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INJECTIVITY OF THE QUOTIENT BERS EMBEDDING OF TEICHMÜLLER SPACES

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Abstract. The Bers embedding of the Teichmüller space is a homeomorphism into the Banach space of certain holomorphic automorphic forms. For a subspace of the universal Teichmüller space and its corresponding Banach subspace, we consider whether the Bers embedding can project down between their quotient spaces. If this is the case, it is called the quotient Bers embedding. Injectivity of the quotient Bers embedding is the main problem in this paper. Alternatively, we can describe this situation as the universal Teichmüller space having an affine foliated structure induced by this subspace. We give several examples of subspaces for which the injectivity holds true, including the Teichmüller space of circle diffeomorphisms with Hölder continuous derivative. As an application, the regularity of conjugation between representations of a Fuchsian group into the group of circle diffeomorphisms is investigated.

1. Introduction

The universal Teichmüller space T is the ambient space of any other Teichmüller spaces. An affine foliated structure of T is induced by its certain subspace through the Bers embedding β of T into the Banach space $B(\mathbf{D}^*)$ of hyperbolically bounded holomorphic quadratic automorphic forms on the disk $\mathbf{D}^* = \widehat{\mathbf{C}} - \overline{\mathbf{D}}$ in the Riemann sphere centered at the infinity. This was first investigated by Gardiner and Sullivan [11] for the little Teichmüller subspace T_0, which consists of the asymptotically conformal elements of T. This subspace is embedded by β into the Banach subspace $B_0(\mathbf{D}^*)$ of $B(\mathbf{D}^*)$ consisting of all elements vanishing at the boundary. They proved that the foliated structure of T given by the right translations of T_0 in T corresponds to the affine foliation of $B(\mathbf{D}^*)$ by the subspace $B_0(\mathbf{D}^*)$ under the Bers embedding β. In other words, the Bers embedding is compatible with the coset decompositions $T_0\backslash T$ and $B_0(\mathbf{D}^*)\backslash B(\mathbf{D}^*)$.

The asymptotic Teichmüller space AT was introduced in [11] as the quotient space $T_0\backslash T$. The compatibility of the Bers embedding β with the coset decompositions as mentioned above yields a well-defined quotient map

$$\widehat{\beta}\colon T_0\backslash T \to B_0(\mathbf{D}^*)\backslash B(\mathbf{D}^*),$$

by which the complex structure modeled on the quotient Banach space $B_0(\mathbf{D}^*)\backslash B(\mathbf{D}^*)$ was provided for AT. Later, the argument was simplified by showing that $\widehat{\beta}$ is also injective. This is due to Kahn (see Gardiner and Lakic [10, Section 16.8]). Earle,

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Markovic and Saric [8, Theorem 4] generalized the injectivity of the quotient Bers embedding to the Teichmüller space $T(\mathbf{D}/\Gamma)$ of a Riemann surface $\mathbf{D}/\Gamma$ for a Fuchsian group Γ with respect to the corresponding little Teichmüller subspace $T_0(\mathbf{D}/\Gamma)$.

In this paper, we show other examples of affine foliated structures of the universal Teichmüller space T ensuring the well-definedness and the injectivity of the quotient Bers embeddings. The subspaces of T we handle here are the p-integrable Teichmüller space T^p (see Cui [5], Guo [13], Shen [25], Tang [28] and Yanagishita [29]) and the Teichmüller space $T_0^{>0}$ of circle diffeomorphisms of Hölder continuous derivative of an arbitrary exponent (see [17, 21]). The definitions of these Teichmüller spaces and the precise statements of our main theorems are given in Sections 4 and 5, respectively. For the 2-integrable Teichmüller space T^2, Takhtajan and Teo [27] have shown the well-definedness of the quotient Bers embedding of $T^2\backslash T$, but the injectivity seems a new result.

The proofs of the injectivity mentioned above are based on a common argument. In Section 3, we summarize it as a general principle. The injectivity of the quotient Bers embedding has been also proved in a different setting and in a different method. See a recent paper by Wei and Zinsmeister [30].

Affine foliated structures can be also defined on other Teichmüller spaces than T. In Sections 6, we consider such situations, and in particular, we prove the affine foliated structure of $T_0^{>0}$ induced by the Teichmüller space T_0^{α} of circle diffeomorphisms of α-Hölder continuous derivative for $\alpha \in (0,1)$. We have obtained in [21] a complex structure on T_0^{α} modeled on a certain Banach space via the Bers embedding. Our result in particular shows that $T_0^{>0}$ admits such a Banach manifold structure with the decomposition into mutually disjoint but equivalent components, and each component corresponds injectively to an affine subspace of the Bers embedding.

As an application of one of our main theorems, we can represent the deformation space $DT(\Gamma)$ of a Fuchsian group $\Gamma \subset \text{Möb}(\mathbf{S}) \cong \text{PSL}(2,\mathbf{R})$ in the group $\text{Diff}_+^{>1}(\mathbf{S})$ of all circle diffeomorphisms with Hölder continuous derivatives of any exponent as a subspace of the Teichmüller space $AT(\Gamma)$ of Γ-invariant symmetric structures on $\mathbf{S}$. Here, $AT(\Gamma)$ is the closed subspace of AT consisting of all elements of AT fixed by the action of every $\gamma \in \Gamma$. This space was studied in [19]. To show the injectivity of $DT(\Gamma) \to AT(\Gamma)$, we also need a rigidity theorem for the representation of Γ in $\text{Diff}_+^{1+\alpha}(\mathbf{S})$ given in [20]. Applying this theorem, we finally prove in Section 7 that if two representations of Γ in $\text{Diff}_+^{r}(\mathbf{S})$ for $r > 1$ are conjugate by a symmetric homeomorphism f representing an element of T_0, then f actually belongs to $\text{Diff}_+^{r}(\mathbf{S})$.

2. Preliminaries and background results

An orientation-preserving homeomorphism w of a domain in the complex plane $\mathbf{C}$ is said to be *quasiconformal* if partial derivatives ∂w and $\bar\partial w$ in the distribution sense exist and if the *complex dilatation* $\mu_w(z) = \bar\partial w(z)/\partial w(z)$ satisfies $\|\mu_w\|_\infty < 1$. Let

$$\text{Bel}(\mathbf{D}) = \{\mu \in L^\infty(\mathbf{D}) \mid \|\mu\|_\infty < 1\}$$

be the space of such measurable functions on the unit disk $\mathbf{D}$, which are called *Beltrami coefficients*. We denote the group of all quasiconformal self-homeomorphisms of $\mathbf{D}$ by $\text{QC}(\mathbf{D})$. By the measurable Riemann mapping theorem (see [1]), for every $\mu \in \text{Bel}(\mathbf{D})$, there is $w \in \text{QC}(\mathbf{D})$ satisfying $\mu_w = \mu$ uniquely up to the post-composition of elements of $\text{Möb}(\mathbf{D}) \cong \text{PSL}(2,\mathbf{R})$, the group of all Möbius transformations of $\mathbf{D}$. This gives the identification

$$\text{Möb}(\mathbf{D})\backslash \text{QC}(\mathbf{D}) \cong \text{Bel}(\mathbf{D}).$$

Every $w \in \text{QC}(\mathbf{D})$ extends continuously to a *quasisymmetric* self-homeomorphism of $\mathbf{S} = \partial \mathbf{D}$. Let QS be the group of all quasisymmetric self-homeomorphisms of $\mathbf{S}$. We denote the boundary extension map by

$$q\colon \text{QC}(\mathbf{D}) \to \text{QS},$$

which is a surjective homomorphism. The *universal Teichmüller space* is defined by

$$T = \text{Möb}(\mathbf{S})\backslash \text{QS},$$

where $\text{Möb}(\mathbf{S}) = q(\text{Möb}(\mathbf{D}))$. Then, q induces the Teichmüller projection $\pi\colon \text{Bel}(\mathbf{D}) \to T$. The quotient topology of T is induced from the norm on $\text{Bel}(\mathbf{D})$ by π. In fact, the Teichmüller distance can be defined by using the hyperbolic distance on $\text{Bel}(\mathbf{D})$.

For every $\mu \in \text{Bel}(\mathbf{D})$, we extend it to a Beltrami coefficient $\widehat{\mu}$ on the Riemann sphere $\widehat{\mathbf{C}}$ by setting $\widehat{\mu}(z) \equiv 0$ for $z \in \mathbf{D}^* = \widehat{\mathbf{C}} - \overline{\mathbf{D}}$. We denote a quasiconformal homeomorphism of $\widehat{\mathbf{C}}$ with the complex dilatation $\widehat{\mu}$ by f_μ. The measurable Riemann mapping theorem guarantees the existence of such f_μ and the uniqueness of f_μ up to the post-composition of Möbius transformations of $\widehat{\mathbf{C}}$. We take the Schwarzian derivative $S_{f_\mu}\colon \mathbf{D}^* \to \widehat{\mathbf{C}}$ of the conformal homeomorphism $f_\mu|_{\mathbf{D}^*}$, which parametrizes the complex projective structures on $\mathbf{D}^*$. By the Nehari–Kraus theorem, S_{f_μ} belongs to the complex Banach space of hyperbolically bounded holomorphic quadratic automorphic forms

$$B(\mathbf{D}^*) = \{\varphi(z)\, dz^2 \mid \|\varphi\|_\infty := \sup_{z\in\mathbf{D}^*} \rho_{\mathbf{D}^*}^{-2}(z)|\varphi(z)| < \infty\},$$

where $\rho_{\mathbf{D}^*}(z) = 2/(|z|^2-1)$ is the hyperbolic density on $\mathbf{D}^*$. By this correspondence $\mu \mapsto S_{f_\mu}$, a map

$$\Phi\colon \text{Bel}(\mathbf{D}) \to B(\mathbf{D}^*)$$

is defined to be a holomorphic split submersion, which is called the *Bers projection* (onto the image).

For the Teichmüller projection $\pi\colon \text{Bel}(\mathbf{D}) \to T$ and the Bers projection $\Phi\colon \text{Bel}(\mathbf{D}) \to B(\mathbf{D}^*)$, we can show that $\Phi\circ\pi^{-1}$ is well-defined and injective, which defines a map $\beta\colon T \to B(\mathbf{D}^*)$ called the *Bers embedding*. In fact, β is a homeomorphism onto the image $\beta(T) = \Phi(\text{Bel}(\mathbf{D}))$ and $\beta(T)$ is a bounded domain in $B(\mathbf{D}^*)$. This provides a complex Banach manifold structure for T.

There is a global continuous section for the Teichmüller projection $\pi\colon \text{Bel}(\mathbf{D}) \to T$. This is defined by a canonical quasiconformal extension $e\colon \text{QS} \to \text{QC}(\mathbf{D})$ for each quasisymmetric self-homeomorphism g of $\mathbf{S}$. Douady and Earle [6] introduced the *barycentric extension* $e_{\text{DE}}\colon \text{QS} \to \text{QC}(\mathbf{D})$ having the *conformal naturality*

$$e_{\text{DE}}(\phi_1 \circ g \circ \phi_2) = e_{\text{DE}}(\phi_1) \circ e_{\text{DE}}(g) \circ e_{\text{DE}}(\phi_2)$$

for any $\phi_1, \phi_2 \in \text{Möb}(\mathbf{S})$ and any $g \in \text{QS}$, where $e_{\text{DE}}(\phi_1)$ and $e_{\text{DE}}(\phi_2)$ are in $\text{Möb}(\mathbf{D})$. Taking the quotient by $\text{Möb}(\mathbf{S}) = \text{Möb}(\mathbf{D})$ in both sides, we have a continuous map $s\colon T \to \text{Bel}(\mathbf{D})$ such that $\pi \circ s = \text{id}_T$. The existence of a global continuous section implies that T is contractible. Let $\sigma\colon \text{Bel}(\mathbf{D}) \to \text{Bel}(\mathbf{D})$ be defined by the correspondence of μ to $s(\pi(\mu))$ for the section s. The image $\sigma(\text{Bel}(\mathbf{D}))$ is the set of all Beltrami coefficients obtained by the barycentric extension.

For any $\nu \in \text{Bel}(\mathbf{D})$, let $f^\nu \in \text{QC}(\mathbf{D})$ be a normalized element having the complex dilatation ν, where the *normalization* is given by fixing three boundary points 1, i and -1 on $\mathbf{S}$. The subgroup of $\text{QC}(\mathbf{D})$ consisting of all normalized elements is denoted by $\mathrm{QC}_*(\mathbf{D})$. Applying this normalization, we can define a group structure on $\mathrm{Bel}(\mathbf{D})$ as follows. For any $\nu_1, \nu_2 \in \mathrm{Bel}(\mathbf{D})$, let $\nu_1 * \nu_2$ be the complex dilatation of the composition $f^{\nu_1} \circ f^{\nu_2}$. Then, $\mathrm{Bel}(\mathbf{D})$ is a group with this operation $*$. In other words, by the identification of $\mathrm{Bel}(\mathbf{D})$ with $\mathrm{QC}_*(\mathbf{D})$, we regard $\mathrm{Bel}(\mathbf{D})$ as a subgroup of $\mathrm{QC}(\mathbf{D})$. We denote the inverse element of $\nu \in \mathrm{Bel}(\mathbf{D})$ by ν^{-1}, which is the complex dilatation of $(f^\nu)^{-1}$. The chain rule of partial differentials yields a formula

$$\nu_1 * \nu_2^{-1}(\zeta) = \frac{\nu_1(z) - \nu_2(z)}{1 - \overline{\nu_2(z)}\nu_1(z)} \cdot \frac{\partial f^{\nu_2}(z)}{\overline{\partial f^{\nu_2}(z)}} \quad (\zeta = f^{\nu_2}(z)).$$

Each $\nu \in \mathrm{Bel}(\mathbf{D})$ induces the right translation $r_\nu \colon \mathrm{Bel}(\mathbf{D}) \to \mathrm{Bel}(\mathbf{D})$ by $\mu \mapsto \mu * \nu^{-1}$. By the above formula, we see that r_ν and $(r_\nu)^{-1} = r_{\nu^{-1}}$ are continuous, and hence r_ν is a homeomorphism of $\mathrm{Bel}(\mathbf{D})$. In fact, this is a biholomorphic automorphism of $\mathrm{Bel}(\mathbf{D})$.

