Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 44, 2019, 635–655

SIERPIŃSKI-TYPE FRACTALS ARE DIFFERENTIABLY TRIVIAL

Estibalitz Durand-Cartagena, Jasun Gong and Jesús A. Jaramillo

UNED, ETSI Industriales, Departamento de Matemática Aplicada
28040 Madrid, Spain; edurand@ind.uned.es

Fordham University, Department of Mathematics
441 E. Fordham Rd, Bronx, NY 10458, United States; jgong7@fordham.edu

Universidad Complutense de Madrid, Facultad de Ciencias Matemáticas, IMI and Departamento de Análisis Matemático y Matemática Aplicada, 28040 Madrid, Spain; jaram@mat.ucm.es

Abstract. In this note we study generalized differentiability of functions on a class of fractals in Euclidean spaces. Such sets are not necessarily self-similar, but satisfy a weaker "scale-similar" property; in particular, they include the non self similar carpets introduced by Mackay–Tyson–Wildrick [12] but with different scale ratios. Specifically we identify certain geometric criteria for these fractals and, in the case that they have zero Lebesgue measure, we show that such fractals cannot support nonzero derivations in the sense of Weaver [16]. As a result (Theorem 26) such fractals cannot support Alberti representations and in particular, they cannot be Lipschitz differentiability spaces in the sense of Cheeger [3] and Keith [9].

1. Motivation

First order differentiable calculus has been extended from smooth manifolds to abstract metric spaces in many ways, by many authors. In this context, one important property of a metric space is the validity of Rademacher's theorem, i.e. that *Lipschitz functions are almost everywhere (a.e.) differentiable* with respect to a choice of coordinates on that space. (For this reason, such spaces are known as *Lipschitz differentiability spaces* in the recent literature, e.g. [1, 2, 4] and said to have a *measurable differentiable structure* in earlier literature, e.g. [9, 11, 14].)

The search for such a property naturally leads to questions of compatibility between a metric space and the choice of a Borel measure on that space. Even the case of Euclidean spaces has been addressed only recently. A result of De Phillipis and Rindler [5, Thm. 1.14] states that if Rademacher's Theorem is true for a Radon measure μ on $\mathbf{R}^m$, then μ must be absolutely continuous to m-dimensional Lebesgue measure.

Here we address the case when μ is singular. As we will see, there is a large class of fractal sets, which we call *Sierpiński-type fractals*, for which Lipschitz functions do not even enjoy *partial* a.e. differentiability on the support of their natural measures.

https://doi.org/10.5186/aasfm.2019.4460
2010 Mathematics Subject Classification: 31E05, 28A80, 13N15.
Key words: Sierpiński-type fractals, doubling measure, metric derivations.
The research of the first and third author is supported by the grant MTM2015-65825-P (Spain). The research of the first author is also supported by the grant 2019-MAT12 (ETSI Industriales, UNED). The second author was supported by grant 259590 from the Academy of Finland. This research began during the visit of the second author to Universidad Complutense de Madrid in the Fall 2012, and continued during the time the three authors visited IPAM (UCLA) in Spring 2013. They wish to thank those institutions for their kind hospitality.

Postponing the technical formulation for now, it is worth noting that this class includes, in all dimensions,

(A) standard self-similar fractals, such as Cantor sets, Sierpiński carpets, and Menger sponges;
(B) non-self-similar constructions, such as carpets complementary to those of Mackay, Tyson, and Wildrick [12];
(C) entirely new, *random* constructions of fractals, which exist under mild symmetry conditions.

1.1. Fractals and derivations. To motivate our result, recall that Bate [1] has recently characterized Lipschitz differentiability spaces in terms of *Alberti representations* of measures—that is, by disintegrating the underlying measure of a space into a family of measures, each of which is supported on nontrivial fragments of rectifiable curves. (Differentiability in this sense therefore corresponds to directional differentiability in a spanning set of directions.)

Prior to Bate's result, Weaver [16] introduced *(metric) derivations* as generalized notions of partial differential operators. This approach is functional analytic in nature, causing derivations to be well-defined on all metric spaces X equipped with Borel measures μ, and to enjoy the algebraic structure of a module over the ring of functions $L^\infty(X, \mu)$. Schioppa [15] further showed that Alberti representations are examples of derivations, a key tool in this paper.

Our main result, Theorem 45, asserts that Sierpiński-type fractals have a trivial module of derivations, that is: as a linear operator on Lipschitz functions, the only derivation is the zero operator. Coupled with Schioppa's result above, we have an immediate conclusion:

Corollary 1. *Sierpiński-type fractals do not support Alberti representations.*

The novelty here is that the structural conditions on certain fractal sets can be further weakened and therefore treated with different techniques. It is known [16, Sect 5E], for example, that self-similar fractals such as the Sierpiński carpet or the Sierpiński gasket have a trivial module of derivations; the proof there exploits their geometric properties of self-similarity and porosity, but it does not extend to non-self-similar fractals.

Our approach is new, in that it relies on a notion of dimension, or *rank*, associated to a module of derivations on a metric measure space. In particular, the notion of rank allows for directional information from derivations, like that of vector fields on a smooth manifold. It therefore exploits different geometric features of fractals, such as symmetries at each scale of their construction, to which the methods in [16] are insensitive.

1.2. The case of carpets. For the sake of clarity, we prove our main result for an explicit example first, in order to motivate the definition of a Sierpiński-type fractal later.

In particular, in each dimension we opt for a non-self-similar example with zero Lebesgue measure. Not only does it highlight new techniques not appearing in the self-similar setting, but in dimension 2 it complements nicely the case of "carpets" treated by Mackay, Tyson, and Wildrick [12]. There the authors show that for non-self-similar constructions of Sierpiński carpets, the condition of positive (2-dimensional) Lebesgue measure is equivalent to the validity of a $(1, p)$-*Poincaré inequality* for all $1 < p \leq 2$.

Recall that in a seminal work, Cheeger [3] proved that metric spaces with doubling measures and supporting $(1,p)$-Poincaré inequalities, for $1 \le p < \infty$, are Lipschitz differentiability spaces and therefore have a nontrivial module of derivations.

Remark 2. Though the fractals treated here are non-smooth sets with no manifold points, we emphasize that the underlying metric is the Euclidean one and the underlying measures are the corresponding Hausdorff measures with respect to this metric.

In contrast, the constructions treated in the *analysis on fractals* make use of a so-called resistance metric that is induced by probabilistic methods (specifically, via the theory of *Dirichlet forms*) on the given fractal and that is known **not** to be comparable with the Euclidean metric. We will not discuss such methods here but refer to the survey of Kigami [10] and the references contained therein.

This paper is outlined as follows. In Section §2 we construct the carpets mentioned above, recall basic facts about derivations, and survey what is already known about derivations on Euclidean spaces. Section §3 consists of a series of lemmas, leading to Theorem 26 which covers the model case of non-self similar Sierpiński carpets. In §4 we define Sierpiński-type fractals and state a more general result, Theorem 45, that will follow essentially from the same proof as Theorem 26. Lastly, Section §5 is a short appendix, where we handle a technical lemma about Sierpiński-type fractals and their associated measures.

2. Setup

We first fix the notation and some basic notions. Given a set X, a subset $A \subseteq X$, and a function $f\colon X \to \mathbf{R}$, the restriction of f to A is denoted $f_{|A}$. Similarly, if μ is a measure on X, then $\mu_{|A}$ refers to the restriction of μ to A, defined as

$$\mu_{|A}(E) := \mu(A \cap E)$$

for all μ-measurable subsets E of X.

If X and Y are topological spaces, if $F\colon X \to Y$ is a Borel map, and if μ is a Borel measure on X, then the PUSHFORWARD of μ under F is a Borel measure on Y, defined on all Borel subsets E of Y as

$$F_{\#}\mu(E) := \mu(F^{-1}(E)).$$

If $X = (X,d)$ is a metric space, then the Lipschitz constant of a function $f\colon X \to \mathbf{R}$ is denoted by

$$L(f) := \sup\left\{\frac{|f(x) - f(y)|}{d(x,y)}\,;\, x \neq y \text{ in } X\right\}$$

and we will often use the following classes of functions:

$$\begin{aligned}\mathrm{Lip}(X) &:= \{\text{all Lipschitz functions on } X\},\\ \mathrm{Lip}_b(X) &:= \{\text{all bounded Lipschitz functions on } X\}.\end{aligned}$$

Note that $\|f\|_{\mathrm{Lip}} = \max\{\|f\|_\infty, L(f)\}$ is a norm on $\mathrm{Lip}_b(X)$. For a sequence $(f_n)_{n=1}^\infty$ in $\mathrm{Lip}_b(X)$, we also write $f_n \xrightarrow{\star} f$ if

$$\sup_n L(f_n) < \infty \quad\text{and}\quad f_n \to f \quad\text{pointwise in } X.$$

With limits of bounded linear operators in mind, let V and W be Banach spaces and consider the space $\mathcal{L}(V, W^*)$ of all bounded linear operators from V into the dual space W^*. The *weak-star operator topology* on $\mathcal{L}(V,W^*)$ is the linear topology generated by the seminorms $p_{x,y}$, with $x \in V$ and $y \in W$, where we define

$$p_{x,y}(T) = |\langle T(x), y\rangle|,$$

for every operator $T \in \mathcal{L}(V,W^*)$. Moreover, we denote the operator norm of each $L \in \mathcal{L}(V,W^*)$ by

$$\|L\|_{\mathrm{op}} \;=\; \sup\{\|Le\|_{W^*}\,;\, \|e\|_V \;\leq\; 1\}.$$

The next lemma is folklore; for a reference, see Theorem 5.3.4 from the second author's Ph.D. thesis.

Lemma 3. *Let V and W be Banach spaces, and let $\mathbf{B}(V,W^*)$ denote the closed unit ball of the space $\mathcal{L}(V,W^*)$.*

(a) $\mathbf{B}(V,W^*)$ *is compact for the weak-star operator topology.*
(b) *If V and W are both separable, then $\mathbf{B}(V,W^*)$ is metrizable for the weak-star operator topology, and therefore it is sequentially compact.*

Indeed (a) is folklore, being a standard consequence of Tychonov's theorem. For the idea for (b), let $\{x_n\}$ and $\{y_n\}$ be dense sequences in V and W, respectively. It is not difficult to see that the expression

$$\rho(R,T) = \sum_{n,m=1}^{\infty} \frac{1}{2^{n+m}} |\langle (R-T)(x_n), y_m\rangle|$$

defines a metric on $\mathbf{B}(V,W^*)$ that induces the weak-star operator topology.

