

EXISTENCE OF POSITIVE SOLUTIONS FOR NONLINEAR ROBIN PROBLEMS WITH GRADIENT DEPENDENCE

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Abstract. We consider a nonlinear Robin problem driven by the p -Laplace differential operator and with a reaction term which depends also on the gradient (convection). Using a topological approach based on the Leray–Schauder alternative principle, we show that the problem has a positive smooth solution.

1. Introduction

Let $\Omega \subseteq \mathbf{R}^N$ be a bounded domain with a C^2 -boundary $\partial\Omega$. In this paper we study the following nonlinear Robin problem with gradient dependence (convection):

$$(1.1) \quad \begin{cases} -\Delta_p u(z) + \xi(z)|u(z)|^{p-2}u(z) = f(z, u(z), Du(z)) & \text{in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2}u = 0 & \text{on } \partial\Omega. \end{cases}$$

In this problem Δ_p denotes the p -Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}(|Du|^{p-2}Du) \quad \text{for all } u \in W^{1,p}(\Omega), \quad 1 < p < \infty.$$

The reaction term $f(z, x, y)$ is a Caratheodory perturbation (that is, for all $(x, y) \in \mathbf{R} \times \mathbf{R}^N$ $z \rightarrow f(z, x, y)$ is measurable and for a.a. $z \in \Omega$ $(x, y) \rightarrow f(z, x, y)$ is continuous). The dependence of f on the gradient of the unknown function u , makes the problem nonvariational. In the boundary condition $\frac{\partial u}{\partial n_p}$ denotes the conormal derivative of u for the p -Laplace differential operator. It is defined by extension of the map

$$C^1(\overline{\Omega}) \ni u \longrightarrow |Du|^{p-2}(Du, n)_{\mathbf{R}^N} = |Du|^{p-2} \frac{\partial u}{\partial n}.$$

Here $n(\cdot)$ denotes the outward unit normal on $\partial\Omega$.

The nonvariational character of (1.1), requires a topological approach based on the fixed point theory. Assuming that $f(z, \cdot, y)$ is $(p-1)$ -sublinear near $+\infty$, we show that there exists a positive smooth solution for problem (1.1).

In the past, positive solutions for elliptic problems with convection were proved primarily for Dirichlet problems. In this direction we mention the works of de Figueiredo–Girardi–Matzeu [6], Girardi–Matzeu [12] (semilinear problems) and Bai

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[2], Faraci–Motreanu–Puglisi [4], Faria–Miyagaki–Motreanu–Tanaka [5], García Melián–Sabina de Lis–Takáč [7], Huy–Quan–Khanh [14], Ruiz [26], Tanaka [27] (nonlinear problems) together with Motreanu–Vetro–Vetro [16, 17] (nonlinear system problems). For Neumann and Robin problems, there are the recent works of Gasinski–Papageorgiou [10], Papageorgiou–Radulescu–Repovs [24] (semilinear Neumann problems with a differential operator of the form $\operatorname{div}(a(u)Du)$, $u \in W^{1,p}(\Omega)$) and Bai–Gasinski–Papageorgiou [3], Papageorgiou–Radulescu–Repovs [25] (nonlinear Robin problems). In these papers the hypotheses on the reaction are different and in many respects more restrictive.

2. Preliminaries and hypotheses

As we already mentioned our approach is topological and employs the Leray–Schauder Alternative Principle (see Gasinski–Papageorgiou [8], p. 890). So, let X, Y be two Banach spaces, $D \subseteq X$ nonempty and $f: D \rightarrow Y$. We say that $f(\cdot)$ is “compact”, if it is continuous and maps bounded subsets of D to relatively compact sets in Y . The Leray–Schauder Alternative Principle reads as follows:

Theorem 2.1. *If X is a Banach space, $C \subseteq X$ is nonempty, convex with $0 \in C$, $f: C \rightarrow C$ is compact and $E(f) = \{u \in C: u = \lambda f(u) \text{ for some } \lambda \in (0, 1)\}$, then either $E(f)$ is unbounded or f admits a fixed point.*

The following spaces are important in the analysis of problem (1.1)

$$W^{1,p}(\Omega), C^1(\overline{\Omega}) \quad \text{and} \quad L^p(\partial\Omega).$$

By $\|\cdot\|$ we denote the norm of the Sobolev space $W^{1,p}(\Omega)$. It is defined by

$$\|u\| = [\|u\|_p^p + \|Du\|_p^p]^{1/p} \quad \text{for all } u \in W^{1,p}(\Omega).$$

The space $C^1(\overline{\Omega})$ is an ordered Banach space with positive (order) cone given by

$$C_+ = \{u \in C^1(\overline{\Omega}): u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$D_+ = \{u \in C_+: u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

Note that D_+ is also the interior of C_+ when this space is equipped with the weaker $C(\overline{\Omega})$ -norm topology.

On $\partial\Omega$ we consider the $(N - 1)$ -dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Using this measure on $\partial\Omega$, we can define in the usual way the boundary Lebesgue spaces $L^q(\partial\Omega)$ ($1 \leq q \leq \infty$). From the theory of Sobolev spaces, we know that there exists a unique continuous linear map $\gamma_0: W^{1,p}(\Omega) \rightarrow L^p(\partial\Omega)$, which is known as the “trace map”, such that

$$\gamma_0(u) = u|_{\partial\Omega} \quad \text{for all } u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}).$$

So, the trace map extends the notion of boundary values to all Sobolev functions. The trace map is compact into $L^q(\partial\Omega)$ for all $q \in [1, \frac{(N-1)p}{N-p}]$ if $p < N$ and into $L^q(\partial\Omega)$ for all $1 \leq q < \infty$ if $N \leq p$. Moreover, we have

$$\operatorname{im}\gamma_0 = W^{\frac{1}{p'}, p}(\partial\Omega) \quad \text{and} \quad \ker\gamma_0 = W_0^{1,p}(\Omega) \quad \left(\frac{1}{p} + \frac{1}{p'} = 1\right).$$

In what follows, for the sake of notational simplicity, we drop the use of the trace map $\gamma_0(\cdot)$. All restrictions of Sobolev functions on $\partial\Omega$ are understood in the sense of traces.

We introduce the hypotheses on the potential function $\xi(\cdot)$ and on the boundary coefficient $\beta(\cdot)$.

- $H(\xi)$: $\xi \in L^\infty(\Omega), \xi(z) \geq 0$ for a.a. $z \in \Omega$.
- $H(\beta)$: $\beta \in C^{0,\alpha}(\partial\Omega)$ with $\alpha \in (0, 1), \beta(z) \geq 0$ for all $z \in \partial\Omega$.
- H_0 : $\xi \neq 0$ or $\beta \neq 0$.

Remark 2.2. When $\beta \equiv 0$, we recover the usual Neumann problem.

