example 4. Fix $\gamma > 1$ and choose $\alpha = \alpha(\gamma) > 1$ such that $\alpha > \frac{\gamma}{\gamma-1}$. Let $\{z_n\}$ be such that

$$|z_n| = 1 - n^{-\alpha} \quad \text{and} \quad |1 - z_n| = n^{-\alpha/\gamma}, \quad n \in \mathbf{N}.$$

Since the sequence $\{z_n\}$ is a subset of $R(\gamma, 1, 1)$, all points of $\{z_n\}$ lie in $\mathbf{D}$. Moreover, it is clear that $\{z_n\}$ satisfies the Blaschke condition $\sum_n (1 - |z_n|) < \infty$ and (1.1) in Theorem 1. Hence the Blaschke product B with zeros $\{z_n\}$ is well-defined. Furthermore,

$$\sum_{n=1}^{\infty} \frac{1-|z_n|}{|1-z_n|} = \sum_{n=1}^{\infty} n^{\alpha/\gamma - \alpha} < \infty;$$

and thus, B has a unimodular radial limit at 1 by Frostman's result. Since condition (1.2) is trivially valid for every $\xi \in \mathbf{T} \setminus \{1\}$, B is an infinite Frostman Blaschke product. Consequently, it is not a one-component inner function.

Recall that a singular inner function takes the form

$$S_\sigma(z) = \exp\left(\int_{\mathbf{T}} \frac{z+\xi}{z-\xi}\, d\sigma(\xi)\right), \quad z \in \mathbf{D},$$

where σ is a positive measure on $\mathbf{T}$, singular with respect to the Lebesgue measure. If the measure σ is atomic, then this definition reduces to the form

$$S(z) = \exp\left(\sum_n \gamma_n \frac{z + e^{i\theta_n}}{z - e^{i\theta_n}}\right), \quad z \in \mathbf{D},$$

where $\theta_n \in [0, 2\pi)$ are distinct points and $\gamma_n > 0$ satisfy $\sum_n \gamma_n < \infty$. These functions are known as atomic singular inner functions associated with $\{e^{i\theta_n}\}$ and $\{\gamma_n\}$.

An atomic singular inner function associated with a measure having only finitely many mass points is a one-component inner function; see [13, Corollary 17]. In the literature, one cannot find any example of a one-component singular inner function associated with a measure having infinitely many mass points. However, the following result gives a way to construct such functions.

Theorem 5. *Let S be the atomic singular inner function associated with $\{e^{i\theta_n}\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$. Moreover, assume that the following conditions are valid:*

(i) $\theta_0 = 0$, $\{\theta_n\}_{n=1}^{\infty} \subset (0, 1)$ *is strictly decreasing and* $\lim_{n\to\infty} \theta_n = 0$.

(ii) *There exists a constant $C = C(S) > 0$ such that $|\theta_{n-1} - \theta_{n+1}| \le C\gamma_n^2$ for all sufficiently large $n \in \mathbf{N}$.*

Then S is a one-component inner function.

Next we give a concrete example of a one-component singular inner function. This example is a direct consequence of Theorem 5.

Example 6. Let $\theta_0 = 0$, $\theta_n = 2^{-n}$, $\gamma_0 = 1$ and $\gamma_n = n^{-2}$ for $n \in \mathbf{N}$. Then the atomic singular inner function S associated with $\{e^{i\theta_n}\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ is a one-component inner function.

The remainder of this note is organized as follows. Sections 2 and 3 consist of the proofs of Theorems 1 and 5, respectively. In Section 4, we study one-component inner functions whose derivatives belong to the Hardy or Bergman spaces. In particular, we give partial improvements for [1, Theorem 6.2] and [19, Theorem 3.10].

2. Proof of Theorem 1

We begin by stating a modification of [5, Theorem 1.11], which is due to [5, p. 2915, Remark 2]. This result offers two practical characterizations for one-component inner functions and plays an important role in the proofs of Theorems 1 and 5. Before it we recall that the spectrum $\rho(\Theta)$ of an inner function Θ is the set of all points on $\mathbf{T}$ in which Θ does not have an analytic continuation. It is a well-known fact that the spectrum of a Blaschke product consists of the accumulation points of zeros. By [18, Chapter 2, Theorem 6.2], the spectrum of a singular inner function S_σ is the closed support of the associated measure σ.

