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NONLINEAR NONHOMOGENEOUS ROBIN PROBLEMS WITH CONVECTION

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Abstract. We consider a Robin problem driven by a nonlinear, nonhomogeneous differential operator with a drift term (convection) and a Carathéodory perturbation. Assuming that the drift coefficient is positive and using a topological approach based on the Leray–Schauder alternative principle, we show that the problem has a positive smooth solution.

1. Introduction

Let $\Omega \subseteq \mathbf{R}$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following nonlinear nonhomogeneous Robin problem with gradient dependence (convection):

$$(1.1)\qquad \begin{cases} -\operatorname{div} a(Du(z))+\xi(z)u(z)^{p-1} = f(z,u(z))+r(z)|Du(z)|^{p-1} & \text{in } \Omega,\\ \frac{\partial u}{\partial n_a}+\beta(z)u^{p-1}=0 & \text{on } \partial\Omega,\ u>0. \end{cases}$$

In this problem $a\colon \mathbf{R}^N \longrightarrow \mathbf{R}^N$ is continuous and strictly monotone and satisfies certain regularity and growth properties listed in hypotheses $H(a)$ below. These hypotheses are general enough to incorporate in our framework many differential operators of interest. The potential function $\xi \in L^\infty(\Omega)$ and $\xi(z) \geqslant 0$ for a.a. $z \in \Omega$. The drift coefficient $r \in L^\infty(\Omega)$ is nonnegative and the perturbation term $f(z,x)$ is a Carathéodory function (that is, for all $x \in \mathbf{R}$, $z \longmapsto f(z,x)$ is measurable and for a.a. $z \in \Omega$, $x \longmapsto f(z,x)$ is continuous) which exhibits $(p-1)$-linear growth near $+\infty$. In the boundary condition $\frac{\partial u}{\partial n_a}$ denotes the conormal derivative defined by extension of the map

$$C^1(\overline{\Omega}) \ni u \longmapsto (a(Du),n)_{\mathbf{R}^N},$$

with n being the outward unit normal on $\partial\Omega$.

The existence of positive solutions for elliptic problems with convection was studied by de Figueiredo–Girardi–Matzeu [4], Girardi–Matzeu [11] (semilinear problems driven by the Dirichlet Laplacian) and by Faraci–Motreanu–Puglisi [2], Faria–Miyagaki–Motreanu [3], Papageorgiou–Vetro–Vetro [19], Tanaka [21] (nonlinear Dirichlet problems). For Neumann problems, we have the recent works of Gasiński–Papageorgiou [8] and Papageorgiou–Rădulescu-Repovš [18] (semilinear problems).

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For Robin problems, there are the works of Bai–Gasinski–Papageorgiou [1] and Papageorgiou–Rădulescu–Repovš [17]. In these two works the gradient term is not decoupled from the perturbation. This leads to different hypotheses which do not cover the present setting (see hypotheses $H(f)$(ii) and (iii) in [1] and $H(f)$ (iii) in [17]). Moreover, in [17] the differential operator is the p-Laplacian. Finally for Robin problems but without convection term we have the works of Gasiński–O'Regan–Papageorgiou [6] and Gasiński–Papageorgiou [9].

The presence of the drift term $u \longmapsto r(z)|Du|^{p-1}$ makes problem (1.1) nonvariational. So, our approach is topological based on the Leray–Schauder alternative principle (fixed point theory).

2. Mathematical background – hypotheses

Let X and Y be Banach spaces and let $K\colon X \longrightarrow Y$ be a map. We say that K is "completely continuous", if $x_n \xrightarrow{w} x$ in X, implies that $K(x_n) \longrightarrow K(x)$ in Y. We say that K is "compact", if it is continuous and maps bounded set in X to relatively compact sets in Y.

The Leray–Schauder Alternative Principle says the following:

Theorem 2.1. *If V is a Banach space, $L\colon V \longrightarrow V$ is a compact map and*

$$S = \{v \in V \colon v = \lambda L(v) \text{ for some } 0 < \lambda < 1\},$$

then exactly one of the following holds:

(a) *S is unbounded; or*
(b) *L has a fixed point.*

The following spaces will be used in the analysis of problem (1.1): the Sobolev space $W^{1,p}(\Omega)$, the Banach space $C^1(\overline{\Omega})$ and the boundary Lebesgue space $L^p(\partial\Omega)$. By $\|\cdot\|$ we denote the norm of $W^{1,p}(\Omega)$ defined by

$$\|u\| = \left(\|u\|_p^p + \|Du\|_p^p\right)^{\frac{1}{p}} \quad \text{for all } u \in W^{1,p}(\Omega).$$

The Banach space $C^1(\overline{\Omega})$ is an ordered Banach space with positive (order) cone

$$C_+ = \{u \in C^1(\overline{\Omega}) \colon u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \Big\{u \in C_+ \colon u(z) > 0 \text{ for all } z \in \Omega,\ \frac{\partial u}{\partial n}\Big|_{\partial\Omega \cap u^{-1}(0)} < 0\Big\}.$$

In fact D_+ is also the interior of C_+ when $C^1(\overline{\Omega})$ is endowed with the $C(\overline{\Omega})$-norm topology.

On $\partial\Omega$ we define the $(N-1)$-dimensional Hausdorff (surface) measure σ. Using this measure we can define in the usual way the "boundary" Lebesgue space $L^r(\partial\Omega)$ $(1 \leqslant r \leqslant +\infty)$. We know that there exists a unique continuous, linear map $\gamma_0\colon W^{1,p}(\Omega) \longrightarrow L^p(\partial\Omega)$, known as the "trace map", such that

$$\gamma_0(u) = u|_{\partial\Omega} \quad \text{for all } u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}).$$

Hence the trace map extends the notion of "boundary values" to all Sobolev functions. The map γ_0 is compact into $L^r(\partial\Omega)$ for all $r \in [1, \frac{(N-1)p}{N-p})$ if $p < N$ and into $L^r(\partial\Omega)$ for all $1 \leqslant r < +\infty$ if $p \geqslant N$. In addition we have

$$\operatorname{im} \gamma_0 = W^{\frac{1}{p'},p}(\partial\Omega) \quad \text{and} \quad \ker \gamma_0 = W_0^{1,p}(\Omega)$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ (that is, γ_0 is not a surjection).