From Mugnai–Papageorgiou [18] (Lemma 4.11), we have:

Lemma 2.3. *If $\xi \in L^\infty(\Omega), \xi(z) \geq 0$ for a.a. $z \in \Omega$ and $\xi \not\equiv 0$, then there exists $c_1 > 0$ such that*

$$\|Du\|_p^p + \int_\Omega \xi(z)|u|^p dz \geq c_1\|u\|^p \quad \text{for all } u \in W^{1,p}(\Omega).$$

Also, from Gasinski–Papageorgiou [11] (Proposition 2.4), we have:

Lemma 2.4. *If $\beta \in L^\infty(\partial\Omega), \beta(z) \geq 0$ for σ -a.a. $z \in \partial\Omega, \beta \not\equiv 0$, then the map*

$$u \rightarrow |u| = \left[\|Du\|_p^p + \int_{\partial\Omega} \beta(z)|u|^p d\sigma \right]^{1/p}$$

defines an equivalent norm on $W^{1,p}(\Omega)$.

In the sequel we assume that hypotheses $H(\xi), H(\beta), H_0$ hold. Let $\gamma: W^{1,p}(\Omega) \rightarrow \mathbf{R}$ be the C^1 -functional defined by

$$\gamma(u) = \|Du\|_p^p + \int_\Omega \xi(z)|u|^p dz + \int_{\partial\Omega} \beta(z)|u|^p d\sigma \quad \text{for all } u \in W^{1,p}(\Omega).$$

On account of Lemmata 2.3 and 2.4, we see that we can find $c_2 > 0$ such that

$$(2.1) \quad \gamma(u) \geq c_2\|u\|^p \quad \text{for all } u \in W^{1,p}(\Omega).$$

We consider the following nonlinear eigenvalue problem:

$$(2.2) \quad \begin{cases} -\Delta_p u(z) + \xi(z)|u(z)|^{p-2}u(z) = \hat{\lambda}|u(z)|^{p-2}u(z) & \text{in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2}u = 0 & \text{on } \partial\Omega. \end{cases}$$

We say that $\hat{\lambda} \in \mathbf{R}$ is an “eigenvalue” of (2.2), if the problem admits a nontrivial solution $\hat{u} \in W^{1,p}(\Omega)$, known as an “eigenfunction” corresponding to $\hat{\lambda}$. From the nonlinear regularity theory of Lieberman [15] (Theorem 2), we have that $\hat{u} \in C^1(\bar{\Omega})$. There is a smallest eigenvalue $\hat{\lambda}_1$ with the following properties:

- (a) $\hat{\lambda}_1$ is isolated (that is, there exists $\varepsilon > 0$ such that the open interval $(\hat{\lambda}_1, \hat{\lambda}_1 + \varepsilon)$ does not contain eigenvalues of (2.2)).
- (b) $\hat{\lambda}_1$ is simple (that is, if \hat{u}, \hat{v} are eigenfunctions corresponding to $\hat{\lambda}_1$, then $\hat{u} = \eta\hat{v}$ for some $\eta \in \mathbf{R} \setminus \{0\}$).
- (c)

$$(2.3) \quad \hat{\lambda}_1 = \inf \left[\frac{\gamma(u)}{\|u\|_p^p} : u \in W^{1,p}(\Omega), u \neq 0 \right] > 0 \quad \text{(see (2.1)).}$$

The infimum in (2.3) is realized on the corresponding one-dimensional eigenspace (see property (b)). From the above properties it follows easily that the elements of this eigenspace do not change sign and as we already said belong in $C^1(\bar{\Omega})$. Let \hat{u}_1 denote the positive, L^p -normalized (that is, $\|\hat{u}_1\|_p = 1$) eigenfunction corresponding to $\hat{\lambda}_1$. We have $\hat{u}_1 \in C_+ \setminus \{0\}$ and by the nonlinear maximum principle (see, for

example, Gasinski–Papageorgiou [8, p. 738]), we have that $\hat{u}_1 \in D_+$. We mention that every eigenfunction \hat{u} corresponding to an eigenvalue $\hat{\lambda} \neq \hat{\lambda}_1$ is nodal (that is, sign changing).

As a simple consequence of the above properties we have the following lemma (see Papageorgiou–Radulescu–Repovš [22, Lemma 14]).

Lemma 2.5. *If $\theta \in L^\infty(\Omega)$, $\theta(z) \leq \hat{\lambda}_1$ for a.a. $z \in \Omega$, $\theta \not\equiv \hat{\lambda}_1$, then there exists $c_3 > 0$ such that*

$$c_3 \|u\|^p \leq \gamma(u) - \int_{\Omega} \theta(z)|u|^p dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

Next we introduce the hypotheses on the reaction term $f(z, x, y)$.

- H(f): $f: \Omega \times \mathbf{R} \times \mathbf{R}^N$ is a Caratheodory function such that $f(z, 0, 0) = 0$ for a.a. $z \in \Omega$, $f(z, x, y) > 0$ for a.a. $z \in \Omega$, all $x > 0$, all $y \in \mathbf{R}^N$ and
- (i) $f(z, x, y) \leq a(z)[1 + x^{p-1}] + \mu|y|^{p-1}$ for a.a. $z \in \Omega$, all $x \geq 0$, all $y \in \mathbf{R}^N$ with $a \in L^\infty(\Omega)$ and $0 < \mu < \hat{\lambda}_1^{1/p}$;
 - (ii) there exists a function $\theta \in L^\infty(\Omega)$ such that if $\mu_0 = 1 - \frac{\mu}{\hat{\lambda}_1^{1/p}} > 0$, then

$$\theta(z) \leq \hat{\lambda}_1 \mu_0 \quad \text{for a.a. } z \in \Omega, \quad \theta \not\equiv \hat{\lambda}_1 \mu_0$$

and for every $\varepsilon > 0$, there exists $M_\varepsilon > 0$ such that

$$f(z, x, y) \leq [\theta(z) + \varepsilon]x^{p-1} + \mu|y|^{p-1}$$

for a.a. $z \in \Omega$, all $x \geq M_\varepsilon$ and all $y \in \mathbf{R}^N$;

- (iii) for every $M > 0$, there exists $\hat{\eta}_M \in L^\infty(\Omega)$ such that

$$\hat{\eta}_M(z) \geq \hat{\lambda}_1 \quad \text{for a.a. } z \in \Omega, \quad \hat{\eta}_M \not\equiv \hat{\lambda}_1,$$

$$\liminf_{x \rightarrow 0^+} \frac{f(z, x, y)}{x^{p-1}} \geq \hat{\eta}_M(z) \quad \text{uniformly for a.a. } z \in \Omega, \text{ all } |y| \leq M;$$

- (iv) for every $\rho > 0$ and every $M > 0$, there exists $\hat{\xi}_{\rho, M} > 0$ such that for a.a. $z \in \Omega$ and all $|y| \leq M$, the function

$$x \rightarrow f(z, x, y) + \hat{\xi}_{\rho, M} x^{p-1}$$

is nondecreasing on $[0, \rho]$ and for a.a. $z \in \Omega$, all $y \in \mathbf{R}^N$, the quotient function

$$x \rightarrow \frac{f(z, x, y)}{x^{p-1}}$$

is nonincreasing on $(0, +\infty)$.