Theorem A. *Let Θ be an inner function. Then the following statements are equivalent:*

(a) *Θ is a one component inner function.*

(b) *There exists a constant $C = C(\Theta) > 0$ such that*

$$|\Theta''(\zeta)| \le C|\Theta'(\zeta)|^2, \quad \zeta \in \mathbf{T} \setminus \rho(\Theta), \tag{2.1}$$

and

$$\liminf_{r\to 1^-} |\Theta(r\xi)| < 1, \quad \xi \in \rho(\Theta). \tag{2.2}$$

(c) *There exists a constant $C = C(\Theta) > 0$ such that* (2.1) *holds, the Lebesgue measure of $\rho(\Theta)$ is zero and Θ' is not bounded on any arc $\Gamma \subset \mathbf{T} \setminus \rho(\Theta)$ with $\overline{\Gamma} \cap \rho(\Theta) \neq \emptyset$.*

Write $f \lesssim g$ if there exists a constant $C > 0$ such that $f \le Cg$, while $f \gtrsim g$ is understood in an analogous manner. If $f \lesssim g$ and $f \gtrsim g$, then the notation $f \asymp g$ is used. With these preparations we are ready to prove Theorem 1.

Proof of Theorem 1. If B is an arbitrary Blaschke product with zeros $\{z_n\}$, then

$$\frac{B'(z)}{B(z)} = \sum_{n=1}^{\infty} \frac{|z_n|^2 - 1}{(1 - \overline{z}_n z)(z_n - z)} \quad \text{and} \quad |B(z)| \le \frac{|z_n - z|}{|1 - \overline{z}_n z|}.$$

Using these estimates, one can easily verify

$$|B''(z)| \le \frac{|B'(z)|^2}{|B(z)|} + 2|B(z)|^{-1} \sum_{n=1}^{\infty} \frac{1 - |z_n|^2}{|1 - \overline{z}_n z|^3}, \quad z \in \mathbf{D}.$$

In particular,

$$|B''(\zeta)| \le |B'(\zeta)|^2 + 2\sum_{n=1}^{\infty} \frac{1 - |z_n|^2}{|1 - \overline{z}_n \zeta|^3} \asymp |B'(\zeta)|^2 + \sum_{n=1}^{\infty} \frac{1 - |z_n|^2}{|z_n - \zeta|^3}$$

for every $\zeta \in \mathbf{T} \setminus \rho(B)$. Using [4, Theorem 2], we deduce that (2.1) (with $\Theta = B$) is valid if

$$\sum_{n=1}^{\infty} \frac{1-|z_n|^2}{|z_n-\zeta|^3} \lesssim \left(\sum_{n=1}^{\infty} \frac{1-|z_n|^2}{|z_n-\zeta|^2}\right)^2 \tag{2.3}$$

holds for every $\zeta \in \mathbf{T} \setminus \rho(B)$.

Assume without loss of generality that zeros $\{z_n\}$ of B lie in a Stolz domain $R(1,1,C)$, and remind that $\{z_n\}$ is ordered by non-decreasing moduli. Then $\rho(B) = \{1\}$, and the functions f and g, defined by

$$f(x) = \begin{cases} 1, & 0 \le x < 1, \\ \min_{n \le x}(1-|z_n|), & 1 \le x < \infty, \end{cases}$$

and

$$g(\theta) = \inf\{x \colon f(x) \le \theta\}, \quad 0 < \theta \le 1,$$

are non-increasing. Since $f(w) = 1 - |z_n|$ for $n \in \mathbf{N}$ and $n \le w < n+1$, it is clear that $g\colon (0,1] \to \mathbf{N} \cup \{0\}$, $f(x) > \theta$ for $x < g(\theta)$, and $f(x) \le \theta$ for $x \ge g(\theta)$. Write $\zeta = e^{i\theta}$, and assume without loss of generality that $\theta > 0$ is close enough to zero. Using [4, Lemma 3] together with some standard estimates, we obtain

$$\begin{aligned} \left(\sum_{n=1}^{\infty} \frac{1-|z_n|^2}{|z_n-\zeta|^3}\right)^{1/2} &\asymp \left(\sum_{n=1}^{\infty} \frac{1-|z_n|^2}{||z_n|-\zeta|^3}\right)^{1/2} \asymp \left(\sum_{n=1}^{\infty} \frac{1-|z_n|}{[(1-|z_n|)^2+\theta^2]^{3/2}}\right)^{1/2} \\ &\le \left(\sum_{n<g(\theta)} f(n)^{-2} + \theta^{-3} \sum_{n \ge g(\theta)} f(n)\right)^{1/2} \\ &\le \left(\sum_{n<g(\theta)} f(n)^{-2}\right)^{1/2} + \theta^{-3/2} \left(\sum_{n \ge g(\theta)} f(n)\right)^{1/2} \\ &\le \sum_{n<g(\theta)} f(n)^{-1} + \theta^{-3/2} \left(\sum_{n \ge g(\theta)} f(n)\right)^{1/2}. \end{aligned} \tag{2.4}$$