In the sequel, for the sake of simplicity we drop the use of the trace map γ_0. All restrictions of Sobolev functions on $\partial\Omega$ are understood in the sense of traces.

Let $k \in C^1(0,+\infty)$ and assume that it satisfies the following growth condition

$$0 < \widehat{c} \leqslant \frac{tk'(t)}{k(t)} \leqslant c_0 \quad \text{and} \quad c_1 t^{p-1} \leqslant k(t) \leqslant c_2(t^{\tau-1} + t^{p-1}) \quad \forall t > 0, \tag{2.1}$$

with $c_1, c_2 > 0$ and $1 \leqslant \tau < p$.

We introduce the conditions on the map a.

$H(a)$: $a(y) = a_0(|y|)y$ for all $y \in \mathbf{R}^N$ with $a_0(t) > 0$ for all $t > 0$ and

(i) $a_0 \in C^1(0,+\infty)$, $t \longmapsto a_0(t)t$ is strictly increasing on $(0,+\infty)$, $a_0(t)t \longrightarrow 0^+$ as $t \to 0^+$ and

$$\lim_{t\to 0^+} \frac{a_0'(t)t}{a_0(t)} > -1;$$

(ii) there exists $c_3 > 0$ such that $|\nabla a(y)| \leqslant c_3 \frac{k(|y|)}{|y|}$ for all $y \in \mathbf{R}^N \setminus \{0\}$;

(iii) $\frac{k(|y|)}{|y|}|\xi|^2 \leqslant (\nabla a(y)\xi, \xi)_{\mathbf{R}^N}$ for all $y \in \mathbf{R}^N \setminus \{0\}$, all $\xi \in \mathbf{R}^N$;

(iv) if $G_0(t) = \int_0^t a_0(s)s\,ds$, then there exists $1 < q \leqslant p$ such that

$$t \longmapsto G_0(t^{\frac{1}{q}}) \text{ is convex on } (0,+\infty)$$

and

$$\limsup_{t\to 0^+} \frac{qG_0(t)}{t^q} \leqslant \widetilde{c}.$$

Remark 2.2. Hypotheses $H(a)$(i)–(iii) are dictated by the nonlinear regularity theory of Lieberman [12] and the nonlinear maximum principle of Pucci–Serrin [20]. Hypothesis $H(a)$(iv) addresses the particular needs of our problem. However, it is a mild requirement and it is satisfied in all cases of interest. Similar conditions were also used in Bai–Gasinski–Papageorgiou [1].

Note that G_0 is strictly increasing and strictly convex. If we set

$$G(y) = G_0(|y|) \quad \forall y \in \mathbf{R}^N,$$

then G is convex, $G(0) = 0$ and

$$\nabla G(y) = G_0'(|y|)\frac{y}{|y|} = a_0(|y|)y = a(y) \quad \forall y \in \mathbf{R}^N \setminus \{0\},\ \nabla G(0) = 0.$$

So, G is the primitive of a and on account of the convexity of G we have

$$G(y) \leqslant (a(y), y)_{\mathbf{R}^N} \quad \forall y \in \mathbf{R}^N. \tag{2.2}$$

The next lemma summarizes the main properties of the map a. It follows from hypotheses $H(a)$.

Lemma 2.3. *If hypotheses* $H(a)$(i), (ii) *and* (iii) *hold, then*

(a) $y \longmapsto a(y)$ *is continuous, monotone (hence maximal monotone too);*

(b) *there exists* $c_4 > 0$*, such that* $|a(y)| \leqslant c_4\big(|y|^{\tau-1} + |y|^{p-1}\big)$ *for all* $y \in \mathbf{R}^N$*;*

(c) $(a(y), y)_{\mathbf{R}^N} \geqslant \frac{c_1}{p-1}|y|^p$ *for all* $y \in \mathbf{R}^N$.

From this lemma and (2.1), (2.2), we have the following growth estimates for the primitive G.

Corollary 2.4. *If hypotheses* $H(a)$ (i), (ii) *and* (iii) *hold, there exists* $c_5 > 0$ *such that*

$$\frac{c_1}{p(p-1)}|y|^p \leqslant G(y) \leqslant c_5(1+|y|^p) \quad \forall y \in \mathbf{R}^N.$$

The p-Laplacian

$$\Delta_p u = \operatorname{div}(|Du|^{p-2}Du) \quad \forall u \in W^{1,p}(\Omega),$$

with $1 < p < +\infty$ and the (p,q)-Laplacian

$$\Delta_p u + \Delta_q u \quad \forall u \in W^{1,p}(\Omega),$$

with $1 < r < p < +\infty$ are within the framework corresponding to hypotheses $H(a)$. More about this set of conditions can be found in Papageorgiou–Rădulescu [16].

The hypotheses on the potential ξ and the boundary coefficient β are the following:

$H(\xi)$: $\xi \in L^\infty(\Omega)$ and $\xi(z) \geqslant 0$ for a.a. $z \in \Omega$.
$H(\beta)$: $\beta \in C^{0,\alpha}(\partial\Omega)$ for some $\alpha \in (0,1)$ and $\beta(z) \geqslant 0$ for all $z \in \partial\Omega$.
H_0: $\xi \not\equiv 0$ or $\beta \not\equiv 0$.

Remark 2.5. If $\beta \equiv 0$, then we recover the Neumann problem for the operator $-\operatorname{div} a(Du) + \xi(z)|u|^p$.

From Gasiński–Papageorgiou [10], for any $r \in (1,+\infty)$, we have the following result.

