

EXISTENCE OF POSITIVE SOLUTION FOR KIRCHHOFF TYPE PROBLEM WITH CRITICAL DISCONTINUOUS NONLINEARITY

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Abstract. In this paper we are concerned with existence of positive solution to the class of nonlinear problems of the Kirchhoff type given by

$$\begin{aligned} L_\epsilon(u) &= H(u - \beta)f(u) + u^{2^*-1} \quad \text{in } \mathbf{R}^N, \\ u &\in H^1(\mathbf{R}^N) \cap W^{2, \frac{q}{q-1}}(\mathbf{R}^N), \end{aligned}$$

where $N \geq 3$, $q \in (2, 2^*)$, $\epsilon, \beta > 0$ are positive parameters, $f: \mathbf{R} \rightarrow \mathbf{R}$ is a continuous function, H is the Heaviside function, i.e., $H(t) = 0$ if $t \leq 0$, $H(t) = 1$ if $t > 0$ and

$$L_\epsilon(u) := \left[M \left(\frac{1}{\epsilon^{N-2}} \int_{\mathbf{R}^N} |\nabla u|^2 dx + \frac{1}{\epsilon^N} \int_{\mathbf{R}^N} V(x)|u|^2 dx \right) \right] [-\epsilon^2 \Delta u + V(x)u].$$

The function M is a general continuous function. The function V is a positive potential that satisfies following hypothesis: or V satisfies the Palais-Smale condition or there is a bounded domain Ω in \mathbf{R}^N such that V has no critical point in $\partial\Omega$. Here we use a suitable truncation to apply a version of the penalization method of Del Pino and Felmer [16] combined with the Mountain Pass Theorem for locally Lipschitz functional.

1. Introduction

Problem of the type

$$(1) \quad -(a + b \int_{\Omega} |\nabla u|^2 dx) \Delta u = g(x, u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded domain in \mathbf{R}^N is very known as Kirchhoff problem. Its hyperbolic equation version which is given by

$$\begin{cases} u_{tt} - (a + b \int_{\Omega} |\nabla u|^2 dx) \Delta u = g(x, u) & \text{in } \Omega \times (0, T), \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x) & \text{in } \Omega, \end{cases}$$

was studied in the first time by Kirchhoff in [26] with $N = 1$. This equation extends the classical d'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations.

Many authors have studied problem (1) using Variational methods and exploring the compactness of $H_0^1(\Omega)$ in $L^s(\Omega)$ with $1 \leq s < 2^*$. Another widely used situation is the dimension $N = 3$. In general the dimension $N = 3$ avoids the competition of the

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term $(a + b \int_{\Omega} |\nabla u|^2 dx)$ with the growth of nonlinearity. For more information on this competition see [20, Introduction]. Since it is impossible to cite all the authors that study problem (1) with $N = 3$, we cite the most recent publications, such as [6], [15], [27], [29] and [31].

Problem (1) in \mathbf{R}^N is more difficult due to the lack of compactness of Sobolev immersions. In general, in order to overcome this difficult, the author put a potential which the geometry allows to recover the compactness. See here the latest publications with these arguments, [11], [18], [24], [25], [32] and [33].

In this paper we are concerned with existence of positive solution to the class of nonlinear problems of the Kirchhoff type given by

$$(2) \quad \begin{cases} L_{\epsilon}(u) = H(u - \beta)f(u) + u^{2^*-1} & \text{in } \mathbf{R}^N, \\ u \in H^1(\mathbf{R}^N) \cap W^{2, \frac{q}{q-1}}(\mathbf{R}^N), \end{cases}$$

where $N \geq 3$, $q \in (2, 2^*)$, $\epsilon, \beta > 0$ are positive parameters, $f: \mathbf{R} \rightarrow \mathbf{R}$ is a continuous function, H is the Heaviside function, i.e., $H(t) = 0$ if $t \leq 0$, $H(t) = 1$ if $t > 0$ and

$$L_{\epsilon}(u) := \left[M \left(\frac{1}{\epsilon^{N-2}} \int_{\mathbf{R}^N} |\nabla u|^2 dx + \frac{1}{\epsilon^N} \int_{\mathbf{R}^N} V(x)|u|^2 dx \right) \right] [-\epsilon^2 \Delta u + V(x)u].$$

In order to enunciate the main result, we need to give some hypotheses on the functions M , V and f . The hypothesis on the continuous function $M: [0, +\infty) \rightarrow (0, \infty)$ is the following:

(M_1) The function M is nondecreasing.

Now we give some examples of functions that satisfy the hypothesis (M_1). The example $M(t) = m_0 + bt$ with $m_0, b > 0$ was considered in [26]. Another examples are $M(t) = \log(t + 1) + 1$ or $M(t) = \exp t^2$.

The hypotheses on the continuous function $V: \mathbf{R}^N \rightarrow \mathbf{R}$ are the following:

- (V_0) There is $V_0 > 0$ such that $V(x) \geq V_0$, for all $x \in \mathbf{R}^N$.
- (V_1) $V \in C^2(\mathbf{R}^N)$ and $V, \frac{\partial V}{\partial x_i}, \frac{\partial^2 V}{\partial x_i \partial x_j}$ are bounded in \mathbf{R}^N , $i, j = 1, 2, \dots, N$.
- (V_2) V verifies the Palais–Smale condition, that is, if $(x_n) \subset \mathbf{R}^N$ is such that, $(V(x_n))$ is bounded and $\nabla V(x_n) \rightarrow 0$, then, (x_n) possesses a convergent subsequence in \mathbf{R}^N .
- (V_3) There is a bounded domain $\Omega \subset \mathbf{R}^N$, such that $\nabla V(x) \neq 0$, for all $x \in \partial\Omega$.

The hypotheses on the continuous function $f: \mathbf{R} \rightarrow \mathbf{R}$ are the following:

(f_1) For all $t \in \mathbf{R}$, there are $C > 0$ and $q \in (2, 2^*)$ such that

$$|f(t)| \leq C(1 + |t|^{q-1}).$$

(f_2) For all $t \in \mathbf{R}$, there exists $\theta \in (2, 2^*)$ such that

$$0 < \theta F(t) = \theta \int_0^t f(s) ds \leq t f(t).$$

(f_3) There is $\beta > 0$, that will be fixed later, such that

$$H(t - 2\beta) \leq f(t), \quad \text{for all } t \in \mathbf{R}.$$

(f_4) $\limsup_{t \rightarrow 0^+} \frac{f(t)}{t} = 0$ and $f(t) = 0$ if $t \leq 0$.