For the base point [id] of $T = \mathrm{M\ddot{o}b}(\mathbf{S}) \backslash \mathrm{QS}$, the inverse image of the Teichmüller projection

$$\pi^{-1}([\mathrm{id}]) = \{\nu \in \mathrm{Bel}(\mathbf{D}) \mid q(f^\nu) = \mathrm{id}\}$$

is a normal subgroup of $\mathrm{Bel}(\mathbf{D})$ since $q\colon \mathrm{QC}(\mathbf{D}) \to \mathrm{QS}$ is a homomorphism. Having $T = \mathrm{Bel}(\mathbf{D})/\pi^{-1}([\mathrm{id}])$, we see that T has a group structure with the operation $*$ defined by $\pi(\nu_1) * \pi(\nu_2) = \pi(\nu_1 * \nu_2)$. The projection of the right translation $r_\nu\colon \mathrm{Bel}(\mathbf{D}) \to \mathrm{Bel}(\mathbf{D})$ under π yields a well-defined map $R_{\pi(\nu)}\colon T \to T$ by

$$\pi(\mu) \mapsto \pi(\mu * \nu^{-1}) = \pi(\mu) * \pi(\nu)^{-1}.$$

In this way, we have the base point change $R_\tau\colon T \to T$ sending τ to [id] for every $\tau \in T$. Alternatively, $R_\tau\colon T \to T$ is defined by $[g] \mapsto [g \circ f^{-1}]$ for $\tau = [f] \in T$. We see that the base point change R_τ is also a biholomorphic automorphism of T.

Every element $\gamma \in \mathrm{M\ddot{o}b}(\mathbf{S})$ acts on $B(\mathbf{D}^*)$ linear isometrically through the Bers embedding β. This means that, for any point $[f] \in T$ with $\beta([f]) = \varphi \in \beta(T)$, the Bers embedding $\beta(\gamma^*[f])$ of $\gamma^*[f] := [f \circ \gamma]$ is represented by

$$(\gamma^*\varphi)(z) = \varphi(\gamma(z))\gamma'(z)^2,$$

where we regard γ as the element of $\mathrm{M\ddot{o}b}(\mathbf{D}^*)$, the group of all Möbius transformations of $\mathbf{D}^*$. Namely, $\gamma^*\varphi$ is the pull-back of φ by γ as a quadratic automorphic form. Clearly, this action extends to $B(\mathbf{D}^*)$ and satisfies $\|\gamma^*\varphi\|_\infty = \|\varphi\|_\infty$.

A quasiconformal self-homeomorphism $w \in \mathrm{QC}(\mathbf{D})$ is called *asymptotically conformal* if its complex dilatation vanishes at the boundary, that is, $\operatorname{ess\,sup}_{|z|>1-t} |\mu_w(z)| \to 0$ as $t \to 0$. The subspace of $\mathrm{Bel}(\mathbf{D})$ consisting of all Beltrami coefficients vanishing at the boundary is denoted by $\mathrm{Bel}_0(\mathbf{D})$ and the subgroup of $\mathrm{QC}(\mathbf{D})$ consisting of all asymptotically conformal self-homeomorphisms of $\mathbf{D}$ is denoted by $\mathrm{AC}(\mathbf{D})$. Every $w \in \mathrm{AC}(\mathbf{D})$ extends continuously to a *symmetric* self-homeomorphism of $\mathbf{S}$. The group of all symmetric self-homeomorphisms of $\mathbf{S}$ is denoted by Sym. Then, the restriction of the boundary extension to $\mathrm{AC}(\mathbf{D})$ gives a surjective homomorphism $q\colon \mathrm{AC}(\mathbf{D}) \to \mathrm{Sym}$.

Gardiner and Sullivan [11] studied the *asymptotic Teichmüller space* defined by

$$AT = \mathrm{Sym} \backslash \mathrm{QS},$$

and the *little universal Teichmüller space* defined by

$$T_0 = \mathrm{M\ddot{o}b}(\mathbf{S}) \backslash \mathrm{Sym} = \pi(\mathrm{Bel}_0(\mathbf{D})).$$

They introduced Sym as a particular topological subgroup of QS. The characterizations of symmetric self-homeomorphisms by the Bers embedding of T_0 was also given. The Banach subspace of $B(\mathbf{D}^*)$ consisting of the elements of vanishing at the boundary is denoted by

$$B_0(\mathbf{D}^*) = \{\varphi \in B(\mathbf{D}^*) \mid \lim_{t\to 0} \sup_{|z|<1+t} \rho_{\mathbf{D}^*}^{-2}(z)|\varphi(z)| = 0\}.$$

Proposition 2.1. *For a quasisymmetric homeomorphism $g \in \mathrm{QS}$, the following conditions are equivalent:* (1) $g \in \mathrm{Sym}$; (2) $s([g]) \in \mathrm{Bel}_0(\mathbf{D})$; (3) $\beta([g]) \in B_0(\mathbf{D}^*)$.

We refer to Earle, Markovic and Saric [8] for this result on the barycentric extension. In particular, we see that

$$\beta(T_0) = \Phi(\mathrm{Bel}_0(\mathbf{D})) = \beta(T) \cap B_0(\mathbf{D}^*).$$

Next, we consider how T_0 is mapped into T by the base point change $R_\tau\colon T \to T$ for $\tau \in T$. Since R_τ is a biholomorphic automorphism of T, T_0 is mapped biholomorphically onto the image $R_\tau(T_0)$. We recall that T_0 is a subgroup of $(T, *)$ and R_τ is defined by $R_\tau(\tau') = \tau' * \tau^{-1}$ for every $\tau' \in T$. Then, the coset decomposition of T by the subgroup T_0 is exactly the disjoint union

$$T = \bigsqcup_{[\tau]\in T_0\backslash T} R_\tau^{-1}(T_0)$$

of mutually biholomorphically equivalent subspaces.

Moreover, we find that the image of the decomposition $T = \bigsqcup_{[\tau]\in T_0\backslash T} R_\tau^{-1}(T_0)$ under the Bers embedding $\beta\colon T \to B(\mathbf{D}^*)$ corresponds to the foliation of $\beta(T)$ by the family of Banach affine subspaces $\{B_0(\mathbf{D}^*) + \psi\}_{[\psi]\in B_0(\mathbf{D}^*)\backslash B(\mathbf{D}^*)}$. This compatibility can be formulated as the following theorem.

Theorem 2.2. *For each $\nu \in \mathrm{Bel}(\mathbf{D})$, let $\psi = \Phi(\nu) \in B(\mathbf{D}^*)$. Then,*

$$\Phi \circ r_\nu^{-1}(\mathrm{Bel}_0(\mathbf{D})) = \beta(T) \cap \{B_0(\mathbf{D}^*) + \psi\}.$$

Hence, $\beta \circ R_\tau^{-1}(T_0) = \beta(T) \cap \{B_0(\mathbf{D}^*) + \beta(\tau)\}$ for every $\tau \in T$.

The fact that $\beta(T) \cap \{B_0(\mathbf{D}^*) + \psi\}$ contains $\Phi \circ r_\nu^{-1}(\mathrm{Bel}_0(\mathbf{D}))$ was proved in [11]. The converse inclusion is due to Kahn (see also Gardiner and Lakic [10, Section 16.8]). As a consequence of the conformal naturality, a different proof of Theorem 2.2 by using the barycentric extension, which is also valid taking the action of a Fuchsian group into account, can be also obtained from the arguments in [8].

By this result, we have the decomposition of the Bers embedding as

$$\beta(T) = \bigsqcup_{[\tau]\in T_0\backslash T} \beta \circ R_\tau^{-1}(T_0) = \bigsqcup_{[\psi]\in B_0(\mathbf{D}^*)\backslash B(\mathbf{D}^*)} \beta(T) \cap (B_0(\mathbf{D}^*) + \psi).$$

Each component $\beta(T) \cap (B_0(\mathbf{D}^*) + \psi)$ is biholomorphically equivalent to $T_0 \cong \beta(T_0)$. We call this decomposition of $T \cong \beta(T)$ the *affine foliated structure* of T induced by T_0.

We consider the embedding of the quotient Teichmüller space $AT = T_0\backslash T$ into the quotient Banach space $B_0(\mathbf{D}^*)\backslash B(\mathbf{D}^*)$. Concerning the equation of Theorem 2.2, the inclusion $\subset$ implies that the quotient map $\widehat{\beta}\colon T_0\backslash T \to B_0(\mathbf{D}^*)\backslash\ B(\mathbf{D}^*)$ is well-defined. This map is called the *quotient Bers embedding*. By showing that $\widehat{\beta}$ is a local homeomorphism, a complex structure on AT modeled on $B_0(\mathbf{D}^*)\backslash B(\mathbf{D}^*)$ was given in [11]. The converse inclusion $\supset$ in Theorem 2.2 implies the stronger result that $\widehat{\beta}$ is a homeomorphism onto the image, and in particular $\widehat{\beta}$ is injective.

Corollary 2.3. *The quotient Bers embedding $\widehat{\beta}\colon T_0\backslash T \to B_0(\mathbf{D}^*)\backslash B(\mathbf{D}^*)$ is well-defined to be a homeomorphism onto the image.*

3. General principle

In this section, we prepare a basic argument to prove the injectivity of the quotient Bers embedding. This is carried out based on Theorem 2.2. We keep using the following notations throughout this section.

We fix $\psi = \Phi(\nu) \in \beta(T) \subset B(\mathbf{D}^*)$ for any $\nu \in \mathrm{Bel}(\mathbf{D})$, and take $\varphi \in B_0(\mathbf{D}^*)$ such that $\varphi + \psi \in \beta(T)$. For a quasiconformal homeomorphism f_ν of $\widehat{\mathbf{C}}$ that is conformal on $\mathbf{D}^*$, we set $\Omega = f_\nu(\mathbf{D})$ and $\Omega^* = f_\nu(\mathbf{D}^*)$. Under these circumstances, Theorem 2.2 implies the following. Asymptotic conformality on Ω can be defined in the same way as on $\mathbf{D}$.

Proposition 3.1. *There exists a quasiconformal homeomorphism $\widehat{f}\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ conformal on Ω^* and asymptotically conformal on Ω such that $S_{\widehat{f}\circ f_\nu|_{\mathbf{D}^*}} = \varphi + \psi$.*

Proof. By Theorem 2.2, there is some $\mu \in \mathrm{Bel}_0(\mathbf{D})$ such that $\Phi \circ r_\nu^{-1}(\mu) = \Phi(\mu * \nu) = \varphi + \psi$. We consider $f_{\mu*\nu}$, the quasiconformal self-homeomorphism of $\widehat{\mathbf{C}}$ conformal on $\mathbf{D}^*$ and quasiconformal on $\mathbf{D}$ with the complex dilatation $\mu * \nu$. Then, we define $\widehat{f} = f_{\mu*\nu} \circ f_\nu^{-1}$, which satisfies $S_{\widehat{f}\circ f_\nu|_{\mathbf{D}^*}} = S_{f_{\mu*\nu}|_{\mathbf{D}^*}} = \varphi + \psi$. Moreover, $\widehat{f}$ is conformal on Ω^* and quasiconformal on Ω with a complex dilatation $\widehat{\mu}(z) := (g_\nu^*\mu)(z) = \mu(g_\nu(z))\overline{g_\nu'(z)}/g_\nu'(z)$, where $g_\nu = f^\nu \circ f_\nu^{-1}\colon \Omega \to \mathbf{D}$ is the conformal homeomorphism (Riemann map). Hence, $\widehat{f}$ is asymptotically conformal on Ω. $\square$

The complex dilatation $\widehat{\mu}$ of $\widehat{f}$ on Ω vanishes at the boundary $\partial\Omega$. In particular, we can choose a compact subset $\Omega_0 \subset \Omega$ such that

$$3\,\|\widehat{\mu}|_{\Omega-\Omega_0}\|_\infty < \delta(\|\nu\|_\infty),$$

where $\delta(\|\nu\|_\infty) > 0$ is a constant given later in Lemma 3.2 depending only on $\|\nu\|_\infty$.

We decompose $\widehat{f}$ into $\widehat{f}_0 \circ \widehat{f}_1$ as follows. The quasiconformal homeomorphism $\widehat{f}_1\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ is chosen so that its complex dilatation coincides with $\widehat{\mu}$ on $\Omega - \Omega_0$ and zero elsewhere. Then, $\widehat{f}_0$ is defined to be $\widehat{f} \circ \widehat{f}_1^{-1}$, whose complex dilatation has a support on the compact subset $\widehat{f}_1(\Omega_0) \subset \widehat{f}_1(\Omega)$. We take $\varphi_1 \in B(\mathbf{D}^*)$ so that

$$S_{\widehat{f}_1\circ f_\nu|_{\mathbf{D}^*}} = \varphi_1 + \psi.$$

This satisfies $\|\varphi_1\|_\infty < \delta(\|\nu\|_\infty)$. Indeed,

$$\rho_{\mathbf{D}^*}^{-2}(z)|\varphi_1(z)| = \rho_{\mathbf{D}^*}^{-2}(z)|S_{\widehat{f}_1\circ f_\nu|_{\mathbf{D}^*}}(z) - S_{f_\nu|_{\mathbf{D}^*}}(z)| = \rho_{\Omega^*}^{-2}(\zeta)|S_{\widehat{f}_1|_{\Omega^*}}(\zeta)|$$

for $\zeta = f_\nu(z)$ and this is bounded by $3\|\widehat{\mu}|_{\Omega-\Omega_0}\|_\infty$ (see [14, Theorem II.3.2]).