2.1. Carpets. Let $\mathbf{a} = (a_n)_{n=1}^{\infty}$ be non-negative numbers of the form

$$a_n := \frac{p_n}{q_n},$$

where $p_n, q_n \in \mathbf{N}$ with $p_n + q_n$ even and with $p_n < q_n$ and where $\mathbf{a} \in \ell^\infty \setminus \ell^2$, that is: the series $\sum_n a_n^2$ diverges, yet $\sup_n |a_n| < \infty$.

We now construct a compact subset $\mathbf{S_a}$ of $\mathbf{R}^2$ by a process analogous to the usual Sierpiński carpet, and where the parameters a_n are used instead of ratios of $\frac{1}{3}$. The basic idea is that, at the nth step, one divides the existing squares into $q_n \times q_n$ new subsquares and removes the middle $p_n \times p_n$ of them.

Step 0: Put $S^0_{\mathbf{a}} := [0,1] \times [0,1]$ and $\mathcal{C}_0 = \{S^0_{\mathbf{a}}\}$ and $\mathcal{C}^0_0 = \emptyset$ first.

Step 1: Divide $S^0_{\mathbf{a}}$ into $q_1 \times q_1$ closed subsquares with sides parallel to the coordinate axes and with lengths $l_1 := q_1^{-1}$, i.e.

$$Q^1_{ij} := \Big[\frac{i-1}{q_1}, \frac{i}{q_1}\Big] \times \Big[\frac{j-1}{q_1}, \frac{j}{q_1}\Big], \tag{4}$$

for $i,j \in \{1,2,\ldots,q_1\}$. Enumerating them as $\mathcal{C}_1 := \{Q^1_{ij}\}_{i,j=1}^{q_1}$, we have

$$S^0_{\mathbf{a}} \;=\; \bigcup_{i,j=1}^{q_1} Q^1_{ij}.$$

Now let $\mathcal{C}^0_1$ be the subcollection of the p_1^2 many "middle" subsquares from $\mathcal{C}_1$. More precisely, let $r_1 = \frac{1}{2}(q_1 - p_1)$, put

$$\mathcal{C}^0_1 := \big\{Q^1_{i,j} \in \mathcal{C}_1\,;\, r_1 + 1 \leq i,j \leq p_1 + r_1\big\},$$
$$\mathcal{C}^+_1 := \mathcal{C}_1 \setminus \mathcal{C}^0_1,$$

and write the union of the remaining squares as

$$S^1_{\mathbf{a}} := \bigcup_{Q \in \mathcal{C}^+_1} Q.$$

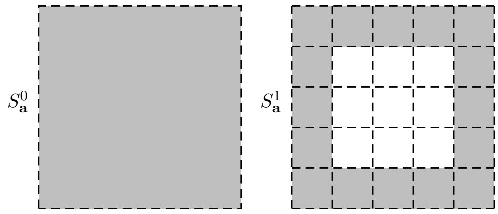

Figure 1. E.g. of $p_1 = 3$ and $q_1 = 5$, so $r_1 = \frac{q_1 - p_1}{2} = \frac{5-3}{2} = 1$.

As a suggestive terminology,

- subsquares in $\mathcal{C}_1$ are called *first-order* subsquares,
- subsquares in $\mathcal{C}^0_1$ are called *first-order middle* subsquares,

and we will use analogous notation for steps 2 and beyond.

Step $n \geq 2$: We proceed inductively. Let $\mathcal{C}_{n-1}$ be the collection of $(n-1)$th-order subsquares with pairwise disjoint interiors, with side length

$$l_{n-1} := (q_1 \cdots q_{n-1})^{-1},$$

and with all sides parallel to the axes. Suppose the sub-collection of $(n-1)$th-order non-middle subsquares $\mathcal{C}^+_{n-1}$ has already been defined. Now sub-divide each $Q \in \mathcal{C}^+_{n-1}$ into $q_n \times q_n$ squares of side length $l_n := q_n^{-1} l_{n-1}$, analogously as in (4), and write the collection as

$$\mathcal{C}_n(Q) := \{Q^n_{ij}\}^{q_n}_{i,j=1}.$$

Again, we remove the middle subsquares; for $r_n := \frac{1}{2}(q_n - p_n)$, put

$$\mathcal{C}^0_n(Q) := \big\{Q_{ij} \in \mathcal{C}_n(Q)\,;\, 1 + r_n \leq i, j \leq p_n + r_n\big\},$$
$$\mathcal{C}^+_n(Q) := \mathcal{C}_n(Q) \setminus \mathcal{C}^0_n(Q)$$

and write the union of these selections as

$$S^n_{\mathbf{a}} := \bigcup_{Q' \in \mathcal{C}^+_n} Q', \quad \text{where} \quad \mathcal{C}^+_n := \bigcup_{Q \in \mathcal{C}^+_{n-1}} \mathcal{C}^+_n(Q).$$

Since $S^n_{\mathbf{a}} \subset S^{n-1}_{\mathbf{a}}$ holds for all $n \in \mathbf{N}$, the limit set

$$\mathbf{S}_{\mathbf{a}} := \bigcap_{n=1}^{\infty} S^n_{\mathbf{a}}$$

is well-defined; we call it the *non-self similar Sierpiński carpet* generated by $\mathbf{a}$, or simply a *carpet*.

Remark 5. (Area) Observe that the classes of carpets we are taking into consideration have no area in the sense of 2-dimensional Lebesgue (Hausdorff) measure:

$$\mathcal{H}^2(\mathbf{S}_{\mathbf{a}}) = 0.$$

Notice that this holds if and only if $\mathbf{a} \notin \ell^2$.

Remark 6. (Geometry) We now list some properties of $\mathbf{S}_{\mathbf{a}}$ that follow from the construction:

(A) For each $n \geq 2$, each $Q \in \mathcal{C}_n^+$ is a subsquare of a unique $Q' \in \mathcal{C}_{n-1}^+$. As a result, there is a unique vector $v_{nQ} \in \mathbf{R}^2$ so that the similitude

$$\sigma^{nQ}(x) := q_n x + v_{nQ}$$

maps Q onto Q' and preserves orientation of edges. In particular, every point not lying on a square boundary—that is, every

$$x \notin \partial^+ \mathbf{S_a} := \left(\bigcup_{n=1}^{\infty} \bigcup_{Q \in \mathcal{C}_n^+} \partial Q \right)$$

has a unique sequence of closed subsquare neighborhoods $(\mathcal{N}_x^n)_{n=1}^{\infty}$, where $\mathcal{N}_x^n \in \mathcal{C}_n^+$ for each $n \in \mathbf{N}$.

(B) The carpet endowed with the Euclidean metric is quasiconvex; recall that a metric space (X, d) is *C-quasiconvex* (with $C \geq 1$) if any pair of points $x, y \in X$ can be joined by a rectifiable path whose length does not exceed $Cd(x, y)$.

(C) There is a canonical measure μ that is supported on $\mathbf{S_a}$. Indeed, consider the sequence of probability measures, each supported on $S_{\mathbf{a}}^n$, as defined by

$$\mu_0 := \mathcal{H}^2|_{S_{\mathbf{a}}^0} \quad \text{and} \quad \mu_n := \sum_{Q \in \mathcal{C}_n^+} \frac{\sigma_{\#}^{nQ'}(\mu_{n-1|Q})}{q_n^2 - p_n^2} \quad \text{for each } n \in \mathbf{N}$$

and hence by Banach–Alaoglu there is a weak-star *sub*limit measure μ that is concentrated on $\mathbf{S_a}$. We claim

(C.1) that μ is both unique and the full (weak-star) limit of $(\mu_n)_{n=1}^{\infty}$; for a proof, see Appendix in §5.

(C.2) that $\mu(\partial Q) = 0$ for every $n \in \mathbf{N}$ and every $Q \in \mathcal{C}_n^+$. Indeed, given any line segment ℓ in ∂Q and any neighborhood O_N of ℓ consisting of subsquares Q' in $\mathcal{C}_{n+N}^+$ with $Q' \cap \ell \neq \emptyset$, lower-semicontinuity of weak-star convergence yields

$$\mu(\ell) \leq \mu(O_N) \leq \liminf_{n \to \infty} \mu_n(O_N) = 0.$$

As a result, $\mu(Q_{ij} \cap Q_{kl}) = 0$ for every $i \neq k$ or $j \neq l$ with $Q_{ij} \in \mathcal{C}_n^+$ in the previous construction.

(C.3) that μ is *doubling*, which means that there exists a constant $C \geq 1$ such that

$$\mu(B(x, 2r)) \leq C\mu(B(x, r))$$

for all balls $B(x, r)$ in X with centers $x \in X$ and radii $r > 0$. The proof is essentially the same as that of [12, Proposition 3.1].

Remark 7. (Higher dimensions) Analogous constructions apply to $\mathbf{R}^m$ for all $m \in \mathbf{N}$, where m replaces the dimension 2 and where we subdivide m-dimensional cubes into $(q_n)^m$ many sub-cubes and omit the middle $(p_n)^m$ of them. We call these limit sets *(Sierpiński) sponges*[1] and we denote them by $\mathbf{S}_{\mathbf{a}}^m$. (In particular, $\mathbf{S}_{\mathbf{a}}^2 = \mathbf{S_a}$ are the carpets from before.)

In this case, we assume that $\mathbf{a} \in \ell^\infty \backslash \ell^m$ and a similar computation as in Remark 5 shows that $\mathcal{H}^m(\mathbf{S}_{\mathbf{a}}^m) = 0$. Moreover, there are canonical measures that are associated

[1] In contrast, Menger sponges are constructed not by omitting subcubes but by omitting cubewise "tunnels" perpendicular to the codimension-1 faces.

to sponges, constructed analogously, and satisfy analogous geometric properties as in Remark 6. We denote them by $\mu_{\mathbf{a}}^m$.

For later purposes we will give a general version of the Lebesgue differentiation theorem for doubling measures. In place of balls it suffices to have subsets of balls with a positive lower bound on its measure density.

To fix notation, let $\Omega \subset \mathbf{R}^m$. For $c \geq 1$ and $x \in \Omega$ define $\mathcal{F}_c(x,r)$ as the family of all measurable sets $E \subset \Omega$ such that $E \subset B(x,r)$ and $\mu(B(x,r)) \leq c\mu(E)$. We say that a sequence of measurable sets $\{E_i\}_{i=1}^{\infty}$ *converges to a point* x if there exists a sequence of radii $r_i > 0$ such that $E_i \subset B(x,r_i)$ and $r_i \to 0$ as $i \to \infty$.