A typical example of a function satisfying the conditions $(f_1), (f_2), (f_3)$ and (f_4) is given by

$$f(t) = \sum_{k=1}^l \frac{|t|^{q_k-1}}{2\beta^{q_k-1}} \text{ if } t \geq 0 \text{ and } q_k \in (2, 2^*) \text{ and } f(t) = 0 \text{ if } t \leq 0.$$

Before our main result, we define weak solution and strong solution for problem (2). A weak solution for (2), we understand as a function $u_\epsilon \in H^1(\mathbf{R}^N) \cap W^{2, \frac{q}{q-1}}(\mathbf{R}^N)$ verifying

$$L_\epsilon(u_\epsilon(x)) - u_\epsilon^{2^*-1}(x) \in [\underline{f}(u_\epsilon(x)), \overline{f}(u_\epsilon(x))] \text{ a.e. in } \mathbf{R}^N,$$

where $\underline{f}(t)$ and $\overline{f}(t)$ are N -measurable functions (see [9]) defined by

$$\underline{f}(t) := \lim_{\delta \downarrow 0} \text{ess inf}_{|t-s| < \delta} H(s - \beta)f(s) \text{ and } \overline{f}(t) := \lim_{\delta \downarrow 0} \text{ess sup}_{|t-s| < \delta} H(s - \beta)f(s).$$

By a strong solution for (2), we understand as a function $u_\epsilon \in H^1(\mathbf{R}^N) \cap W^{2, \frac{q}{q-1}}(\mathbf{R}^N)$ verifying

$$(3) \quad L_\epsilon(u_\epsilon(x)) = H(u_\epsilon(x) - \beta)f(u_\epsilon(x)) + u_\epsilon^{2^*-1}(x) \text{ a.e. in } \mathbf{R}^N.$$

Our main result is as follows:

Theorem 1.1. *Suppose that M satisfies (M_1) , f satisfies $(f_1), (f_2), (f_3), (f_4)$ and V satisfies $(V_0), (V_1)$ and (V_2) or $(V_0), (V_1)$ and (V_3) . Then, there exists $\epsilon^* > 0$ small, such that for each $\epsilon \in (0, \epsilon^*)$, the problem (2) has a weak positive solution u_ϵ . Moreover:*

- (i) *The set $\{x \in \mathbf{R}^N : u_\epsilon(x) = \beta\}$ has null measure and so u_ϵ satisfies (3).*
- (ii) *The set $\{x \in \mathbf{R}^N : u_\epsilon(x) > \beta\}$ has positive measure.*

Some physical problems are related to discontinuous surface

$$\Gamma_\beta(u) = \{x \in \Omega ; u(x) = \beta\}$$

which causes difficulties in analyzing this kind of problems. The interest in the study of nonlinear partial differential equations with discontinuous nonlinearities has increased because many free boundary problems arising in mathematical physics may be stated in this form. Among these problems, we have the obstacle problem, the seepage surface problem, and the Elenbaas equation, see for example [10, 8, 9]. Still related to elliptic problems with discontinuous nonlinearity, we cite the papers of Alves, Bertone and Gonçalves [2], Alves, Santos and Gonçalves [3], Badiale [4, 5], Dinu [17], Radulescu [30], Gazzola and Radulescu [21] and their references. Several techniques have been developed or applied in their study, such as variational methods for nondifferentiable functionals, lower and upper solutions, global branching, and the theory of multivalued mappings.

Our result is related with the articles [1], [14] and [19]. Alves in [1] used in the first time the hypotheses on the potential V in the problem

$$(4) \quad -\epsilon^2 \Delta u + V(x)u = g(u) \text{ in } \mathbf{R}^N,$$

where g was continuous and subcritical. The version of this problem with discontinuous nonlinearity was studied in [19]. In [14] was studied a problem with a nonlocal operator and discontinuous nonlinearity in bounded domain. But in this paper, the nonlocal term does not contemplate the Kirchhoff function.

We have completed some previous results in the following sense:

- i) Here, we can not use Variational Methods for C^1 functionals, because in our case, the energy functional is only locally Lipschitz continuous, and so, we have to use Variational Methods for nondifferentiable functionals, see Section 2 for more details.
- ii) In [1] was necessary some estimates involving the boundedness of solution of an auxiliary problem (see Lemma 2.1 in [1]). Since our problema has a nonlocal term and the functional associated is only locally Lipschitz continuous it was necessary to develop new arguments to get the boundedness of solution of auxiliary problems (see Lemma 3.6).
- iii) This paper is a version of [19] considering the Kirchhoff term and critical growth.
- iv) In this paper we consider $N \geq 3$. In order to overcome the competition between the operator and the nonlinearity, we make a truncation the operator like [20].

This paper is organized as follows. In the Section 2 we make a revision about functional Lip_{loc} . In the Section 3 we make a truncation of the operator and another truncation in the nonlinearity. After that we show existence of positive solution of an auxiliary problem. Finally, we show that the solution of auxiliary problem is a solution of original problem in the Sections 4 and 5.

2. Variational framework and preliminary results

In this section, for the reader's convenience, we recall some definitions and basic results on the critical point theory of locally Lipschitz continuous functionals as developed by Chang [10], Clarke [12, 13] and Grossinho and Tersian [23].

Let X be a real Banach space. A functional $I: X \rightarrow \mathbf{R}$ is locally Lipschitz continuous, $I \in \text{Lip}_{\text{loc}}(X, \mathbf{R})$ for short, if given $u \in X$ there is an open neighborhood $V := V_u \subset X$ and some constant $K = K_V > 0$ such that

$$|I(v_2) - I(v_1)| \leq K \|v_2 - v_1\|, \quad v_i \in V, \quad i = 1, 2.$$

The directional derivative of I at u in the direction of $v \in X$ is defined by

$$I^0(u; v) = \limsup_{h \rightarrow 0, \sigma \downarrow 0} \frac{I(u + h + \sigma v) - I(u + h)}{\sigma}.$$

Hence $I^0(u; \cdot)$ is continuous, convex and its subdifferential at $z \in X$ is given by

$$\Upsilon^0(u; z) = \{\mu \in X^*; I^0(u; v) \geq I^0(u; z) + \langle \mu, v - z \rangle, v \in X\},$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between X^* and X . The generalized gradient of I at u is the set

$$\partial I(u) = \{\mu \in X^*; \langle \mu, v \rangle \leq I^0(u; v), v \in X\}.$$

Since $I^0(u; 0) = 0$, $\partial I(u)$ is the subdifferential of $I^0(u; 0)$. A few definitions and properties will be recalled below.