Proposition 2.6. (a) *If* $\xi \in L^\infty(\Omega)$, $\xi(z) \geqslant 0$ *for a.a.* $z \in \Omega$ *and* $\xi \not\equiv 0$, *then*

$$\|Du\|_r^r + \int_\Omega \xi(z)|u|^r\,dz \geqslant c_6\|u\|^r \quad \forall u \in W^{1,r}(\Omega),$$

for some $c_6 > 0$;

(b) *If* $\beta \in C^{0,\alpha}(\partial\Omega)$, $\beta(z) \geqslant 0$ *for all* $z \in \partial\Omega$ *and* $\beta \not\equiv 0$, *then*

$$\|Du\|_r^r + \int_{\partial\Omega} \beta(z)|u|^r\,d\sigma \geqslant c_7\|u\|^r \quad \forall u \in W^{1,r}(\Omega),$$

for some $c_7 > 0$.

Remark 2.7. If $\gamma_r(u) = \|Du\|_r^r + \int_\Omega \xi(z)|u|^r + \int_{\partial\Omega}\beta(z)|u|^r\,d\sigma$ for all $u \in W^{1,p}(\Omega)$, then Proposition 2.6 implies that

$$\gamma_r(u) \geqslant \widehat{c}_0\|u\|^r \quad \forall u \in W^{1,r}(\Omega),$$

for some $\widehat{c}_0 > 0$.

Let $r \in (1,+\infty)$ and consider the following nonlinear eigenvalue problem:

$$\begin{cases} -\Delta_r u(z) + \xi(z)|u(z)|^{r-2}u(z) = \widehat{\lambda}|u(z)|^{r-2}u(z) & \text{in } \Omega, \\ \frac{\partial u}{\partial n_r} + \beta(z)|u|^{r-2}u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Here $\frac{\partial u}{\partial n_r} = |Du|^{r-2}(Du,n)_{\mathbf{R}^N}$. We say that $\widehat{\lambda}$ is an "eigenvalue", if problem (2.3) admits a nontrivial solution $\widehat{u} \in W^{1,r}(\Omega)$, known as an "eigenfunction" corresponding to $\widehat{\lambda}$. Nonlinear regularity theory (see Lieberman [12]), implies that $\widehat{u} \in C^1(\overline{\Omega})$. There is a smallest eigenvalue $\widehat{\lambda}_1(r,\xi,\beta)$ which has the following properties:

- $\widehat{\lambda}_1(r,\xi,\beta) > 0$ (see Proposition 2.6);
- $\widehat{\lambda}_1(r,\xi,\beta)$ is isolated in the spectrum $\widehat{\sigma}(r)$ of (2.3) (that is, there exists $\varepsilon > 0$ such that $(\widehat{\lambda}_1(r,\xi,\beta), \widehat{\lambda}_1(r,\xi,\beta)+\varepsilon) \cap \widehat{\sigma}(r) = \emptyset$);

- $\widehat{\lambda}_1(r,\xi,\beta)$ is simple (that is, if $\widehat{u},\widehat{v}\in C^1(\overline{\Omega})$ are eigenfunctions corresponding to $\widehat{\lambda}_1(r,\xi,\beta)$, then $\widehat{u}=\eta\widehat{v}$ for some $\eta\in\mathbf{R}\setminus\{0\}$);
- if $\gamma_r(u)=\|Du\|_r^r+\int_\Omega\xi(z)|u|^r\,dz+\int_{\partial\Omega}\beta(z)|u|^r\,d\sigma$ for all $u\in W^{1,r}(\Omega)$, then

$$\widehat{\lambda}_1(r,\xi,\beta)=\inf_{u\in W^{1,r}(\Omega)\setminus\{0\}}\frac{\gamma_r(u)}{\|u\|_r^r}. \tag{2.4}$$

The above properties imply that the elements of the one-dimensional eigenspace corresponding to $\widehat{\lambda}_1(r,\xi,\beta)>0$, do not change sign. By $\widehat{u}_1(r,\xi,\beta)$ we denote the positive, L^r-normalized (that is $\|\widehat{u}_1(r,\xi,\beta)\|_r=1$) eigenfunction corresponding to $\widehat{\lambda}_1(r,\xi,\beta)>0$. We have $\widehat{u}_1(r,\xi,\beta)\in D_+$ (see Gasiński–Papageorgiou [7, p. 739]). More about the eigenvalue problem (2.3) can be found in Fragnelli–Mugnai–Papageorgiou [5] and Papageorgiou–Rădulescu [14].

Using above properties, we can easily prove the following lemma (see Mugnai–Papageorgiou [13, Lemma 4.11]).

Lemma 2.8. *If $\vartheta\in L^\infty(\Omega)$ and $\vartheta(z)\leqslant\widehat{\lambda}_1(r,\xi,\beta)$ for a.a. $z\in\Omega$ with strict inequality on a set of positive measure, then there exists $c_8>0$ such that*

$$c_8\|u\|^r\leqslant\gamma_r(u)-\int_\Omega\vartheta(z)|u|^r,dz\quad\forall u\in W^{1,r}(\Omega).$$

In what follows, we set

$$\xi_*=\frac{p-1}{c_1}\xi\quad\text{and}\quad\beta_*=\frac{p-1}{c_1}\beta,\quad\xi_0=\frac{1}{\widetilde{c}}\xi\quad\text{and}\quad\beta_0=\frac{1}{\widetilde{c}}\beta.$$

Both pairs satisfy hypotheses $H(\xi)$, $H(\beta)$ and H_0.

The hypotheses on the drift coefficient r are the following.

$H(r)$: $r\in L^\infty(\Omega)$, $r(z)\geqslant0$ for a.a. $z\in\Omega$ and $\tau_0=\frac{c_1}{p-1}-\frac{\|r\|_\infty}{\widehat{\lambda}_1(p,\xi_*,\beta_*)}>0$.

Remark 2.9. The last part of the above hypothesis impose a bound on the drift coefficient r.

Finally we introduce the hypotheses on the perturbation $f(z,x)$.