Theorem 8. [7, Theorem 14.15] *Let μ be doubling on $\Omega \subset \mathbf{R}^m$ and $u \in L^1_{loc}(\Omega,\mu)$. Then for μ-a.e. $x \in \Omega$ we have*

$$\lim_{r\to 0} \fint_{B(x,r)} u(y)\, d\mu(y) = u(x).$$

More generally, if $c \geq 1$ then for μ-a.e $x \in \Omega$ and every sequence of sets $\{E_i\}_i$ that converge to x with $E_i \in \mathcal{F}_c(x,r_i)$ we have that

$$\lim_{i\to\infty} \fint_{E_i} u(y)\, d\mu(y) = u(x).$$

2.2. Derivations: basic facts. The following notion is due to Weaver [16] for so-called *measurable metrics*; for the case of (pointwise) metrics in the usual sense, see the survey of Heinonen [8] as well as [6], [14], and [15].

Fix a Radon measure μ on a separable metric space (X,d). We refer to the collection (X,d,μ) as a *metric measure space.*

Definition 9. (Weaver) A bounded linear operator

$$\delta\colon \operatorname{Lip}_b(X) \to L^\infty(X,\mu)$$

is called a *(metric) derivation* if it satisfies

- the *product rule*: $\delta(fg) = f\,\delta g + g\delta f$ holds for all $f,g \in \operatorname{Lip}_b(X)$;
- *weak-star continuity*: if $(f_n)_{n=1}^\infty$ and f in $\operatorname{Lip}_b(X)$ satisfy $f_n \xrightarrow{\star} f$, then $\delta f_n \xrightarrow{\star} \delta f$ in $L^\infty(X,\mu)$, i.e.,

$$\int_X \varphi\, \delta f_n\, d\mu \to \int_X \varphi\, \delta f\, d\mu \tag{10}$$

  holds for all $\varphi \in L^1(X,\mu)$.

Let $\Upsilon(X,\mu)$ denote the space of derivations with respect to μ on X.

Note that $\Upsilon(X,\mu)$ is an $L^\infty(X,\mu)$-module, where the scalar action is

$$(\lambda\delta)f = \lambda(\delta f),$$

for all $\lambda \in L^\infty(X,\mu)$. As defined in [8, pp. 216], call a metric measure space (X,d,μ) *differentiably trivial* if it has a trivial module of derivations, i.e. that $\Upsilon(X,\mu) = 0$.

Moreover, we call a set $\{\delta_i\}_{i=1}^k$ *linearly dependent* in $\Upsilon(X,\mu)$ if there exist $\{\lambda_i\}_{i=1}^k$ in $L^\infty(X,\mu)$, not all zero, so that

$$\lambda_1\delta_1 + \cdots + \lambda_k\delta_k = 0.$$

Otherwise we say that $\{\delta_i\}_{i=1}^k$ are *linearly independent.*

Lastly, we say that $\Upsilon(X,\mu)$ has *rank-k* if it contains a linearly independent set of k derivations and if every set of $k+1$ derivations is linearly dependent.

We now turn to basic properties of derivations. The first lemma combines Lemma 27 and Theorem 29 in [16].

Lemma 11. (Locality) *Let μ be Radon on X. If A is a μ-measurable subset of X, then as sets and modules,*

$$\chi_A \Upsilon(X,\mu_{|A}) := \{\chi_A\delta\,;\ \delta \in \Upsilon(X,\mu)\} = \Upsilon(A,\mu_{|A})$$

and in particular, if $f|_A$ is constant, then $\delta f = 0$ μ-a.e. on A.

Remark 12. As a consequence, every derivation $\delta \in \Upsilon(\mathbf{R}^m,\mu)$ has a well-defined linear extension to each $f \in \mathrm{Lip}(\mathbf{R}^m)$, also denoted by δ, that satisfies

$$\|\delta f\|_{L^\infty} \le \|\delta\|_{\mathrm{op}} L(f).$$

Remark 13. For the sponges $\mathbf{S}^m_{\mathbf{a}}$ from Remark 7, the measures $\mu^m_{\mathbf{a}}$ satisfy

$$\mu^m_{\mathbf{a}}(\mathbf{R}^m \setminus \mathbf{S}^m_{\mathbf{a}}) \;=\; 0$$

by construction, the locality property gives

$$\chi_{\mathbf{S}^m_{\mathbf{a}}} \Upsilon(\mathbf{R}^m,\mu^m_{\mathbf{a}}) \;=\; \Upsilon(\mathbf{S}^m_{\mathbf{a}},\mu^m_{\mathbf{a}}).$$

So in terms of derivations with respect to $\mu^m_{\mathbf{a}}$, the sets $\mathbf{R}^m$ and $\mathbf{S}^m_{\mathbf{a}}$ are treated the same *analytically*, even though they differ *geometrically* as metric spaces. (For example, $\mu^m_{\mathbf{a}}$ is doubling on $\mathbf{S}^m_{\mathbf{a}}$ but its zero extension is not doubling on all of $\mathbf{R}^m$.)

2.3. Derivations on Euclidean spaces. Due to the locality property (Lemma 11), every derivation on $\mathbf{R}^m$ is well-defined on polynomials and other locally Lipschitz functions. Roughly speaking, the action of such derivations is completely determined by their action on the standard coordinate functions.

Of the next four results, the first is a direct consequence of [6, Lemma 27] and [13, Theorem 1.19], the second is [6, Lemma 2.19], and the third and fourth are consequences of the second. To fix notation, $\mathbf{x} := (x_1, x_2, \dots, x_m)$ denotes the identity map on $\mathbf{R}^m$, so x_i is the usual ith linear coordinate.

Lemma 14. (Change of variables) *Let X and Y be metric spaces, let $F\colon X \to Y$ be a proper Lipschitz map, and let μ be a Radon measure on X. For each $\delta \in \Upsilon(X,\mu)$, there is a unique derivation $F_\#\delta \in \Upsilon(Y, F_\#\mu)$ called* the pushforward of δ under F *that satisfies*

$$\int_Y g\,(F_\#\delta) f\, d(F_\#\mu) = \int_X (g\circ F)\delta(f\circ F)\, d\mu$$

for all $f \in \mathrm{Lip}(Y)$ and all $g \in L^1(Y, F_\#\mu)$. If moreover F^{-1} exists and is Lipschitz, then for μ-a.e. $x \in X$, it holds that

$$\delta(f\circ F)(x) = (F_\#\delta) f\big(F(x)\big)$$

Lemma 15. (Chain rule) *For every $f \in \mathrm{Lip}(\mathbf{R}^m)$, there exists $\mathbf{v}^f \in L^\infty(\mathbf{R}^m; \mathbf{R}^m, \mu)$ so that every $\delta \in \Upsilon(\mathbf{R}^m,\mu)$ satisfies the μ-a.e. inequalities*

$$\delta f = \mathbf{v}^f \cdot \delta\mathbf{x} = \sum_{i=1}^n v_i^f\, \delta x_i \quad \text{and} \quad \|\mathbf{v}^f\|_{L^\infty} \le L(f).$$

If moreover f is C^1-smooth, then $\mathbf{v}^f = \nabla f$.

Corollary 16. *Fix a Radon measure μ on $\mathbf{R}^m$. For all $f \in \mathrm{Lip}_b(\mathbf{R}^m)$ and all C^1-smooth biLipschitz embeddings $F\colon \mathbf{R}^m \to \mathbf{R}^m$, the identity*

$$\delta(f\circ F)(x) = \mathbf{v}^f(F(x))^T \cdot DF(x)\cdot \delta\mathbf{x}(x)$$

holds for μ-a.e. $x \in \mathbf{R}^m$ and for every $\delta \in \Upsilon(\mathbf{R}^m,\mu)$.

Proof. Approximating f in $\mathrm{Lip}_b(\mathbf{R}^m)$ by convolutions of the form

$$(f \circ \eta_\epsilon) \circ F \overset{\star}{\rightharpoonup} f \circ F$$

we obtain, as weak-star limits in $L^\infty(\mathbf{R}^m, \mu)$, the identities

$$\begin{aligned}\mathbf{v}^{f\circ F}(x) &= \lim_{\epsilon\to 0} \nabla((f * \eta_\epsilon) \circ F)(x) \\ &= \lim_{\epsilon\to 0} \nabla(f * \eta_\epsilon)(F(x)) \cdot DF(x) = \mathbf{v}^f(F(x)) \cdot DF(x)\end{aligned}$$

for μ-a.e. $x \in \mathbf{R}^m$ and hence, by Lemma 15,

$$\delta(f \circ F)(x) = \mathbf{v}^{f\circ F}(x) \cdot \delta\mathbf{x}(x) = \mathbf{v}^f(F(x))^T \cdot DF(x) \cdot \delta\mathbf{x}(x)$$

holds as desired. □

Theorem 17. *If μ is a Radon measure on $\mathbf{R}^m$, then bounded subsets of $\Upsilon(\mathbf{R}^m, \mu)$ are closed for the weak-star operator topology.*

Proof. Let $\{\delta_\alpha\}_\alpha$ be a bounded net in $\Upsilon(\mathbf{R}^m, \mu)$ that converges in the weak-star operator topology to $\delta \in \mathcal{L}(\mathrm{Lip}_b(\mathbf{R}^m), L^\infty(\mathbf{R}^m, \mu))$. To show that $\delta \in \Upsilon(\mathbf{R}^m, \mu)$, it suffices to check the Leibniz rule and weak-star continuity.

First, note that for every $f, g \in \mathrm{Lip}_b(\mathbf{R}^m)$ and every $\varphi \in L^1(\mathbf{R}^m, \mu)$, we have $f\varphi, g\varphi \in L^1(\mathbf{R}^m, \mu)$. Each δ_α satisfies the Leibniz rule, so

$$\begin{aligned}\int_{\mathbf{R}^m} \varphi\, \delta(fg)\, d\mu &= \lim_\alpha \int_{\mathbf{R}^m} \varphi\, \delta_\alpha(fg)\, d\mu = \lim_\alpha \int_{\mathbf{R}^m} \varphi\, (f\delta_\alpha g \,+\, g\delta_\alpha f)\, d\mu \\ &= \lim_\alpha \int_{\mathbf{R}^m} f\varphi\, \delta_\alpha g\, d\mu \,+\, \lim_\alpha \int_{\mathbf{R}^m} g\varphi\, \delta_\alpha f\, d\mu \\ &= \int_{\mathbf{R}^m} f\varphi\, \delta g\, d\mu + \int_{\mathbf{R}^m} g\varphi\, \delta f\, d\mu = \int_{\mathbf{R}^m} \varphi\, (f\, \delta g \,+\, g\, \delta f)\, d\mu\end{aligned}$$

where $\lim_\alpha$ refers to limits of nets. Thus δ satisfies the Leibniz rule.