$\partial I(u) \subset X^*$ is convex, non-empty and weak*-compact,

$$(5) \quad \Lambda(u) = \min \{ \|\mu\|_{X^*}; \mu \in \partial I(u) \}$$

and

$$\partial I(u) = \{I'(u)\} \text{ if } I \in C^1(X, \mathbf{R}).$$

A critical point of I is an element $u_0 \in X$ such that $0 \in \partial I(u_0)$ and a critical value of I is a real number c such that $I(u_0) = c$ for some critical point $u_0 \in X$.

We say that $I \in \text{Lip}_{\text{loc}}(X, \mathbf{R})$ satisfies the Palais–Smale condition at level $c \in \mathbf{R}$ ($(PS)_c$ -condition for short), if the following holds: Every sequence $(u_n) \subset X$, such that $I(u_n) \rightarrow c$ and $\Lambda(u_n) \rightarrow 0$ has a strongly convergent subsequence.

Theorem 2.1. *Let X be a Banach space and let $I \in \text{Lip}_{\text{loc}}(X, \mathbf{R})$ with $I(0) = 0$. Suppose there are numbers $\varrho, r > 0$ and $e \in X$, such that*

- (i) $I(u) \geq \varrho$, for all $u \in X$; $\|u\| = r$,
- (ii) $I(e) < 0$ and $\|e\| > r$.

Let

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) \quad \text{and} \quad \Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = 0 \text{ and } \gamma(1) = e\}.$$

Then $c \geq \varrho$ and there is a sequence $(u_n) \subset X$ (named a $(PS)_c$ -sequence) satisfying

$$I(u_n) \rightarrow c \quad \text{and} \quad \Lambda(u_n) \rightarrow 0.$$

If, in addition, I satisfies the $(PS)_c$ -condition, then c is a critical value of I .

3. An auxiliary problem

Using the change variable $v(x) = u(\epsilon x)$, it is possible to prove that (2) is equivalent to the following problem

$$(6) \quad \begin{aligned} \tilde{L}_\epsilon(u) &= H(u - \beta)f(u) + u^{2^*-1} \quad \text{in } \mathbf{R}^N, \\ u &\in H^1(\mathbf{R}^N) \cap W^{2, \frac{q}{q-1}}(\mathbf{R}^N), \end{aligned}$$

where

$$\tilde{L}_\epsilon(u) := \left[M \left(\int_{\mathbf{R}^N} |\nabla u|^2 + \int_{\mathbf{R}^N} V(\epsilon x)|u|^2 \right) \right] [-\Delta u + V(\epsilon x)u].$$

In this section, we fix some notations and an auxiliary problem, which are key points in our arguments. To establish the existence of positive solutions, we will make a truncation on the function M as in [20] and we will adapt the penalization method due to Del Pino and Felmer [16], which consists in considering an auxiliary problem. To this end, we need to fix some notations.

In this section, on the function V , we assume only (V_0) and (V_1) . From (M_1) , there exists $t_0 > 0$ such that $M(0) \leq M(t_0) < \frac{\theta}{2}M(0)$ where θ is given by (f_2) . We set

$$(7) \quad M_0(t) := \begin{cases} M(t), & \text{if } 0 \leq t \leq t_0, \\ M(t_0), & \text{if } t \geq t_0. \end{cases}$$

Using (M_1) and (7), we get

$$(8) \quad M(0) \leq M_0(t) < \frac{\theta}{2}M(0), \quad \text{for all } t \geq 0.$$

Note that by (f_4) we have

$$\lim_{t \rightarrow 0} \left[\frac{f(t)}{t} + t^{2^*-2} \right] = 0 \quad \text{and} \quad \lim_{t \rightarrow +\infty} \left[\frac{f(t)}{t} + t^{2^*-2} \right] = +\infty.$$

Then there exists $a > 0$ such that

$$(9) \quad \frac{f(a)}{a} + a^{2^*-2} = \frac{V_0}{k},$$

where V_0 is given in (V_0) and

$$(10) \quad k > \frac{2\theta}{\theta M(0) - 2M(t_0)},$$

where θ is given in (f_2) . In (f_3) we choose $\beta > 0$ such that $\beta < a < 2\beta$.

Using the above numbers, let us define the function

$$\tilde{f}(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ H(t - \beta)f(t) + t^{2^*-1} & \text{if } 0 < t < a, \\ \frac{V_0}{k}t & \text{if } t \geq a. \end{cases}$$

Fixing $\Omega \subset \mathbf{R}^N$ a bounded domain and using the function \tilde{f} , let us consider the function

$$(11) \quad g(x, t) = \chi_\Omega(x)(H(t - \beta)f(t) + t^{2^*-1}) + (1 - \chi_\Omega(x))\tilde{f}(t),$$

where χ_Ω is the characteristic function related to Ω . Now consider the auxiliary problem

$$(P_\epsilon)_a \quad \begin{cases} M_0 \left(\int_{\mathbf{R}^N} (|\nabla u|^2 + V(\epsilon x)|u|^2) \right) (-\Delta u + V(\epsilon x)u) = g(\epsilon x, u), \\ u \in H^1(\mathbf{R}^N) \cap W^{2, \frac{q}{q-1}}(\mathbf{R}^N). \end{cases}$$

A weak solution for $(P_\epsilon)_a$, we understand as a function $u_\epsilon \in H^1(\mathbf{R}^N) \cap W^{2, \frac{q}{q-1}}(\mathbf{R}^N)$ verifying

$$\begin{aligned} & M_0 \left(\int_{\mathbf{R}^N} (|\nabla u_\epsilon|^2 + V(\epsilon x)|u_\epsilon|^2) \right) (-\Delta u_\epsilon(x) + V(\epsilon x)u_\epsilon(x)) \\ & \in [\underline{g}(\epsilon x, u_\epsilon(x)), \overline{g}(\epsilon x, u_\epsilon(x))] \quad \text{a.e. in } \mathbf{R}^N, \end{aligned}$$

where

$$\underline{g}(x, t) := \lim_{\delta \downarrow 0} \text{ess inf}_{|t-s| < \delta} g(x, s) \quad \text{and} \quad \overline{g}(x, t) := \lim_{\delta \downarrow 0} \text{ess sup}_{|t-s| < \delta} g(x, s),$$

which are N -measurable.