Remark 2.6. Since we are looking for positive solutions and the above hypotheses concern $x \geq 0$, without any loss of generality we may assume that $f(z, x, y) = 0$ for a.a. $z \in \Omega$, all $x \leq 0$, all $y \in \mathbf{R}^N$. Hypothesis H(f)(ii) implies that

$$\limsup_{x \rightarrow +\infty} \frac{f(z, x, y)}{x^{p-1}} \leq \theta(z)$$

uniformly for a.a. $z \in \Omega$ and all $y \in \mathbf{R}^N$ in a bounded set. So, this hypothesis is in fact a nonuniform nonresonance condition for $f(z, \cdot, y)$ at $+\infty$.

Example 2.7. The following functions satisfy hypotheses H(f). For the sake of simplicity, we drop the z -dependence

$$f_1(x, y) = \begin{cases} \hat{\eta}x^{p-1} + x^{q-1}|y|^{p-1} & \text{if } 0 \leq x \leq 1 \\ \hat{\eta}x^{\tau-1} + |y|^{p-1} & \text{if } 1 < x \end{cases} \quad \text{with } \hat{\eta} > \hat{\lambda}_1, 1 < \tau, q < p,$$

$$f_2(x, y) = \begin{cases} \hat{\eta}x^{p-1} + \mu|y|^{p-1} & \text{if } 0 \leq x \leq 1 \\ \theta x^{p-1} + (\hat{\eta} - \theta)|x|^{q-1} + \mu|y|^{p-1} & \text{if } 1 < x \end{cases} \quad \text{with } 0 < \mu < \hat{\lambda}_1^{1/p},$$

$$\theta < \hat{\lambda}_1\mu_0, \hat{\lambda}_1 < \hat{\eta} \text{ and } 1 < q < p.$$

Now let $B \subseteq C^1(\bar{\Omega})$ be bounded and let $M = \sup_{v \in B} \|v\|_{C^1(\bar{\Omega})}$. On account of hypotheses H(f)(i), (iii) and since $\hat{\eta}_M \in L^\infty(\Omega)$, given $\varepsilon > 0$ and $r \in (p, p^*)$ (recall $p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } N \leq p \end{cases}$), we can find $c_4 = c_4(\varepsilon, r) > 0$ such that

$$(2.4) \quad f_v(z, x) = f(z, x, Dv(z)) \geq [\hat{\eta}_M(z) - \varepsilon]x^{p-1} - c_4x^{r-1}$$

for a.a. $z \in \Omega$, all $x \geq 0$, all $v \in B$.

Motivated by the unilateral growth condition (2.4) on $f_v(z, \cdot)$ for all $v \in B$, we consider the following auxiliary nonlinear Robin problem

$$(2.5) \quad \begin{cases} -\Delta_p u(z) + \xi(z)u(z)^{p-1} = [\hat{\eta}_M(z) - \varepsilon]u(z)^{p-1} - c_4u(z)^{r-1} & \text{in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z)u^{p-1} = 0 & \text{on } \partial\Omega, \quad u > 0. \end{cases}$$

Proposition 2.8. *If hypotheses H(ξ), H(β), H₀ hold, then for all ε > 0 small problem (2.5) admits a unique positive solution $\tilde{u} \in D_+$.*

Proof. We consider the C^1 -functional $\psi: W^{1,p}(\Omega) \rightarrow \mathbf{R}$ defined by

$$\psi(u) = \frac{1}{p}\gamma(u) + \frac{c_4}{r}\|u^+\|_r^r - \frac{1}{p} \int_{\Omega} [\hat{\eta}_M(z) - \varepsilon](u^+)^p dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

Since $r > p$, we see that

$$\psi(\cdot) \text{ is coercive.}$$

Also using the Sobolev embedding theorem and the compactness of the trace map, we have that

$$\psi(\cdot) \text{ is sequentially weakly lower semicontinuous.}$$

So, by the Weierstrass–Tonelli theorem, we can find $\tilde{u} \in W^{1,p}(\Omega)$ such that

$$(2.6) \quad \psi(\tilde{u}) = \inf [\psi(u) : u \in W^{1,p}(\Omega)].$$

For $t > 0$, we have

$$\begin{aligned} \psi(t\hat{u}_1) &= \frac{t^p}{p}\hat{\lambda}_1 + \frac{t^r}{r}c_4\|\hat{u}_1\|_r^r - \frac{t^p}{p} \int_{\Omega} \hat{\eta}_M(z)\hat{u}_1^p + \frac{\varepsilon t^p}{p} \quad (\text{recall that } \|\hat{u}_1\|_p = 1) \\ &\leq \frac{t^p}{p} \left[\int_{\Omega} (\hat{\lambda}_1 - \hat{\eta}_M(z))\hat{u}_1^p dz + \varepsilon \right] + \frac{t^r}{r}c_4\|\hat{u}_1\|_r^r. \end{aligned}$$

From the hypothesis on $\hat{\eta}_M(\cdot)$ (see H(f)(iii)) and since $\hat{u}_1 \in D_+$, we have

$$c_M = \int_{\Omega} (\hat{\eta}_M(z) - \hat{\lambda}_1)\hat{u}_1^p dz > 0.$$

So, we have

$$\psi(t\hat{u}_1) \leq \frac{t^p}{p}(-c_M + \varepsilon) + \frac{t^r}{r}c_4\|\hat{u}_1\|_r^r.$$

Choosing $\varepsilon \in (0, c_M)$ it follows that

$$\psi(t\hat{u}_1) \leq -\frac{t^p}{p}c_5 + \frac{t^r}{r}c_4\|\hat{u}_1\|_r^r \quad \text{for some } c_5 > 0, \text{ all } t > 0.$$

Since $r > p$, choosing $t \in (0, 1)$ small, we conclude that

$$\begin{aligned} \psi(t\hat{u}_1) &< 0, \\ \implies \psi(\tilde{u}) &< 0 = \psi(0) \quad (\text{see (2.6)}), \\ \implies \tilde{u} &\neq 0. \end{aligned}$$