As for weak-star continuity, let $(f_i)_{i=1}^\infty$ be a sequence of 1-Lipschitz functions on $\mathbf{R}^m$ that converge pointwise to 0 and let $\epsilon > 0$ be given. It suffices to prove, for each $g \in L^1(\mathbf{R}^m, \mu)$, that there exists $i_0 \in \mathbf{N}$ such that

$$\left| \int_{\mathbf{R}^m} g\, \delta f_i\, d\mu \right| \le \epsilon$$

holds for all $i \ge i_0$. Without loss of generality, assume $\delta \ne 0$.

As a first case, assume $\mathrm{spt}(\mu)$ is compact, so μ is finite and each coordinate function x_i lies in $\mathrm{Lip}_b(\mathrm{spt}(\mu))$, for each $i \in 1, \cdots, m$. Let E_0 be the closed subspace of $\mathrm{Lip}_b(\mathrm{spt}(\mu))$ generated by both $(f_i)_{i=1}^\infty$ and the coordinate functions $\{x_i\}_{i=1}^m$. Then E_0 is separable, and so is $Y = L^1(\mathbf{R}^m, \mu)$, since μ is (σ-)finite. Thus, by Lemma 3, the unit ball $\mathbf{B}(E_0, Y^*)$ is sequentially compact.

On the other hand, the restrictions $(\delta_\alpha|_{E_0})_\alpha$ converge to $\delta|_{E_0}$ in the weak-star operator topology of $\mathcal{L}(E_0, Y^*)$, whose bounded sets are metrizable by Lemma 3, Part (b). Then there exists a sequence $(\delta_j)_j = (\delta_{\alpha_j})_j$ which converges to δ in $\mathcal{L}(E_0, Y^*)$ too; without loss of generality, assume for convenience that $\alpha_j \le \alpha_{j+1}$ for all j. In particular, $\delta_j f_i \overset{\star}{\rightharpoonup} 0$ as $i \to \infty$ in $L^\infty(\mathbf{R}^m, \mu)$ and $\delta_j \mathbf{x} \overset{\star}{\rightharpoonup} \delta\mathbf{x}$ as $j \to \infty$ in $L^\infty(\mathbf{R}^m; \mathbf{R}^m, \mu)$.

Since μ is finite, it follows that $L^p(\mathbf{R}^m, \mu)$ is dense in $L^1(\mathbf{R}^m, \mu)$ for all $p \in (1, \infty)$ and hence $\delta_j \mathbf{x} \rightharpoonup \delta\mathbf{x}$ in $L^q(\mathbf{R}^m, \mu)$, where $q := \frac{p}{p-1}$ is the Hölder conjugate of p.

Letting $g \in L^1(\mathbf{R}^m, \mu)$ be arbitrary, choose $g_\epsilon \in L^p(\mathbf{R}^m, \mu)$ so that

$$\|g - g_\epsilon\|_{L^1} < \frac{\epsilon}{3\|\delta\|_{\mathrm{op}}}. \tag{18}$$

We now apply a variant of Mazur's lemma to obtain convex combinations

$$\widetilde{\delta_j \mathbf{x}} := \sum_{l=j}^{N_j} \lambda_{jl} \delta_l \mathbf{x} \ \to \ \delta \mathbf{x} \tag{19}$$

that converge in L^q-norm. In particular, there exists $j = j(\epsilon, g_\epsilon) \in \mathbf{N}$ so that

$$\|\widetilde{\delta_j \mathbf{x}} - \delta \mathbf{x}\|_{L^q} \le \frac{\epsilon}{3\|g_\epsilon\|_{L^p}} \tag{20}$$

and define derivations with the same convex combinations:

$$\widetilde{\delta_j} := \sum_{l=j}^{N_j} \lambda_{jl} \delta_l.$$

We claim $\{\widetilde{\delta_j}\}_j$ converges to δ in the weak-star operator topology, too. To see why, by definition for each $\varphi \in L^1(\mathbf{R}^m, \mu)$, each $\psi \in \mathrm{Lip}_b(\mathbf{R}^m)$, and each $s > 0$, there exists $k \in \mathbf{N}$ so that the original convergence yields

$$\Big| \int_{\mathbf{R}^m} \varphi \, (\delta_j - \delta) \psi \, d\mu \Big| < s$$

for all $j \ge k$; so for $j \in \mathbf{N} \cap [k, N_k]$ the previous estimate yields

$$\Big| \int_{\mathbf{R}^m} \varphi \, (\widetilde{\delta_j} - \delta) \psi \, d\mu \Big| \le \sum_{l=j}^{N_j} \lambda_{jl} \Big| \int_{\mathbf{R}^m} \varphi \, (\delta_l - \delta) \psi \, d\mu \Big| \ < \ \sum_{l=j}^{N_j} \lambda_{jl} s = s.$$

Fixing j as in (20), observe that $\widetilde{\delta_j}$ is a (finite) linear combination in $\Upsilon(\mathbf{R}^m, \mu)$, so $\widetilde{\delta_j} \in \Upsilon(\mathbf{R}^m, \mu)$ and hence $\widetilde{\delta_j} f_i \overset{\star}{\rightharpoonup} 0$ in $L^\infty(\mathbf{R}^m, \mu)$. Testing further with $g_\epsilon \in L^1(\mathbf{R}^m, \mu)$ there exists $i_0 = i_0(j, \epsilon, g_\epsilon) \in \mathbf{N}$ so that

$$\left| \int_{\mathbf{R}^m} g_\epsilon \widetilde{\delta_j} f_i \, d\mu \right| < \frac{\epsilon}{3} \tag{21}$$

whenever $i \ge i_0$. The chain rule (Lemma 15) implies that there exists $\mathbf{v}^i = \mathbf{v}^{f_i} \in L^\infty(\mathbf{R}^m; \mathbf{R}^m, \mu)$ with $|\mathbf{v}^i| \le 1$ μ-a.e. on $\mathbf{R}^m$ such that

$$\widetilde{\delta_j} f_i = \mathbf{v}^i \cdot \widetilde{\delta_j} \mathbf{x}.$$

So from this and the convergence $\widetilde{\delta_j} \overset{\star}{\rightharpoonup} \delta$, we have on the one hand that

$$\begin{aligned} \int_{\mathbf{R}^m} g_\epsilon \widetilde{\delta_j} f_i \, d\mu = \int_{\mathbf{R}^m} g_\epsilon \mathbf{v}^i \cdot \widetilde{\delta_j} \mathbf{x} \, d\mu &= \sum_{k=1}^m \int_{\mathbf{R}^m} g_\epsilon v_k^i \widetilde{\delta_j} x_k \, d\mu \\ &\longrightarrow \sum_{k=1}^m \int_{\mathbf{R}^m} g_\epsilon v_k^i \delta_\mu x_k \, d\mu = \int_{\mathbf{R}^m} g_\epsilon \mathbf{v}^i \cdot \delta \mathbf{x} \, d\mu \end{aligned}$$

and on the other hand that

$$\int_{\mathbf{R}^m} g_\epsilon \widetilde{\delta_j} f_i \, d\mu \longrightarrow \int_{\mathbf{R}^m} g_\epsilon \delta f_i \, d\mu.$$

Since weak-star limits are unique, it follows that

$$\delta f_i = \mathbf{v}^i \cdot \delta \mathbf{x},$$
$$(\delta - \widetilde{\delta}_j) f_i = \mathbf{v}^i \cdot (\delta - \widetilde{\delta}_j)\mathbf{x}$$

and using Hölder's inequality and previous estimates, it further follows that

$$\begin{aligned}\left|\int_{\mathbf{R}^m} g\,\delta f_i\,d\mu\right| &\le \|g - g_\epsilon\|_{L^1}\|\delta f_i\|_{L^\infty} + \left|\int_{\mathbf{R}^m} g_\epsilon\,\delta f_i\,d\mu\right| \\ &\overset{(18)}{\le} \frac{\epsilon}{3} + \left|\int_{\mathbf{R}^m} g_\epsilon(\delta - \widetilde{\delta}_j) f_i\,d\mu\right| + \left|\int_{\mathbf{R}^m} g_\epsilon\,\widetilde{\delta}_j f_i\,d\mu\right| \\ &\overset{(21)}{\le} \frac{\epsilon}{3} + \left|\int_{\mathbf{R}^m} g_\epsilon\,\mathbf{v}^i \cdot (\delta - \widetilde{\delta}_j)\mathbf{x}\,d\mu\right| + \frac{\epsilon}{3} \\ &\le \frac{2\epsilon}{3} + \|g_\epsilon\|_{L^p}\|\mathbf{v}^i\|_{L^\infty}\|(\delta - \widetilde{\delta}_j)\mathbf{x}\|_{L^q} \overset{(20)}{\le} \epsilon.\end{aligned}$$

Since ϵ and g were arbitrary, it follows that $\delta \in \Upsilon(\mathbf{R}^m, \mu)$.

As for the general case, since $g \in L^1(\mathbf{R}^m, \mu)$ there exists $R > 0$ so that

$$\int_{\mathbf{R}^m \setminus B(0,R)} |g|\,d\mu \le \frac{\epsilon}{2\|\delta\|_{\mathrm{op}}}.$$

By Lemma 11 the previous case applies to $\mu_{|B(0,R)}$ with $\frac{\epsilon}{2}$ in place of ϵ, so there exists $i_0 = i_0(j, \epsilon, (g_{|B(0,R)})_\epsilon) \in \mathbf{N}$ so that if $i \ge i_0$ then

$$\left|\int_{B(0,R)} g\,\delta f_i\,d\mu\right| \le \frac{\epsilon}{2},$$
$$\left|\int_{\mathbf{R}^m} g\,\delta f_i\,d\mu\right| \le \|\delta\|_{\mathrm{op}} \int_{\mathbf{R}^m \setminus B(0,R)} |g|\,d\mu + \left|\int_{B(0,R)} g\,\delta f_i\,d\mu\right| \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

as desired. □

The final lemma is straightforward but not easily found in the literature; for completeness, a proof sketch is included below.

Lemma 22. *Let μ be Radon on $\mathbf{R}^m$ and let $\mathbf{d} = \{\delta_i\}_{i=1}^m$ be a subset of $\Upsilon(\mathbf{R}^m, \mu)$. If the Jacobi-type matrix*

$$\mathbf{dx}(z) := \begin{bmatrix} \delta_1 x_1(z) & \delta_2 x_1(z) & \cdots & \delta_n x_1(z) \\ \delta_1 x_2(z) & \delta_2 x_2(z) & \cdots & \delta_n x_2(z) \\ \vdots & \vdots & \ddots & \vdots \\ \delta_1 x_m(z) & \delta_2 x_m(z) & \cdots & \delta_m x_m(z) \end{bmatrix} \tag{23}$$

is invertible for μ-a.e. $z \in \mathbf{R}^m$, then $\mathbf{d}$ is linearly independent.