We utilize a local section for the Bers projection $\Phi\colon \mathrm{Bel}(\mathbf{D}) \to \beta(T) \cap B(\mathbf{D}^*)$, which is a generalization of the Ahlfors–Weill section defined in a neighborhood of the origin, and was constructed by using a quasiconformal reflection originally due to Ahlfors [2]. This was improved later with the aid of the barycentric extension by Earle and Nag [9] to hold compatibility with the action of Möbius transformations. See [22, Section 3.8]. The following assertion can be also proved by the arguments in [10, Section 14.4] and [14, Section II.4.2].

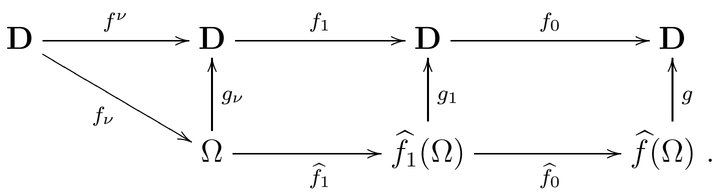
Lemma 3.2. *Let $f_\nu\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ be the quasiconformal homeomorphism with complex dilatation $\nu \in \sigma(\mathrm{Bel}(\mathbf{D}))$ obtained by the barycentric extension, which is conformal on $\mathbf{D}^*$ with $S_{f_\nu|_{\mathbf{D}^*}} = \psi$. Then, there exists a constant $\delta = \delta(\|\nu\|_\infty) > 0$ depending only on $\|\nu\|_\infty$ such that for every $\varphi \in B(\mathbf{D}^*)$ with $\|\varphi\|_\infty < \delta$, there is a quasiconformal homeomorphism $\widehat{f}\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ conformal on Ω^* such that $S_{\widehat{f}\circ f_\nu|_{\mathbf{D}^*}} = \varphi + \psi$ and* *the complex dilatation* $\widehat{\mu}$ *of* $\widehat{f}$ *on* Ω *satisfies*

$$|\widehat{\mu}(f_\nu(z))| \leq \frac{1}{\delta}\rho_{\mathbf{D}^*}^{-2}(z^*)|\varphi(z^*)|$$

for every $z \in \mathbf{D}$. *Here,* $z^* = (\bar{z})^{-1}$ *is the reflection of* $z \in \mathbf{D}$ *with respect to* $\mathbf{S}$.

We apply this lemma to the previous ν with $\psi = \Phi(\nu)$ and φ_1 with $S_{\widehat{f}_1 \circ f_\nu|_{\mathbf{D}^*}} = \varphi_1 + \psi$. We may always assume that ν is obtained by the barycentric extension. Replacing $\widehat{f}_1$ with the quasiconformal homeomorphism obtained in Lemma 3.2, we can further assume that the complex dilatation $\widehat{\mu}_1$ of $\widehat{f}_1$ satisfies the above inequality. We remark that $\widehat{f}_1|_{\Omega^*}$ does not change by this replacement. We use the same $\widehat{f}_0$ as before; correspondingly, $\widehat{f}|_\Omega$ changes but $\widehat{f}|_{\Omega^*}$ does not.

Having f_ν, $\widehat{f}_1$ and $\widehat{f}_0$ already, we take the normalized quasiconformal homeomorphisms $f^\nu\colon \mathbf{D} \to \mathbf{D}$, $f_1\colon \mathbf{D} \to \mathbf{D}$ and $f_0\colon \mathbf{D} \to \mathbf{D}$, and the conformal homeomorphisms (Riemann mappings) $g_\nu\colon \Omega \to \mathbf{D}$, $g_1\colon \widehat{f}_1(\Omega) \to \mathbf{D}$ and $g\colon \widehat{f}(\Omega) \to \mathbf{D}$ so that the following commutative diagram holds:



We note that g_ν, g_1 and g are uniquely determined. Set $f = f_0 \circ f_1$. Then, the complex dilatation of $\widehat{f} \circ f_\nu$ on $\mathbf{D}$ coincides with that of $f \circ f^\nu$. Hence, its image under the Bers projection Φ is $\varphi + \psi$. This is also true for f_1 and φ_1 instead of f and φ.

We consider $\varphi - \varphi_1 = (\varphi + \psi) - (\varphi_1 + \psi)$ for $z \in \mathbf{D}^*$, which is equal to

$$S_{\widehat{f} \circ f_\nu|_{\mathbf{D}^*}}(z) - S_{\widehat{f}_1 \circ f_\nu|_{\mathbf{D}^*}}(z) = S_{\widehat{f}_0 \circ \widehat{f}_1 \circ f_\nu|_{\mathbf{D}^*}}(z) - S_{\widehat{f}_1 \circ f_\nu|_{\mathbf{D}^*}}(z).$$

The complex dilatation μ_0 of f_0 is equal to the push-forward $(g_1)_*\widehat{\mu}_0$ of the complex dilatation $\widehat{\mu}_0$ of $\widehat{f}_0$ by the conformal homeomorphism g_1. Since the support of $\widehat{\mu}_0$ is on the compact subset of $\widehat{f}_1(\Omega)$, the support of μ_0 is on a compact subset of $\mathbf{D}$. Similarly, the complex dilatation μ_1 of f_1 is equal to $(g_\nu)_*\widehat{\mu}_1$. Then, we see that $\varphi - \varphi_1$ coincides with $\Phi(\mu_0 * \mu_1 * \nu) - \Phi(\mu_1 * \nu)$.

The above arguments are summarized as follows.

Proposition 3.3. *Let* $f_\nu\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ *be a quasiconformal homeomorphism with complex dilatation* $\nu \in \sigma(\mathrm{Bel}(\mathbf{D}))$, *which is conformal on* $\mathbf{D}^*$ *with* $S_{f_\nu|_{\mathbf{D}^*}} = \psi$. *Let* $\widehat{f}\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ *be a quasiconformal homeomorphism with complex dilatation* $\widehat{\mu}$ *on* $\Omega = f_\nu(\mathbf{D})$ *vanishing at the boundary that is conformal on* $\Omega^* = f_\nu(\mathbf{D}^*)$ *with* $S_{\widehat{f} \circ f_\nu|_{\mathbf{D}^*}} = \varphi + \psi$. *Then,* $\widehat{f}$ *is decomposed into two quasiconformal homeomorphisms* $\widehat{f}_0$ *and* $\widehat{f}_1$ *of* $\widehat{\mathbf{C}}$ *with* $\widehat{f} = \widehat{f}_0 \circ \widehat{f}_1$, *where* $\widehat{f}_1$ *is conformal on* Ω^* *with* $S_{\widehat{f}_1 \circ f_\nu|_{\mathbf{D}^*}} = \varphi_1 + \psi$, *satisfying the following properties:*

(1) *the complex dilatation* $\widehat{\mu}_1$ *of* $\widehat{f}_1$ *on* Ω *satisfies*

$$|\widehat{\mu}_1 \circ f_\nu(z)| \leq \frac{1}{\delta}\,\rho_{\mathbf{D}^*}^{-2}(z^*)|\varphi_1(z^*)|$$

for some $\delta > 0$ *and for every* $z \in \mathbf{D}$;

(2) *the support of the complex dilatation* μ_0 *of the normalized quasiconformal homeomorphism* $f_0\colon \mathbf{D} \to \mathbf{D}$, *which is conformally conjugate to* $\widehat{f}_0\colon \widehat{f}_1(\Omega) \to \widehat{f}(\Omega)$, *is contained in a compact subset of* $\mathbf{D}$;

(3) *for the complex dilatation* μ_1 *of the normalized quasiconformal homeomorphism* $f_1\colon \mathbf{D} \to \mathbf{D}$, *which is conformally conjugate to* $\widehat{f}_1\colon \Omega \to \widehat{f}_1(\Omega)$, *we have*

$$\varphi - \varphi_1 = \Phi(\mu_0 * \mu_1 * \nu) - \Phi(\mu_1 * \nu).$$

In the remainder of this section, we state two results, which provides a foundation for the argument on the affine foliated structure given by the Bers embedding.

Proposition 3.4. *Let* f_μ *and* f_ν *be the quasiconformal homeomorphisms of* $\widehat{\mathbf{C}}$ *that are conformal on* $\mathbf{D}^*$ *and have complex dilatations* μ *and* ν *respectively on* $\mathbf{D}$. *We assume that* $\lim_{z\to\infty}(f_\mu(z) - z) = 0$ *and* $\lim_{z\to\infty}(f_\nu(z) - z) = 0$. *Let* $\Omega = f_\nu(\mathbf{D})$ *and* $\Omega^* = f_\nu(\mathbf{D}^*)$. *Then,*

$$|S_{f_\mu \circ f_\nu^{-1}|_{\Omega^*}}(\zeta)| \le \frac{3\rho_{\Omega^*}(\zeta)}{\sqrt{\pi}} \left(\int_\Omega \frac{|\mu(f_\nu^{-1}(w)) - \nu(f_\nu^{-1}(w))|^2}{(1-|\mu(f_\nu^{-1}(w))|^2)(1-|\nu(f_\nu^{-1}(w))|^2)} \frac{du\,dv}{|w-\zeta|^4} \right)^{1/2}$$

holds for $\zeta \in \Omega^*$, *where* $\rho_{\Omega^*}(\zeta)$ *is the hyperbolic density on* Ω^*.

Proof. It was shown in Yanagishita [29, Lemma 3.1] applying the argument of Astala and Zinsmeister [3] that

$$|S_{f_\mu \circ f_\nu^{-1}|_{\Omega^*}}(\zeta)| = \frac{3}{2\pi}\rho_{\Omega^*}^2(\zeta) \left| \int_{\lambda^{-1}(\Omega)} \bar{\partial}_z G(z,\zeta)\, dx\, dy \right|,$$

where $G(\cdot,\zeta) = \kappa(\cdot,\zeta) \circ f_\mu \circ f_\nu^{-1} \circ \lambda(\cdot,\zeta)$ and

$$\kappa(z,\zeta) = -\frac{(|f_\nu^{-1}(\zeta)|^2 - 1) f'_\mu(f_\nu^{-1}(\zeta))}{z - f_\mu(f_\nu^{-1}(\zeta))}, \quad \lambda(z,\zeta) = \zeta - \frac{(|f_\nu^{-1}(\zeta)|^2 - 1) f'_\nu(f_\nu^{-1}(\zeta))}{z}.$$

Then, by the estimate using the Cauchy–Schwarz inequality and the area theorem as in the proof of [29, Proposition 3.2], we have that

$$\left| \int_{\lambda^{-1}(\Omega)} \bar{\partial}_z G(z,\zeta)\, dx\, dy \right|^2 \le 4\pi \rho_{\Omega^*}^{-2}(\zeta) \int_\Omega \frac{|\alpha(w)|^2}{1-|\alpha(w)|^2} \frac{du\,dv}{|w-\zeta|^4},$$

where α is the complex dilatation of $f_\mu \circ f_\nu^{-1}$. This yields the required inequality. □

Proposition 3.5. *For* $\mu_1, \mu_2, \nu \in \mathrm{Bel}(\mathbf{D})$, *we have*

$$|r_\nu(\mu_1)(\zeta) - r_\nu(\mu_2)(\zeta)| \le \frac{|\mu_1(z) - \mu_2(z)|}{\sqrt{(1-|\mu_1(z)|^2)(1-|\mu_2(z)|^2)}}$$

for $\zeta = f^\nu(z)$ *with* $z \in \mathbf{D}$.

Proof. A simple computation shows that

$$\begin{aligned} |r_\nu(\mu_1)(\zeta) - r_\nu(\mu_2)(\zeta)| &= |\mu_1 * \nu^{-1}(\zeta) - \mu_2 * \nu^{-1}(\zeta)| \\ &= \left| \frac{\mu_1(z) - \nu(z)}{1 - \overline{\nu(z)}\mu_1(z)} - \frac{\mu_2(z) - \nu(z)}{1 - \overline{\nu(z)}\mu_2(z)} \right| \\ &= \frac{|\mu_1(z) - \mu_2(z)|(1-|\nu(z)|^2)}{|1 - \overline{\nu(z)}\mu_1(z)||1 - \overline{\nu(z)}\mu_2(z)|} \end{aligned}$$

$$\leq \frac{|\mu_1(z) - \mu_2(z)|}{\sqrt{(1-|\mu_1(z)|^2)(1-|\mu_2(z)|^2)}}$$

for $\zeta = f^\nu(z)$. See [29, Proposition 5.1]. □

4. The p-integrable Teichmüller space

In this section, we prove the affine foliated structure of the universal Teichmüller space T induced by the p-integrable Teichmüller space T^p for $p \geq 2$. Later in this section, this is also extended to the Teichmüller space $T(\mathbf{D}/\Gamma)$ and the p-integrable Teichmüller space $T^p(\mathbf{D}/\Gamma)$ of a Riemann surface $\mathbf{D}/\Gamma$ for a certain Fuchsian group Γ.