$H(f)$: $f\colon\Omega\times\mathbf{R}\longrightarrow\mathbf{R}$ is a Carathéodory function, $f(z,0)=0$ for a.a. $z\in\Omega$ and
- (i) $|f(z,x)|\leqslant a_0(z)(1+x^{r-1})$ for a.a. $z\in\Omega$, all $x\geqslant0$, with $a_0\in L^\infty(\Omega)_+$, $p<r<p^*$;
- (ii) there exists a function $\vartheta\in L^\infty(\Omega)_+$ such that

$$\vartheta(z)\leqslant\tau_0\widehat{\lambda}_1(p,\xi_*,\beta_*)\quad\text{a.e. in }\Omega,\quad\vartheta\not\equiv\tau_0\widehat{\lambda}_1(p,\xi_*,\beta_*),$$

$$\limsup_{x\to+\infty}\frac{f(z,x)}{x^{p-1}}\leqslant\vartheta(z)\quad\text{uniformly for a.a. }z\in\Omega;$$

- (iii) there exists a function $\eta\in L^\infty(\Omega)$ such that

$$\eta(z)\geqslant\widehat{\lambda}_1(q,\xi_0,\beta_0)\quad\text{for a.a. }z\in\Omega,\quad\eta\not\equiv\widehat{\lambda}_1(q,\xi_0,\beta_0),$$

$$\liminf_{x\to0^+}\frac{f(z,x)}{x^{q-1}}\geqslant\eta(z)\quad\text{uniformly for a.a. }z\in\Omega$$

(here $1<q\leqslant p$ is as in hypothesis $H(a)$(iv)).

Remark 2.10. Since our aim is to find positive solutions and the above hypotheses concern the positive semiaxis $\mathbf{R}_+=[0,+\infty)$, without any loss of generality, we may assume that

$$f(z,x)=0\quad\text{for a.a. }z\in\Omega,\ \text{all }x\leqslant0. \tag{2.5}$$

In what follows $A\colon W^{1,p}(\Omega) \longrightarrow W^{1,p}(\Omega)^*$ is the nonlinear map defined by

$$\langle A(u), h\rangle = \int_\Omega (a(Du), Dh)_{\mathbf{R}^N}\, dz \quad \forall u, h \in W^{1,p}(\Omega).$$

This map is monotone, continuous, hence maximal monotone. Also, if $x \in \mathbf{R}$, we set $x^\pm = \max\{\pm x, 0\}$. Then for $u \in W^{1,p}(\Omega)$ we define $u^\pm(\cdot) = u(\cdot)^\pm$. We know that, if $u \in W^{1,p}(\Omega)$, then

$$u^\pm \in W^{1,p}(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ + u^-.$$

3. Positive solution

On account of hypotheses $H(f)$, given $\varepsilon > 0$, we can find $c_9 = c_9(\varepsilon) > 0$ such that

$$f(z,x) \geqslant (\widehat{\eta}(z) - \varepsilon)x^{q-1} - c_9 x^{r-1} \quad \text{for a.a. } z \in \Omega, \text{ all } x \geqslant 0. \tag{3.1}$$

We consider the following auxiliary Robin problem:

$$\begin{cases} -\operatorname{div} a(Du(z)) + \xi(z)u(z)^{p-1} = (\widehat{\eta}(z) - \varepsilon)u(z)^{q-1} - c_9 u(z)^{r-1} & \text{in } \Omega, \\ \frac{\partial u}{\partial n_a} + \beta(z)u^{p-1} = 0 & \text{on } \partial\Omega,\ u > 0. \end{cases} \tag{3.2}$$

Proposition 3.1. *If hypotheses $H(a)$, $H(\xi)$, $H(\beta)$ and H_0 hold, then for all $\varepsilon > 0$ small, problem (3.2) admits a unique solution $u_* \in D_+$.*

Proof. We consider the C^1-functional $\psi_\varepsilon\colon W^{1,p}(\Omega) \longrightarrow \mathbf{R}$, $\varepsilon > 0$, defined by

$$\begin{aligned} \psi_\varepsilon(u) = &\int_\Omega G(Du)\, dz + \frac{1}{p}\int_\Omega \xi(z)|u|^p\, dz + \frac{1}{p}\int_{\partial\Omega} \beta(z)|u|^p\, d\sigma \\ &- \frac{1}{q}\int_\Omega (\eta(z) - \varepsilon)(u^+)^q\, dz + \frac{c_9}{r}\|u^+\|_r^r. \end{aligned}$$

Using hypothesis H_0, Proposition 2.6 and recalling that $q \leqslant p < r$, we have

$$\begin{aligned} \psi_\varepsilon(u) &\geqslant c_{10}\|u\|^p + \frac{c_9}{r}\|u^+\|_r^r - c_{11}\|u^+\|_q^q \\ &\geqslant c_{10}\|u\|^p + c_{12}\|u^+\|_p^r - c_{13}\|u^+\|_p^q \\ &= c_{10}\|u\|^p + (c_{12}\|u^+\|_p^{r-q} - c_{13})\|u^+\|_p^q \quad \forall u \in W^{1,p}(\Omega), \end{aligned}$$

for some $c_{10}, c_{11}, c_{12}, c_{13} > 0$, so, ψ_ε is coercive.

Also using the Sobolev embedding theorem and the compactness of the trace map, we infer that ψ_ε is sequentially weakly lower semicontinuous. So, by the Weierstrass–Tonelli theorem, we can find $u_* \in W^{1,p}(\Omega)$ such that

$$\psi_\varepsilon(u_*) = \inf_{u \in W^{1,p}(\Omega)} \psi_\varepsilon(u). \tag{3.3}$$

On account of hypothesis $H(a)$(iv), given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) \in (0,1)$ such that

$$G(y) \leqslant \frac{1}{q}(\widetilde{c} + \varepsilon)|y|^q \quad \forall |y| \leqslant \delta. \tag{3.4}$$

Let $t \in (0,1)$ be small such that

$$0 < t\widehat{u}_1(q, \xi_0, \beta_0)(z) \leqslant \delta \quad \forall z \in \overline{\Omega} \tag{3.5}$$