Applying hypothesis (1.1) and the above-mentioned properties of f and g, we find $C = C(B) > 0$ such that

$$\sum_{n \ge g(\theta)} f(n) \ge C f(g(\theta)-1) \ge C\theta.$$

It follows that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1-|z_n|^2}{|z_n-\zeta|^2} &\asymp \sum_{n=1}^{\infty} \frac{1-|z_n|}{(1-|z_n|)^2+\theta^2} \\ &\ge \frac{1}{2} \sum_{n<g(\theta)} f(n)^{-1} + \frac{\theta^{-2}}{2} \sum_{n \ge g(\theta)} f(n) \\ &\ge \frac{1}{2} \sum_{n<g(\theta)} f(n)^{-1} + \frac{\sqrt{C}\theta^{-3/2}}{2} \left(\sum_{n \ge g(\theta)} f(n)\right)^{1/2}. \end{aligned} \tag{2.5}$$

Using estimates (2.4) and (2.5), it is easy to see that condition (2.3) is valid for $\zeta \in \mathbf{T} \setminus \{1\}$. Consequently, B satisfies (2.1).

Let B_0 be the Blaschke product with zeros $\{|z_n|\}$. It is obvious that $\liminf_{r\to 1^-} |B_0(r)| = 0$. Hence, by the deduction above, it is clear that B_0 satisfies condition (b) in Theorem A, and thus also the other conditions are valid. Since B_0 satisfies (c) in Theorem A, also B satisfies it. This is due to [4, Lemma 3], which asserts that $|B'(\xi)| \asymp |B_0'(\xi)|$ for $\xi \in \mathbf{T} \setminus \{1\}$. Hence B is a one-component inner function by Theorem A. This completes the proof. □

3. Proof of Theorem 5

Let us prove Theorem 5.

Proof of Theorem 5. Due to hypothesis (i), the set of mass points $\{e^{i\theta_n}\}_{n=0}^\infty$ is closed. Consequently, the spectrum $\rho(S)$ consists of points $\{e^{i\theta_n}\}_{n=0}^\infty$. Hence, by [18, Chapter 2, Theorem 6.2], we have

$$\lim_{r\to 1^-} |S(r\xi)| = 0, \quad \xi \in \rho(S).$$

This means that S satisfies condition (2.2) (with $\Theta = S$) in Theorem A. Consequently, it suffices to show that S fulfills also (2.1).

By a straightforward calculation, one can check that

$$S''(z) = 4\left(\sum_{n=0}^\infty \frac{\gamma_n e^{i\theta_n}}{(z-e^{i\theta_n})^3} + \left(\sum_{m=0}^\infty \frac{\gamma_m e^{i\theta_m}}{(z-e^{i\theta_m})^2}\right)^2\right)\exp\left(\sum_{k=0}^\infty \gamma_k \frac{z+e^{i\theta_k}}{z-e^{i\theta_k}}\right), \quad z \in \mathbf{D}.$$

Since

$$|S'(\zeta)| = 2\sum_{n=0}^\infty \frac{\gamma_n}{|\zeta - e^{i\theta_n}|^2}, \quad \zeta \in \mathbf{T} \setminus \rho(S),$$

by [4, Theorem 2], we obtain

$$|S''(\zeta)| \le 4\sum_{n=0}^\infty \frac{\gamma_n}{|\zeta - e^{i\theta_n}|^3} + |S'(\zeta)|^2, \quad \zeta \in \mathbf{T} \setminus \rho(S).$$

Consequently, it suffices to show

$$\sum_{n=0}^\infty \frac{\gamma_n}{|\zeta - e^{i\theta_n}|^3} \lesssim \left(\sum_{n=0}^\infty \frac{\gamma_n}{|\zeta - e^{i\theta_n}|^2}\right)^2, \quad \zeta \in \mathbf{T} \setminus \rho(S). \tag{3.1}$$

Assume without loss of generality that $\zeta \in \mathbf{T} \setminus \rho(S)$ is close enough to one, and write $\zeta = e^{i\theta}$. Choose $j = j(\theta, S) \in \mathbf{N} \cup \{0\}$ such that $|\theta - \theta_j|$ is as small as possible. Then standard estimates yield

$$\begin{aligned}\left(\sum_{n=0}^\infty \frac{\gamma_n}{|\zeta - e^{i\theta_n}|^3}\right)^{1/2} &\asymp \left(\sum_{n=0}^\infty \frac{\gamma_n}{|\theta - \theta_n|^3}\right)^{1/2} \\ &\le |\theta - \theta_j|^{-3/2}\left(\sum_{n=0}^\infty \gamma_n\right)^{1/2} \asymp |\theta - \theta_j|^{-3/2}\end{aligned} \tag{3.2}$$

and

$$\sum_{n=0}^\infty \frac{\gamma_n}{|\zeta - e^{i\theta_n}|^2} \asymp \sum_{n=0}^\infty \frac{\gamma_n}{|\theta - \theta_n|^2} \ge \frac{\gamma_j}{|\theta - \theta_j|^2}. \tag{3.3}$$