From (2.6) we have

$$\begin{aligned} \psi'(\tilde{u}) &= 0, \\ \implies \langle A(\tilde{u}), h \rangle &+ \int_{\Omega} \xi(z)|\tilde{u}|^{p-2}\tilde{u}h \, dz + \int_{\partial\Omega} \beta(z)|\tilde{u}|^{p-2}\tilde{u}h \, d\sigma \\ (2.7) \quad &= \int_{\Omega} (\hat{\eta}_M(z) - \varepsilon)(\tilde{u}^+)^{p-1}h \, dz - c_4 \int_{\Omega} (\tilde{u}^+)^{r-1}h \, dz \quad \text{for all } h \in W^{1,p}(\Omega). \end{aligned}$$

In (2.7) we choose $h = -\tilde{u}^- \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} \gamma(\tilde{u}^-) &= 0, \\ \implies \hat{\lambda}_1\|\tilde{u}^-\|_p^p &\leq 0 \quad (\text{see (2.3)}), \\ \implies \tilde{u} &\geq 0, \tilde{u} \neq 0. \end{aligned}$$

Then from (2.7) it follows that

$$(2.8) \quad \begin{cases} -\Delta_p \tilde{u}(z) + \xi(z)\tilde{u}(z)^{p-1} = [\hat{\eta}_M(z) - \varepsilon]\tilde{u}(z)^{p-1} - c_4\tilde{u}(z)^{r-1} & \text{for a.a. } z \in \Omega, \\ \frac{\partial \tilde{u}}{\partial n_p} + \beta(z)\tilde{u}^{p-1} = 0 & \text{on } \partial\Omega \end{cases}$$

(see Papageorgiou–Radulescu [19]).

From (2.8) and Proposition 2.10 of Papageorgiou–Radulescu [20], we have

$$\tilde{u} \in L^\infty(\Omega).$$

Invoking Theorem 2 of Lieberman [15], we have

$$\tilde{u} \in C_+ \setminus \{0\}.$$

From (2.5), hypothesis $H(\xi)$ and by taking $\varepsilon > 0$ even smaller if necessary (so that $\varepsilon < \hat{\lambda}_1$), we have

$$\begin{aligned} \Delta_p \tilde{u}(z) &\leq [\|\xi\|_\infty + c_4\|\tilde{u}\|_\infty^{r-p}] \tilde{u}(z)^{p-1} \quad \text{for a.a. } z \in \Omega, \\ \implies \tilde{u} &\in D_+ \end{aligned}$$

(by the nonlinear maximum principle, see [8, p. 738]).

We show that this positive solution is unique. Indeed, suppose that $\tilde{v} \in W^{1,p}(\Omega)$ is another positive solution of (2.5). Again we have $\tilde{v} \in D_+$. Let $t > 0$ be the biggest positive real such that

$$(2.9) \quad t\tilde{v} \leq \tilde{u}.$$

Suppose that $t < 1$. If $\rho = \|\tilde{u}\|_\infty$, then we can find $\tilde{\xi}_\rho > 0$ such that for a.a. $z \in \Omega$, the function

$$x \rightarrow [\hat{\eta}_M(z) - \varepsilon]x^{p-1} - c_4x^{r-1} + \tilde{\xi}_\rho x^{p-1}$$

in nondecreasing on $[0, \rho]$. We have

$$\begin{aligned}
 & -\Delta_p(t\tilde{v}) + [\xi(z) + \tilde{\xi}_\rho](t\tilde{v})^{p-1} \\
 &= t^{p-1}[-\Delta_p\tilde{v} + (\xi(z) + \tilde{\xi}_\rho)\tilde{v}^{p-1}] \\
 &= t^{p-1}[(\hat{\eta}_M(z) - \varepsilon)\tilde{v}^{p-1} - c_4\tilde{v}^{r-1} + \tilde{\xi}_\rho\tilde{v}^{p-1}] \\
 &\leq [\hat{\eta}_M(z) - \varepsilon](t\tilde{v})^{p-1} - c_4(t\tilde{v})^{r-1} + \tilde{\xi}_\rho(t\tilde{v})^{p-1} \quad (\text{since } t < 1 \text{ and } r > p) \\
 &\leq [\hat{\eta}_M(z) - \varepsilon]\tilde{u}^{p-1} - c_4\tilde{u}^{r-1} + \tilde{\xi}_\rho\tilde{u}^{p-1} \quad (\text{see (2.9)}) \\
 (2.10) \quad &= -\Delta_p\tilde{u} + [\xi(z) + \tilde{\xi}_\rho]\tilde{u}^{p-1} \quad \text{for a.a. } z \in \Omega.
 \end{aligned}$$

Note that since $\tilde{v} \in D_+$, we have $\tilde{m} = \min_{\bar{\Omega}} \tilde{v} > 0$. Then

$$c_4[t^{p-1} - t^{r-1}]\tilde{v}(z) \geq c_4[t^{p-1} - t^{r-1}]\tilde{m} > 0 \quad (\text{since } t < 1 \text{ and } r > p).$$

Then from (2.10) and Proposition 2.10 of Papageorgiou–Radulescu–Repovs [23] it follows that

$$\tilde{u} - t\tilde{v} \in \text{int } C_+,$$

which contradicts the maximality of $t > 0$. Hence $t \geq 1$ and so

$$\tilde{v} \leq \tilde{u} \quad (\text{see (2.9)}).$$

In the above argument we reverse the roles of \tilde{u} and \tilde{v} and obtain

$$\begin{aligned}
 & \tilde{u} \leq \tilde{v}, \\
 & \implies \tilde{u} = \tilde{v}.
 \end{aligned}$$

This proves the uniqueness of the positive solution of problem (2.5). □

3. Positive solution

We employ the “freezing method”. Namely, we fix (freeze) the gradient term in the reaction $f(z, x, y)$ and study the resulting problem which has variational structure. We solve this problem and we show that it has a positive solution. Moreover, we show that it has a smallest positive solution (minimal positive solution). So, we have a canonical way to choose a solution of the “frozen problem”. We show that this minimal solution map has a fixed point. This is the desired positive solution of (1.1).

We start implementing this approach. We fix $v \in C^1(\bar{\Omega})$ and consider the following nonlinear Robin problem

$$(3.1) \quad \begin{cases} -\Delta_p u(z) + \xi(z)u(z)^{p-1} = f(z, u(z), Dv(z)) & \text{in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z)u^{p-1} = 0 & \text{on } \partial\Omega, \quad u > 0. \end{cases}$$

Proposition 3.1. *If hypotheses $H(\xi), H(\beta), H_0, H(f)$ hold and $v \in C^1(\bar{\Omega})$, then problem (3.1) admits a positive solution $\hat{u}_v \in D_+$.*

Proof. Consider the C^1 -functional $\hat{\varphi}_v : W^{1,p}(\Omega) \rightarrow \mathbf{R}$ defined by

$$\hat{\varphi}_v(u) = \frac{1}{p}\gamma(u) - \int_{\Omega} F_v(z, u^+(z)) dz \quad \text{for all } u \in W^{1,p}(\Omega),$$

where $F_v(z, x) = \int_0^x f(z, x, Dv(z)) dz$.