A Beltrami coefficient $\mu \in \mathrm{Bel}(\mathbf{D})$ is *p-integrable* for $p \geq 1$ if

$$\|\mu\|_p^p := \int_{\mathbf{D}} |\mu(z)|^p \rho_{\mathbf{D}}^2(z)\, dx\, dy < \infty,$$

where $\rho_{\mathbf{D}}(z) = 2/(1-|z|^2)$ is the hyperbolic density on $\mathbf{D}$. The space of all p-integrable Beltrami coefficients on $\mathbf{D}$ is denoted by $\mathrm{Ael}^p(\mathbf{D})$. The p-integrable Teichmüller spaces defined below have been studied by Cui [5], Guo [13], Shen [25], Takhtajan and Teo [27], Tang [28] and Yanagishita [29] among others. See also a recent paper by Shen and Tang [26].

Definition. A quasisymmetric homeomorphism $g\colon \mathbf{S} \to \mathbf{S}$ belongs to Sym^p for $p \geq 2$ if g has a quasiconformal extension $\widetilde{g}\colon \mathbf{D} \to \mathbf{D}$ whose complex dilatation $\mu_{\widetilde{g}}$ belongs to $\mathrm{Ael}^p(\mathbf{D})$. The *p-integrable Teichmüller space* T^p is defined by

$$T^p = \pi(\mathrm{Ael}^p(\mathbf{D})) = \mathrm{M\ddot{o}b}(\mathbf{S})\backslash \mathrm{Sym}^p \subset T.$$

The topology on T^p is induced by a norm $\|\cdot\|_p + \|\cdot\|_\infty$ on $\mathrm{Ael}^p(\mathbf{D})$.

We also consider the space of all p-integrable holomorphic quadratic automorphic forms on $\mathbf{D}^*$:

$$A^p(\mathbf{D}^*) = \{\varphi \in B(\mathbf{D}^*) \mid \|\varphi\|_p^p := \int_{\mathbf{D}^*} \rho_{\mathbf{D}^*}^{2-2p}(z)|\varphi(z)|^p\, dx\, dy < \infty\}.$$

It is known that $A^p(\mathbf{D}^*) \subset B_0(\mathbf{D}^*)$. It was proved in [5, Theorem 2] and [13, Theorem 2] that the Bers embedding β of T^p is a homeomorphism onto the image and satisfies

$$\beta(T^p) = \beta(T) \cap A^p(\mathbf{D}^*)$$

for $p \geq 2$. This in particular implies that $T^p \subset T_0$ and hence $\mathrm{Sym}^p \subset \mathrm{Sym}$.

For the map $\sigma : \mathrm{Bel}(\mathbf{D}) \to \mathrm{Bel}(\mathbf{D})$ given by the barycentric extension, we have that $\sigma(\mathrm{Ael}^p(\mathbf{D})) \subset \mathrm{Ael}^p(\mathbf{D})$ (see [5, Theorem 6], [28, Theorem 2.1], [29, Theorem 2.4]). Moreover, if $\nu \in \sigma(\mathrm{Ael}^p(\mathbf{D}))$, then r_ν preserves $\mathrm{Ael}^p(\mathbf{D})$ and gives a biholomorphic automorphism of $\mathrm{Ael}^p(\mathbf{D})$. Therefore, for arbitrary μ and ν in $\mathrm{Ael}^p(\mathbf{D})$, we have that $\pi(r_\nu(\mu)) \in T^p$. If $\tau \in T^p$ then the base point change R_τ preserves T^p and gives a biholomorphic automorphism of T^p (see [5, Theorem 4], [27, Corollary 2.12 and Lemma 3.4], [29, Proposition 5.1]).

We state the main result in this section.

Theorem 4.1. *For each $\nu \in \mathrm{Bel}(\mathbf{D})$, let $\psi = \Phi(\nu) \in B(\mathbf{D}^*)$. Then,*

$$\Phi \circ r_\nu^{-1}(\mathrm{Ael}^p(\mathbf{D})) = \beta(T) \cap \{A^p(\mathbf{D}^*) + \psi\}.$$

Hence, $\beta \circ R_\tau^{-1}(T^p) = \beta(T) \cap \{A^p(\mathbf{D}^*) + \beta(\tau)\}$ for every $\tau \in T$.

Proof. The inclusion $\subset$ in the case of $p = 2$ was shown by Takhtajan and Teo [27, Theorem 2.13]. For a general $p \geq 2$, we can prove this by using the following claims based on Propositions 3.4 and 3.5. They were shown in Lemma 6.4 and Proposition 7.1 of [20] and the remarks after them, respectively. We note that the inequalities below include the case where the right-hand side is ∞.

Claim 1. Let $\mu \in \mathrm{Bel}(\mathbf{D})$ be arbitrary and let $\mu' \in \sigma(\mathrm{Bel}(\mathbf{D}))$ be obtained by the barycentric extension. Then,

$$\|\Phi(\mu) - \Phi(\mu')\|_p \leq \frac{C_1\|\mu - \mu'\|_p}{\sqrt{(1 - \|\mu\|_\infty^2)(1 - \|\mu'\|_\infty^2)}},$$

where $C_1 > 0$ is a constant depending only on $\|\mu'\|_\infty$.

Claim 2. For $\mu_1, \mu_2 \in \mathrm{Bel}(\mathbf{D})$ and $\nu^{-1} \in \sigma(\mathrm{Bel}(\mathbf{D}))$, we have that

$$\|r_\nu(\mu_1) - r_\nu(\mu_2)\|_p \leq C_2\|\mu_1 - \mu_2\|_p,$$

where $C_2 > 0$ is a constant depending only on $\|\nu\|_\infty$, $\|\mu_1\|_\infty$, and $\|\mu_2\|_\infty$.

For any $\mu \in \mathrm{Ael}^p(\mathbf{D})$ and $\nu \in \sigma(\mathrm{Bel}(\mathbf{D}))$, we have only to show that $\Phi(r_\nu^{-1}(\mu)) - \Phi(\nu) \in A^p(\mathbf{D}^*)$. By Claim 2 applied to the right translation $r_{\nu^{-1}} = r_\nu^{-1}$, we have that

$$\|r_\nu^{-1}(\mu) - \nu\|_p = \|r_\nu^{-1}(\mu) - r_\nu^{-1}(0)\|_p \leq C_2\|\mu\|_p < \infty.$$

Then, Claim 1 yields that

$$\|\Phi(r_\nu^{-1}(\mu)) - \Phi(\nu)\|_p \leq \frac{C_1\|r_\nu^{-1}(\mu) - \nu\|_p}{\sqrt{(1 - \|r_\nu^{-1}(\mu)\|_\infty^2)(1 - \|\nu\|_\infty^2)}} < \infty.$$

This proves the inclusion $\subset$.

For the other inclusion $\supset$, we take $\varphi \in A^p(\mathbf{D}^*)$ such that $\varphi + \psi \in \beta(T)$. Since $A^p(\mathbf{D}^*) \subset B_0(\mathbf{D}^*)$, Proposition 3.1 asserts that there is a quasiconformal homeomorphism $\widehat{f}\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ conformal on Ω^* and asymptotically conformal on Ω such that $S_{\widehat{f} \circ f_\nu|_{\mathbf{D}^*}} = \varphi + \psi$. According to Proposition 3.3, we consider the decomposition $\widehat{f} = \widehat{f}_0 \circ \widehat{f}_1$ together with the maps that appear in this proposition. We use the properties shown in this proposition.

Since $\varphi \in A^p(\mathbf{D}^*)$, if $\varphi - \varphi_1 \in A^p(\mathbf{D}^*)$, then $\varphi_1 \in A^p(\mathbf{D}^*)$. By property (2), μ_0 in particular belongs to $\mathrm{Ael}^p(\mathbf{D})$, and then by property (3) and the consequence from Claims 1 and 2 just proved above, we have that $\varphi - \varphi_1 \in A^p(\mathbf{D}^*)$. Hence, $\varphi_1 \in A^p(\mathbf{D}^*)$.

We estimate the p-norm of $\widehat{\mu}_1 \circ f_\nu$. Property (1) yields that

$$\int_{\mathbf{D}} |\widehat{\mu}_1 \circ f_\nu(z)|^p \rho_{\mathbf{D}}^2(z)\, dx\, dy \leq \frac{1}{\delta^p} \int_{\mathbf{D}} (\rho_{\mathbf{D}^*}^{-2}(z^*)|\varphi_1(z^*)|)^p \rho_{\mathbf{D}}^2(z)\, dx\, dy.$$

Here, we change the variables from $z = x + iy$ to $z^* = (\bar{z})^{-1} = x^* + iy^*$. By $\rho_{\mathbf{D}}^2(z)\, dx\, dy = \rho_{\mathbf{D}^*}^2(z^*)\, dx^*\, dy^*$, the last integral is equal to

$$\int_{\mathbf{D}^*} |\varphi_1(z^*)|^p \rho_{\mathbf{D}^*}^{2-2p}(z^*)\, dx^*\, dy^*,$$

which is finite by $\varphi_1 \in A^p(\mathbf{D}^*)$. Hence, $\widehat{\mu}_1 \circ f_\nu \in \mathrm{Ael}^p(\mathbf{D})$.

We will show that the complex dilatation μ_1 of $f_1\colon \mathbf{D} \to \mathbf{D}$ belongs to $\mathrm{Ael}^p(\mathbf{D})$. Since $|\widehat{\mu}_1 \circ f_\nu| = |\mu_1 \circ f^\nu|$, we have that $\mu_1 \circ f^\nu \in \mathrm{Ael}^p(\mathbf{D})$. Then,

$$\int_{\mathbf{D}} |\mu_1(\zeta)|^p \rho_{\mathbf{D}}^2(\zeta)\, d\xi\, d\eta = \int_{\mathbf{D}} |\mu_1(f^\nu(z))|^p \rho_{\mathbf{D}}^2(f^\nu(z)) J_{f^\nu}(z)\, dx\, dy$$

for $\zeta = f^{\nu}(z)$. We may assume that f^{ν} is obtained by the barycentric extension. In this case, the Jacobian $J_{f^{\nu}}$ is estimated as

$$\rho^2_{\mathbf{D}}(f^{\nu}(z))J_{f^{\nu}}(z) \leq C\rho^2_{\mathbf{D}}(z),$$

where $C > 0$ is a constant depending only on $\|\nu\|_\infty$ ([6, Theorem 2]). Thus, we see that the above integral is finite.

By property (2), the support of the complex dilatation μ_0 of $f_0\colon \mathbf{D} \to \mathbf{D}$ is contained in a compact subset of $\mathbf{D}$. Hence, we see that the complex dilatation $\mu_f = \mu_0 * \mu_1$ of $f = f_0 \circ f_1$ belongs to $\mathrm{Ael}^p(\mathbf{D})$. Since the complex dilatation on $\mathbf{D}$ of the quasiconformal homeomorphism $\widehat{f} \circ f_\nu$ is $r_\nu^{-1}(\mu_f)$, we have that

$$\varphi + \psi = \Phi(r_\nu^{-1}(\mu_f)) \in \Phi \circ r_\nu^{-1}(\mathrm{Ael}^p(\mathbf{D})),$$

which shows the inclusion $\supset$. □

This result can be generalized to the p-integrable Teichmüller space $T^p(\mathbf{D}/\Gamma)$ of a Riemann surface $\mathbf{D}/\Gamma$ for a Fuchsian group Γ whose hyperbolic elements have translation lengths uniformly bounded away from 0. We say that such a Fuchsian group Γ satisfies the *Lehner condition*. This Teichmüller space $T^p(\mathbf{D}/\Gamma)$ was introduced by Yanagishita [29].

A Beltrami coefficient $\mu \in \mathrm{Bel}(\mathbf{D})$ is Γ-invariant if

$$(\gamma^*\mu)(z) := \mu(\gamma(z))\frac{\overline{\gamma'(z)}}{\gamma'(z)} = \mu(z) \quad (\text{a.e. } z \in \mathbf{D})$$

for every $\gamma \in \Gamma$. Such a μ projects down to a Beltrami differential on the Riemann surface $\mathbf{D}/\Gamma$, and the space of all Γ-invariant Beltrami coefficients $\mu \in \mathrm{Bel}(\mathbf{D})$ is denoted by $\mathrm{Bel}(\mathbf{D}/\Gamma)$. We define the space of all Γ-invariant p-integrable Beltrami coefficients on $\mathbf{D}$ as

$$\mathrm{Ael}^p(\mathbf{D}/\Gamma) = \{\mu \in \mathrm{Bel}(\mathbf{D}/\Gamma) \mid \|\mu\|_p^p := \int_{\mathbf{D}/\Gamma} |\mu(z)|^p \rho^2_{\mathbf{D}}(z)\,dx\,dy < \infty\}.$$

A holomorphic quadratic automorphic form $\varphi \in B(\mathbf{D}^*)$ is Γ-invariant if

$$(\gamma^*\varphi)(z) := \varphi(\gamma(z))\gamma'(z)^2 = \varphi(z) \quad (\forall z \in \mathbf{D})$$

for every $\gamma \in \Gamma$. Such a φ projects down to a holomorphic quadratic differentials on the Riemann surface $\mathbf{D}^*/\Gamma$, and the Banach space of all Γ-invariant bounded holomorphic quadratic automorphic forms $\varphi \in B(\mathbf{D}^*)$ is denoted by $B(\mathbf{D}^*/\Gamma)$. We define the Banach space of all Γ-invariant p-integrable holomorphic quadratic automorphic forms on $\mathbf{D}^*$ as

$$A^p(\mathbf{D}^*/\Gamma) = \{\varphi \in B(\mathbf{D}^*/\Gamma) \mid \|\varphi\|_p^p := \int_{\mathbf{D}^*/\Gamma} |\varphi(z)|^p \rho^{2-2p}_{\mathbf{D}^*}(z)\,dx\,dy < \infty\}.$$

We apply the Teichmüller projection $\pi\colon \mathrm{Bel}(\mathbf{D}) \to T$ and the Bers projection $\Phi\colon \mathrm{Bel}(\mathbf{D}) \to B(\mathbf{D}^*)$ to $\mathrm{Bel}(\mathbf{D}/\Gamma)$ and $\mathrm{Ael}^p(\mathbf{D}/\Gamma)$, respectively. The Teichmüller space $T(\mathbf{D}/\Gamma)$ of the Riemann surface $\mathbf{D}/\Gamma$ can be defined as $T(\mathbf{D}/\Gamma) = \pi(\mathrm{Bel}(\mathbf{D}/\Gamma))$. The topology on $T(\mathbf{D}/\Gamma)$ is induced by the norm $\|\cdot\|_\infty$ on $\mathrm{Bel}(\mathbf{D}/\Gamma)$. It is well-known (see [10, 14, 22]) that the Bers embedding β restricted to $T(\mathbf{D}/\Gamma)$ is a homeomorphism onto the image

$$\beta(T(\mathbf{D}/\Gamma)) = \Phi(\mathrm{Bel}(\mathbf{D}/\Gamma)) = \beta(T) \cap B(\mathbf{D}^*/\Gamma),$$

which is a bounded domain in $B(\mathbf{D}^*/\Gamma)$. By the conformal naturality of the barycentric extension, we have $\sigma(\mathrm{Bel}(\mathbf{D}/\Gamma)) \subset \mathrm{Bel}(\mathbf{D}/\Gamma)$.