If $\theta < 0$, then $j = 0$; and hence, (3.1) is a direct consequence of (3.2) and (3.3). Let $\theta > 0$. By hypothesis (i), we have $\theta_{j+1} < \theta < \theta_{j-1}$, where $j \in \mathbf{N}$ is large enough. Consequently, hypothesis (ii) gives

$$\frac{\gamma_j}{|\theta - \theta_j|^2} \geq \frac{\gamma_j}{|\theta - \theta_j|^{3/2}|\theta_{j-1} - \theta_{j+1}|^{1/2}} \gtrsim |\theta - \theta_j|^{-3/2}.$$

According to this estimate, (3.1) is a consequence of (3.2) and (3.3). Finally the assertion follows from Theorem A. □

4. Derivatives of one-component inner functions in function spaces

We begin by fixing the notation. Let $\mathcal{H}(\mathbf{D})$ be the space of all analytic functions in $\mathbf{D}$. For $0 < p < \infty$, the Hardy space H^p consists of those $f \in \mathcal{H}(\mathbf{D})$ such that

$$\|f\|_{H^p} = \sup_{0 \leq r < 1} M_p(r, f) < \infty, \quad \text{where} \quad M_p^p(r, f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta.$$

For $0 < p < \infty$ and $-1 < \alpha < \infty$, the Bergman space A^p_α consists of those $f \in \mathcal{H}(\mathbf{D})$ such that

$$\|f\|^p_{A^p_\alpha} = \int_{\mathbf{D}} |f(z)|^p (1 - |z|)^\alpha \, dA(z) < \infty,$$

where $dA(z) = dx\, dy$ is the Lebesgue area measure on $\mathbf{D}$.

By [23, Theorem 5] and [28, Lemma 1.4], we have

$$\{f \colon f' \in A^p_{p-1}\} \subset H^p, \quad 0 < p \leq 2. \tag{4.1}$$

and

$$H^p \subset \{f \colon f' \in A^p_{p-1}\}, \quad 2 \leq p < \infty. \tag{4.2}$$

It is clear that $\{f \colon f' \in A^2_1\} = H^2$, while otherwise the inclusions are strict. For instance, an example showing the strictness of inclusions (4.1) and (4.2) can be given by using gap series; see details in [7]. Nevertheless, we have the following result, which is essentially a consequence of [1, Theorem 6.2] and [19, Theorem 3.10].

Theorem 7. *Let $\frac{1}{2} < p < \infty$ and Θ be an inner function. Then the following statements are equivalent:*

(a) $\Theta' \in H^p$,
(b) $\Theta' \in A^{2p}_{p-1}$,
(c) $\Theta'' \in A^p_{p-1}$.

Before the proof of Theorem 7, we note that, for $f \in \mathcal{H}(\mathbf{D})$, $n \in \mathbf{N}$ and $0 < p < \infty$, we have $M_p(r, f^{(n)}) \asymp M_p(r, D^n f)$ with comparison constants independent of r [16]. Here D^n is the fractional derivative of order n. This fact is exploited when we apply some results in the literature.

Proof. The equivalence (a) ⇔ (c) is a consequence of [19, Theorem 3.10]. For $\frac{1}{2} < p < 1$, the equivalence (a) ⇔ (b) can be verified, for instance, using [1, Theorem 6.2] together with [26, Corollary 7]. It is a well-known fact the only inner functions whose derivative belongs to H^p for some $p \geq 1$ are finite Blaschke products. Using this fact together with [22, Theorem 7(c)] and the equivalence (a) ⇔ (c), it is easy to deduce that an inner function Θ is a finite Blaschke product if it satisfies any of conditions (a)–(c) for some $p \geq 1$. In addition, it is clear that every finite Blaschke product Θ satisfies conditions (a)–(c) for all $p > 0$. Finally the assertion follows by combining the above-mentioned facts. □

By [3, Lemma 2], there exists a Blaschke product B such that $B' \in A^1_{-1/2} \setminus H^{1/2}$. This means that, for $p = \frac{1}{2}$, condition (b) in Theorem 7 does not always imply (a). Nevertheless, it is an open question whether the equivalence (a) $\Leftrightarrow$ (c) is valid also for $0 < p \le \frac{1}{2}$. This question was earlier posed in [27]. The next result shows that the statement of Theorem 7 is valid for all $p > 0$ if Θ is a one-component inner function. Consequently, we obtain a partial answer to the question.

Theorem 8. *Let $0 < p < \infty$ and Θ be a one-component inner function. Then conditions* (a)–(c) *in Theorem 7 are equivalent.*

Next we recall [5, Theorem 1.9], which consists of a strengthened Schwarz–Pick lemma for one-component inner functions. This result plays a key role in the proof of Theorem 8.