Hypotheses $H(f)(i), (ii)$, imply that given $\varepsilon > 0$, we can find $c_6 > 0$ such that

$$(3.2) \quad F_v(z, x) \leq \frac{1}{p}[\theta(z) + \varepsilon]x^p + c_6 \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq 0 \quad (\text{recall } v \in C^1(\bar{\Omega})).$$

We have

$$\begin{aligned} \hat{\varphi}_v(u) &\geq \frac{1}{p}\gamma(u) - \frac{1}{p} \int_{\Omega} [\theta(z) + \varepsilon]|u|^p dz - c_7 \quad \text{for some } c_7 > 0 \\ &\geq \frac{1}{p}[c_3 - \varepsilon]\|u\|^p - c_7 \quad (\text{see Lemma 2.5}). \end{aligned}$$

Choosing $\varepsilon \in (0, c_3)$, we infer that

$$\hat{\varphi}_v(\cdot) \text{ is coercive.}$$

Also, from the Sobolev embedding theorem and the compactness of the trace map, we have that

$$\hat{\varphi}_v(\cdot) \text{ is sequentially weakly lower semicontinuous.}$$

So, we can find $\hat{u}_v \in W^{1,p}(\Omega)$ such that

$$(3.3) \quad \hat{\varphi}_v(\hat{u}_v) = \inf [\hat{\varphi}_v(u) : u \in W^{1,p}(\Omega)].$$

Let $M = \|v\|_{C^1(\bar{\Omega})}$. Then on account of hypothesis H(f)(iii), given $\varepsilon > 0$ we can find $\delta > 0$ such that

$$(3.4) \quad F_v(z, x) \geq \frac{1}{p}[\hat{\eta}_M(z) - \varepsilon]x^p \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq \delta.$$

Recall that $\hat{u}_1 \in D_+$. So, we can find $t \in (0, 1)$ small such that

$$(3.5) \quad t\hat{u}_1(z) \in (0, \delta] \quad \text{for all } z \in \bar{\Omega}.$$

As in the proof of Proposition 2.8, using (3.4) and (3.5), we show that for $t \in (0, 1)$ small we have

$$\begin{aligned} &\hat{\varphi}_v(t\hat{u}_1) < 0, \\ \implies &\hat{\varphi}_v(\hat{u}_v) < 0 = \hat{\varphi}_v(0) \quad (\text{see (3.3)}), \\ \implies &\hat{u}_v \neq 0. \end{aligned}$$

From (3.3), we have

$$\begin{aligned} &\hat{\varphi}'_v(\hat{u}_v) = 0, \\ \implies &\langle A(\hat{u}_v), h \rangle + \int_{\Omega} \xi(z)|\hat{u}_v|^{p-2}\hat{u}_v h dz + \int_{\partial\Omega} \beta(z)|\hat{u}_v|^{p-2}\hat{u}_v h d\sigma \\ (3.6) \quad &= \int_{\Omega} f_v(z, \hat{u}_v^+)h dz \quad \text{for all } h \in W^{1,p}(\Omega). \end{aligned}$$

In (3.6) we choose $h = -\hat{u}_v^- \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} &\gamma(\hat{u}_v^-) = 0, \\ \implies &\hat{\lambda}_1\|\hat{u}_v^-\|_p^p \leq 0 \quad (\text{see (2.3)}), \\ \implies &\hat{u}_v \geq 0, \quad \hat{u}_v \neq 0. \end{aligned}$$

From (3.6) we have

$$(3.7) \quad \begin{cases} -\Delta_p \hat{u}_v(z) + \xi(z)\hat{u}_v(z)^{p-1} = f(z, \hat{u}_v(z), Dv(z)) & \text{for a.a. } z \in \Omega, \\ \frac{\partial \hat{u}_v}{\partial n_p} + \beta(z)\hat{u}_v^{p-1} = 0 & \text{on } \partial\Omega \end{cases}$$

(see Papageorgiou–Radulescu [19]).

As before from (3.7) and the nonlinear regularity theory we infer that

$$\hat{u}_v \in C_+ \setminus \{0\}.$$

From (3.7) and since $f(z, \hat{u}_v(z), Dv(z)) \geq 0$ for a.a. $z \in \Omega$, we have

$$\begin{aligned} \Delta_p \hat{u}_v(z) &\leq \|\xi\|_\infty \hat{u}_v(z)^{p-1} \quad \text{for a.a. } z \in \Omega, \\ \implies \hat{u}_v &\in D_+ \quad (\text{see Gasinski-Papageorgiou [8, p. 738]}). \quad \square \end{aligned}$$

We will show the existence of a minimal positive solution. Let S_v^+ be the set of positive solutions of problem (3.1). We have just seen (see Proposition 3.1) that $\emptyset \neq S_v^+ \subseteq D_+$. Moreover, from Papageorgiou–Radulescu–Repovs [21] (see the proof of Proposition 7), we know that S_v^+ is downward directed (that is, if $u_1, u_2 \in S_v^+$, then there exists $u \in S_v^+$ such that $u \leq u_1, u \leq u_2$).

Proposition 3.2. *If hypotheses $H(\xi), H(\beta), H_0, H(f)$ hold and $v \in C^1(\bar{\Omega})$, then there exists $u_v^* \in S_v^+$ such that $u_v^* \leq u$ for all $u \in S_v^+$.*

Proof. Invoking Lemma 3.10, p. 178, of Hu–Papageorgiou [13], we can find $\{u_n\}_{n \geq 1} \subseteq S_v^+$ decreasing such that

$$\inf S_v^+ = \inf_{n \geq 1} u_n.$$

We have

$$(3.8) \quad \langle A(u_n), h \rangle + \int_\Omega \xi(z) u_n^{p-1} h \, dz + \int_{\partial\Omega} \beta(z) u_n^{p-1} h \, d\sigma = \int_\Omega f(z, u_n, Dv) h \, dz$$

for all $h \in W^{1,p}(\Omega)$, all $n \in \mathbf{N}$.

Since $0 \leq u_n \leq u_1 \in D_+$ for all $n \in \mathbf{N}$, from (3.8), hypothesis $H(f)(i)$ and Lemmata 2.3 and 2.4 it follows that

$$\{u_n\}_{n \geq 1} \subseteq W^{1,p}(\Omega) \text{ is bounded.}$$

So, we may assume that

$$(3.9) \quad u_n \xrightarrow{w} \hat{u}_v^* \text{ in } W^{1,p}(\Omega) \text{ and } u_n \rightarrow \hat{u}_v^* \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega).$$

In (3.8) we choose $h = u_n - \hat{u}_v^* \in W^{1,p}(\Omega)$ and pass to the limit as $n \rightarrow +\infty$. Then

$$(3.10) \quad \begin{aligned} \lim_{n \rightarrow +\infty} \langle A(u_n), u_n - \hat{u}_v^* \rangle &= 0, \\ \implies u_n &\rightarrow \hat{u}_v^* \text{ in } W^{1,p}(\Omega) \end{aligned}$$

(see Papageorgiou–Radulescu–Repovs [23, Proposition 2.7]).