(recall that $\widehat{u}_1(q,\xi_0,\beta_0) \in D_+$). To simplify the notation, let $\widehat{u}_1(q) = \widehat{u}_1(q,\xi_0,\beta_0)$ and $\widehat{\lambda}_1(q) = \widehat{\lambda}_1(q,\xi_0,\beta_0)$. Since $\delta \in (0,1)$ and $q \leqslant p$, we have

$$\begin{aligned}\psi_\varepsilon(t\widehat{u}_1(q)) &\leqslant \frac{\widetilde{c}+\varepsilon}{q} t^q \|D\widehat{u}_1(q)\|_q^q + \frac{\widetilde{c}}{q}\int_\Omega \xi_0(t\widehat{u}_1(q))^q\,dz + \frac{\widetilde{c}}{q}\int_{\partial\Omega} \beta_0(t\widehat{u}_1(q))^q\,d\sigma \\ &\quad - \frac{1}{q}\int_\Omega \eta(z)(t\widehat{u}_1(q))^q\,dz + \frac{\varepsilon t^q}{q} + \frac{c_6 t^r}{r}\|\widehat{u}_1(q)\|_r^r \\ &= \frac{\widetilde{c}t^q}{q}\int_\Omega (\widehat{\lambda}_1(q) - \eta(z))\widehat{u}_1(q)^q\,dz + \frac{\varepsilon t^q}{q}(\widehat{\lambda}_1(q)+1) + c_{14}t^r, \end{aligned} \tag{3.6}$$

for some $c_{14} > 0$ (see (3.4), (3.5) and recall that $\|\widehat{u}_1(q)\|_q = 1$).

Note that

$$\int_\Omega (\widehat{\lambda}_1(q) - \eta(z))\widehat{u}_1(q)^q\,dz < 0$$

(see hypothesis $H(f)$(iii)). Therefore choosing $\varepsilon > 0$ small and since $t \in (0,1)$, $q < r$, from (3.6) we infer that

$$\psi_\varepsilon(t\widehat{u}_1(q)) < 0 \quad \forall \varepsilon > 0 \text{ small},$$

so

$$\psi_\varepsilon(u_*) < 0 = \psi_\varepsilon(0)$$

(see (3.3)) and thus $u_* \neq 0$. From (3.3) we have

$$\psi_\varepsilon'(u_*) = 0,$$

so

$$\begin{aligned}\langle A(u_*), h\rangle &+ \int_\Omega \xi(z)|u_*|^{p-2}u_* h\,dz + \int_{\partial\Omega} \beta(z)|u_*|^{p-2}u_* h\,d\sigma \\ &= \int_\Omega (\eta(z)-\varepsilon)(u_*^+)^{q-1}h\,dz - c_9\int_\Omega (u_*^+)^{r-1}h\,dz \quad \forall h \in W^{1,p}(\Omega).\end{aligned} \tag{3.7}$$

In (3.7) we choose $h = -u_*^- \in W^{1,p}(\Omega)$. Then

$$\frac{c_1}{p-1}\|Du_*^-\|_p^p + \int_\Omega \xi(z)(u_*^-)^p\,dz + \int_{\partial\Omega} \beta(z)(u_*^-)^p\,d\sigma \leqslant 0,$$

so

$$c_{15}\|u_*^-\|^p \leqslant 0,$$

for some $c_{15} > 0$ (see Proposition 2.6), thus

$$u_* \geqslant 0, \quad u_* \neq 0. \tag{3.8}$$

From (3.7) and (3.8), we have

$$\begin{cases} -\operatorname{div} a(Du_*(z)) + \xi(z)u_*(z)^{p-1} = (\eta(z)-\varepsilon)u_*(z)^{q-1} + c_9 u_*(z)^{r-1} & \text{in } \Omega, \\ \frac{\partial u_*}{\partial n_a} + \beta(z)u_*^{p-1} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.9}$$

(see Papageorgiou–Rădulescu [14]).

From (3.9) and Proposition 2.10 of Papageorgiou–Rădulescu [15], we have

$$u_* \in L^\infty(\Omega).$$

Then from the nonlinear regularity theory of Lieberman [12], we have that

$$u_* \in C_+ \setminus \{0\}.$$

From (3.9) we obtain

$$\operatorname{div} a(Du_*(z)) \leqslant (c_9\|u_*\|_\infty^{r-p} + \|\xi\|_\infty)u_*(z)^{p-1} \quad \text{for a.a. } z \in \Omega,$$

so $u_* \in D_+$ (see Pucci–Serrin [20, pp. 111, 120]).

In fact this positive solution of (3.2) is unique. To show this, we introduce the integral functional $j\colon L^1(\Omega) \longrightarrow \overline{\mathbf{R}} = \mathbf{R} \cup \{+\infty\}$ defined by

$$j(u) = \begin{cases} \int_\Omega G(Du^{\frac{1}{q}})\,dz + \frac{1}{p}\int_\Omega \xi(z)u^{\frac{p}{q}}\,dz, & \\ \qquad\qquad + \frac{1}{p}\int_{\partial\Omega}\beta(z)u^{\frac{p}{q}}\,d\sigma & \text{if } u \geqslant 0,\ u^{\frac{1}{q}} \in W^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

As in Papageorgiou–Rădulescu [16, proof of Proposition 3.5], we show that

$$j \text{ is convex} \tag{3.10}$$

and

$$j'(u_*^q)(h) = \frac{1}{q}\int_\Omega \frac{-\operatorname{div} a(Du_*) + \xi(z)u_*^{p-1}}{u_*^{q-1}} h\,dz \quad \forall h \in C^1(\overline{\Omega}). \tag{3.11}$$

Here we use the fact that given $h \in C^1(\overline{\Omega})$, for $|t| < 1$ small we have $u_*^q + th \in \operatorname{dom} j$.