By a strong solution for $(P_\epsilon)_a$, we understand as a function $u_\epsilon \in H^1(\mathbf{R}^N) \cap W^{2, \frac{q}{q-1}}(\mathbf{R}^N)$ verifying

$$M_0 \left(\int_{\mathbf{R}^N} (|\nabla u_\epsilon|^2 + V(\epsilon x)|u_\epsilon|^2) \right) (-\Delta u_\epsilon(x) + V(\epsilon x)u_\epsilon(x)) = g(\epsilon x, u_\epsilon(x)) \quad \text{a.e. in } \mathbf{R}^N.$$

Remark 3.1. The problem $(P_\epsilon)_a$ is strongly related to (6), because if u_ϵ is a solution of $(P_\epsilon)_a$ verifying $\|u_\epsilon\| \leq t_0$ and $u_\epsilon(x) \leq a$ in $\mathbf{R}^N \setminus \Omega_\epsilon$ where $\Omega_\epsilon = \Omega/\epsilon$, then by definitions of M_0 and g , u_ϵ will be a solution for (6).

From (f_1) – (f_4) , the function g satisfies the following conditions uniformly in $x \in \mathbf{R}^N$:

- (g_1) $g(x, t) = 0$ for all $t \leq 0$ and $\limsup_{|t| \rightarrow 0} \frac{g(x, t)}{|t|} = 0$.
- (g_2) $g(x, t) = H(t - \beta)f(t) + t^{2^*-1}$, for all $x \in \Omega$, $t > 0$, or $x \in \Omega^c$ and $t \in [0, a]$.
- (g_3) $g(x, t) \leq f(t) + t^{2^*-1}$ for all $x \in \mathbf{R}^N$, $t \in \mathbf{R}$.
- (g_4) $0 < \theta G(x, t) = \theta \int_0^t g(x, s) ds \leq \underline{g}(x, t)t$, for all $x \in \Omega$, $t > 0$, and $0 < 2G(x, t) \leq \underline{g}(x, t)t \leq \overline{g}(x, t)t \leq \frac{1}{k}V_0t^2$, for all $x \notin \Omega$, $t \geq 0$.

We note that from (V_0) and (V_1) , we can work in $H^1(\mathbf{R}^N)$ with the norm

$$\|u\| := \left(\int_{\mathbf{R}^N} |\nabla u|^2 + V(\epsilon x)|u|^2 \right)^{1/2}, \quad u \in H^1(\mathbf{R}^N),$$

which is equivalent to the usual norm.

Let $\widehat{\Psi}_\epsilon$ be a functional given

$$\widehat{\Psi}_\epsilon(u) = \int_{\mathbf{R}^N} G(\epsilon x, u), \quad \text{for all } u \in L^{2^*}(\mathbf{R}^N).$$

Using (g_2) , (f_1) and (11), we obtain a constant $C = C(\beta) > 0$, such that

$$(12) \quad |g(\epsilon x, t)| \leq C|t|^{2^*-1}, \quad \text{for all } x \in \mathbf{R}^N, t \in \mathbf{R}.$$

As an immediate consequence of (12), we obtain the following version of Theorems 2.1 and 2.2 due to Chang [10] and its proof will be omitted.

Lemma 3.2. *The functional $\widehat{\Psi}_\epsilon$ satisfies:*

(i) $\widehat{\Psi}_\epsilon \in \text{Lip}_{\text{loc}}(L^{2^*}(\mathbf{R}^N), \mathbf{R})$ and

$$\partial\widehat{\Psi}_\epsilon(u) \subseteq \int_{\mathbf{R}^N} \partial G(\epsilon x, u), \quad \text{for all } u \in L^{2^*}(\mathbf{R}^N),$$

in the sense that for every $\xi^* \in \partial\widehat{\Psi}_\epsilon(u)$ there exists $\xi \in L^{\frac{2^*}{2^*-1}}(\mathbf{R}^N)$, such that, $\xi(x) \in \partial G(\epsilon x, u(x)) \subset [\underline{g}(\epsilon x, u(x)), \bar{g}(\epsilon x, u(x))]$ a.e. in \mathbf{R}^N and

$$\langle \xi^*, \varphi \rangle = \int_{\mathbf{R}^N} \xi \varphi, \quad \text{for all } \varphi \in L^{2^*}(\mathbf{R}^N).$$

(ii) If $\Psi_\epsilon = \widehat{\Psi}_\epsilon|_{H^1(\mathbf{R}^N)}$, then $\partial\Psi_\epsilon(u) = \partial\widehat{\Psi}_\epsilon(u)$, for all $u \in H^1(\mathbf{R}^N)$.

The functional associated with $(P_\epsilon)_a$ is given by $J_\epsilon(u) = Q_\epsilon(u) - \Psi_\epsilon(u)$, $u \in H^1(\mathbf{R}^N)$, where

$$Q_\epsilon(u) = \frac{1}{2}\widehat{M}_0(\|u\|^2), \quad \widehat{M}_0(t) = \int_0^t M_0(s) ds \quad \text{and} \quad \Psi_\epsilon(u) = \int_{\mathbf{R}^N} G(\epsilon x, u).$$

The functional $Q_\epsilon \in C^1(H^1(\mathbf{R}^N), \mathbf{R})$, $J_\epsilon \in \text{Lip}_{\text{loc}}(H^1(\mathbf{R}^N), \mathbf{R})$ with

$$\langle Q'_\epsilon(u), \varphi \rangle = M_0(\|u\|^2) \int_{\mathbf{R}^N} (\nabla u \nabla \varphi + V(\epsilon x)u\varphi), \quad \text{for all } u, \varphi \in H^1(\mathbf{R}^N)$$

and

$$\partial J_\epsilon(u) = \{Q'_\epsilon(u)\} - \partial\Psi_\epsilon(u), \quad \text{for all } u \in H^1(\mathbf{R}^N).$$

Lemma 3.3. *Let (u_n) be a $(PS)_c$ sequence for J_ϵ , then (u_n) is bounded in $H^1(\mathbf{R}^N)$.*