The p-integrable Teichmüller space is defined by $T^p(\mathbf{D}/\Gamma) = \pi(\mathrm{Ael}^p(\mathbf{D}/\Gamma))$. The topology on $T^p(\mathbf{D}/\Gamma)$ is induced by the norm $\|\cdot\|_p + \|\cdot\|_\infty$ on $\mathrm{Ael}^p(\mathbf{D}/\Gamma)$. It was shown in [29, Theorem 4.4] that β restricted to $T^p(\mathbf{D}/\Gamma)$ is a homeomorphism onto the image

$$\beta(T^p(\mathbf{D}/\Gamma)) = \Phi(\mathrm{Ael}^p(\mathbf{D}/\Gamma)).$$

which is a domain in $A^p(\mathbf{D}^*/\Gamma)$. Properties of $\sigma\colon \mathrm{Ael}^p(\mathbf{D}/\Gamma) \to \mathrm{Ael}^p(\mathbf{D}/\Gamma)$ induced by the barycentric extension, the right translation $r_\nu\colon \mathrm{Ael}^p(\mathbf{D}/\Gamma) \to \mathrm{Ael}^p(\mathbf{D}/\Gamma)$, and the base point change $R_\tau\colon T^p(\mathbf{D}/\Gamma) \to T^p(\mathbf{D}/\Gamma)$ are the same as in the case where Γ is trivial, which are all given in [29].

The Γ-invariant version of the previous theorem is as follows.

Theorem 4.2. *We assume that a Fuchsian group Γ satisfies the Lehner condition. For each $\nu \in \mathrm{Bel}(\mathbf{D}/\Gamma)$, let $\psi = \Phi(\nu) \in B(\mathbf{D}^*/\Gamma)$. Then,*

$$\Phi \circ r_\nu^{-1}(\mathrm{Ael}^p(\mathbf{D}/\Gamma)) = \beta(T(\mathbf{D}/\Gamma)) \cap \{A^p(\mathbf{D}^*/\Gamma) + \psi\}.$$

Hence, $\beta \circ R_\tau^{-1}(T^p(\mathbf{D}/\Gamma)) = \beta(T(\mathbf{D}/\Gamma)) \cap \{A^p(\mathbf{D}^*/\Gamma) + \beta(\tau)\}$ for every $\tau \in T(\mathbf{D}/\Gamma)$.

Proof. We only refer to the points where the compatibility with Γ is required in addition to the proof of Theorem 4.1. The other places are similarly carried out by the previous arguments. (1) The corresponding statement to Theorem 2.2 is valid for $T(\mathbf{D}/\Gamma)$ and $T_0(\mathbf{D}/\Gamma)$ by Earle, Markovic and Saric [8, Theorem 4]. (2) Lemma 3.2 and Proposition 3.3 respecting the compatibility with Γ are already valid as they are since the Earle–Nag quasiconformal reflection can be used for constructing a local section of the Bers projection (see also [10, Section 14.3] and [14, Section V.4.8]). (3) The corresponding statement to Claim 1 can be obtained by also using the proof of [29, Proposition 3.2]. (4) The corresponding statement to Claim 2 can be obtained by also using the proof of [29, Proposition 5.1]. □

Theorem 4.2 in particular implies that $\beta(T(\mathbf{D}/\Gamma)) \cap \{A^p(\mathbf{D}^*/\Gamma) + \beta(\tau)\}$ is connected for each $\tau \in T(\mathbf{D}/\Gamma)$, and any such components are biholomorphically equivalent to each other, which admits the Banach manifold structure.

We consider the projections $T(\mathbf{D}/\Gamma) \to T^p(\mathbf{D}/\Gamma)\backslash T(\mathbf{D}/\Gamma)$ and $B(\mathbf{D}^*/\Gamma) \to A^p(\mathbf{D}^*/\Gamma)\backslash B(\mathbf{D}^*/\Gamma)$. These maps are continuous and open with respect to the quotient topology. Then, we see that the projection of the Bers embedding β to these quotient spaces is not only well-defined to be an injection but also a homeomorphism onto the image.

Corollary 4.3. *The quotient Bers embedding*

$$\widehat{\beta}^p\colon T^p(\mathbf{D}/\Gamma)\backslash T(\mathbf{D}/\Gamma) \to A^p(\mathbf{D}^*/\Gamma)\backslash B(\mathbf{D}^*/\Gamma)$$

is well-defined to be a homeomorphism onto the image for any Fuchsian group Γ satisfying the Lehner condition.

5. The Teichmüller space of circle diffeomorphisms of Hölder continuous derivative

In this section, we prove the affine foliated structure of the universal Teichmüller space T induced by the Teichmüller space $T_0^{>0}$ of circle diffeomorphisms of Hölder continuous derivative.

For a constant $\alpha \in (0,1)$, we denote by $\mathrm{Diff}_+^{1+\alpha}(\mathbf{S})$ the group of all orientation-preserving diffeomorphisms g of the unit circle $\mathbf{S}$ whose derivatives are α-Hölder continuous; there is a constant $c \geq 0$ such that

$$|g'(x) - g'(y)| \leq c|x-y|^\alpha$$

for any $x, y \in \mathbf{S} = \mathbf{R}/\mathbf{Z}$. We give a characterization of $\mathrm{Diff}_+^{1+\alpha}(\mathbf{S})$ analogously to Proposition 2.1 by considering the following spaces:

$$T_0^\alpha = \mathrm{M\ddot{o}b}(\mathbf{S}) \backslash \mathrm{Diff}_+^{1+\alpha}(\mathbf{S});$$
$$\mathrm{Bel}_0^\alpha(\mathbf{D}) = \{\mu \in \mathrm{Bel}_0(\mathbf{D}) \mid \|\mu\|_{\infty,\alpha} := \operatorname{ess\,sup}_{z\in\mathbf{D}} \rho_{\mathbf{D}}^\alpha(z)|\mu(z)| < \infty\};$$
$$B_0^\alpha(\mathbf{D}^*) = \{\varphi \in B_0(\mathbf{D}^*) \mid \|\varphi\|_{\infty,\alpha} := \sup_{z\in\mathbf{D}^*} \rho_{\mathbf{D}^*}^{-2+\alpha}(z)|\varphi(z)| < \infty\}.$$

We define T_0^α to be the *Teichmüller space of circle diffeomorphisms* of α-Hölder continuous derivatives ([16, 17, 18, 21]).

Proposition 5.1. *For a quasisymmetric homeomorphism $g \in \mathrm{QS}$, the following conditions are equivalent:* (1) $g \in \mathrm{Diff}_+^{1+\alpha}(\mathbf{S})$; (2) $s([g]) \in \mathrm{Bel}_0^\alpha(\mathbf{D})$; (3) $\beta([g]) \in B_0^\alpha(\mathbf{D}^*)$.

We consider the Teichmüller space $T_0^{>\alpha} = \bigcup_{\varepsilon>0} T_0^{\alpha+\varepsilon}$ of circle diffeomorphisms of Hölder continuous derivative with exponent greater than $\alpha \in [0,1)$. In other words, for $\mathrm{Diff}_+^{>1+\alpha}(\mathbf{S}) = \bigcup_{\varepsilon>0} \mathrm{Diff}_+^{1+\alpha+\varepsilon}(\mathbf{S})$, we define $T_0^{>\alpha} = \mathrm{M\ddot{o}b}(\mathbf{S}) \backslash \mathrm{Diff}_+^{>1+\alpha}(\mathbf{S})$. Correspondingly, we set the unions of increasing subspaces as

$$\mathrm{Bel}_0^{>\alpha}(\mathbf{D}) = \bigcup_{\varepsilon>0} \mathrm{Bel}_0^{\alpha+\varepsilon}(\mathbf{D}); \quad B_0^{>\alpha}(\mathbf{D}^*) = \bigcup_{\varepsilon>0} B_0^{\alpha+\varepsilon}(\mathbf{D}^*).$$

The norms on these spaces can be given by $\lim_{\varepsilon\to 0} \|\mu\|_{\infty,\alpha+\varepsilon}$ and $\lim_{\varepsilon\to 0} \|\varphi\|_{\infty,\alpha+\varepsilon}$, respectively. We see that $T_0^{>\alpha} = \pi(\mathrm{Bel}_0^{>\alpha}(\mathbf{D}))$ and $\beta(T_0^{>\alpha}) = \beta(T) \cap B_0^{>\alpha}(\mathbf{D}^*)$. In particular, for $\alpha = 0$, we have the Teichmüller space $T_0^{>0}$ of circle diffeomorphisms of Hölder continuous derivative with an arbitrary exponent.

The following lemma plays the role of the combination of Claims 1 and 2 in the previous proof, and used to show that the quotient Bers embedding is well-defined. This lemma also tells us that the exact T_0^α does not seem to induce an affine foliated structure for T.

Lemma 5.2. *Let $\nu \in \mathrm{Bel}(\mathbf{D})$ hold $\|\nu\|_\infty < k < 1$ and put $K = (1+k)/(1-k)$. Then,*

$$\|\Phi(r_\nu(\mu_1)) - \Phi(r_\nu(\mu_2))\|_{\infty,\alpha/K^3} \leq C\|\mu_1 - \mu_2\|_{\infty,\alpha}$$

is satisfied for any $\mu_1, \mu_2 \in \mathrm{Bel}_0^\alpha(\mathbf{D})$ with $\alpha \in (0,1)$, where $C > 0$ depends only on k, $\|\mu_1\|_\infty$, and $\|\mu_2\|_{\infty,\alpha}$.

Proof. The Mori theorem (see [1, Section III.C]) implies that there is some constant $C_0 \geq 1$ depending only on k such that

$$\frac{1}{C_0}(1-|z|)^K \leq 1 - |\zeta|$$

for $\zeta = f^\nu(z)$. Then,

$$\rho_{\mathbf{D}}^{\alpha/K}(\zeta) = \left(\frac{2}{1-|\zeta|^2}\right)^{\alpha/K} \leq 2C_0 \left(\frac{2}{1-|z|^2}\right)^\alpha = 2C_0 \rho_{\mathbf{D}}^\alpha(z),$$

and it follows from Proposition 3.5 that

$$\rho_{\mathbf{D}}^{\alpha/K}(\zeta)|r_\nu(\mu_1)(\zeta) - r_\nu(\mu_2)(\zeta)| \leq C_1 \rho_{\mathbf{D}}^\alpha(z)|\mu_1(z) - \mu_2(z)|$$

for $C_1 = 2C_0/\sqrt{(1-\|\mu_1\|_\infty^2)(1-\|\mu_2\|_\infty^2)}$. Thus, we have

$$\|r_\nu(\mu_1) - r_\nu(\mu_2)\|_{\infty,\alpha/K} \le C_1\|\mu_1 - \mu_2\|_{\infty,\alpha}.$$

We see from the proof of [20, Lemma 3.3] based on Proposition 3.4 that

$$\|\Phi(\mu_1') - \Phi(\mu_2')\|_{\infty,\,\alpha/\widetilde{K}^2} \le C_2\|\mu_1' - \mu_2'\|_{\infty,\alpha}$$

for $\widetilde{K} = (1+\tilde{k})/(1-\tilde{k})$ with $\tilde{k} = \|\mu_2'\|_\infty$, where $C_2 > 0$ is a constant given by $\|\mu_1'\|_\infty$ and $\|\mu_2'\|_\infty$. We apply this inequality for $\mu_1' = r_\nu(\mu_1)$ and $\mu_2' = r_\nu(\mu_2)$ to obtain

$$\|\Phi(r_\nu(\mu_1)) - \Phi(r_\nu(\mu_2))\|_{\infty,\,\alpha/(K\widetilde{K}^2)} \le C_2\|r_\nu(\mu_1) - r_\nu(\mu_2)\|_{\infty,\alpha/K}.$$

However, since $\mu_2 \in \mathrm{Bel}_0^\alpha(\mathbf{D})$, we can assume $\tilde{k} = \|r_\nu(\mu_2)\|_\infty$ to be arbitrarily close to $\|\nu\|_\infty$ by allowing C_2 to depend on $\|\mu_2\|_{\infty,\alpha}$. Indeed, to see this, we use the decomposition $\mu_2 = \mu_{20} * \mu_{21}$ such that μ_{20} has a compact support in $\mathbf{D}$ and $\|\mu_{21}\|_\infty$ is arbitrarily small. Then, we apply the previous estimate for $\|r_\nu(\mu_1) - r_\nu(\mu_2)\|_{\infty,\alpha/K}$ and conclude that

$$\|\Phi(r_\nu(\mu_1)) - \Phi(r_\nu(\mu_2))\|_{\infty,\,\alpha/K^3} \le C_1C_2\|\mu_1 - \mu_2\|_{\infty,\alpha}.$$

Here, the constant C can be chosen as $C = C_1C_2$ which depends only on k, $\|\mu_1\|_\infty$, and $\|\mu_2\|_{\infty,\alpha}$. □

We state the main result in this section. The arguments are parallel to those for Theorem 4.1 in some parts.