So, if in (3.8) we pass to the limit as $n \rightarrow +\infty$ and use (3.10), then

$$\langle A(\hat{u}_v^*), h \rangle + \int_\Omega \xi(z) (\hat{u}_v^*)^{p-1} h \, dz + \int_{\partial\Omega} \beta(z) (\hat{u}_v^*)^{p-1} h \, d\sigma = \int_\Omega f(z, \hat{u}_v^*, Dv) h \, dz$$

for all $h \in W^{1,p}(\Omega)$.

Therefore $\hat{u}_v^* \in C_+$ is a solution of (3.1). We will show that $\hat{u}_v^* \neq 0$. To this end, let $\hat{u} \in S_v^+$ and consider the Caratheodory function $k(z, x)$ defined by

$$(3.11) \quad k(z, x) = \begin{cases} 0 & \text{if } x < 0, \\ [\hat{\eta}_M(z) - \varepsilon] x^{p-1} - c_4 x^{r-1} & \text{if } 0 \leq x \leq \hat{u}(z), \\ [\hat{\eta}_M(z) - \varepsilon] \hat{u}(z)^{p-1} - c_4 \hat{u}(z)^{r-1} & \text{if } \hat{u}(z) < x. \end{cases}$$

We set $K(z, x) = \int_0^x k(z, s) \, ds$ and consider the C^1 -functional $\hat{\psi}: W^{1,p}(\Omega) \rightarrow \mathbf{R}$ defined by

$$\hat{\psi}(u) = \frac{1}{p} \gamma(u) - \int_\Omega K(z, u) \, dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

Clearly $\hat{\psi}(\cdot)$ is coercive (see (3.11)). Also, it is sequentially weakly lower semicontinuous. So, we can find $\bar{u} \in W^{1,p}(\Omega)$ such that

$$(3.12) \quad \hat{\psi}(\bar{u}) = \inf [\hat{\psi}(u) : u \in W^{1,p}(\Omega)].$$

As in the proof of Proposition 2.8, since $r > p$, we have

$$\begin{aligned} \hat{\psi}(\bar{u}) &< 0 = \hat{\psi}(0), \\ \implies \bar{u} &\neq 0. \end{aligned}$$

From (3.12) we have

$$(3.13) \quad \begin{aligned} \hat{\psi}'(\bar{u}) &= 0, \\ \implies \langle A(\bar{u}), h \rangle + \int_{\Omega} \xi(z)|\bar{u}|^{p-2}\bar{u}h \, dz + \int_{\partial\Omega} \beta(z)|\bar{u}|^{p-2}\bar{u}h \, d\sigma &= \int_{\Omega} k(z, \bar{u})h \, dz \end{aligned}$$

for all $h \in W^{1,p}(\Omega)$.

In (3.13) we choose $h = -\bar{u}^- \in W^{1,p}(\Omega)$. Then using (3.11) we obtain

$$\begin{aligned} \gamma(\bar{u}^-) &= 0, \\ \implies \hat{\lambda}_1 \|\bar{u}^-\|_p^p &\leq 0 \quad (\text{see (2.3)}), \\ \implies \bar{u} &\geq 0, \bar{u} \neq 0. \end{aligned}$$

Next in (3.13) we choose $h = (\bar{u} - \hat{u})^+ \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} &\langle A(\bar{u}), (\bar{u} - \hat{u})^+ \rangle + \int_{\Omega} \xi(z)\bar{u}^{p-1}(\bar{u} - \hat{u})^+ \, dz + \int_{\partial\Omega} \beta(z)\bar{u}^{p-1}(\bar{u} - \hat{u})^+ \, d\sigma \\ &= \int_{\Omega} [(\hat{\eta}_M(z) - \varepsilon)\hat{u}^{p-1} - c_4\hat{u}^{r-1}](\bar{u} - \hat{u})^+ \, dz \quad (\text{see (3.11)}) \\ &\leq \int_{\Omega} f(z, \hat{u}, Dv)(\bar{u} - \hat{u})^+ \, dz \quad (\text{see (2.4)}) \\ &= \langle A(\hat{u}), (\bar{u} - \hat{u})^+ \rangle + \int_{\Omega} \xi(z)\hat{u}^{p-1}(\bar{u} - \hat{u})^+ \, dz \\ &\quad + \int_{\partial\Omega} \beta(z)\hat{u}^{p-1}(\bar{u} - \hat{u})^+ \, d\sigma \quad (\text{since } \hat{u} \in S_v^+), \\ \implies \bar{u} &\leq \hat{u}. \end{aligned}$$

Therefore we have proved that

$$(3.14) \quad \bar{u} \in [0, \hat{u}] = \{y \in W^{1,p}(\Omega) : 0 \leq y(z) \leq \hat{u}(z) \text{ for a.a. } z \in \Omega\}, \quad \bar{u} \neq 0.$$

From (3.13), (3.11) and (3.14), we infer that

$$\begin{aligned} -\Delta_p \bar{u}(z) + \xi(z)\bar{u}(z)^{p-1} &= [\hat{\eta}_M(z) - \varepsilon]\bar{u}(z)^{p-1} - c_4\bar{u}(z)^{r-1} \quad \text{for a.a. } z \in \Omega, \\ \frac{\partial \bar{u}}{\partial n_p} + \beta(z)\bar{u}^{p-1} &= 0 \quad \text{on } \partial\Omega, \\ \implies \bar{u} = \tilde{u} \in D_+ &\quad (\text{see Proposition 2.8}), \\ \implies \tilde{u} \leq \hat{u} &\quad \text{for all } \hat{u} \in S_v^+. \end{aligned}$$

Therefore we have

$$\begin{aligned} \tilde{u} &\leq u_n \quad \text{for all } n \in \mathbf{N}, \\ \implies \tilde{u} &\leq \hat{u}_v^* \quad (\text{see (3.10)}), \\ \implies \hat{u}_v^* &\neq 0. \end{aligned}$$

We conclude that $u_v^* \in S_v^+ \subseteq D_+$ and $u_v^* = \inf S_v^+$. □

So, we can define the minimal solution map $\tau: C_+ \rightarrow C_+$ by

$$\tau(v) = \hat{u}_v^*.$$

Proposition 3.3. *If hypotheses $H(\xi), H(\beta), H_0, H(f)$ hold, then the minimal solution map $\tau: C_+ \rightarrow C_+$ is compact.*