Suppose that v_* is another positive solution of (3.2). Similarly we have

$$v_* \in D_+$$

and

$$j'(v_*^q)(h) = \frac{1}{q}\int_\Omega \frac{-\operatorname{div} a(Dv_*) + \xi(z)v_*^{p-1}}{v_*^{q-1}} h\,dz \quad \forall h \in C^1(\overline{\Omega}).$$

From (3.10), it follows that j' is monotone. Therefore

$$\begin{aligned} 0 &\leqslant \int_\Omega \bigg(\frac{-\operatorname{div} a(Du_*) + \xi(z)u_*^{p-1}}{u_*^{q-1}} h\,dz - \frac{-\operatorname{div} a(Dv_*) + \xi(z)v_*^{p-1}}{v_*^{q-1}} h\,dz\bigg)(u_*^q - v_*^q)\,dz \\ &= c_9 \int_\Omega (v_*^{r-1} - u_*^{r-1})(u_*^q - v_*^q)\,dz \leqslant 0, \end{aligned}$$

so $u_* = v_*$. This proves the uniqueness of the positive solution of (3.2). □

For $h \in L^\infty(\Omega)$, we consider the following auxiliary Robin problem:

$$\begin{cases} -\operatorname{div} a(Du(z)) + \xi(z)|u(z)|^{p-2}u(z) = h(z) & \text{in } \Omega, \\ \frac{\partial u}{\partial n_a} + \beta(z)|u|^{p-2}u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.12}$$

Proposition 3.2. *If hypotheses $H(a)$, $H(\xi)$, $H(\beta)$ and H_0 hold, then problem (3.12) admits a unique solution $K(h) \in C^1(\overline{\Omega})$.*

Proof. Consider the C^1-functional $\mu\colon W^{1,p}(\Omega) \longrightarrow \mathbf{R}$ defined by

$$\begin{aligned} \mu(u) = \int_\Omega G(Du)\,dz + \frac{1}{p}\int_\Omega \xi(z)|u|^p\,dz + \frac{1}{p}\int_{\partial\Omega}\beta(z)|u|^p\,d\sigma \\ - \int_\Omega hu\,dz \quad \forall u \in W^{1,p}(\Omega). \end{aligned}$$

Using Corollary 2.4 and Proposition 2.6, we see that μ is coercive. Also, it is sequentially lower semicontinuous. So, by the Weierstrass–Tonelli theorem, we can find $K(h) = \widehat{u} \in W^{1,p}(\Omega)$ such that

$$\mu(\widehat{u}) = \inf_{u \in W^{1,p}(\Omega)} \mu(u),$$

so $\mu'(\widehat{u}) = 0$ and thus

$$\langle A(\widehat{u}), v\rangle + \int_\Omega \xi(z)|\widehat{u}|^{p-2}\widehat{u}v\,dz + \int_{\partial\Omega}\beta(z)|\widehat{u}|^{p-2}\widehat{u}v\,d\sigma = \int_\Omega hv\,dz \tag{3.13}$$

for all $u \in W^{1,p}(\Omega)$, so $K(h) = \widehat{u}$ is a solution of (3.12). The nonlinear regularity theory implies that

$$K(h) = \widehat{u} \in C^1(\overline{\Omega}).$$

The uniqueness of this positive solution follows as in the proof of Proposition 3.1. □

Remark 3.3. If $h \in L^\infty(\Omega)$ satisfies $h(z) \geqslant 0$ for a.a. $z \in \Omega$, $h \not\equiv 0$, then $K(h) \in D_+$. To see this, in (3.13) we choose $v = -\widehat{u}^- \in W^{1,p}(\Omega)$ and obtain

$$c_{16}\|\widehat{u}^-\|^p \leqslant 0$$

for some $c_{16} > 0$ (see Lemma 2.3 and Proposition 2.6), so

$$\widehat{u} \geqslant 0, \quad \widehat{u} \neq 0$$

(since $h \not\equiv 0$).

So, we have $\widehat{u} = K(h) \in C_+$ and

$$\operatorname{div} a(D\widehat{u}(z)) \leqslant \|\xi\|_\infty \widehat{u}(z)^{p-1} \quad \text{for a.a. } z \in \Omega$$

(since $h \geqslant 0$), thus

$$\widehat{u} = K(h) \in D_+$$

(see Pucci–Serrin [20, p. 111, 120]).

We consider the solution map $K\colon L^\infty(\Omega) \longrightarrow C^1(\overline{\Omega})$.

Proposition 3.4. *If hypotheses $H(a)$, $H(\xi)$, $H(\beta)$ and H_0 hold, then the map K is sequentially continuous from $L^\infty(\Omega)$ with the w^*-topology into $C^1(\overline{\Omega})$ with the norm topology.*

Proof. Let $h_n \xrightarrow{w^*} h$ in $L^\infty(\Omega)$ and let $\widehat{u}_n = K(h_n)$ for all $n \in \mathbf{N}$. We have

$$\begin{aligned} &\langle A(\widehat{u}_n), v\rangle + \int_\Omega \xi(z)|\widehat{u}_n|^{p-2}\widehat{u}_n v\,dz + \int_{\partial\Omega} \beta(z)|\widehat{u}_n|^{p-2}\widehat{u}_n v\,d\sigma \\ &\quad = \int_\Omega h_n v\,dz \quad \forall v \in W^{1,p}(\Omega),\ n \in \mathbf{N}. \end{aligned} \tag{3.14}$$

In (3.14) we choose $v = \widehat{u}_n \in W^{1,p}(\Omega)$. Then

$$\frac{c_1}{p-1}\|D\widehat{u}_n\|_p^p + \int_\Omega \xi(z)|\widehat{u}_n|^p\,dz + \int_{\partial\Omega} \beta(z)|\widehat{u}_n|^p\,d\sigma \leqslant c_{17}\|\widehat{u}_n\| \quad \forall n \in \mathbf{N},$$

for some $c_{17} > 0$ (see Lemma 2.3), so

$$c_{18}\|\widehat{u}_n\|^p \leqslant c_{17}\|\widehat{u}_n\| \quad \forall n \in \mathbf{N},$$

for some $c_{18} > 0$ (see Proposition 2.6) and thus the sequence $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega)$ is bounded.