Theorem 5.3. *For each $\nu \in \mathrm{Bel}(\mathbf{D})$, let $\psi = \Phi(\nu) \in B(\mathbf{D}^*)$. Then,*

$$\Phi \circ r_\nu^{-1}(\mathrm{Bel}_0^{>0}(\mathbf{D})) = \beta(T) \cap \{B_0^{>0}(\mathbf{D}^*) + \psi\}.$$

Hence, $\beta \circ R_\tau^{-1}(T_0^{>0}) = \beta(T) \cap \{B_0^{>0}(\mathbf{D}^*) + \beta(\tau)\}$ for every $\tau \in T$.

Proof. For one inclusion $\subset$, we take an arbitrary $\mu \in \mathrm{Bel}_0^{>0}(\mathbf{D})$. There is some $\alpha \in (0,1)$ such that $\mu \in \mathrm{Bel}_0^\alpha(\mathbf{D})$. We apply Lemma 5.2 for $\mu_1 = \mu$ and $\mu_2 = 0$ with the right translation $r_{\nu^{-1}} = r_\nu^{-1}$. Then, we obtain

$$\|\Phi(r_\nu^{-1}(\mu)) - \Phi(\nu)\|_{\infty,\,\alpha/K^3} = \|\Phi(r_\nu^{-1}(\mu)) - \Phi(r_\nu^{-1}(0))\|_{\infty,\,\alpha/K^3} \le C\|\mu\|_{\infty,\alpha} < \infty.$$

This implies that $\Phi(r_\nu^{-1}(\mu)) \in B_0^{>0}(\mathbf{D}^*) + \Phi(\nu)$, and hence $\Phi \circ r_\nu^{-1}(\mathrm{Bel}_0^{>0}(\mathbf{D}))$ is contained in $B_0^{>0}(\mathbf{D}^*) + \psi$.

For the other inclusion $\supset$, we take $\varphi \in B_0^{>0}(\mathbf{D}^*)$ such that $\varphi + \psi \in \beta(T)$. Since $B_0^{>0}(\mathbf{D}^*) \subset B_0(\mathbf{D}^*)$, Proposition 3.1 asserts that there is a quasiconformal homeomorphism $\widehat{f}\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ conformal on Ω^* and asymptotically conformal on Ω such that $S_{\widehat{f}\circ f_\nu|_{\mathbf{D}^*}} = \varphi + \psi$. According to Proposition 3.3, we consider the decomposition $\widehat{f} = \widehat{f}_0 \circ \widehat{f}_1$ with related maps. Since $\varphi \in B_0^{>0}(\mathbf{D}^*)$, if $\varphi - \varphi_1 \in B_0^{>0}(\mathbf{D}^*)$, then $\varphi_1 \in B_0^{>0}(\mathbf{D}^*)$. By property (2), μ_0 in particular belongs to $\mathrm{Bel}_0^{>0}(\mathbf{D})$, and then by property (3) and the consequence from Lemma 5.2 just proved above, we have that $\varphi - \varphi_1 \in B_0^{>0}(\mathbf{D}^*)$. Hence, $\varphi_1 \in B_0^{>0}(\mathbf{D}^*)$. Property (1) yields that

$$|\widehat{\mu}_1 \circ f_\nu(z)| \le \frac{1}{\delta}\,\rho_{\mathbf{D}^*}^{-2}(z^*)|\varphi_1(z^*)|.$$

This implies that $\widehat{\mu}_1 \circ f_\nu \in \mathrm{Bel}_0^{>0}(\mathbf{D})$.

We will show that the complex dilatation μ_1 of $f_1\colon \mathbf{D} \to \mathbf{D}$ belongs to $\mathrm{Bel}_0^{>0}(\mathbf{D})$. Since $|\widehat{\mu}_1 \circ f_\nu| = |\mu_1 \circ f^\nu|$, we have that $\mu_1 \circ f^\nu \in \mathrm{Bel}_0^{>0}(\mathbf{D})$. By the Mori theorem at the boundary, we know that

$$\frac{1}{C_0}(1-|z|)^K \le 1 - |f^\nu(z)| \quad (z \in \mathbf{D})$$

for some constant $C_0 \geq 1$, where $K = (1 + \|\nu\|_\infty)/(1 - \|\nu\|_\infty)$. From this estimate, we see that $\mu_1 \in \mathrm{Bel}_0^{>0}(\mathbf{D})$.

By property (2), the support of μ_0 is contained in a compact subset of $\mathbf{D}$. Hence, we see that the complex dilatation $\mu_f = \mu_0 * \mu_1$ of $f = f_0 \circ f_1$ belongs to $\mathrm{Bel}_0^{>0}(\mathbf{D})$. Since the complex dilatation on $\mathbf{D}$ of the quasiconformal homeomorphism $\widehat{f} \circ f_\nu$ is $r_\nu^{-1}(\mu_f)$, we have that

$$\varphi + \psi = \Phi(r_\nu^{-1}(\mu_f)) \in \Phi \circ r_\nu^{-1}(\mathrm{Bel}_0^{>0}(\mathbf{D})),$$

which shows the inclusion $\supset$. □

Corollary 5.4. *The quotient Bers embedding* $\widehat{\beta}_0^{>0} : T_0^{>0} \backslash T \to B_0^{>0}(\mathbf{D}^*) \backslash B(\mathbf{D}^*)$ *is well-defined to be a homeomorphism onto the image.*

6. Stratification of foliated structures

We can also consider the affine foliated structure for a certain subspace of the universal Teichmüller space T induced by a smaller subspace. We consider again the affine foliated structure for this smaller subspace, and repeat this process. Then, we obtain a stratification of affine foliated structures of T. In this section, we observe such an example.

First, we show the affine foliated structure for the little Teichmüller space T_0 induced by $T_0^{>\alpha}$, the Teichmüller space of circle diffeomorphisms of Hölder continuous derivative of exponent greater than $\alpha \in [0, 1)$. We in particular have the affine foliated structure for the little subspace T_0 by $T_0^{>0}$.

We prepare the asymptotically conformal version of the Mori theorem at the boundary. The corresponding result under a stronger assumption that the complex dilatation μ has an explicit decay order as in $\mathrm{Bel}_0^\alpha(\mathbf{D})$ is given in [21, Theorem 6.4].

Lemma 6.1. *Let* $f^\nu \in \mathrm{AC}(\mathbf{D})$ *be a normalized asymptotically conformal homeomorphism of* $\mathbf{D}$ *with the complex dilatation* $\nu \in \mathrm{Bel}_0(\mathbf{D})$*. Let* $\varepsilon > 0$ *be an arbitrary positive constant. Then, there is a constant* $A \geq 1$ *depending only on* ν *and* ε *such that*

$$\frac{1}{A}(1 - |z|)^{1+\varepsilon} \leq 1 - |f^\nu(z)| \leq A(1 - |z|)^{1-\varepsilon}$$

for every $z \in \mathbf{D}$.

Proof. Since $\nu \in \mathrm{Bel}_0(\mathbf{D})$, we can find $t_0 \in (0, 1/4)$ so that $|\nu(\zeta)| \leq \varepsilon/2$ for almost every $\zeta \in \mathbf{D}$ with $|\zeta| > 1 - \sqrt{t_0}$. This depends on ν and ε. We define a Beltrami coefficient $\nu_0(\zeta)$ by setting $\nu_0(\zeta) = \nu(\zeta)$ on $|\zeta| \leq 1 - \sqrt{t_0}$ and $\nu_0(\zeta) \equiv 0$ elsewhere. Let f_0 be the normalized quasiconformal homeomorphism of $\mathbf{D}$ with the complex dilatation ν_0. Let f_1 be the quasiconformal homeomorphism of $\mathbf{D}$ such that $f^\nu = f_1 \circ f_0$. For $K = (1 + \varepsilon/2)/(1 - \varepsilon/2)$, we see that f_1 is a K-quasiconformal homeomorphism of $\mathbf{D}$. Here, we have

$$\frac{1}{K} = \frac{1 - \varepsilon/2}{1 + \varepsilon/2} \geq 1 - \varepsilon.$$

First, we apply a distortion theorem to the conformal homeomorphism $f_0(\zeta)$ restricted to $\zeta \in \mathbf{D}$ with $|\zeta| > 1 - \sqrt{t_0}$. In fact, we may assume that f_0 is a conformal homeomorphism of an annulus $\{1 - \sqrt{t_0} < |\zeta| < 1/(1 - \sqrt{t_0})\}$ by the reflection principle. Since $\mathbf{S}$ is compact, there is some constant $L \geq 1$ such that the modulus of the derivative $|f_0'(\xi)|$ at any $\xi \in \mathbf{S}$ is bounded by L, which is depending only on

t_0. The Koebe distortion theorem (see [24, Theorem 1.3]) in the disk of radius $\sqrt{t_0}$ and center $\xi = z/|z|$ yields

$$1-|f_0(z)| \le \frac{L(1-|z|)}{\{1-(1-|z|)/\sqrt{t_0}\}^2} \le \frac{L(1-|z|)}{(1-\sqrt{t_0})^2} \le 4L(1-|z|)$$

for every $z \in \mathbf{D}$ with $1-|z| < t_0$.

Next, we apply the Mori theorem at the boundary to the quasiconformal homeomorphism f_1 of $\mathbf{D}$. It implies that there is a constant $C_0 = C_0(\nu) \ge 1$ such that

$$1-|f_1(w)| \le C_0(1-|w|)^{1/K} \le C_0(1-|w|)^{1-\varepsilon}$$

for every $w \in \mathbf{D}$. Then, by setting $w = f_0(z)$, we have that

$$1-|f^\nu(z)| \le C_0\{4L(1-|z|)\}^{1-\varepsilon} \le 4C_0L(1-|z|)^{1-\varepsilon}.$$

If $1-|z| \ge t_0$, we simply obtain

$$1-|f^\nu(z)| \le 1 \le \frac{1}{t_0}(1-|z|)^{1-\varepsilon}.$$

Combined with the previous estimate, this gives the right side inequality in the statement. For the left side inequality, we apply the lower estimates in both the Koebe and the Mori theorems, or apply the above arguments to the inverse map $(f^\nu)^{-1}$. □

Theorem 6.2. *Let $\alpha \in [0,1)$ be an arbitrary exponent. For each $\nu \in \mathrm{Bel}_0(\mathbf{D})$, let $\psi = \Phi(\nu) \in B_0(\mathbf{D}^*)$. Then,*

$$\Phi \circ r_\nu^{-1}(\mathrm{Bel}_0^{>\alpha}(\mathbf{D})) = \beta(T_0) \cap \{B_0^{>\alpha}(\mathbf{D}^*) + \psi\}.$$

Hence, $\beta \circ R_\tau^{-1}(T_0^{>\alpha}) = \beta(T) \cap \{B_0^{>\alpha}(\mathbf{D}^*) + \beta(\tau)\}$ for every $\tau \in T_0$.

Proof. For one inclusion $\subset$, we take an arbitrary $\mu \in \mathrm{Bel}_0^{>\alpha}(\mathbf{D})$. Then, there is some $\alpha' \in (0,1)$ such that $\alpha < \alpha'$ and $\mu \in \mathrm{Bel}_0^{\alpha'}(\mathbf{D})$. We choose $\varepsilon > 0$ such that $\alpha + \varepsilon < \alpha'$. We apply Lemma 6.1 to the argument of Lemma 5.2 to obtain that

$$\|\Phi(r_\nu^{-1}(\mu)) - \Phi(\nu)\|_{\infty,\alpha'-\varepsilon} = \|\Phi(r_\nu^{-1}(\mu)) - \Phi(r_\nu^{-1}(0))\|_{\infty,\alpha'-\varepsilon} \le C\|\mu\|_{\infty,\alpha'} < \infty$$

for some constant $C > 0$. This implies that $\Phi(r_\nu^{-1}(\mu)) - \Phi(\nu) \in B_0^{>\alpha}(\mathbf{D}^*)$, and hence $\Phi \circ r_\nu^{-1}(\mathrm{Bel}_0^{>\alpha}(\mathbf{D}))$ is contained in $B_0^{>\alpha}(\mathbf{D}^*) + \psi$.

For the other inclusion $\supset$, we take $\varphi \in B_0^{>\alpha}(\mathbf{D}^*)$ such that $\varphi + \psi \in \beta(T)$. By Proposition 3.1, there is a quasiconformal homeomorphism $\widehat{f} : \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ conformal on Ω^* and asymptotically conformal on Ω such that $S_{\widehat{f} \circ f_\nu|_{\mathbf{D}^*}} = \varphi + \psi$. We consider the decomposition $\widehat{f} = \widehat{f}_0 \circ \widehat{f}_1$ as in Proposition 3.3.