Proof. First we show that $\tau(\cdot)$ is continuous. So, let $v_n \rightarrow v$ in $C_+ \subseteq C^1(\bar{\Omega})$ and let $\hat{u}_n^* = \tau(v_n)$ for all $n \in \mathbf{N}$. We have

$$\begin{aligned} & \langle A(\hat{u}_n^*), h \rangle + \int_{\Omega} \xi(z)(\hat{u}_n^*)^{p-1} h \, dz + \int_{\partial\Omega} \beta(z)(\hat{u}_n^*)^{p-1} h \, d\sigma \\ (3.15) \quad & = \int_{\Omega} f(z, \hat{u}_n^*, Dv_n) h \, dz \quad \text{for all } h \in W^{1,p}(\Omega). \end{aligned}$$

Let $\rho = \max_{n \geq 1} \|v_n\|_{C^1(\bar{\Omega})}$ and $B_0 = \bar{B}_\rho(0) = \{y \in \mathbf{R}^N : |y| \leq \rho\}$. On account of hypotheses $H(f)(i),(ii)$, given $\varepsilon > 0$, we can find $c_8 = c_8(\varepsilon) > 0$ such that

$$\begin{aligned} & f(z, x, y) \leq [\theta(z) + \varepsilon]x^{p-1} + c_8 \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq 0, \text{ all } y \in B_0, \\ (3.16) \quad & \Rightarrow f(z, \hat{u}_n^*(z), Dv_n(z)) \leq [\theta(z) + \varepsilon]\hat{u}_n^*(z)^{p-1} + c_8 \quad \text{for a.a. } z \in \Omega, \text{ all } n \in \mathbf{N}. \end{aligned}$$

In (3.15) we choose $h = \hat{u}_n^* \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} & \gamma(\hat{u}_n^*) = \int_{\Omega} f(z, \hat{u}_n^*, Dv_n) \hat{u}_n^* \, dz \leq \int_{\Omega} [\theta(z) + \varepsilon](\hat{u}_n^*)^p \, dz + c_8 \int_{\Omega} \hat{u}_n^* \, dz, \\ & \Rightarrow [c_3 - \varepsilon]\|\hat{u}_n^*\|^{p-1} \leq c_9 \quad \text{for some } c_9 > 0, \text{ all } n \in \mathbf{N} \quad (\text{see Lemma 2.5}), \\ & \Rightarrow \{\hat{u}_n^*\}_{n \geq 1} \subseteq W^{1,p}(\Omega) \text{ is bounded} \quad (\text{choose } \varepsilon \in (0, c_3)). \end{aligned}$$

Invoking Proposition 2.10 of Papageorgiou–Radulescu [20], we can find $c_{10} > 0$ such that

$$\|\hat{u}_n^*\|_{\infty} \leq c_{10} \quad \text{for all } n \in \mathbf{N}.$$

Then using Theorem 2 of Lieberman [15], we can find $s \in (0, 1)$ and $c_{11} > 0$ such that

$$\hat{u}_n^* \in C^{1,s}(\bar{\Omega}), \quad \|\hat{u}_n^*\|_{C^{1,s}(\bar{\Omega})} \leq c_{11} \quad \text{for all } n \in \mathbf{N}.$$

Exploiting the compact embedding of $C^{1,s}(\bar{\Omega})$ into $C^1(\bar{\Omega})$, at least for a subsequence we have

$$(3.17) \quad \hat{u}_n^* \rightarrow \hat{u}^* \quad \text{in } C^1(\bar{\Omega}) \quad \text{and} \quad \hat{u}^* \in S_v^+ \quad (\text{see (3.15)}).$$

We will show that $\hat{u}^* = \tau(v) = \hat{u}_v^*$ and this will imply that the original sequence converges in $C^1(\bar{\Omega})$ to $\hat{u}^* = \tau(v) = \hat{u}_v^*$ and so we have the continuity of $\tau(\cdot)$.

Claim. We can find a sequence $\{\hat{u}_n\}_{n \geq 1}$ with $\hat{u}_n \in S_{v_n}^+$ for all $n \in \mathbf{N}$ such that

$$\hat{u}_n \rightarrow \tau(v) = u_v^* \quad \text{in } C^1(\bar{\Omega}).$$

To establish the Claim, we start by considering the following nonlinear Robin problem

$$\begin{cases} -\Delta_p u(z) + \xi(z)|u(z)|^{p-2}u(z) = f(z, u_v^*(z), Dv_n(z)) & \text{in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2}u = 0 & \text{on } \partial\Omega. \end{cases}$$

This problem has a unique solution (note that the map $u \rightarrow A(u) + \xi(z)|u|^{p-2}u$ is continuous, strictly monotone (hence maximal monotone) and coercive (see Lemma 2.3)).

Let $u_n^0 \in D_+(n \in \mathbf{N})$ be this unique solution. The nonlinear regularity theory of Lieberman [15] implies that there exist $\eta \in (0, 1)$ and $c_{12} > 0$ such that

$$u_n^0 \in C^{1,\eta}(\overline{\Omega}) \quad \text{and} \quad \|u_n^0\|_{C^{1,\eta}(\overline{\Omega})} \leq c_{12} \quad \text{for all } n \in \mathbf{N}.$$

So, by passing to a subsequence if necessary, we have

$$(3.18) \quad u_n^0 \rightarrow \hat{u}^0 \quad \text{in } C^1(\overline{\Omega}) \quad \text{as } n \rightarrow +\infty.$$

In the limit we have

$$\begin{aligned} -\Delta_p \hat{u}^0(z) + \xi(z) \hat{u}^0(z)^{p-1} &= f(z, u_v^*(z), Dv(z)) \quad \text{for a.a. } z \in \Omega, \\ \frac{\partial \hat{u}^0}{\partial n_p} + \beta(z) (\hat{u}^0)^{p-1} &= 0 \quad \text{on } \partial\Omega, \\ \implies \hat{u}^0 &= u_v^*. \end{aligned}$$

Hence for the original sequence we have

$$u_n^0 \rightarrow u_v^* \quad \text{in } C^1(\overline{\Omega}).$$

Next we consider the following nonlinear Robin problem

$$(3.19) \quad \begin{cases} -\Delta_p u(z) + \xi(z) |u(z)|^{p-2} u(z) = f(z, u_n^0(z), Dv_n(z)) & \text{in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z) |u|^{p-2} u = 0 & \text{on } \partial\Omega. \end{cases}$$

As above this problem too has a unique solution $u_n^1 \in D_+, n \in \mathbf{N}$ and

$$u_n^1 \rightarrow u_v^* \quad \text{in } C^1(\overline{\Omega}) \quad \text{as } n \rightarrow +\infty.$$