Proof. Let $(w_n) \subset H^{-1}(\mathbf{R}^N)$ be such that $\Lambda(u_n) = \|w_n\|_* = o_n(1)$, where Λ appeared in (5). Since $w_n \in \partial J_\epsilon(u_n)$, there exists $\xi_n^* \in \partial\Psi_\epsilon(u_n)$ verifying

$$(13) \quad \langle w_n, \varphi \rangle = \langle Q'_\epsilon(u_n), \varphi \rangle - \langle \xi_n^*, \varphi \rangle, \quad \text{for all } \varphi \in H^1(\mathbf{R}^N).$$

From (13), (M_1) and (g_4) , we have

$$(14) \quad J_\epsilon(u_n) - \frac{1}{\theta} \langle w_n, u_n \rangle \geq \left(\frac{M(0)}{2} - \frac{M(t_0)}{\theta} \right) \|u_n\|^2 + \int_{\Omega_\epsilon^c} \left[\frac{1}{\theta} \xi_n u_n - G(\epsilon x, u_n) \right],$$

where $\xi_n \in [\underline{g}(\epsilon x, u_n), \bar{g}(\epsilon x, u_n)]$.

Since $\theta > 2$, the relation (g_4) imply that

$$(15) \quad \int_{\Omega_\epsilon^c} \left[\frac{1}{\theta} \xi_n u_n - G(\epsilon x, u_n) \right] \geq \frac{2-\theta}{\theta} \frac{1}{2k} \int_{\Omega_\epsilon^c} V(\epsilon x) |u_n|^2.$$

Using (14) and (15), we obtain

$$C + C\|u_n\| \geq \left(\frac{M(0)}{2} - \frac{M(t_0)}{\theta} \right) \|u_n\|^2 + \frac{2 - \theta}{\theta} \frac{1}{2k} \|u_n\|^2.$$

By choice of k , see (10), we have,

$$C + C\|u_n\| \geq \frac{1}{2} \left(\frac{M(0)}{2} - \frac{M(t_0)}{\theta} \right) \|u_n\|^2.$$

Therefore, (u_n) is bounded in $H^1(\mathbf{R}^N)$. □

Lemma 3.4. *Let (u_n) be a $(PS)_c$ sequence for J_ϵ . Then for each $\delta > 0$, there exists $R = R(\delta) > 0$, such that*

$$\limsup_{n \rightarrow \infty} \int_{\mathbf{R}^N \setminus B_R(0)} [|\nabla u_n|^2 + V(\epsilon x)|u_n|^2] < \delta.$$

Proof. Let (w_n) , ξ_n^* and ξ_n used in the proof of the previous lemma and $\eta_R \in C^\infty(\mathbf{R}^N, [0, 1])$, such that $\eta_R(x) = 0$ in $B_R(0)$, $\eta_R(x) = 1$ in $B_{2R}(0)^c$ and $|\nabla \eta_R(x)| \leq C/R$ in \mathbf{R}^N , where $C > 0$ is a constant independent on R .

By Lemma 3.3, (u_n) is bounded in $H^1(\mathbf{R}^N)$ and $\langle w_n, \eta_R u_n \rangle = o_n(1)$. From (13), we obtain

$$M(0) \int_{\mathbf{R}^N} \eta_R [|\nabla u_n|^2 + V(\epsilon x)|u_n|^2] \leq \int_{\mathbf{R}^N} \xi_n \eta_R u_n - M_0(\|u_n\|^2) \int_{\mathbf{R}^N} u_n \nabla \eta_R \nabla u_n + o_n(1).$$

Fixed $R > 0$, such that $\Omega_\epsilon \subset B_{\frac{R}{2}}(0)$. Since $\xi_n \in [g(\epsilon x, u_n), \bar{g}(\epsilon x, u_n)]$, by (g_4) ,

$$\int_{\mathbf{R}^N} \xi_n \eta_R u_n \leq \frac{1}{k} \int_{\mathbf{R}^N} \eta_R V(\epsilon x)|u_n|^2,$$

combining this with (8), we find

$$\left(M(0) - \frac{1}{k} \right) \int_{\mathbf{R}^N} \eta_R [|\nabla u_n|^2 + V(\epsilon x)|u_n|^2] \leq M(0) \frac{\theta C}{2R} \|u_n\|_{L^2} \|\nabla u_n\|_{L^2} + o_n(1).$$

Since $k > \frac{1}{M(0)}$, see (10), we conclude the proof. □

Let S be the best constant for Sobolev’s embedding $D^{1,2}(\mathbf{R}^N) \hookrightarrow L^{2^*}(\mathbf{R}^N)$.

Proposition 3.5. *The functional J_ϵ verifies the $(PS)_c$ condition in $H^1(\mathbf{R}^N)$ for $c < \left(\frac{M(0)}{2} - \frac{M(t_0)}{\theta} \right) (M(0)S^{\frac{N-2}{N-2}})^{\frac{N-2}{2}}$.*

Proof. By Lemma 3.3, (u_n) is bounded in $H^1(\mathbf{R}^N)$. Taking a subsequence, we may assume that $u_n \rightharpoonup u$ in $H^1(\mathbf{R}^N)$, $u_n(x) \rightarrow u(x)$ a.e. in \mathbf{R}^N ,

$$|\nabla u_n|^2 \rightharpoonup |\nabla u|^2 + \mu \quad \text{and} \quad |u_n|^{2^*} \rightharpoonup |u|^{2^*} + \nu \quad (\text{weak}^*\text{-sense of measure}).$$

Using the concentration compactness-principle by Lions [26, Lemma 2.1], we obtain at most countable index set Σ , sequences $(x_i) \subset \mathbf{R}^N$, $(\mu_i), (\nu_i) \subset [0, \infty)$, such that

$$(16) \quad \nu = \sum_{i \in \Sigma} \nu_i \delta_{x_i}, \quad \mu \geq \sum_{i \in \Sigma} \mu_i \delta_{x_i} \quad \text{and} \quad S \nu_i^{\frac{2}{2^*}} \leq \mu_i,$$

for all $i \in \Sigma$, where δ_{x_i} is the Dirac mass at x_i .

Now we claim that $\Sigma = \emptyset$. Arguing by contradiction, assume that $\Sigma \neq \emptyset$ and fix $i \in \Sigma$. Let w_n, ξ_n^* and ξ_n used in the proof of the Lemma 3.3.