Then from Proposition 2.10 of Papageorgiou–Rădulescu [15], we know that we can find $c_{19} > 0$ such that

$$\|\widehat{u}_n\|_\infty \leqslant c_{19} \quad \forall n \in \mathbf{N}.$$

The nonlinear regularity theory of Lieberman [12] implies that

$$\widehat{u}_n \in C^{1,\alpha}(\overline{\Omega}), \quad \|\widehat{u}_n\|_{C^{1,\alpha}(\overline{\Omega})} \leqslant c_{20} \quad \forall n \in \mathbf{N}, \tag{3.15}$$

for some $\alpha \in (0,1)$ and some $c_{20} > 0$. Exploiting the compactness of the embedding $C^{1,\alpha}(\overline{\Omega}) \subseteq C^1(\overline{\Omega})$, from (3.15) we see that, at least for a subsequence, we have

$$\widehat{u}_n \longrightarrow \widehat{u} \quad \text{in } C^1(\overline{\Omega}).$$

Passing to the limit as $n \to +\infty$ in (3.14), we obtain

$$\widehat{u} = K(h).$$

So, for the origin sequence, we have

$$\widehat{u}_n = K(h_n) \longrightarrow K(h) = \widehat{u} \quad \text{in } C^1(\overline{\Omega}),$$

so $K\colon L^\infty(\Omega) \longrightarrow C^1(\overline{\Omega})$ is sequentially (w^*, s)-continuous. □

Let $u_* \in D_+$ be the unique positive solution of problem (3.2) produced in Proposition 3.1. We introduce the following truncation of $f(z,\cdot)$:

$$\widehat{f}(z,x) = \begin{cases} f(z,u_*(z)) & \text{if } x \leqslant u_*(z), \\ f(z,x) & \text{if } u_*(z) < x. \end{cases} \tag{3.16}$$

This is a Carathéodory function. Let $N_{\widehat{f}}$ be the Nemytski (superposition) map corresponding to $\widehat{f}$, that is,

$$N_{\widehat{f}}(u)(\cdot) = \widehat{f}(\cdot, u(\cdot)) \quad \forall u \in W^{1,p}(\Omega).$$

We consider the map $N\colon C^1(\overline{\Omega}) \longrightarrow L^p(\Omega)$ defined by

$$N(u) = N_{\widehat{f}}(u) + r(z)|Du^+|^{p-1} \quad \forall u \in C^1(\overline{\Omega}).$$

We know that $u \longmapsto u^+$ is continuous from $W^{1,p}(\Omega)$ into itself. Moreover, note that N has values in L^∞ (see hypotheses $H(f)(i)$ and $H(r)$). In fact, N maps bounded sets in $C^1(\overline{\Omega})$ to bounded sets in $L^\infty(\Omega)$. So, by Krasnoselskii's theorem (see Gasiński–Papageorgiou [7, Theorem 3.4.4, p. 407]), the Nemytskii map N is continuous.

Now, consider the map $L = K \circ N\colon C^1(\overline{\Omega}) \longrightarrow C^1(\overline{\Omega})$. We see that L is continuous. Also, if $D \subseteq C^1(\overline{\Omega})$ is bounded, then $N(D) \subseteq L^\infty(\Omega)$ is bounded and so it is relatively sequentially w^*-compact (since $L^\infty(\Omega) = L^1(\Omega)^*$ and the space $L^1(\Omega)$ is separable). Therefore, using Proposition 3.4, we obtain that $L(D) \subseteq C^1(\overline{\Omega})$ is relatively compact. We conclude that the map $u \longmapsto L(u) = (K \circ N)(u)$ is compact.

Consider the set

$$S = \{u \in C^1(\overline{\Omega}) \colon u = \lambda L(u),\ 0 < \lambda < 1\}.$$

Proposition 3.5. *If hypotheses $H(a)$, $H(\xi)$, $H(\beta)$, $H(r)$ and $H(f)$ hold, then the set $S \subseteq C^1(\overline{\Omega})$ is bounded.*

Proof. Let $u \in S$. Then

$$\frac{1}{\lambda}u = L(u) = (K \circ N)(u) = K(N(u)),$$

so

$$\begin{cases} -\operatorname{div} a\big(\frac{1}{\lambda}Du(z)\big) + \frac{1}{\lambda^{p-1}}\xi(z)|u(z)|^{p-2}u(z) \\ \quad = \widehat{f}(z,u(z)) + r(z)|Du^+(z)|^{p-1} & \text{in } \Omega \\ \frac{\partial(\frac{1}{\lambda}u)}{\partial n_a} + \frac{1}{\lambda^{p-1}}\beta(z)|u^{p-2}|u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.17}$$

On (3.17) we act with u and obtain

$$\begin{aligned} &\frac{c_1}{\lambda^{p-1}(p-1)}\|Du\|_p^p + \frac{1}{\lambda^{p-1}}\int_\Omega \xi(z)|u|^p\,dz + \frac{1}{\lambda}\int_{\partial\Omega} \beta(z)|u|^p\,d\sigma \\ &\leqslant \int_\Omega \widehat{f}(z,u)u\,dz + \int_\Omega r(z)|Du^+|u^+\,dz \end{aligned}$$

(see Lemma 2.3 and recall that $Du^+ = (Du)\chi_{\{u>0\}}$), so

$$\frac{c_1}{p-1}\Big(\|Du\|_p^p + \int_\Omega \xi_*(z)|u|^p\,dz + \int_{\partial\Omega} \beta_*(z)|u|^p\,d\sigma\Big)$$

$$\leqslant \int_\Omega \widehat{f}(z,u)u\,dz + \int_\Omega r(z)|Du^+|u^+\,dz \tag{3.18}$$

(since $0 < \lambda < 1$). From (3.16) and hypotheses $H(f)$(i) and (ii) we see that given $\varepsilon > 0$, we can find $c_{21} = c_{21}(\varepsilon) > 0$ such that

$$\widehat{f}(z,x)x \leqslant (\vartheta(z)+\varepsilon)|x|^p + c_{21} \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbf{R},$$

so

$$\int_\Omega \widehat{f}(z,u)u\,dz \leqslant \int_\Omega (\vartheta(z)+\varepsilon)|u|^p\,dz + c_{22}, \tag{3.19}$$

for some $c_{22} > 0$. Also, if

$$\gamma_p(u) = \|Dv\|_p^p + \int_\Omega \xi_*(z)|v|^p\,dz + \int_{\partial\Omega} \beta_*(z)|v|^p\,d\sigma \quad \forall v \in W^{1,p}(\Omega)$$

and $\widehat{\lambda}_1(p) = \widehat{\lambda}_1(p,\xi_*,\beta_*) > 0$, then

$$\int_\Omega r(z)|Du^+|^{p-1}u^+\,dz \ \leqslant\ \|r\|_\infty \|Du^+\|_p^{p-1}\|u^+\|_p$$

$$\leqslant \|r\|_\infty \gamma_p(u)\|u\|_p \ \leqslant\ \frac{\|r\|_\infty}{\widehat{\lambda}_1(p)^{\frac{1}{p}}}\gamma_p(u) \tag{3.20}$$