We continue this way and generate a sequence $\{u_n^k\}_{k,n \in \mathbf{N}}$ such that

$$(3.20) \quad \begin{cases} -\Delta_p u_n^k(z) + \xi(z) u_n^k(z)^{p-1} = f(z, u_n^{k-1}(z), Dv_n(z)) & \text{for a.a. } z \in \Omega, \\ \frac{\partial u_n^k}{\partial n_p} + \beta(z) (u_n^k)^{p-1} = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$(3.21) \quad u_n^k \rightarrow u_v^* \quad \text{in } C^1(\overline{\Omega}) \quad \text{as } n \rightarrow +\infty \quad \text{for all } k \in \mathbf{N}.$$

For every fixed $n \in \mathbf{N}$, from (3.20) and the nonlinear regularity theory (see [20, Proposition 2.10] and [15, Theorem 2]), we have that

$$\{u_n^k\}_{k \in \mathbf{N}} \subseteq C^1(\overline{\Omega}) \quad \text{is relatively compact.}$$

So, we can find a subsequence $\{u_n^{k_m}\}_{m \in \mathbf{N}}$ of $\{u_n^k\}_{k \in \mathbf{N}}$ such that

$$(3.22) \quad u_n^{k_m} \rightarrow \hat{u}_n \quad \text{in } C^1(\overline{\Omega}) \quad \text{as } m \rightarrow +\infty.$$

From (3.20) we obtain

$$(3.23) \quad \begin{cases} -\Delta_p \hat{u}_n(z) + \xi(z) \hat{u}_n(z)^{p-1} = f(z, \hat{u}_n(z), Dv_n(z)) & \text{for a.a. } z \in \Omega, \\ \frac{\partial \hat{u}_n}{\partial n_p} + \beta(z) \hat{u}_n^{p-1} = 0 & \text{on } \partial\Omega, \quad n \in \mathbf{N}. \end{cases}$$

From (3.23) we have that $\{\hat{u}_n\}_{n \geq 1} \subseteq C^1(\overline{\Omega})$ is relatively compact. So, from (3.21), (3.22) and the double limit lemma (see Aubin [1] and Gasinski–Papageorgiou [9, Problem 1.175, p. 61]), we have

$$\hat{u}_n \in S_{v_n}^+, n \in \mathbf{N} \quad (\text{see (3.23)}) \quad \text{and} \quad \hat{u}_n \rightarrow u_v^* \quad \text{in } C^1(\overline{\Omega}) \quad \text{as } n \rightarrow +\infty.$$

This proves the Claim.

On account of the Claim, we can find $\hat{u}_n \in S_{v_n}^+, n \in \mathbf{N}$ such that

$$(3.24) \quad \hat{u}_n \rightarrow u_v^* = \tau(v) \quad \text{in } C^1(\overline{\Omega}).$$

We have

$$\begin{aligned} u_n^* &= \tau(v_n) \leq \hat{u}_n \quad \text{for all } n \in \mathbf{N}, \\ \implies \hat{u}^* &\leq u_v^* \quad (\text{see (3.17), (3.24)}), \\ \implies \hat{u}^* &= u_v^* = \tau(v), \\ \implies \tau(\cdot) &\text{ is continuous.} \end{aligned}$$

From the above arguments it is also clear that, if $E \subseteq C_+$ is bounded, then

$$\tau(E) \subseteq C_+ \text{ is relatively compact.}$$

We conclude that the map $\tau(\cdot)$ is compact. □

We introduce the following set

$$L = \{u \in C_+ : u = t\tau(u), 0 < t < 1\}.$$

For this set we have the following result.

Proposition 3.4. *If hypotheses $H(\xi), H(\beta), H_0, H(f)$ hold, then $L \subseteq C_+$ is bounded.*

Proof. Let $u \in L$. We have

$$\frac{1}{t}u = \tau(u), \quad 0 < t < 1.$$

Therefore we have

$$(3.25) \quad \begin{cases} -\Delta_p(\frac{1}{t}u) + \xi(z)(\frac{1}{t}u)^{p-1} = f(z, \frac{1}{t}u, Du) & \text{for a.a. } z \in \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z)u^{p-1} = 0 & \text{on } \partial\Omega. \end{cases}$$

From (3.25) we obtain

$$(3.26) \quad \gamma(u) \leq t^{p-1} \int_{\Omega} f(z, \frac{1}{t}u, Du)u \, dz \leq \int_{\Omega} f(z, u, Du)u \, dz$$

(see hypothesis $H(f)(iv)$ and recall that $0 < t < 1$).

On account of hypotheses $H(f)(i),(ii)$, given $\varepsilon > 0$, we can find $c_{13} = c_{13}(\varepsilon) > 0$ such that

$$(3.27) \quad f(z, x, y)x \leq [\theta(z) + \varepsilon]x^p + c_{13}x + \mu|y|^{p-1}x$$

for a.a. $z \in \Omega$, all $x \geq 0$, all $y \in \mathbf{R}^N$.

Using (3.27) in (3.26), via Hölder’s inequality, we obtain

$$\begin{aligned} \gamma(u) - \int_{\Omega} \theta(z)|u|^p \, dz - \varepsilon\|u\|^p - \frac{\mu}{\hat{\lambda}_1^{1/p}}\gamma(u) &\leq c_{14}\|u\| \text{ for some } c_{14} > 0 \text{ (see (2.3)),} \\ \implies \mu_0\gamma(u) - \int_{\Omega} \theta(z)|u|^p \, dz - \varepsilon\|u\|^p &\leq c_{14}\|u\|, \\ \implies [c_{15} - \varepsilon]\|u\|^{p-1} &\leq c_{14} \quad (\text{see Lemma 2.5 and hypothesis } H(f)(ii)). \end{aligned}$$

Choosing $\varepsilon \in (0, c_{15})$, we infer that

$$L \subseteq W^{1,p}(\Omega) \text{ is bounded.}$$

From this as before, using the nonlinear regularity theory ([20, Proposition 2.10] and [15, Theorem 2]), we conclude that $L \subseteq C_+$ is bounded. □

Now we can state the following existence theorem.

Theorem 3.5. *If hypotheses $H(\xi)$, $H(\beta)$, H_0 , $H(f)$ hold, then problem (1.1) admits a positive solution $u_0 \in D_+$.*

Proof. Propositions 3.3 and 3.4 permit the use of Theorem 2.1 (the Leray-Schauder Alternative Theorem). So, we can find $u_0 \in C_+$ such that

$$\begin{aligned} u_0 &= \tau(u_0), \\ \implies u_0 &\in S_{u_0}^+ \subseteq D_+, \\ \implies u_0 &\in D_+ \text{ is a positive solution of (1.1).} \quad \square \end{aligned}$$

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