Consider $\varphi \in C_0^\infty(\mathbf{R}^N, [0, 1])$, such that, $\varphi \equiv 1$ in $B_1(0)$, $\varphi \equiv 0$ in $\mathbf{R}^N \setminus B_2(0)$ and $|\nabla\varphi|_{L^\infty} \leq 2$. Defining $\varphi_\varrho(x) = \varphi\left(\frac{x-x_i}{\varrho}\right)$, where $\varrho > 0$, we have that $(\varphi_\varrho u_n)$ is bounded in $H^1(\mathbf{R}^N)$, that implies

$$M_0(\|u_n\|^2) \int_{\mathbf{R}^N} (\nabla u_n \nabla(\varphi_\varrho u_n) + V(\epsilon x)\varphi_\varrho u_n^2) = \int_{\Omega_\epsilon} \xi_n \varphi_\varrho u_n + \int_{\Omega_\xi} \xi_n \varphi_\varrho u_n + o_n(1).$$

Using (M_1) , (g_3) and (g_4) , we obtain

$$(17) \quad \begin{aligned} M(0) \int_{\mathbf{R}^N} \varphi_\varrho |\nabla u_n|^2 &\leq -M_0(\|u_n\|^2) \int_{\mathbf{R}^N} (u_n \nabla u_n \nabla \varphi_\varrho + V(\epsilon x)\varphi_\varrho u_n^2) \\ &+ C \int_{\mathbf{R}^N} |u_n|^q \varphi_\varrho + \int_{\mathbf{R}^N} |u_n|^{2^*} \varphi_\varrho + C \int_{\mathbf{R}^N} |u_n|^2 \varphi_\varrho. \end{aligned}$$

Since (u_n) is bounded in $H^1(\mathbf{R}^N)$, the support of φ_ϱ is contained in $B_{2\varrho}(x_i)$,

$$\begin{aligned} \lim_{\varrho \rightarrow 0} \lim_{n \rightarrow \infty} M_0(\|u_n\|^2) \int_{\mathbf{R}^N} (u_n \nabla u_n \nabla \varphi_\varrho + V(\epsilon x)\varphi_\varrho u_n^2) &= 0, \\ \lim_{\varrho \rightarrow 0} \lim_{n \rightarrow \infty} \int_{\mathbf{R}^N} |u_n|^2 \varphi_\varrho = 0 \quad \text{and} \quad \lim_{\varrho \rightarrow 0} \lim_{n \rightarrow \infty} \int_{\mathbf{R}^N} |u_n|^q \varphi_\varrho u_n &= 0. \end{aligned}$$

Thus, by (17)

$$M(0) \int_{\mathbf{R}^N} \varphi_\varrho d\mu \leq \int_{\mathbf{R}^N} \varphi_\varrho d\nu + o_\varrho(1).$$

Letting $\varrho \rightarrow 0$ and using standard theory of Radon measures, we conclude that $M(0)\mu_i \leq \nu_i$. From (16), $\mu_i \geq (M(0)S^{\frac{N}{N-2}})^{\frac{N-2}{2}}$.

Once that (u_n) is a sequence $(PS)_c$, arguing as in the proof of Lemma 3.3,

$$\begin{aligned} c &= J_\epsilon(u_n) - \frac{1}{\theta} \langle w_n, u_n \rangle + o_n(1) \\ &\geq \left(\frac{M(0)}{2} - \frac{M(t_0)}{\theta} \right) \int_{\mathbf{R}^N} |\nabla u_n|^2 \varphi_\varrho + C_0 \int_{\mathbf{R}^N} V(\epsilon x)|u_n|^2 \varphi_\varrho + o_n(1). \end{aligned}$$

Thus, we conclude that $c \geq \left(\frac{M(0)}{2} - \frac{M(t_0)}{\theta} \right) (M(0)S^{\frac{N}{N-2}})^{\frac{N-2}{2}}$, which is a contradiction. Therefore, Σ is empty and it follows that

$$(18) \quad u_n \rightarrow u \text{ in } L_{loc}^{2^*}(\mathbf{R}^N).$$

Recalling that

$$(19) \quad M_0(\|u_n\|^2)\|u_n\|^2 = \int_{\mathbf{R}^N} \xi_n u_n + o_n(1)$$

and since $\xi_n \in [\underline{g}(\epsilon x, u_n), \bar{g}(\epsilon x, u_n)]$, by (12) and the boundedness of (u_n) in $H^1(\mathbf{R}^N)$, we obtain that (ξ_n) is bounded in $L^{\frac{2^*}{2^*-1}}(\mathbf{R}^N)$. Thus, up to a subsequence,

$$(20) \quad \|u_n\| \rightarrow \sigma \text{ in } \mathbf{R}, \quad \xi_n \rightharpoonup \xi \text{ in } L^{\frac{2^*}{2^*-1}}(\mathbf{R}^N) \text{ and } u_n \rightharpoonup u \text{ in } H^1(\mathbf{R}^N).$$

From (13),

$$(21) \quad M_0(\sigma^2)\|u\|^2 = \int_{\mathbf{R}^N} \xi u.$$

Since

$$\left| \int_{B_R(0)} \xi_n u_n - \int_{B_R(0)} \xi u \right| \leq \|u_n - u\|_{L^{2^*}(B_R(0))} \|\xi_n\|_{L^{\frac{2^*}{2^*-1}}(\mathbf{R}^N)} + \left| \int_{B_R(0)} (\xi_n - \xi) u \right|,$$

using (18), (20) and the Riesz representation theorem, we have

$$(22) \quad \int_{B_R(0)} \xi_n u_n \rightarrow \int_{B_R(0)} \xi u.$$

The Lemma 3.4 implies that

$$(23) \quad \limsup_{n \rightarrow \infty} \left| \int_{B_R^c(0)} \xi_n u_n \right| = o_R(1).$$

Noting that

$$\left| \int_{\mathbf{R}^N} \xi_n u_n - \int_{\mathbf{R}^N} \xi u \right| \leq \left| \int_{B_R(0)} \xi_n u_n - \int_{B_R(0)} \xi u \right| + \left| \int_{B_R^c(0)} \xi_n u_n \right| + o_R(1),$$

using (22) and (23), we conclude that

$$(24) \quad \int_{\mathbf{R}^N} \xi_n u_n \rightarrow \int_{\mathbf{R}^N} \xi u.$$

Combining (19), (21) and (24), we obtain $M(\|u_n\|^2)\|u_n\|^2 = M(\sigma^2)\|u\|^2 + o_n(1)$. Therefore, $u_n \rightarrow u$ in $H^1(\mathbf{R}^N)$. \square

Lemma 3.6. *Let (u_n) be a $(PS)_c$ sequence for $J_{\epsilon,\beta}$ with $c < \frac{1}{2} \left(\frac{M(0)}{2} - \frac{M(t_0)}{\theta} \right) t_0^2$. Then, there exists $n_0 > 0$, such that*

$$\|u_n\| \leq t_0, \quad \text{for all } n \geq n_0,$$

where t_0 is given by (7).