(by Hölder's inequality and by hypotheses $H(\xi)$, $H(\beta)$). We return to (3.18) and use (3.19) and (3.20). Then

$$\Big(\frac{c_1}{p-1} - \frac{\|r\|_\infty}{\widehat{\lambda}_1(p)^{\frac{1}{p}}}\Big)\gamma_p(u) - \int_\Omega \vartheta(z)|u|^p\,dz - \varepsilon\|u\|^p \leqslant c_{22},$$

so

$$\tau_0\gamma_p(u) - \int_\Omega \vartheta(z)|u|^p\,dz - \varepsilon\|u\|^p \leqslant c_{22},$$

thus

$$(c_{23} - \varepsilon)\|u\|^p \leqslant c_{22},$$

for some $c_{23} > 0$ (see Lemma 2.8 and hypothesis $H(f)$(ii)).

Choosing $\varepsilon \in (0, c_{23})$, we conclude that the set $S \subseteq W^{1,p}(\Omega)$ is bounded. Then from (3.17) and Proposition 7 of Papageorgiou–Rădulescu [15], we infer that

$$\Big\|\frac{1}{\lambda}u\Big\|_\infty \leqslant c_{24} \quad \forall \lambda \in (0,1),\ u \in S,$$

for some $c_{24} > 0$. Therefore from Lieberman [12], we have that

$$\frac{1}{\lambda}u \in C^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad \Big\|\frac{1}{\lambda}u\Big\|_{C^{1.\alpha}(\overline{\Omega})} \leqslant c_{25} \quad \text{with } \lambda \in (0,1),\ u \in S, \tag{3.21}$$

for some $\alpha \in (0,1)$ and $c_{25} > 0$.

Since $\lambda c_{25} \leqslant c_{25}$ ($\lambda \in (0,1)$) and $C^{1,\alpha}(\overline{\Omega}) \subseteq C^1(\overline{\Omega})$, from (3.21), we conclude that the set $S \subseteq C^1(\overline{\Omega})$ is bounded. □

Now, we are ready to prove the existence of a positive solution for problem (1.1).

Theorem 3.6. *If hypotheses $H(a)$, $H(\xi)$, $H(\beta)$, H_0 and $H(f)$ hold, then problem (1.1) has a solution $u_0 \in D_+$.*

Proof. Proposition 3.5 permit the use of Theorem 2.1 (the Leray–Schauder alternative principle). So, we can find $u_0 \in C^1(\overline{\Omega})$ such that

$$u_0 = L(u_0) = K(N(u_0)),$$

so

$$\begin{aligned} &\langle A(u_0), h\rangle + \int_\Omega \xi(z)|u_0|^{p-2}u_0 h\, dz + \int_{\partial\Omega} \beta(z)|u_0|^{p-2}u_0 h\, d\sigma \\ &= \int_\Omega (\widehat{f}(z,u_0) + r(z)|Du_0^+|^{p-1})h\, dz \quad \forall h \in W^{1,p}(\Omega). \end{aligned} \tag{3.22}$$

In (3.22) we choose $h = (u_* - u_0)^+ \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} &\langle A(u_0), (u_* - u_0)^+\rangle + \int_\Omega \xi(z)|u_0|^{p-2}u_0(u_* - u_0)^+\, dz + \int_{\partial\Omega} \beta(z)|u_0|^{p-2}u_0(u_* - u_0)^+\, d\sigma \\ &= \int_\Omega (f(z,u_*) + r(z)|Du_0^+|^{p-1})(u_* - u_0)^+\, dz \geqslant \int_\Omega f(z,u_*)(u_* - u_0)^+\, dz \\ &\geqslant \int_\Omega ((\eta(z) - \varepsilon)u_*^{q-1} - c_9 u_*^{r-1})(u_* - u_0)^+\, dz \\ &= \langle A(u_*), (u_* - u_0)^+\rangle + \int_\Omega \xi(z)u_*^{p-1}(u_* - u_0)^+\, dz + \int_{\partial\Omega} \beta(z)u_*^{p-1}(u_* - u_0)^+\, d\sigma \end{aligned}$$

(see (3.16), hypothesis $H(r)$, (3.1) and use the fact that $r \geqslant 0$), so

$$u_0 \geqslant u_*,$$

thus $u_0 \in D_+$ and u_0 solves problem (1.1) (see (3.16) and (3.22)). ☐

Remark 3.7. A similar existence theorem can be proved for the Dirichlet problem. In fact on account of the Poincaré inequality, the estimations in the proofs are easier and we can also have $\xi \equiv 0$. Suppose that the differential operator is the Dirichlet p-Laplacian (that is, $a(y) = |y|^{p-2}y$ for all $y \in \mathbf{R}^N$, $1 < p < +\infty$), $r(z) \equiv r_0 > 0$ and $\vartheta(z) \equiv \vartheta_0 > 0$. In this case $c_1 = p - 1$ and the condition in hypothesis $H(r)$ becomes

$$\widehat{\lambda}_1 > r_0 + \vartheta_1 \widehat{\lambda}_1^{\frac{1}{p}},$$

for some $\vartheta_1 \in (\vartheta_0, \widehat{\lambda}_1)$. This is exactly the growth hypothesis in Faraci–Motreanu–Puglisi [2].

It would be interesting to have Theorem 3.6 without the hypothesis that $r \geqslant 0$.

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