Proof. We argue by contraposition. For $m = 1$ this follows from the chain rule above (Lemma 15); indeed, if the singleton $\{\delta_1\}$ were linearly dependent in $\Upsilon(\mathbf{R}, \mu)$, then there would exist a nonzero $\lambda \in L^\infty(\mathbf{R}, \mu)$ so that

$$\lambda(z)\delta_1 x_1(z) \;=\; 0$$

holds for μ-a.e. $z \in \mathbf{R}$. In particular, $\delta_1 x_1 = 0$ holds on the (μ-essential) support of λ and hence the 1×1 matrix $[\delta_1 x_1]$ would be non-invertible on $\mathrm{supp}(\lambda)$, which is a positive μ-measured subset.

For $m = 2$, if $\mathbf{d}$ were linearly dependent, there would exist $\lambda_1, \lambda_2 \in L^\infty(\mathbf{R}^2, \mu)$ not both zero (and without loss $\lambda_2 \neq 0$ μ-a.e.) so that

$$\delta_2 = -\frac{\lambda_1}{\lambda_2}\delta_1 \tag{24}$$

holds μ-a.e. on $\mathbf{R}^2$. As a result, the Jacobi matrix $\mathbf{dx}$ becomes

$$\mathbf{dx} := \det \begin{bmatrix} \delta_1 x_1 & \delta_2 x_1 \\ \delta_1 x_2 & \delta_2 x_2 \end{bmatrix} = \begin{bmatrix} \delta_1 x_1 & -\frac{\lambda_1}{\lambda_2}\delta_1 x_1 \\ \delta_1 x_2 & -\frac{\lambda_1}{\lambda_2}\delta_1 x_2 \end{bmatrix}$$

which clearly has zero determinant.

For $m \in \mathbf{N}$, an identity analogous to (24) holds, where δ_m can be written as a linear combination of $\delta_1, \cdots, \delta_{m-1}$ for some choice of scalars $\lambda_1, \cdots, \lambda_{m-1}$. The subsequent $m \times m$ Jacobi matrix will contain a column that is a linear combination of the other $m-1$ columns, which gives the lemma. □

The following theorem, regarding rigidity of derivations on Euclidean spaces, is a consequence of the main results from [5] and [15]. More precisely, in [5] the conclusion of absolute continuity was proven in the case of *independent collections of Alberti representations*, whereas in [15], it is shown that every Alberti representation determines a derivation in the previous sense, and independence induces linear independence. The case of $\mathbf{R}^2$ was treated in [6].

Theorem 25. *Let $m \in \mathbf{N}$ and let μ be a Radon measure on $(\mathbf{R}^m, |\cdot|)$. Then the module of derivations on $\mathbf{R}^m$ with respect to μ has rank-m if and only if μ is absolutely continuous with respect to the Lebesgue measure. Moreover, derivations with respect to μ are linear combinations of the differential operators $\{\partial/\partial x_i\}_{i=1}^m$ with coefficients in $L^\infty(\mathbf{R}^m, \mu)$.*

3. Differentiably trivial carpets

3.1. Previous results on carpets. For $c \in \mathbf{N}$ odd denote by $S_c = \mathbf{S}^2_{\mathbf{a}}$ the self-similar Sierpiński carpet defined by the constant sequence $\mathbf{a} := \left(\frac{1}{c}, \frac{1}{c}, \frac{1}{c}, \ldots\right)$. Note that S_3 is the standard Sierpiński carpet.

In [16, Theorem 40], Weaver proved that $\Upsilon(S_3, \mu) = 0$ and the same argument applies to any self-similar Sierpiński carpet with respect to a constant sequence. Moreover, the argument can be extended to any sequence $\mathbf{a} \in \ell^\infty \setminus c_0$ and to associated Sierpiński sponges in any dimension m, in that $\limsup \mathbf{a} > 0$ implies $\Upsilon(\mathbf{S_a}, \mu) = 0$.

On the other hand, in [12], the authors considered the class of non-self similar Sierpiński carpets in the particular case when $p_n = 1$ for each $n \in \mathbf{N}$. They prove that the class of non-self similar Sierpiński carpets $\mathbf{S_a}$ support Poincaré inequalities if and only if $\mathbf{a} \in \ell^2$. One can check that if $\mathbf{a} \in \ell^2$, the measure μ is comparable to the restriction of the Lebesgue measure to $\mathbf{S_a}$. By Theorem 25, $(\mathbf{S_a}, \mu)$ induces a rank-2 module of derivations, so $\mathbf{S_a}$ is a Lipschitz differentiability space.

Actually, the associated measurable differentiable structure is the restriction of the standard differentiable structure from $\mathbf{R}^2$.

The next theorem is new and is the main result of this section. It covers the remaining case, that is, when $\mathbf{a} \in c_0 \setminus \ell^2$, thereby covering the full range of possible sequences in ℓ^p, $1 \le p \le \infty$.

Theorem 26. *If $\mathbf{S_a}$ is a carpet with $\mathbf{a} \in \ell^\infty \setminus \ell^2$ and with the canonical measure μ as in Remark 6.C, then $(\mathbf{S_a}, \mu)$ is differentiably trivial.*

In what follows, we will prove in fact a more general result, Theorem 39, in an arbitrary dimension $m \geq 2$, for Sierpiński sponges $\mathbf{S}^m_{\mathbf{a}}$ as defined in Remark 7.

The proof will be divided into three steps:

- Any nonzero derivation $\delta \in \Upsilon(\mathbf{R}^m, \mu)$ induces a derivation $\delta_\mu \in \Upsilon(\mathbf{R}^m, \mu)$ that is supported everywhere, in the sense that $\delta\mathbf{x}$ is μ-a.e. nonzero. See Subsection 3.2.
- For any derivation that is supported everywhere, there is a full set of derivations that are linearly independent to it. See Subsection 3.3.
- If μ has rank-m then μ is absolutely continuous with respect to the (m-dimensional) Lebesgue measure. See Theorem 25.

3.2. If one derivation, then one everywhere.

Theorem 27. *Let $\mathbf{S}^m_{\mathbf{a}}$ be a Sierpiński sponge in $\mathbf{R}^m$ with $\mathbf{a} \in \ell^\infty \setminus \ell^m$ and let $\mu = \mu^m_{\mathbf{a}}$ be the canonical measure. If $\Upsilon(\mathbf{R}^m, \mu) \neq 0$, then there exists $\delta_\mu \in \Upsilon(\mathbf{R}^m, \mu)$ so that the vectorfield $\delta_\mu \mathbf{x} = (\delta_\mu x_1, \delta_\mu x_2, \ldots, \delta_\mu x_m)$ is nonzero (and in fact constant) μ-a.e. on $\mathbf{R}^m$.*

The proof proceeds in two steps: (1) finding a candidate for δ_μ, and then (2) checking nondegeneracy.

Proof. Fix a nonzero derivation $\delta \in \Upsilon(\mathbf{R}^m, \mu)$ with $\|\delta\|_{\mathrm{op}} \leq 1$. Observe that since $\Upsilon(\mathbf{R}^m, \mu) \neq 0$, the chain rule (Lemma 15) implies that there exists $i \in \{1, \ldots, m\}$ such that $\|\delta x_i\|_{L^\infty} > 0$.

Moreover, the set $\partial^+ \mathbf{S}^m_{\mathbf{a}}$ is a countable union of $(m-1)$-dimensional cubes parallel to coordinate hyperplanes, so from Remark 6.C.2 it follows that $\mu(\partial^+ \mathbf{S}^m_{\mathbf{a}}) = 0$. It therefore suffices to prove the theorem for μ-a.e. point in $\mathbf{S}^m_{\mathbf{a}} \setminus \partial^+ \mathbf{S}^m_{\mathbf{a}}$ instead.

Step 1: A candidate operator. Let $x_0 \in \mathbf{S}^m_{\mathbf{a}} \setminus \partial^+ \mathbf{S}^m_{\mathbf{a}}$ be a point of μ-density for $\delta\mathbf{x}$ with

$$\delta\mathbf{x}(x_0) \;=\; (\delta x_1(x_0), \delta x_2(x_0), \ldots, \delta x_n(x_0)) \;\neq\; \mathbf{0}$$

and as given in Remark 6.A, let $(\mathcal{N}^n_0)_{n=1}^\infty$ be the unique sequence of (closed) subsquare neighborhoods satisfying $x_0 \in \mathcal{N}^n_0 \in \mathcal{C}^+_n$ for all $n \in \mathbf{N}$.

For each $Q \in \mathcal{C}^+_n$, let $\tau^{nQ} \colon \mathbf{R}^m \to \mathbf{R}^m$ denote the unique translation that maps $\mathcal{N}^n_0$ isometrically onto Q and consider the sequence of operators

$$\delta_{nQ} := \tau^{nQ}_{\#}(\chi_{\mathcal{N}^n_0} \delta)$$

as well as the derivations $\delta_n \in \Upsilon(\mathbf{R}^m, \mu)$ defined by the action

$$\delta_n f(x) := \sum_{Q \in \mathcal{C}^+_n} \delta_{nQ} f\Big((\tau^{nQ})^{-1}(x)\Big). \tag{28}$$

Notice that each $f \in \mathrm{Lip}(\mathbf{S}^m_{\mathbf{a}})$ can be expressed as

$$f = \sum_{Q \in \mathcal{C}^+_n} \chi_{Q \cap \mathbf{S}^m_{\mathbf{a}}} f_{|Q},$$

and by the locality property, the action of δ_n gives

$$\delta_n f \;=\; \sum_{Q \in \mathcal{C}^+_n} \tau^{nQ}_{\#}(\chi_{\mathcal{N}^n_0} \delta)(f_{|Q}).$$

Since τ^{nQ} is 1-biLipschitz, the change of variables formula (Corollary 16) implies for μ-a.e. $y \in Q$ with

$$x = (\tau^{nQ})^{-1}(y)$$

and for all $f \in \mathrm{Lip}(\mathbf{S}^m_{\mathbf{a}})$ with $\|f\|_{\mathrm{Lip}} \leq 1$ that

$$\begin{aligned}\delta_n f(y) &= \sum_{Q\in\mathcal{C}_n^+} \tau_{\#}^{nQ}(\chi_{\mathcal{N}^n}\delta) f(x) = \sum_{Q\in\mathcal{C}_n^+} \chi_{\mathcal{N}^n}(x)\,\delta(f\circ\tau^{nQ})(x)\\ &= \sum_{Q\in\mathcal{C}_n^+} \chi_Q(y)\,\mathbf{v}^f(\tau^{nQ}(x))\cdot D\tau^{nQ}(x)\cdot\delta\mathbf{x}(x) = \mathbf{v}^f\cdot\delta\mathbf{x}\Big((\tau^{nQ})^{-1}(y)\Big),\end{aligned}$$

and moreover

$$\|\delta_n f\|_{L^\infty} \leq \|\mathbf{v}^f(y)\|_{L^\infty}\|\delta\mathbf{x}\|_{L^\infty} \leq L(f)\|\delta\mathbf{x}\|_{L^\infty},$$

so $(\delta_n)_{n=1}^\infty$ is bounded in $\Upsilon(\mathbf{R}^m,\mu)$ with $\|\delta_n\|_{\mathrm{op}} \leq \|\delta\|_{\mathrm{op}}$. From Lemma 3 there exists a subnet $(\delta_{n_\alpha})_{\alpha\in\Lambda}$ that converges to an operator

$$\delta_\mu \in \mathcal{L}(\mathrm{Lip}_b(\mathbf{S}^m_{\mathbf{a}}), L^\infty(\mathbf{R}^m,\mu)) \tag{29}$$

in the weak-star operator topology, that is, in the sense that

$$\int_{\mathbf{R}^m} \varphi\,\delta_{n_\alpha} f\,d\mu \;\to\; \int_{\mathbf{R}^m}\varphi\,\delta_\mu f\,d\mu$$

holds for all $\varphi \in L^1(\mathbf{R}^m,\mu)$ and all $f\in\mathrm{Lip}_b(\mathbf{S}^m_{\mathbf{a}})$. It therefore follows from Theorem 17 that $\delta_\mu \in \Upsilon(\mathbf{R}^m,\mu)$.