Theorem B. *Let $n \in \mathbf{N}$ and Θ be a one-component inner function. Then there exists $C = C(n, \Theta) > 0$ such that*

$$|\Theta^{(n)}(z)| \le C \left(\frac{1 - |\Theta(z)|}{1 - |z|} \right)^n \tag{4.3}$$

for all $z \in \mathbf{D}$.

For the proof of Theorem 8, we need also a generalization of [1, Theorem 6.1].

Lemma 9. *Let $0 < p < 1$, $-1 < \alpha < \infty$ and Θ be an inner function. Then there exists $C = C(p, \alpha) > 0$ such that*

$$\int_0^1 |\Theta'(re^{i\theta})|^{p+\alpha+1}(1-r)^\alpha \, dr \le C|\Theta'(e^{i\theta})|^p, \quad e^{i\theta} \in \mathbf{T} \setminus \rho(\Theta).$$

In particular, $\|\Theta'\|^{p+\alpha+1}_{A^{p+\alpha+1}_\alpha} \le 2\pi C \|\Theta'\|^p_{H^p}$.

Proof. Let $e^{i\theta} \in \mathbf{T} \setminus \rho(\Theta)$. By [1, Lemma 6.1], we know that $|\Theta'(re^{i\theta})| \le 4|\Theta'(e^{i\theta})|$ for all $r \in [0, 1)$. Using this fact together with the Schwarz–Pick lemma, we obtain

$$\begin{aligned}\int_0^1 |\Theta'(re^{i\theta})|^{p+\alpha+1}(1-r)^\alpha dr &\le \int_0^x (1-r)^{-p-1} dr + (4|\Theta'(e^{i\theta})|)^{p+\alpha+1} \int_x^1 (1-r)^\alpha dr \\ &\lesssim (1-x)^{-p} - 1 + (1-x)^{\alpha+1}|\Theta'(e^{i\theta})|^{p+\alpha+1}\end{aligned}$$

for every $x \in [0, 1]$. Now it suffices to show that

$$(1-x)^{-p} - 1 + (1-x)^{\alpha+1}|\Theta'(e^{i\theta})|^{p+\alpha+1} \lesssim |\Theta'(e^{i\theta})|^p \tag{4.4}$$

for some x. If $|\Theta'(e^{i\theta})| \le 1$, then this true for $x = 0$. In the case where $|\Theta'(e^{i\theta})| > 1$, the choice $x = 1 - 1/|\Theta'(e^{i\theta})|$ implies (4.4). Since the last assertion is a direct consequence of the first assertion, Hardy's convexity and the mean convergence theorems [15], the proof is complete. □

Now we are ready to prove Theorem 8.

Proof of Theorem 8. By Theorem 7, we may assume $0 < p < 1$ (or even $p \le \frac{1}{2}$). Using Theorem B with $n = 2$, [2, Theorem 6] and Lemma 9 with $\alpha = p - 1$, we obtain

$$\|\Theta''\|^p_{A^p_{p-1}} \lesssim \int_{\mathbf{D}} \left(\frac{1 - |\Theta(z)|}{1 - |z|} \right)^{2p} (1 - |z|)^{p-1} dA(z) \asymp \|\Theta'\|^{2p}_{A^{2p}_{p-1}} \lesssim \|\Theta'\|^p_{H^p}. \tag{4.5}$$

The assertion follows from (4.1) and (4.5). □

It is a well-known fact that, for $0 < p < \infty$ and $-1 < \alpha < \infty$, the Bergman space A^p_α coincides with $\{f\colon f' \in A^p_{\alpha+p}\}$ [16]. Using this result, it is easy to generalize condition (c) in Theorem 8 to the form $\Theta^{(n)} \in A^p_{p(n-1)-1}$ for any/every $n \in \mathbf{N} \setminus \{1\}$. For $0 < p < 1$, we can show this also by modifying the proof of Theorem 8; and as a substitute of this process we obtain an alternative version of Theorem 8.

Theorem 10. *Let $0 < p < \infty$ and Θ be a one-component inner function. Then the following statements are equivalent:*

(a) $\Theta' \in H^p$,
(b) $\Theta' \in A^{p+\alpha+1}_\alpha$ *for some* $\alpha \in (-1, \infty)$,
(c) $\Theta' \in A^{p+\alpha+1}_\alpha$ *for every* $\alpha \in (-1, \infty)$.