By property (2), μ_0 in particular belongs to $\mathrm{Bel}_0^{>\alpha}(\mathbf{D})$, and then by property (3) and the result proved above, we have that $\varphi - \varphi_1 \in B_0^{>\alpha}(\mathbf{D}^*)$. Hence, $\varphi_1 \in B_0^{>\alpha}(\mathbf{D}^*)$ follows from $\varphi \in B_0^{>\alpha}(\mathbf{D}^*)$. Property (1) yields that

$$|\widehat{\mu}_1 \circ f_\nu(z)| \le \frac{1}{\delta}\rho_{\mathbf{D}^*}^{-2}(z^*)|\varphi_1(z^*)|.$$

This implies that $\widehat{\mu}_1 \circ f_\nu \in \mathrm{Bel}_0^{>\alpha}(\mathbf{D})$.

Let μ_1 be the complex dilatation of $f_1\colon \mathbf{D} \to \mathbf{D}$. Since $|\widehat{\mu}_1 \circ f_\nu| = |\mu_1 \circ f^\nu|$, we have that $\mu_1 \circ f^\nu \in \mathrm{Bel}_0^{\alpha'}(\mathbf{D})$ for some $\alpha' > \alpha$. Having $\nu \in \mathrm{Bel}_0(\mathbf{D})$, we apply Lemma 6.1 to $\zeta = f^\nu(z)$. We choose $\varepsilon' > 0$ such that $(1+\varepsilon')^{-1}\alpha' > \alpha$. Then, we have

$$\frac{1}{A}(1-|z|)^{1+\varepsilon'} \le 1-|f^\nu(z)| = 1-|\zeta|$$

for some constant $A \geq 1$. This shows that $\mu_1 \in \mathrm{Bel}_0^{>\alpha}(\mathbf{D})$.

By property (2) again, the complex dilatation $\mu_f = \mu_0 * \mu_1$ of $f = f_0 \circ f_1$ belongs to $\mathrm{Bel}_0^{>\alpha}(\mathbf{D})$. Since the complex dilatation on $\mathbf{D}$ of the quasiconformal homeomorphism $\widehat{f} \circ f_\nu$ is $r_\nu^{-1}(\mu_f)$, we have that

$$\varphi + \psi = \Phi(r_\nu^{-1}(\mu_f)) \in \Phi \circ r_\nu^{-1}(\mathrm{Bel}_0^{>\alpha}(\mathbf{D})),$$

which shows the inclusion $\supset$. □

Next, we consider the affine foliated structure of the Teichmüller space $T_0^{>0}$ of circle diffeomorphisms of Hölder continuous derivative induced by T_0^α.

Theorem 6.3. *For each $\nu \in \mathrm{Bel}_0^{>0}(\mathbf{D})$, let $\psi = \Phi(\nu) \in B_0^{>0}(\mathbf{D}^*)$. Then,*

$$\Phi \circ r_\nu^{-1}(\mathrm{Bel}_0^\alpha(\mathbf{D})) = \beta(T_0^{>0}) \cap \{B_0^\alpha(\mathbf{D}^*) + \psi\}.$$

Hence, $\beta \circ R_\tau^{-1}(T_0^\alpha) = \beta(T_0^{>0}) \cap \{B_0^\alpha(\mathbf{D}^*) + \beta(\tau)\}$ for every $\tau \in T_0^{>0}$.

Theorem 6.3 in particular implies that the Teichmüller space $T_0^{>0}$ of circle diffeomorphisms of Hölder continuous derivatives of any exponent has the complex Banach manifold structure modeled on $B_0^\alpha(\mathbf{D}^*)$ for every $\alpha \in (0,1)$ and each connected component of $T_0^{>0}$ is biholomorphically equivalent to T_0^α. We note that this is not close in the topology of $T_0^{>0}$. Moreover, the Bers embedding realizes the connected components of $T_0^{>0}$ as its affine foliation by $B_0^\alpha(\mathbf{D}^*)$.

Corollary 6.4. *The Bers embedding $\beta\colon T_0^{>0} \to B_0^{>0}(\mathbf{D}^*)$ provides the complex Banach manifold structure for $T_0^{>0}$ modeled on $B_0^\alpha(\mathbf{D}^*)$ together with its affine foliation.*

Proof of Theorem 6.3. For one inclusion $\subset$, we take an arbitrary $\mu \in \mathrm{Bel}_0^\alpha(\mathbf{D})$. By [20, Theorem 3.6], we obtain for $\nu \in \mathrm{Bel}_0^{>0}(\mathbf{D})$ that

$$\|\Phi(r_\nu^{-1}(\mu)) - \Phi(\nu)\|_{\infty,\alpha} = \|\Phi(r_\nu^{-1}(\mu)) - \Phi(r_\nu^{-1}(0))\|_{\infty,\alpha} \leq C\|\mu\|_{\infty,\alpha} < \infty$$

for some constant $C > 0$. This implies that $\Phi(r_\nu^{-1}(\mu)) - \Phi(\nu) \in B_0^\alpha(\mathbf{D}^*)$, and hence $\Phi \circ r_\nu^{-1}(\mathrm{Bel}_0^\alpha(\mathbf{D}))$ is contained in $B_0^\alpha(\mathbf{D}^*) + \psi$.

For the other inclusion $\supset$, we take $\varphi \in B_0^\alpha(\mathbf{D}^*)$ such that $\varphi + \psi \in \beta(T)$. For a quasiconformal homeomorphism $\widehat{f}\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ conformal on Ω^* and asymptotically conformal on Ω such that $S_{\widehat{f} \circ f_\nu|_{\mathbf{D}^*}} = \varphi + \psi$ (Proposition 3.1), we consider the decomposition $\widehat{f} = \widehat{f}_0 \circ \widehat{f}_1$ as in Proposition 3.3. By property (2), μ_0 belongs to $\mathrm{Bel}_0^\alpha(\mathbf{D})$, and then by property (3) with the above result, we have that $\varphi - \varphi_1 \in B_0^\alpha(\mathbf{D}^*)$. Hence, $\varphi_1 \in B_0^\alpha(\mathbf{D}^*)$ follows from $\varphi \in B_0^\alpha(\mathbf{D}^*)$. Property (1) yields that

$$|\widehat{\mu}_1 \circ f_\nu(z)| \leq \frac{1}{\delta}\, \rho_{\mathbf{D}^*}^{-2}(z^*)|\varphi_1(z^*)|,$$

which implies that $\widehat{\mu}_1 \circ f_\nu \in \mathrm{Bel}_0^\alpha(\mathbf{D})$.

We consider the complex dilatation μ_1 of $f_1\colon \mathbf{D} \to \mathbf{D}$. Since $|\widehat{\mu}_1 \circ f_\nu| = |\mu_1 \circ f^\nu|$, we have that $\mu_1 \circ f^\nu \in \mathrm{Bel}_0^\alpha(\mathbf{D})$. We apply the stronger version of the Mori theorem at the boundary [21, Theorem 6.4] to $\zeta = f^\nu(z)$ for $\nu \in \mathrm{Bel}_0^{>0}(\mathbf{D})$. Then

$$\frac{1}{A}(1 - |z|) \leq 1 - |f^\nu(z)| = 1 - |\zeta|$$

for some constant $A \geq 1$. This shows that $\mu_1 \in \mathrm{Bel}_0^\alpha(\mathbf{D})$. By property (2), the complex dilatation $\mu_f = \mu_0 * \mu_1$ of $f = f_0 \circ f_1$ belongs to $\mathrm{Bel}_0^\alpha(\mathbf{D})$. Since the complex dilatation of $\widehat{f} \circ f_\nu$ on $\mathbf{D}$ is $r_\nu^{-1}(\mu_f)$, we have that

$$\varphi + \psi = \Phi(r_\nu^{-1}(\mu_f)) \in \Phi \circ r_\nu^{-1}(\mathrm{Bel}_0^\alpha(\mathbf{D})).$$

This proves the assertion. □

Now we obtain a stratification of the affine foliated structures of the universal Teichmüller space T as follows: the first level is T by T_0; the second level is T_0 by $T_0^{>0}$; and the third level is $T_0^{>0}$ by T_0^{α}.

7. Applications to representation spaces

The universal asymptotic Teichmüller space is given by $AT = \mathrm{Sym} \setminus \mathrm{QS}$ in Gardiner and Sullivan [11]. This admits a complex structure modeled on the quotient Banach space $B_0(\mathbf{D})\backslash B(\mathbf{D})$. See also Earle, Gardiner and Lakic [7] for the asymptotic Teichmüller space $AT(\mathbf{D}/\Gamma)$ of a Riemann surface. Similarly, for the group $\mathrm{Diff}_+^{>1}(\mathbf{S})$ of all circle diffeomorphisms of Hölder continuous derivatives, we can consider the quotient space $DT = \mathrm{Diff}_+^{>1}(\mathbf{S})\setminus \mathrm{QS}$. However, this is no more a Hausdorff space in the quotient topology of QS.

We impose group compatibility on these spaces. Let $\Gamma \subset \mathrm{Möb}(\mathbf{S}) \cong \mathrm{PSL}(2, \mathbf{R})$ be a non-elementary Fuchsian group. The deformation space of Γ in $\mathrm{Möb}(\mathbf{S})$ is given as the Teichmüller space of Γ, which is defined by

$$T(\Gamma) = \mathrm{Möb}(\mathbf{S})\backslash\{f \in \mathrm{QS} \mid f\Gamma f^{-1} \subset \mathrm{Möb}(\mathbf{S})\} \subset T.$$

This is a closed subspace of T, and can be identified with the Teichmüller space $T(\mathbf{D}/\Gamma)$ of the Riemann surface $\mathbf{D}/\Gamma$. In a similar manner, the deformation space of Γ in Sym is given in [19] as the asymptotic Teichmüller space of Γ, which is defined by

$$AT(\Gamma) = \mathrm{Sym} \setminus\{f \in \mathrm{QS} \mid f\Gamma f^{-1} \subset \mathrm{Sym}\} \subset AT.$$

We remark that this is different from the asymptotic Teichmüller space $AT(\mathbf{D}/\Gamma)$ of the Riemann surface $\mathbf{D}/\Gamma$ studied in [7]. We also define here the deformation space of Γ in $\mathrm{Diff}_+^{>1}(\mathbf{S})$:

$$DT(\Gamma) = \mathrm{Diff}_+^{>1}(\mathbf{S})\backslash\{f \in \mathrm{QS} \mid f\Gamma f^{-1} \subset \mathrm{Diff}_+^{>1}(\mathbf{S})\} \subset DT.$$

Clearly, $AT(\Gamma)$ is closed in AT and $DT(\Gamma)$ is closed in DT.

We consider the canonical projections

$$\alpha : T = \mathrm{Möb}(\mathbf{S})\setminus \mathrm{QS} \longrightarrow AT = \mathrm{Sym} \setminus \mathrm{QS};$$
$$\theta : DT = \mathrm{Diff}_+^{>1}(\mathbf{S})\setminus \mathrm{QS} \longrightarrow AT = \mathrm{Sym} \setminus \mathrm{QS}\,.$$

We note that $\alpha|_{T(\Gamma)} \colon T(\Gamma) \to AT(\Gamma)$ is not surjective. More precisely, if $T(\Gamma) \neq \{[\mathrm{id}]\}$, then the image $\alpha T(\Gamma)$ is strictly contained in $AT(\Gamma)$ (see [19, Theorem 1.1]).

On the other hand, the following rigidity theorems are proved in [20, Theorems 2.2 and 4.1].

Theorem 7.1. *Let Γ be a subgroup of $\mathrm{Möb}(\mathbf{S})$ containing a hyperbolic element.*

(1) *If $f \in \mathrm{Sym}$ satisfies $f\Gamma f^{-1} \subset \mathrm{Möb}(\mathbf{S})$, then $f \in \mathrm{Möb}(\mathbf{S})$.*
(2) *If $f \in \mathrm{Sym}$ satisfies $f\Gamma f^{-1} \subset \mathrm{Diff}_+^{1+\alpha}(\mathbf{S})$, then $f \in \mathrm{Diff}_+^{1+\alpha}(\mathbf{S})$.*

Theorem 7.1 (1) implies that the restriction $\alpha|_{T(\Gamma)} \colon T(\Gamma) \to AT(\Gamma)$ is injective for a non-elementary Fuchsian group Γ. As an application of Theorem 7.1 (2) and Theorem 5.3, we will prove below that $\theta|_{DT(\Gamma)} \colon DT(\Gamma) \to AT(\Gamma)$ is also injective. Then, under the identification of the embedded images in $AT(\Gamma)$ by α and θ, we have

$$T(\Gamma) \subset DT(\Gamma) \subset AT(\Gamma).$$

We may ask a problem of which (or both) inclusion is strict.

Theorem 7.2. *For a Fuchsian group* $\Gamma \subset \mathrm{M\ddot{o}b}(\mathbf{S})$ *with a hyperbolic element, the restriction of the projection*

$$\theta|_{DT(\Gamma)} \colon DT(\Gamma) \to AT(\Gamma)$$

is injective. Hence, the deformation space $DT(\Gamma)$ *can be realized in* $AT(\Gamma)$*.*

Proof. Suppose that there are $f_1, f_2 \in \mathrm{QS}$ such that (1) both $f_1 \Gamma f_1^{-1}$ and $f_2 \Gamma f_2^{-1}$ are subgroups of $\mathrm{Diff}_+^{>1}(\mathbf{S})$; (2) $f_2 \circ f_1^{-1} \in \mathrm{Sym}$ but $f_2 \circ f_1^{-1} \notin \mathrm{Diff}_+^{>1}(\mathbf{S})$. For $[f_1], [f_2] \in T$, we consider the Bers embeddings $\varphi_1 = \beta([f_1])$ and $\varphi_2 = \beta([f_2])$ in $B(\mathbf{D}^*)$. By Theorem 5.3, condition (1) can be read as the orbit of φ_1 under Γ is in $B_0^{>0}(\mathbf{D}^*) + \varphi_1$ and the orbit of φ_2 under Γ is in $B_0^{>0}(\mathbf{D}^*) + \varphi_2$. Condition (2) can be read as $\varphi_1 - \varphi_2$ is in $B_0(\mathbf{D}^*)$ but not in $B_0^{>0}(\mathbf{D}^*)$ (we also use Theorem 2.2).