Step 2: Nondegeneracy. Lastly, for each coordinate direction $i \in \{1,2,\dots,m\}$ and for any μ-density point $x \in \mathbf{S}^m_{\mathbf{a}}\setminus\partial^+\mathbf{S}^m_{\mathbf{a}}$ of $\delta_\mu\mathbf{x}$, let $(\mathcal{N}^n_x)_n$ be the sequence of subsquare neighborhoods of x as in Remark 6.A, and the Lebesgue differentiation theorem for Radon measures on $\mathbf{R}^m$ guarantees an index $n\in\mathbf{N}$ so that

$$\left|\delta_\mu x_i(x) - \fint_{\mathcal{N}^n_x}\delta_\mu x_i\,d\mu\right| < \frac{\epsilon}{3} \quad\text{and}\quad \left|\delta x_i(x_0) - \fint_{\mathcal{N}^n_0}\delta x_i\,d\mu\right| < \frac{\epsilon}{3} \tag{30}$$

holds for $i=1,2,\dots,m$. Since μ is compactly supported, it follows that $\mu(\mathcal{N}^n_x)^{-1}\chi_{\mathcal{N}^n_x} \in L^1(\mathbf{R}^m,\mu)$, so with convex combinations $\widetilde{\delta_{n_j}}$ chosen as in Equation (19), there exists $j = j(n,\epsilon)\in\mathbf{N}$ satisfying

$$\left|\fint_{\mathcal{N}^n_x}(\delta_\mu - \widetilde{\delta_{n_j}})x_i\,d\mu\right| < \frac{\epsilon}{3}. \tag{31}$$

So by the previous estimates and by Corollary 16, we therefore obtain

$$\begin{aligned}|\delta_\mu x_i(x) - \delta x_i(x_0)| &\overset{(30)}{\leq} \left|\fint_{\mathcal{N}^n_x}(\delta_\mu x_i - \delta x_i(x_0))\,d\mu\right| + \frac{\epsilon}{3}\\ &\overset{(31)}{\leq} \left|\fint_{\mathcal{N}^n_x}(\widetilde{\delta_{n_j}}x_i - \delta x_i(x_0))\,d\mu\right| + \frac{2\epsilon}{3}\\ &\overset{(19)}{=} \left|\sum_l \fint_{\mathcal{N}^n_x}\lambda_{n_jl}\big(\delta_{n_l}x_i(z) - \delta x_i(x_0)\big)\,d\mu(z)\right| + \frac{2\epsilon}{3}\\ &\overset{(28)}{=} \left|\sum_l \lambda_{n_jl}\fint_{\mathcal{N}^n_x}\big(\tau_{\#}^{n_l\mathcal{N}^x_n}\delta)x_i((\tau^{n_l\mathcal{N}^x_n})^{-1}(z)) - \delta x_i(x_0)\big)\,d\mu(z)\right| + \frac{2\epsilon}{3}\\ &\overset{\text{Lemma 14}}{=} \left|\sum_l\lambda_{n_jl}\fint_{\mathcal{N}^n_0}(\delta x_i - \delta x_i(x_0))\,d\mu\right| + \frac{2\epsilon}{3}\\ &= \left|\fint_{\mathcal{N}^n_0}(\delta x_i - \delta x_i(x_0))\,d\mu\right| + \frac{2\epsilon}{3} \;\leq\; \epsilon.\end{aligned}$$

As $\epsilon > 0$ was arbitrary, it follows that $\delta_\mu \mathbf{x}$ is μ-a.e. constant, where

$$\delta_\mu \mathbf{x} = \delta \mathbf{x}(x_0). \qquad \square$$

Remark 32. We summarise the proof with the following observation: To construct δ_μ from a density point x_0 of μ, it suffices that at every scale l and by enumerating the lth order non-middle subsquares of $\mathbf{S}^m_{\mathbf{a}}$ as $C^+_l = \{Q_k\}_{k\in\mathbf{N}}$, there is a partition of $\mathbf{S}^m_{\mathbf{a}}$ into subsets

$$E^k_n := \tau^{nQ_k}(\mathcal{N}^n_0) \cap \mathbf{S}^m_{\mathbf{a}}$$

with τ^{nQ_k} as in Step 1 of the above proof and with the following property: the Lebesgue differentiation theorem holds true at x_0 for the sequence of sets $E_n = \mathcal{N}^n_0$, as $n \to \infty$. (See Theorem 8.)

This motivates the definition of a Sierpiński-type fractal in the sequel.

3.3. If one derivation everywhere, then a full set everywhere. We begin with a geometric fact about the canonical measure μ from Remark 6.C.

Lemma 33. *For all $\mathbf{a} \in \ell^\infty$, the identity $(T \circ \theta \circ T^{-1})_\# \mu^m_{\mathbf{a}} = \mu^m_{\mathbf{a}}$ holds for all Borel sets in $\mathbf{R}^m$, where T is the translation*

$$T(x_1, x_2, \ldots, x_m) = \left(x_1 + \frac{1}{2}, x_2 + \frac{1}{2}, \ldots, x_m + \frac{1}{2}\right)$$

and θ is either one of the reflections $R^{i,j}$ or S^i about hyperplanes $(\mathbf{e}_i - \mathbf{e}_j)^\perp$ or $x_i = 0$, respectively, for $i, j \in \{1, 2, \ldots, m\}$ with $i \neq j$; equivalently these isometries are defined by the following conditions:

$$R^{i,j}(\mathbf{e}_k) = \begin{cases} \mathbf{e}_j, & \text{if } k = i, \\ \mathbf{e}_i, & \text{if } k = j, \\ \mathbf{e}_k, & \text{if } k \neq i, j, \end{cases} \quad \text{or} \quad S^i(\mathbf{e}_k) = \begin{cases} -\mathbf{e}_i, & \text{if } k = i, \\ \mathbf{e}_i, & \text{if } k \neq i. \end{cases} \tag{34}$$

Proof. By definition, for each step n of the construction in §2.1 the identity

$$(T \circ \theta \circ T^{-1})_\# \mu_n(Q) = \mu_n(Q)$$

holds for all $Q \in \mathcal{C}^+_n$ and all $n \in \mathbf{N}$. If O is an open set in $\mathbf{R}^m$, then let $\epsilon > 0$ be given and take a cover $\mathcal{C}$ of $O \cap [0,1]^m$ by cubes in $\bigcup_n \mathcal{C}^+_n$ with pairwise-disjoint interiors and so that

$$\mu\Big(O \setminus \bigcup_{Q\in\mathcal{C}} Q\Big) \; < \; \epsilon.$$

Since $(T \circ \theta^{-1} \circ T^{-1})(Q) \in \mathcal{C}^+_n$ holds whenever $Q \in \mathcal{C}^+_n$, the desired identity holds true for all open sets O as $\epsilon \to 0$. The lemma then follows from Borel regularity of μ. $\square$

Theorem 35. *If $\mathbf{a} \in \ell^\infty$ and $\Upsilon(\mathbf{R}^m, \mu^m_{\mathbf{a}}) \neq 0$, then $\Upsilon(\mathbf{R}^m, \mu^m_{\mathbf{a}})$ has rank-m.*

Proof. Put $\mu = \mu^m_{\mathbf{a}}$ and let $\delta = \delta_\mu$ be as in Theorem 27, so $\mathbf{v} := \delta\mathbf{x}$ is constant and nonzero μ-a.e. on $\mathbf{R}^m$. By the chain rule (Lemma 15) there exists $j \in \{1, 2, \ldots, m\}$ so that $\delta x_j \neq 0$ μ-a.e. as well. By means of pushforwards of δ by reflections $R^{k,l}$, we may assume there exists $p \in \{1, 2, \ldots, m\}$ so that $\delta x_j \neq 0$ whenever $j \leq p$ and $\delta x_j = 0$ whenever $j > p$. (In particular, $\delta x_1 \neq 0$.)