Proof. By the proof of Theorem 7, we know that, for $1 \le p < \infty$, Θ satisfies any/all of conditions (a)–(c) if and only if it is a finite Blaschke product. Hence we may assume $0 < p < 1$. Moreover, let $-1 < \alpha < \infty$ and $n \in \mathbf{N} \setminus \{1\}$. Then [16, Theorem 3], Theorem B, [2, Theorem 6], the Schwarz–Pick lemma and Lemma 9 yield

$$\begin{aligned} \|\Theta'\|^p_{H^p} &\lesssim \|\Theta^{(n)}\|^p_{A^p_{p(n-1)-1}} \lesssim \int_{\mathbf{D}} \left(\frac{1-|\Theta(z)|}{1-|z|}\right)^{np} (1-|z|)^{p(n-1)-1} dA(z) \\ &\asymp \|\Theta'\|^{np}_{A^{np}_{p(n-1)-1}} \le \|\Theta'\|^{p+\alpha+1}_{A^{p+\alpha+1}_\alpha} \lesssim \|\Theta'\|^p_{H^p} \end{aligned} \tag{4.6}$$

when $p + \alpha + 1 \le np$. Since we may choose $n = n(p, \alpha)$ such that $n \ge (p + \alpha + 1)/p$, the assertion follows from (4.6). □

Note that, applying Theorem 10 and [27, Theorem 3], we can give several characterizations for one-component inner functions Θ whose derivative belongs to H^p for some $p \in (0, 1)$. By [27, Corollary 4], these characterizations for $p \in (\frac{1}{2}, 1)$ are valid even if Θ would be an arbitrary inner function. Next we show a counterpart of Theorem 10 for all members of $\mathcal{R}_1$.

Corollary 11. *Let $0 < p < \infty$ and $B \in \mathcal{R}_1$. Then the following statements are equivalent:*

(a) $B' \in H^p$,
(b) $B' \in A^{p+\alpha+1}_\alpha$ *for some* $\alpha \in (-1, \infty)$,
(c) $B' \in A^{p+\alpha+1}_\alpha$ *for every* $\alpha \in (-1, \infty)$.

Proof. Assume without loss of generality that $0 < p < 1$ and zeros $\{w_n\}$ of B lie in a Stolz domain $R(1, 1, C)$. Let B_0 be the Blaschke product with zeros $x_n = 1 - 2^{-n}$ for $n \in \mathbf{N}$, write $\Theta = BB_0$ and $\{z_n\} = \{w_n\} \cup \{x_n\}$, where $\{z_n\}$ is ordered by non-decreasing moduli. Then, for each $n \in \mathbf{N}$, there exists $k_n \in \mathbf{N}$ such that

$$|x_{k_n}| \le |z_n| < |x_{k_n+1}|.$$

It follows that

$$\frac{\sum_{|z_j|>|z_n|}(1-|z_j|)}{1-|z_n|} \ge \frac{1-|x_{k_n+1}|}{1-|x_{k_n}|} = \frac{1}{2}.$$

Consequently, Θ is a one-component inner function by Theorem 1.

Let $-1 < \alpha < \infty$. By [4, Theorem 5] and a simple modification of [25, Corollary 2.5] based on [2, Theorem 6], we know that

$$\Theta' \in H^p \quad \Leftrightarrow \quad B' \in H^p \quad \text{and} \quad B_0' \in H^p$$

and

$$\Theta' \in A^{p+\alpha+1}_\alpha \quad \Leftrightarrow \quad B' \in A^{p+\alpha+1}_\alpha \quad \text{and} \quad B'_0 \in A^{p+\alpha+1}_\alpha.$$

In addition, [4, Theorem 7] and Lemma 9 imply $B'_0 \in H^p \cap A^{p+\alpha+1}_\alpha$. Finally, the assertion follows by using Theorem 10 together with the above-mentioned facts. □

Since the derivative of an arbitrary $B \in \mathcal{R}_1$ belongs to $H^p \cap A^{p+\alpha+1}_\alpha$ for every $p \in (0, \frac{1}{2})$ and $\alpha \in (-1, \infty)$ by [20, Theorem 2.3] and Lemma 9, the statement of Corollary 11 for $p \neq \frac{1}{2}$ does not come as a surprise. However, the case $p = \frac{1}{2}$ is interesting because it is not easy to find an alternative way to prove this result.

Recall that $f \in \mathcal{H}(\mathbf{D})$ belongs to the Nevalinna class $\mathcal{N}$ if

$$\sup_{0\le r<1} \int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta < \infty,$$

where $\log^+ 0 = 0$ and $\log^+ x = \max\{0, \log x\}$ for $0 < x < \infty$. As a consequence of Theorem B, we can also give sufficient conditions for higher order derivatives of one-component inner functions to be in the Hardy space H^p or Nevanlinna class $\mathcal{N}$.