Proof. Suppose, that the lemma is false, then we obtain a subsequence of (u_n) , that we are still denoting by (u_n) , that satisfies $\|u_n\| > t_0$, for all $n \in \mathbf{N}$. Using $w_n \in \partial J(u_n)$, we get

$$c + o_n(1) = J_\epsilon(u_n) - \frac{1}{\theta} \langle w_n, u_n \rangle \geq \frac{1}{2} \left(\frac{M(0)}{2} - \frac{M(t_0)}{\theta} \right) \|u_n\|^2.$$

Since $M(t_0) < \frac{\theta}{2} M(0)$, we have

$$c \geq \frac{1}{2} \left(\frac{M(0)}{2} - \frac{M(t_0)}{\theta} \right) t_0^2,$$

which is a contradiction. \square

The number

$$(25) \quad c^* := \min \left\{ t_0^2, \left(M(0) S^{\frac{N}{N-2}} \right)^{\frac{N-2}{2}} \right\} \frac{1}{2} \left(\frac{M(0)}{2} - \frac{M(t_0)}{\theta} \right)$$

is important in order to prove that the functional satisfies the $(PS)_c$ condition and that a solution of the auxiliary problem $(P_\epsilon)_a$ is a solution of the original problem (6).

In the next result we choose the constant β of the condition (f_3) .

Lemma 3.7. *Suppose that (f_1) , (f_2) , (f_3) , (f_4) , (V_0) , (V_1) are satisfied. Fixed $\epsilon^*, \beta > 0$ small, for each $\epsilon \in (0, \epsilon^*)$, there are $\gamma_0 > 0$ and $v_0 \in H^1(\mathbf{R}^N)$, which are independent of ϵ^* and β , such that*

- (i) $\max_{t \in [0, \gamma_0]} J_\epsilon(t v_0) < c^*$.
- (ii) There are $r, \varrho > 0$, such that, $J_\epsilon(u) \geq \varrho, \forall u \in H^1(\mathbf{R}^N); \|u\| = r$.
- (iii) $J_\epsilon(\gamma_0 v_0) < 0$ and $\gamma_0 v_0 \in B_r(0)^c$.

Proof. Without any loss of generality we may assume that $0 \in \Omega$. Fixed $\epsilon^* \in (0, 1)$, consider $v_0 \in C_0^\infty(\mathbf{R}^N)$, such that, $\int_{\mathbf{R}^N} (|\nabla v_0|^2 + V_\infty |v_0|^2) = 1$, $v_0 \geq 0$, $\text{supp } v_0 \subset B_R(0) \subset \Omega$. Since $V(\epsilon x) \leq V_\infty$, for each $x \in B_R(0)$ and $\epsilon \in (0, \epsilon^*)$, we have $\epsilon x \in \Omega$, then, using (8) and (g_2) , we get

$$(26) \quad J_\epsilon(tv_0) \leq j(t) - \int_{B_R(0)} \int_0^{tv_0(x)} H(s - \beta)f(s) ds dx \leq j(t), \quad \text{for all } t \geq 0,$$

where $j(t) = \frac{\theta}{2}M(0)t^2 - \frac{t^{2^*}}{2^*} \int_{B_R(0)} |v_0|^{2^*}$.

Note that, the function j is increasing in $(0, t^*)$, for some $t^* > 0$. Since $\lim_{t \rightarrow 0} j(t) = 0$, there exists $\gamma_0 > 0$, independent on ϵ^* and β such that,

$$\gamma_0 < t^* \quad \text{and} \quad \max_{t \in [0, \gamma_0]} J_\epsilon(tv_0) \leq j(\gamma_0) < c^*,$$

where c^* is given in (25), which prove (i).

Let us prove (ii). Note that $J_\epsilon(0) = 0$. By $(g_3), (f_1), (M_1)$ and choosing $\|u\| = r < \gamma_0/2$, there exists $\varrho > 0$ such that $J_\epsilon(u) \geq \varrho, \forall u \in H^1(\mathbf{R}^N); \|u\| = r$. Moreover, using (f_3)

$$\int_{B_R(0)} \int_0^{\gamma_0 v_0(x)} H(s - \beta)f(s) ds dx \geq \int_{B_R(0)} [\gamma_0 v_0 - 2\beta] dx.$$

By (26) we obtain

$$(27) \quad J_\epsilon(\gamma_0 v_0) \leq j(\gamma_0) - \int_{B_R(0)} [\gamma_0 v_0 + 2\beta] dx,$$

for $\gamma_0 > 0$ and $\beta > 0$ small enough. □

Theorem 3.8. *Suppose that $(M_1), (f_1), (f_2), (f_3), (f_4), (V_0)$ and (V_1) are satisfied. Then, there are $\epsilon^*, \beta > 0$ small, such that for each $\epsilon \in (0, \epsilon^*)$, problem $(P_\epsilon)_a$ has a positive solution u_ϵ satisfying:*

- (i) u_ϵ is a weak solution of problem $(P_\epsilon)_a$ and $\|u_\epsilon\| \leq t_0, \forall \epsilon \in (0, \epsilon^*)$.
- (ii) The set $\{x \in \mathbf{R}^N : u_\epsilon(x) = \beta\}$ has null measure, and so u_ϵ is a strong solution of $(P_\epsilon)_a$.
- (iii) The set $\{x \in \mathbf{R}^N : u_\epsilon(x) > \beta\}$ has positive measure.