Denote the identity map on $\mathbf{R}^m$ by $\mathbf{x}$ and define isometries $\theta^i = (\theta^i_1, \ldots, \theta^i_m)\colon \mathbf{R}^m \to \mathbf{R}^m$ of a non-self-similar Sierpiński carpet $\mathbf{S}^m_{\mathbf{a}}$ as follows:

$$\theta^i := \begin{cases} \mathbf{x}, & \text{if } i = 1, \\ S^i \circ \theta^{i-1}, & \text{if } 2 \le i \le p \\ R^{1,i}, & \text{if } p < i \le m; \end{cases} \tag{36}$$

For example, if $p = 3$ then in $\mathbf{R}^4$ we have

$$\theta^4\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_4 \\ x_2 \\ x_3 \\ x_1 \end{bmatrix} \quad \text{and} \quad \theta^3\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \theta^2\left(\begin{bmatrix} x_1 \\ x_2 \\ -x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ -x_2 \\ -x_3 \\ x_4 \end{bmatrix}.$$

Moreover, for each $i = 2, \cdots, m$ put

$$\Theta^i := T \circ \theta^i \circ T^{-1} \quad \text{and} \quad \delta_i := \Theta^i_{\#}\delta \tag{37}$$

where T is as in Lemma 33. So by applying that lemma as well as a change of variables (Lemma 14) each $\varphi \in C_c(\mathbf{R}^m)$ satisfies

$$\begin{aligned} \int_{\mathbf{R}^m} \varphi\, \delta_i x_j \, d\mu &= \int_{\mathbf{R}^m} \varphi(\Theta^i_{\#}\delta) x_j \, d(\Theta^i_{\#}\mu) = \int_{\mathbf{R}^m} (\varphi \circ \Theta^i)\delta(x_j \circ \Theta^i)\, d\mu \\ &= \int_{\mathbf{R}^m} \varphi(\delta(x_j \circ \Theta^i) \circ (\Theta^i)^{-1})\, d\mu \end{aligned}$$

in which case it holds μ-a.e. on $\mathbf{R}^m$ that

$$\begin{aligned} \delta_i x_j = \delta(x_j \circ \Theta^i) \circ (\Theta^i)^{-1} = \big((\nabla x_j \circ \Theta^i)^T D\Theta^i \delta \mathbf{x}\big) \circ (\Theta^i)^{-1} = \mathbf{e}_j^T D\Theta^i(\delta\mathbf{x} \circ (\Theta^i)^{-1}), \\ \text{so} \quad \delta_i \mathbf{x} = D\Theta^i \left(\delta\mathbf{x} \circ (\Theta^i)^{-1}\right). \end{aligned}$$

By Theorem 27 once again, it holds that

$$\mathbf{w} = \delta\mathbf{x} \circ (\Theta^1)^{-1} = \ldots = \delta\mathbf{x} \circ (\Theta^m)^{-1}$$

is constant μ-a.e. on $\mathbf{R}^m$, in which case it further holds that

$$\begin{aligned} \det \mathbf{dx} &= \det \left[D\Theta^1\mathbf{w} | D\Theta^2\mathbf{w} | \cdots | D\Theta^p\mathbf{w} | D\Theta^{p+1}\mathbf{w} | \cdots | D\Theta^m\mathbf{w} \right] \\ &= \det \left[\begin{array}{r|r|c|r|r|c|r} \delta x_1 & \delta x_1 & \cdots & \delta x_1 & 0 & \cdots & 0 \\ \delta x_2 & -\delta x_2 & \cdots & -\delta x_2 & \delta x_2 & \cdots & \delta x_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \delta x_p & \delta x_p & \cdots & -\delta x_p & \delta x_p & \cdots & \delta x_p \\ 0 & 0 & \cdots & 0 & \delta x_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & \delta x_1 \end{array}\right] \end{aligned} \tag{38}$$

$$
= (\delta x_1)^{m-p} \left(\prod_{i=1}^{p} \delta x_i \right) \det \left[\begin{array}{c|c|c|c|c|c|c} 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & -1 & \cdots & -1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & -1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \end{array} \right]
$$

$$
= -(-2)^{p-1} (\delta x_1)^{m-p} \left(\prod_{i=1}^{p} \delta x_i \right)
$$

is also constant and nonzero μ-a.e. on $\mathbf{R}^m$. By Lemma 22, it follows that $\{\delta_i\}_{i=1}^m$ is a linearly independent set in $\Upsilon(\mathbf{R}^m, \mu)$. □

The next result includes Theorem 26 as a special case.

Theorem 39. *If $\mathbf{S}_{\mathbf{a}}^m$ is a (Sierpiński) sponge with $\mathbf{a} \in \ell^\infty \setminus \ell^m$ and with the canonical measure $\mu_{\mathbf{a}}^m$ as in Remark 7, then $(\mathbf{S}_{\mathbf{a}}^m, \mu_{\mathbf{a}}^m)$ is differentiably trivial.*

Proof. Assume by contradiction that $\Upsilon(\mathbf{R}^m, \mu_{\mathbf{a}}^m) \neq 0$ and put $\mu = \mu_{\mathbf{a}}^m$. By Theorem 27, there exists $\delta_\mu \in \Upsilon(\mathbf{R}^m, \mu)$ so that the vectorfield $\delta_\mu \mathbf{x}$ is nonzero μ-a.e. on $\mathbf{R}^m$, which means that $\Upsilon(\mathbf{R}^m, \mu)$ has rank-m, by Theorem 35.

Theorem 25 now applies, so μ is absolutely continuous to the Lebesgue measure which yields a contradiction. (Indeed, $\mathbf{S}_{\mathbf{a}}^m$ has zero Lebesgue measure whereas $\mu(\mathbf{S}_{\mathbf{a}}^m) > 0$). □

4. Sierpiński-type fractals

A careful look to the proof of Theorem 26 reveals that one can actually get the same result for a larger class of fractals, beyond carpets and sponges. The proof also works for subsets $X \subset \mathbf{R}^m$ endowed with the restriction of the Euclidean metric and a non-zero Radon measure μ with the following geometric properties:

(S0) The set X has m-dimensional Lebesgue *measure zero*.

(S1) μ is supported on X and μ is *doubling* on X.

(S2) *Tile partitions at all scales*: There is a collection of subsets $\{E_n\}_{n=1}^\infty$ of X so that

$$
E_{n+1} \subset E_n \quad \text{for each } n \in \mathbf{N} \quad \text{and} \quad \lim_{n\to\infty} \operatorname{diam}(E_n) = 0
$$

both hold, as well as a finite collection of isometries τ^{nk} of $\mathbf{R}^m$, for $k \in \mathbf{N}$, so that the sets $E_n^k := \tau^{nk}(E_n)$, called *nth-order tiles of* X, satisfy

$$
\mu(E_n^k \cap E_n^l) = 0 \quad \text{and} \quad \mu(E_n^k) = \mu(E_n^l)
$$

whenever $l \neq k$, as well as

$$
X \;=\; \bigcup_k \tau^{nk}(E_n).
$$

Furthermore, there exists $c \geq 1$ such that if $x \in \bigcap_n E_n$ then

$$
\mu(B(x, \operatorname{diam}(E_n))) \leq c\mu(E_n). \tag{40}
$$

As an example, for the non-self similar Sierpiński carpets $\mathbf{S}^2_{\mathbf{a}}$ from §2.1 and for $x_0 \in \mathbf{S}^2_{\mathbf{a}}$, it suffices to choose the closed square neighborhoods of x_0 as tiles, i.e. $E_n := \mathbf{S}^2_{\mathbf{a}} \cap \mathcal{N}^n_0$, with translations as isometries $\tau^{nk} := \tau^{nQ_k}$, for an enumeration of squares $\mathcal{C}^+_n = \{Q_k\}_{k=1}^\infty$ as in Remark 32.

(S3) *Isometric invariance for tiles*: For each $n \in \mathbf{N}$ and for $i = 2, \cdots, m$, there exist isometries Θ^i_n of $\mathbf{R}^m$ with the following properties: with the tiles $(E_n)_{n=1}^\infty$ of X as before, for each $n \in \mathbf{N}$ we have

$$\Theta^i_n(E_n) = E_n$$

and, for some constant $C > 0$ independent of n, that the $m \times m$ matrix inequality also holds, just as in (38): for every $v \in \mathbf{R}^m$, it holds that

$$\left| \det \left[v | D\Theta^2_n v | \cdots | D\Theta^m_n v \right] \right| \geq C.$$

Once again, for $\mathbf{S}^m_{\mathbf{a}}$ the compositions of translations and reflections from Equations (34)–(37) give an example of such isometries $\Theta^i_n := \Theta^i$ as above.

Remark 41. By combining (S2) and (S3) it follows that every n-th order tile E^l_n also enjoys a generalized rotational invariance:

$$\left(\tau^{nk} \circ \Theta^i_n \circ (\tau^{nk})^{-1}\right)(E^k_n) = E^k_n. \tag{42}$$

Definition 43. A subset $X = (X, |\cdot|)$ in $\mathbf{R}^m$ equipped with a (non-zero) Radon measure μ is called a *Sierpiński-type fractal* if it enjoys the preceding conditions (S0)–(S3).

4.1. Examples, old and new. As previously announced in the Introduction (§1.2), examples of Sierpiński-type fractals include earlier well-known constructions, such as

- self-similar fractals, such as the standard Sierpiński carpet and gasket, Menger sponges (or *m-dimensional Menger continua* $M(m,1)$), Cantor dust $M(m,0)$, Sierpiński sponges $M(m,m-1)$, etc;
- their non-self-similar counterparts, such as the carpets $\mathbf{S}_{\mathbf{a}}$ from before, with $\mathbf{a} \in c_0 \setminus \ell^2$. Cantor sets in $\mathbf{R}$ with ratios $\mathbf{a} \in \ell^\infty \setminus \ell^1$, sponges in $\mathbf{R}^3$ with $\mathbf{a} \in \ell^\infty \setminus \ell^3$, or constructions in other dimensions are similarly defined as in §2.1.

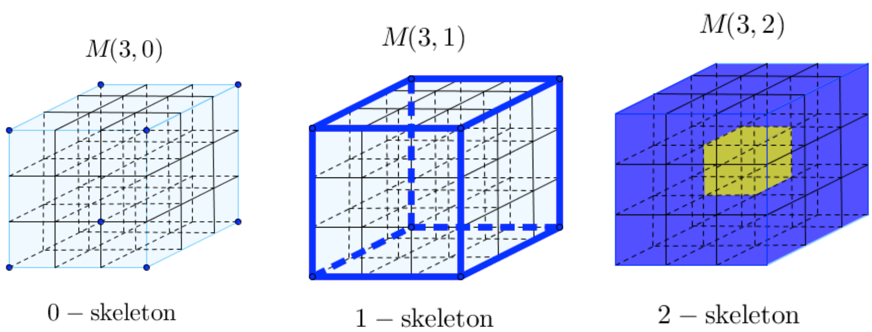

Figure 2. The 0-skeleton of $M_0 = Q^3$ is made up of 8 points so M^0_1 consists of the 8 subcubes containing these points. Iterating this construction on each subcube, we obtain the 3-dimensional Cantor dust $M(3,0)$. The 1-skeleton of M_0 consists of 12 edges so M^1_1 is made up of 12 subcubes. The iterative construction leads to the Menger sponge $M(3,1)$. The 2-skeleton of M_0 would be 6 square faces and M^2_1 would consist of 26 subcubes (*i.e. everything except the central subcube*). This construction yields the Sierpiński sponge $M(3,2)$.

For the sake of clarity, we recall here the construction of the k-dimensional Menger continuum in $\mathbf{R}^m$. Take the m-dimensional unit cube $M_0 = Q^m$ and subdivide it into 3^m congruent subcubes. Let M_1^k be the union of all the subcubes that meet the k-skeleton of M_0. To get M_2^k we repeat the construction on each of the cubes that constitute M_1^k. The *k-dimensional Menger continuum* in $\mathbf{R}^m$ is $M(m,k) = \bigcap_i M_i^k$.

That said, clearly all these metric spaces have a canonically associated doubling measure.