Corollary 12. *Let $0 < p < \infty$, $n \in \mathbf{N}$ and Θ be a one-component inner function. Then the following statements are valid:*

(a) *If $\Theta' \in H^p$, then $\Theta^{(n)} \in H^{p/n}$.*

(b) *If $\Theta' \in \mathcal{N}$, then $\Theta^{(n)} \in \mathcal{N}$.*

Proof. As a consequence of Theorem B [5], we find $C = C(n, \Theta)$ such that

$$|\Theta^{(n)}(\xi)| \le C|\Theta'(\xi)|^n, \quad \xi \in \mathbf{T} \setminus \rho(\Theta). \tag{4.7}$$

Since the spectrum $\rho(\Theta)$ has a Lebesgue measure zero, inequality (4.7), Hardy's convexity and the mean convergence theorems yield

$$\|\Theta^{(n)}\|^{p/n}_{H^{p/n}} = \frac{1}{2\pi}\int_0^{2\pi} |\Theta^{(n)}(e^{i\theta})|^{p/n} d\theta \lesssim \frac{1}{2\pi}\int_0^{2\pi} |\Theta'(e^{i\theta})|^p d\theta = \|\Theta'\|^p_{H^p}.$$

Hence assertion (a) is proved. Since case (b) can be verified in a similar manner, the proof is complete. □

We close this note with two results regarding certain one-component singular inner functions.

Corollary 13. *Let $0 < p < \infty$ and S be the one-component atomic singular inner function associated with $\{e^{i\theta_n}\}$ and $\{\gamma_n\} \in l^{1/2}$. Then S satisfies any/all of conditions* (a)–(c) *in Theorem 7 if and only if $p < \frac{1}{2}$.*

Proof. By [26, Theorem 3], for $\frac{1}{4} \le p < \infty$, the derivative of S belongs to A^{2p}_{p-1} if and only if $p < \frac{1}{2}$. Since $H^{p_1} \subset H^{p_2}$ for $0 < p_2 \le p_1 < \infty$, the assertion follows from this result and Theorem 8. □

The following result shows that Corollary 12(a) is sharp.

Corollary 14. *Let $0 < p < \infty$, $m \in \mathbf{N}$ and S be the one-component atomic singular inner function associated with $\{e^{i\theta_n}\}$ and $\{\gamma_n\} \in l^{1/2}$. Moreover, assume that there exist an index $j = j(S)$ and $\varepsilon = \varepsilon(j) > 0$ such that $|\theta_j - \theta_n| > \varepsilon$ for all $n \neq j$. Then $S^{(m)} \in H^p$ if and only if $p < \frac{1}{2m}$.*

Proof. By Corollary 13, $S' \in H^{mp}$ if and only if $p < \frac{1}{2m}$. Consequently, Corollary 12(a) implies $S^{(m)} \in H^p$ for $p < \frac{1}{2m}$. Hence it suffices to show that $S^{(m)} \in H^p$ only if $p < \frac{1}{2m}$.

Fix $j = j(S)$ to be the smallest index such that $|\theta_j - \theta_n| > \varepsilon$ for all $n \neq j$ and some $\varepsilon = \varepsilon(j) > 0$. Let us represent S in the form $S = S_1 S_2$, where

$$S_1(z) = \exp\left(\gamma_j \frac{z + e^{i\theta_j}}{z - e^{i\theta_j}}\right), \quad z \in \mathbf{D},$$

and $S_2 = S/S_1$. Using this factorization, we obtain

$$|S^{(m)}(e^{i\theta})| = \left|\sum_{k=0}^{m} \binom{m}{k} S_1^{(m-k)}(e^{i\theta}) S_2^{(k)}(e^{i\theta})\right| \asymp \left|S_1^{(m)}(e^{i\theta}) S_2(e^{i\theta})\right| = |S_1^{(m)}(e^{i\theta})|$$

when θ (which is not θ_j) is close enough to θ_j depending on S and m. Consequently, we find a sufficiently small $\alpha = \alpha(p, S, m) > 0$ such that

$$\int_0^{2\pi} |S_1^{(m)}(e^{i\theta})|^p \, d\theta \asymp \int_{\theta_j - \alpha}^{\theta_j + \alpha} |S_1^{(m)}(e^{i\theta})|^p \, d\theta \lesssim \int_0^{2\pi} |S^{(m)}(e^{i\theta})|^p \, d\theta,$$

where the comparison constants depend only on p, S and m. It follows that $S^{(m)} \in H^p$ only if $S_1^{(m)} \in H^p$. Moreover, a simple modification of the main result of [24] shows that $S_1^{(m)} \in H^p$ if and only if $p < \frac{1}{2m}$. Combining these facts, we deduce that $S^{(m)} \in H^p$ (if and) only if $p < \frac{1}{2m}$. This completes the proof. □

Acknowledgements. The author thanks Toshiyuki Sugawa for valuable comments, and the referees for careful reading of the manuscript.

References

[1] Ahern, P.: The mean modulus and the derivative of an inner function. - Indiana Univ. Math. J. 28:2, 1979, 311–347.

[2] Ahern, P.: The Poisson integral of a singular measure. - Canad. J. Math. 35:4, 1983, 735–749.