We set $\varphi = t(\varphi_1 - \varphi_2)$ for a sufficiently small constant $t > 0$ with $\varphi \in \beta(T)$. Since Γ acts on $B(\mathbf{D}^*)$ linearly, the above conditions imply that the orbit of φ under Γ is in $B_0^{>0}(\mathbf{D}^*) + \varphi$ and that φ is in $B_0(\mathbf{D}^*) - B_0^{>0}(\mathbf{D}^*)$. If we choose $f \in \mathrm{Sym}$ such that $\beta([f]) = \varphi$, then these conditions are equivalent to that $f \Gamma f^{-1} \subset \mathrm{Diff}_+^{>1}(\mathbf{S})$ and $f \notin \mathrm{Diff}_+^{>1}(\mathbf{S})$. Here, by choosing any hyperbolic element $\gamma \in \Gamma$, we can find some $\alpha \in (0,1)$ such that $f \langle \gamma \rangle f^{-1} \subset \mathrm{Diff}_+^{1+\alpha}(\mathbf{S})$. However, this contradicts Theorem 7.1 (2). Therefore, there are no such f_1, f_2 satisfying the conditions mentioned at the beginning, which shows that $\theta|_{DT(\Gamma)}$ is injective. □

We have handled so far the class $\mathrm{Diff}_+^{1+\alpha}(\mathbf{S})$ for $\alpha \in (0,1)$, but we can also consider higher regularity of circle diffeomorphisms at the same time. We denote the group of such circle diffeomorphisms by $\mathrm{Diff}_+^r(\mathbf{S})$ for $r > 1$. The following corollary to the above theorem asserts that if we restrict a group of circle homeomorphisms to the one obtained by the quasisymmetric conjugation of a Möbius group, we can extend the rigidity theorem from diffeomorphic conjugation to symmetric conjugation. Here, we refer to a quasisymmetric conjugate of a Möbius group as a *uniformly quasisymmetric group*. Justification of using this terminology stems from the result of Markovic [15].

Corollary 7.3. *Let* Γ_1 *and* Γ_2 *be non-abelian uniformly quasisymmetric subgroups of* $\mathrm{Diff}_+^r(\mathbf{S})$ $(r > 1)$*. If* $f \Gamma_1 f^{-1} = \Gamma_2$ *for* $f \in \mathrm{Sym}$*, then* $f \in \mathrm{Diff}_+^r(\mathbf{S})$*.*

Proof. We may assume that $\Gamma_1 = f_1 \Gamma f_1^{-1}$ and $\Gamma_2 = f_2 \Gamma f_2^{-1}$ for a subgroup Γ of $\mathrm{M\ddot{o}b}(\mathbf{S})$ with a hyperbolic element and that $f = f_2 \circ f_1^{-1} \in \mathrm{Sym}$. From this, we see that f_1 and f_2 modulo $\mathrm{Diff}_+^{>1}(\mathbf{S})$ belong to $DT(\Gamma)$. On the other hand, the assumption $f_2 \circ f_1^{-1} \in \mathrm{Sym}$ implies that the projections of these points in $DT(\Gamma)$ by θ are the same point of $AT(\Gamma)$. By the injectivity due to Theorem 7.2, we obtain that f_1 coincides with f_2 modulo $\mathrm{Diff}_+^{>1}(\mathbf{S})$. In particular, $f = f_2 \circ f_1^{-1}$ is a diffeomorphism. Then, by Theorem 7.4 below, f belongs to $\mathrm{Diff}_+^r(\mathbf{S})$. □

The following result is a special case of the theorem due to Ghys and Tsuboi [12]. Moreover, as remarked by Navas [23, p. 152], their theorem can be generalized to $\mathrm{Diff}_+^{1+\alpha}(\mathbf{S})$ by showing the Sternberg linearization theorem in the corresponding setting. A proof is given in the appendix for the sake of convenience.

Theorem 7.4. *Let* Γ *be a subgroup of* $\mathrm{M\ddot{o}b}(\mathbf{S})$ *with a hyperbolic element. If* $f \in \mathrm{Diff}_+(\mathbf{S})$ *satisfies* $f \Gamma f^{-1} \in \mathrm{Diff}_+^r(\mathbf{S})$ *for* $r > 1$*, then* $f \in \mathrm{Diff}_+^r(\mathbf{S})$*.*

We note that a certain part of the arguments in [20, Section 4] for the proof of Theorem 7.1 (2) can be replaced with this theorem.

8. Appendix: The Sternberg linearization theorem

In this appendix, we give a proof of Theorem 7.4. To this end, following the instruction by Navas [23, p. 150], we will do an exercise in proving a $C^{1+\alpha}$-version of the Sternberg linearization theorem. We note that [23, Theorem 3.6.2] handles the case of C^r for any integer $r \geq 2$ including $r = \infty$, and the non-integer case can be shown similarly to the argument below.

Theorem 8.1. *Let g be a real-valued orientation-preserving $C^{1+\alpha}$-diffeomorphism defined on some neighborhood of $0 \in \mathbf{R}$ such that $g(0) = 0$ and $g'(0) = a \neq 1$. Then, there exists a real-valued orientation-preserving $C^{1+\alpha}$-diffeomorphism h defined on some neighborhood of $0 \in \mathbf{R}$ with $h(0) = 0$ and $h'(0) = 1$ such that $h(g(x)) = ah(x)$. Moreover, such an h is unique. More precisely, if a real-valued orientation-preserving C^1-diffeomorphism h_1 defined on some neighborhood of $0 \in \mathbf{R}$ satisfies the same properties as h, then h_1 coincides with h in some neighborhood of $0 \in \mathbf{R}$.*

Proof. By considering g^{-1} if necessary, we may assume that $0 < a < 1$. For $\delta \in (0, 1)$, we assume that g is defined on $[-\delta, \delta]$. We set

$$c_\delta = \sup_{-\delta \leq x, y \leq \delta} \frac{|g'(x) - g'(y)|}{|x - y|^\alpha}.$$

Then, c_δ is decreasing as $\delta \to 0$. By setting $y = 0$, we in particular have

$$|g'(x) - g'(0)| \leq c_\delta |x|^\alpha \leq c_\delta \delta^\alpha \quad (-\delta \leq x \leq \delta).$$

Therefore, $|g'(x)| \leq a + c_\delta \delta^\alpha$ and $|g(x)| \leq \delta(a + c_\delta \delta^\alpha)$.

Let E_δ be a linear space of real-valued $C^{1+\alpha}$-functions ψ on $[-\delta, \delta]$ such that $\psi(0) = \psi'(0) = 0$. By providing a norm

$$\|\psi\| = \sup_{-\delta \leq x, y \leq \delta} \frac{|\psi'(x) - \psi'(y)|}{|x - y|^\alpha},$$

we see that E_δ is a Banach space. As before, $|\psi'(x)| \leq \|\psi\| \delta^\alpha$ for every $x \in [-\delta, \delta]$. We define a linear operator $S_\delta \colon E_\delta \to E_\delta$ by

$$S_\delta(\psi) = \frac{1}{a}\, \psi \circ g.$$

We will show that this is well-defined and the operator norm satisfies $\|S_\delta\| < 1$ if we choose a sufficiently small $\delta > 0$. Since

$$\begin{aligned}
&|(\psi \circ g)'(x) - (\psi \circ g)'(y)| \\
&= |\psi'(g(x))g'(x) - \psi'(g(y))g'(x) + \psi'(g(y))g'(x) - \psi'(g(y))g'(y)| \\
&\leq |\psi'(g(x)) - \psi'(g(y))| \cdot |g'(x)| + |\psi'(g(y))| \cdot |g'(x) - g'(y)|,
\end{aligned}$$

we have that

$$\begin{aligned}
&\frac{|(\psi \circ g)'(x) - (\psi \circ g)'(y)|}{|x - y|^\alpha} \\
&\leq \frac{|\psi'(g(x)) - \psi'(g(y))|}{|g(x) - g(y)|^\alpha} \cdot \frac{|g(x) - g(y)|^\alpha}{|x - y|^\alpha} \cdot |g'(x)| + |\psi'(g(y))| \cdot \frac{|g'(x) - g'(y)|}{|x - y|^\alpha}.
\end{aligned}$$

We choose $\delta > 0$ so small that both $g(x)$ and $g(y)$ are in $[-\delta, \delta]$. Then, the last formula in the above inequality is bounded by

$$\|\psi\| |g'(\xi)|^\alpha |g'(x)| + \|\psi\| \delta^\alpha c_\delta \leq \|\psi\| \{(a + c_\delta \delta^\alpha)^{1+\alpha} + \delta^\alpha c_\delta\},$$

where ξ is some real number between x and y. Therefore,

$$\|S_\delta\| \le \frac{1}{a}\{(a + c_\delta \delta^\alpha)^{1+\alpha} + c_\delta \delta^\alpha\},$$

which can be made smaller than 1 by $\delta \to 0$ and hence $c_\delta \delta^\alpha \to 0$.

We set $\psi_*(x) = g(x) - ax$, which belongs to E_δ. By fixing ψ_*, we consider a functional equation

$$S_\delta(\psi) + a^{-1}\psi_* = \psi$$

with respect to $\psi \in E_\delta$. If we set the left side as

$$F(\psi) = S_\delta(\psi) + a^{-1}\psi_*,$$

then $F\colon E_\delta \to E_\delta$ satisfies

$$\|F(\psi_1) - F(\psi_2)\| = \|S_\delta(\psi_1 - \psi_2)\| \le \|S_\delta\|\|\psi_1 - \psi_2\|$$

with $\|S_\delta\| < 1$. By the Banach contraction principle, there exists some $\psi_0 \in E_\delta$ uniquely such that $F(\psi_0) = \psi_0$.

We define $h(x) = x + \psi_0(x)$. This satisfies that

$$\begin{aligned} h(g(x)) &= g(x) + \psi_0 \circ g(x) = \psi_*(x) + ax + \psi_0 \circ g(x) \\ &= \psi_*(x) + ax + aS_\delta(\psi_0)(x) = a\psi_0(x) + ax = ah(x). \end{aligned}$$

Thus, this function h is the desired one.

Next, we show the latter statement on the uniqueness. For any x in some neighborhood of $0 \in \mathbf{R}$, we have

$$ah \circ h_1^{-1}(x) = h \circ g \circ h_1^{-1}(x) = h \circ h_1^{-1}(ax).$$

Then, it follows that

$$h \circ h_1^{-1}(x) = \frac{h \circ h_1^{-1}(a^n x)}{a^n} = x\,\frac{h \circ h_1^{-1}(a^n x)}{a^n x}$$

for any $n \in \mathbf{N}$, and this tends to x as $n \to \infty$. Hence, $h \circ h_1^{-1}(x) = x$, that is, $h(x) = h_1(x)$. □

We have the following consequence in the same way as [23, Corollary 3.6.3].

Corollary 8.2. *For $r > 1$, let g_1 and g_2 be C^r-diffeomorphisms satisfying all properties of g as in Theorem 8.1. If a C^1-diffeomorphism φ defined on some neighborhood of $0 \in \mathbf{R}$ with $\varphi(0) = 0$ conjugates g_1 to g_2, then φ is a C^r-diffeomorphism.*

Proof of Theorem 7.4. For a hyperbolic element $\gamma \in \Gamma$, we set $g = f\gamma f^{-1} \in \mathrm{Diff}_+^r(\mathbf{S})$. For the attracting fixed point $\xi \in \mathbf{S}$ of γ, $f(\xi)$ is the attracting fixed point of g and $|g'(f(\xi))| = |\gamma'(\xi)| \neq 1$. In local coordinates around $f(\xi) \in \mathbf{S}$, g is represented by a real-valued orientation-preserving C^r-diffeomorphism $\widetilde{g}$ defined on some neighborhood of $0 \in \mathbf{R}$ such that $\widetilde{g}(0) = 0$ and $\widetilde{g}'(0) = a \neq 1$. We also consider the representation of γ in local coordinates around $\xi \in \mathbf{S}$, which is also a real-valued orientation-preserving C^r-diffeomorphism $\widetilde{\gamma}$ defined on some neighborhood of $0 \in \mathbf{R}$ such that $\widetilde{\gamma}(0) = 0$ and $\widetilde{\gamma}'(0) = a \neq 1$. Then, a C^1-diffeomorphism φ between these neighborhoods, which stems from f, conjugates $\widetilde{\gamma}$ to $\widetilde{g}$. By Corollary 8.2, we see that φ is a C^r-diffeomorphism.

By the above argument, we see that f is in C^r in some neighborhood of $\xi \in \mathbf{S}$. However, the iteration of γ^{-1} expands this neighborhood to $\mathbf{S}$ except the repelling fixed point of γ, and we find that f is in C^r there. Finally, by exchanging the roles of the attracting and the repelling fixed points, we conclude that f is a C^r-diffeomorphism of the entire space $\mathbf{S}$. □

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