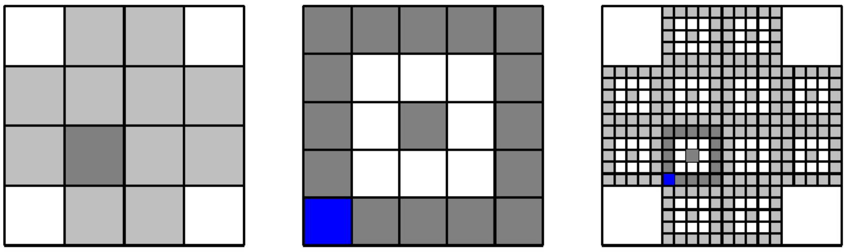

Figure 3. At left, the union X_1 from $E_1 = [\frac{1}{4}, \frac{1}{2}] \times [\frac{1}{4}, \frac{1}{2}]$; in the center, a translated-and-dilated copy of $E_1 \cap X_2$ from $E_2 = [\frac{1}{4}, \frac{3}{10}] \times [\frac{1}{4}, \frac{3}{10}]$, by a dilation factor of 4; at right, the union X_2.

Remark 44. A close look at Definition 43 suggests that more general, even *random*, examples of Sierpiński-type fractals are possible. Indeed, Condition (S3) allows the 'rotations' Θ_n^i to depend on the tile E_n at scale $r_n := \operatorname{diam}(E_n)$—or more accurately, on the union of tiles $X_n := \bigcup_l E_n^l$. Condition (S2) does not require, moreover, that unions X_n and X_{n+1} be geometrically (or even topologically) equivalent.

As one example, consider the following configuration, where at odd-numbered scales, corner subsquares are removed, while at even-numbered scales, square annuli are removed.

Theorem 45. *Sierpiński-type fractals are differentiably trivial.*

Proof. The proof follows the same scheme as that of Theorem 26. We indicate only where conditions (S0)–(S3) play a role. First we prove that if we have one derivation, then we have one everywhere. As done in Theorem 27, by the aid of the partition at all scales provided in (S2), we can "copy and paste" the derivation μ-a.e. on E_n to E_n^l, for any l, which will produce candidate operators as in Equations (28) and (29). Theorem 17 then applies, so the limit operator must be a derivation too.

The key point in order to guarantee the nondegeneracy of this derivation is to be able to apply Lebesgue differentiation theorem at μ-a.e. point. For this purpose, we use the sets $\{E_n\}_{n=1}^{\infty}$ that satisfy property (40) as a neighborhood basis of μ-a.e point.

As done in Subsection 3.3, the next thing to do is to prove that if one derivation exists everywhere, then m of them exist everywhere. In this case we can combine the isometries in (S2) and (S3) to produce m linearly independent derivations in $\Upsilon(\mathbf{R}^m, \mu)$ satisfying (42); see Remark 41.

To finish, because μ enjoys condition (S0), Theorem 25 applies. So if μ has rank-m then μ is absolutely continuous with respect to the (m-dimensional) Lebesgue measure. $\square$

We now indicate the sharpness of each hypothesis from Theorem 45:

(1) *Zero area*: By the classical Rademacher theorem, Lipschitz functions are a.e. differentiable with respect to Lebesgue measure, so the partial derivatives of every Lipschitz function are well-defined a.e. on any positive Lebesgue

measured set $A \subset \mathbf{R}^m$. A variant of Weaver's argument [16] then shows that each partial differential operator determines a derivation on $(A, |\cdot|, \mathcal{H}^m)$.

(2) *Doubling measure*: Let $\{q_1, q_2, \ldots\}$ be an enumeration of $\mathbf{Q}$, and consider a sum of point masses at each rational number: $\nu = \sum_{i=1}^{\infty} \delta_{q_i}$. Let $X = \mathbf{Q} \times [0,1] \cup [0,1] \times \mathbf{Q}$ and let $\mu = \nu \otimes \mathcal{H}^1 + \mathcal{H}^1 \otimes \nu$. Note that μ is not locally finite, hence not doubling, yet the partial differential operators $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ are derivations with respect to μ.

(3) *Tile partitions at all scales*: Consider the middle thirds Cantor set $\mathcal{C}$ in $\mathbf{R}$ endowed with the measure $\mathcal{H}^\alpha$, where $\alpha := \frac{\log 2}{\log 3}$. Put $X = \mathcal{C} \times [0,1]$ and $\mu := \mathcal{H}^\alpha \otimes \mathcal{H}^1$. With the rotation $\Theta(x,y) = (-y, x)$, consider the measure

$$\nu \; := \; \mu + \Theta_{\#}\mu$$

on $Y := X \cup \Theta(X)$. We note that the derivation

$$\delta \; := \; \chi_{\Theta(X)} \frac{\partial}{\partial x} \; + \; \chi_X \frac{\partial}{\partial y}$$

is a nonzero rotationally invariant derivation on $(Y, |\cdot|, \nu)$.

(4) *Isometric invariance for tiles*: Consider the middle thirds Cantor set $\mathcal{C}$ in $\mathbf{R}$, put $X = \mathcal{C} \times [0,1]$ and $\alpha := \frac{\log 2}{\log 3}$ and $\mu := \mathcal{H}^\alpha \otimes \mathcal{H}^1$. Combined with Fubini's theorem, an integration-by-parts argument shows that $\frac{\partial}{\partial y} \in \Upsilon(\mathbf{R}^2, \mu)$.

5. Appendix: Canonical measures on fractals

We now prove item (C.1) of Remark 6:

Proof of uniqueness of μ for $\mathbf{S_a}$. Let μ and μ' be any two sublimits of the sequence of measures $\{\mu_n\}$, with associated sequences of scales $n_k, n'_k \in \mathbf{N}$ so that $q_{n_k} \to 0$ and $q_{n'_k} \to 0$.

Let $m \in \mathbf{N}$ and $Q \in \mathcal{C}_m^+$ be arbitrary. If $k \in \mathbf{N}$ satisfies $\min(n_k, n_{k'}) \geq m$, then by the construction of the measures μ_n, we have

$$\mu_m(Q) = \mu_{m+1}(Q) = \cdots = \mu_{n_k}(Q) \tag{46}$$

and since each $\mu_{n_k} \ll \mathcal{H}^2$, we also have

$$\mu_m(\mathrm{int}(Q)) = \mu_m(Q \setminus \partial Q) = \cdots = \mu_{n_k}(Q \setminus \partial Q) = \mu_{n_k}(\mathrm{int}(Q)).$$

So by semi-continuity of measures, we obtain

$$\mu(Q) \leq \liminf_{k\to\infty} \mu_{n_k}(Q) = \mu_m(Q) = \mu_m(\mathrm{int}(Q)) = \limsup_{k\to\infty} \mu_{n_k}(\mathrm{int}(Q)) \leq \mu(\mathrm{int}(Q))$$

and by Property (C2), it follows that $\mu(Q) = \mu_m(Q)$. Similarly, (46) also holds for $\mu_{n'_k}$, so the same argument gives $\mu'(Q) = \mu_m(Q)$ and hence

$$\mu(Q) = \mu'(Q)$$

holds true for all $Q \in \mathcal{C}_m^+$ and all $m \in \mathbf{N}$.

Note that every open ball B in $\mathbf{R}^2$ is a countable, pairwise-disjoint union of cubes in $\bigcup_{k=1}^{\infty} \mathcal{C}_k$, so the previous identity implies that $\mu(B) = \mu'(B)$. Using the Vitali covering theorem, it is therefore easy to see that $\mu(O) = \mu'(O)$ then holds true for all open sets O in $\mathbf{R}^2$ and therefore all Borel sets.

This shows that all weak-star sublimits of $\{\mu_n\}_{n=1}^{\infty}$ are equal, regardless of the subsequences of scales chosen. Since there always exists at least one sublimit (by weak-star compactness) it follows that $\{\mu_n\}$ has a unique weak-star limit. □

Acknowledgements. The authors would like to thank Prof. Jeremy T. Tyson, Prof. Nageswari Shanmugalingam and Prof. John M. Mackay for fruitful discussions that led to the development of this paper, as well as the anonymous referee for helpful comments that greatly improved this paper.

References

[1] Bate, D.: Structure of measures in Lipschitz differentiability spaces. - J. Amer. Math. Soc. 28:1, 2015, 421–482.

[2] Bate, D., and S. Li: Characterizations of rectifiable metric measure spaces. - Ann. Sci. Éc. Norm. Supér. (4) 50:1, 2017, 1–37.

[3] Cheeger, J.: Differentiability of Lipschitz functions on metric measure spaces. - Geom. Funct. Anal. 9:3, 1999, 428–517.

[4] David, G. C.: Tangents and rectifiability of Ahlfors regular Lipschitz differentiability spaces. - Geom. Funct. Anal. 25:2, 2015, 553–579.

[5] De Philippis, G., and F. Rindler: On the structure of $\mathcal{A}$-free measures and applications. - Ann. of Math. (2) 184:3, 2016, 1017–1039.

[6] Gong, J.: Rigidity of derivations in the plane and in metric measure spaces. - Illinois J. Math. 56:4, 2012, 1109–1147.

[7] Hajłasz, P., and P. Koskela: Sobolev met Poincaré. - Mem. Amer. Math. Soc. 145:668, 2000.

[8] Heinonen, J.: Nonsmooth calculus. - Bull. Amer. Math. Soc. (N.S.) 44:2, 2007, 163–232.

[9] Keith, S.: A differentiable structure for metric measure spaces. - Adv. Math. 183:2, 2004, 271–315.

[10] Kigami, J.: Analysis on fractals. - Cambridge Tracts in Math. 143, Cambridge Univ. Press, Cambridge, 2001.

[11] Kleiner, B., and J. M. Mackay: Differentiable structures on metric measure spaces: a primer. - Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 16:1, 2016, 41–64.

[12] Mackay, J. M., J. T. Tyson, and K. Wildrick: Modulus and Poincaré inequalities on non-self-similar Sierpiński carpets. - Geom. Funct. Anal. 23:3, 2013, 985–1034.

[13] Mattila, P.: Geometry of sets and measures in Euclidean spaces. - Cambridge Stud. Adv. Math. 44, Cambridge Univ. Press, Cambridge, 1995.

[14] Schioppa, A.: On the relationship between derivations and measurable differentiable structures. - Ann. Acad. Sci. Fenn. Math. 39:1, 2014, 275–304.

[15] Schioppa, A.: Derivations and Alberti representations. - Adv. Math. 293, 2016, 436–528.

[16] Weaver, N.: Lipschitz algebras and derivations. II. Exterior differentiation. - J. Funct. Anal. 178:1, 2000, 64–112.

Received 13 July 2018 • Accepted 9 November 2018