[3] Ahern, P. R., and D. N. Clark: On inner functions with B^p derivative. - Michigan Math. J. 23:2, 1976, 107–118.

[4] Ahern, P. R., and D. N. Clark: On inner functions with H^p derivative. - Michigan Math. J. 21, 1974, 115–127.

[5] Aleksandrov, A. B.: Embedding theorems for coinvariant subspaces of the shift operator. II. - Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 262, 1999, Issled. po Lineĭn. Oper. i Teor. Funkts. 27, 5–48, 231; English transl. in J. Math. Sci. (New York) 110:5, 2002, 2907–2929.

[6] Aleman, A., Y. Lyubarskii, E. Malinnikova, and K.-M. Perfekt: Trace ideal criteria for embeddings and composition operators on model spaces. - J. Funct. Anal. 270:3, 2016, 861–883.

[7] Baernstein, A., D. Girela, and J. A. Peláez: Univalent functions, Hardy spaces and spaces of Dirichlet type. - Illinois J. Math. 48:3, 2004, 837–859.

[8] Baranov, A., R. Bessonov, and V. Kapustin: Symbols of truncated Toeplitz operators. - J. Funct. Anal. 261:12, 2011, 3437–3456.

[9] Bessonov, R. V.: Duality theorems for coinvariant subspaces of H^1. - Adv. Math. 271, 2015, 62–90.

[10] Bessonov, R. V.: Fredholmness and compactness of truncated Toeplitz and Hankel operators. - Integral Equations Operator Theory 82:4, 2015, 451–467.

[11] Cargo, G. T.: Angular and tangential limits of Blaschke products and their successive derivatives. - Canad. J. Math. 14, 1962, 334–348.

[12] Chalendar, I., E. Fricain, and D. Timotin: Functional models and asymptotically orthonormal sequences. - Ann. Inst. Fourier (Grenoble) 53:5, 2003, 1527–1549.

[13] Cima, J., and R. Mortini: One-component inner functions. - Complex Anal. Synerg. 3:1, 2017, Paper No. 2, 1–15.

[14] Cohn, B.: Carleson measures for functions orthogonal to invariant subspaces. - Pacific J. Math. 103:2, 1982, 347–364.

[15] Duren, P.: Theory of H^p spaces. - Academic Press, New York–London, 1970.

[16] Flett, T. M.: The dual of an inequality of Hardy and Littlewood and some related inequalities. - J. Math. Anal. Appl. 38, 1972, 746–765.

[17] Frostman, O.: Sur les produits de Blaschke. - Kungl. Fysiografiska Sällskapets i Lund Förhandlingar 12:15, 1942, 169–182.

[18] Garnett, J.: Bounded analytic functions. Revised 1st edition. - Springer, New York, 2007.

[19] Girela, D., C. González, and M. Jevtić: Inner functions in Lipschitz, Besov, and Sobolev spaces. - Abstr. Appl. Anal. 2011, 2011, Art. ID 626254, 1–26.

[20] Girela, D., J. A. Peláez, and D. Vukotić: Integrability of the derivative of a Blaschke product. - Proc. Edinb. Math. Soc. (2) 50:3, 2007, 673–687.

[21] Girela, D., J. A. Peláez, and D. Vukotić: Interpolating Blaschke products: Stolz and tangential approach regions. - Constr. Approx. 27:2, 2008, 203–216.

[22] Gluchoff, A.: On inner functions with derivative in Bergman spaces. - Illinois J. Math. 31:3, 1987, 518–528.

[23] Littlewood, J. E., and R. E. A. C. Paley: Theorems on Fourier series and power series (II). - Proc. London Math. Soc. (2) 42:1, 1936, 52–89.

[24] Mateljević, M., and M. Pavlović: On the integral means of derivatives of the atomic function. - Proc. Amer. Math. Soc. 86:3, 1982, 455–458.

[25] Pérez-González, F., and J. Rättyä: Inner functions in the Möbius invariant Besov-type spaces. - Proc. Edinburgh Math. Soc. 52, 2009, 751–770.

[26] Pérez-González, F., J. Rättyä, and A. Reijonen: Derivatives of inner functions in Bergman spaces induced by doubling weights. - Ann. Acad. Sci. Fenn. Math. 42:2, 2017, 735–753.

[27] Reijonen, A., and T. Sugawa: Characterizations for inner functions in certain function spaces. - Complex Anal. Oper. Theory, 2018, 1–19, https://doi.org/10.1007/s11785-018-0863-9.

[28] Vinogradov, S. A.: Multiplication and division in the space of analytic functions with an area-integrable derivative, and in some related spaces. - Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 222, 1995 (in Russian), Issled. po Linein. Oper. i Teor. Funktsii 23, 45–77; English transl. in J. Math. Sci. (New York) 87, 1997, 3806–3827.

Received 13 May 2018 • Accepted 29 November 2018