Proof. (i) Let β, v_0 and γ_0 be as in the Lemma 3.7. By Lemma 3.7, J_ϵ has the mountain pass geometry, it follows that there exist sequences $(u_n) \subset H^1(\mathbf{R}^N)$, $(w_n) \subset \partial J_\epsilon(u_n)$ and $(\xi_n^*) \subset \partial \Psi_\epsilon(u_n)$, such that, $w_n = Q'_\epsilon(u_n) - \xi_n^*$ in $H^{-1}(\mathbf{R}^N)$, $\|w_n\|_* = o_n(1), J_\epsilon(u_n) = c_\epsilon + o_n(1)$, where $c_\epsilon = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} J_\epsilon(\gamma(t))$ and $\Gamma = \{\gamma \in C([0, 1], H^1(\mathbf{R}^N)) : \gamma(0) = 0 \text{ and } \gamma(1) = \gamma_0 v_0\}$. Since (u_n) is bounded in $H^1(\mathbf{R}^N)$ and $\xi_n \in [g(\epsilon x, u_n), \bar{g}(\epsilon x, u_n)]$, by (12) we obtain (ξ_n) bounded in $L^{\frac{2^*}{2^*-1}}(\mathbf{R}^N)$.

Now, the Proposition 3.5, implies that $u_n \rightarrow u_\epsilon$ in $H^1(\mathbf{R}^N)$ and $\xi_n \rightarrow \xi_\epsilon$ in $L^{\frac{2^*}{2^*-1}}(\mathbf{R}^N)$. Thus,

$$(28) \quad M_0(\|u_\epsilon\|^2) \int_{\mathbf{R}^N} [\nabla u_\epsilon \nabla \varphi + V(\epsilon x)u_\epsilon \varphi] = \int_{\mathbf{R}^N} \xi_\epsilon \varphi, \quad \text{for all } \varphi \in H^1(\mathbf{R}^N)$$

where $\xi_\epsilon \in [g(\epsilon x, u_\epsilon), \bar{g}(\epsilon x, u_\epsilon)]$. By the Lemma 3.6, we get

$$\|u_\epsilon\| \leq t_0 \quad \text{and} \quad M_0(\|u_\epsilon\|) = M(\|u_\epsilon\|), \quad \text{for all } \epsilon \in (0, \epsilon^*).$$

Once that $\xi_\epsilon \in L^{\frac{2^*}{2^*-1}}(\mathbf{R}^N)$, from elliptic regularity theory, $u_\epsilon \in W^{2, \frac{2^*}{2^*-1}}(\mathbf{R}^N)$ and

$$(29) \quad \tilde{L}_\epsilon(u_\epsilon)(x) \in [\underline{g}(\epsilon x, u_\epsilon(x)), \bar{g}(\epsilon x, u_\epsilon(x))] \quad \text{a.e. in } \mathbf{R}^N,$$

where

$$\tilde{L}_\epsilon(u_\epsilon) := M(\|u_\epsilon\|^2)[- \Delta u_\epsilon + V(\epsilon x)u_\epsilon].$$

Taking as test function u_ϵ^- , we obtain $u_\epsilon = u_\epsilon^+ \geq 0$. By Harnack inequality, see [22, Teorema 8.20], we conclude that $u_\epsilon > 0$. Therefore, u_ϵ is positive solution of $(P_\epsilon)_a$ and we prove (i).

Now, suppose that the set $\Omega_{\epsilon, \beta} := \{x \in \mathbf{R}^N : u_\epsilon(x) = \beta\}$ has positive measure. By Stampachia Theorem, see [28], we can deduce that, $-\Delta u_\epsilon(x) = 0$ a.e. in $\Omega_{\epsilon, \beta}$. From (29)

$$(30) \quad M(\|u_\epsilon\|^2)V(\epsilon x)\beta \in [\underline{g}(\epsilon x, \beta), \bar{g}(\epsilon x, \beta)] \quad \text{a.e. in } \Omega_{\epsilon, \beta}.$$

Once that $\beta < a$ and

$$(31) \quad |g(x, t)| \leq V_0 t/k, \quad \text{for all } x \in \mathbf{R}^N, t \in [0, a],$$

thus by (30), $M(0) \leq 1/k$, but this contradicts (10). Thus, $|\Omega_{\epsilon, \beta}| = 0$, which proves (ii).

Now, suppose, by contradiction, that $|\{x \in \mathbf{R}^N : u_\epsilon(x) > \beta\}| = 0$, then

$$(32) \quad u_\epsilon(x) \leq \beta \quad \text{a.e. in } \mathbf{R}^N,$$

it follows from (28), (31) and (32), that $(M(0) - 1/k)\|u_\epsilon\|^2 \leq 0$, then $u_\epsilon = 0$, but this contradicts $J_\epsilon(u_\epsilon) = c_\epsilon > 0$. \square

Lemma 3.9. *Assume the same hypotheses of Theorem 3.8. Let u_{ϵ_n} be a positive solution of $(P_\epsilon)_a$, $\epsilon_n \rightarrow 0^+$ and $(x_n) \subset \mathbf{R}^N$. The sequence $v_n(x) := u_{\epsilon_n}(x + x_n) \in C(\mathbf{R}^N)$ and has a subsequence such that converges uniformly on compact sets to its weak limit $v \in W^{2, \frac{q}{q-1}}(\mathbf{R}^N) \cap C(\mathbf{R}^N)$.*

Proof. First, we adapt for our problem some ideas found in Brezis–Kato [7], for obtain $v_n \in L^p(\mathbf{R}^N)$, for all $p > 1$. For this end, fixed $n \in \mathbf{N}$. Using the change variable $z = x + x_n$, defining $v_n(x) := u_{\epsilon_n}(x + x_n)$ and $\xi_n(x) := \frac{1}{M(\|u_{\epsilon_n}\|^2)}\xi_{\epsilon_n}(x + x_n)$, we have

$$\begin{cases} -\Delta v_n(x) + V(\epsilon_n x + \epsilon_n x_n)v_n(x) = \xi_n(x + x_n) \quad \text{a.e. in } \mathbf{R}^N, \\ v_n \in H^1(\mathbf{R}^N) \cap W^{2, \frac{2^*}{2^*-1}}(\mathbf{R}^N), \end{cases}$$

where $\xi_{\epsilon_n}(z) \in [\underline{g}(\epsilon_n z, u_{\epsilon_n}(z)), \bar{g}(\epsilon_n z, u_{\epsilon_n}(z))]$ a.e. in \mathbf{R}^N .

Consider $s \geq 0, L > 0$, $A_{n,L} = \{x \in \mathbf{R}^N : |v_n|^s < L\}$, $B_{n,L}^c = A_{n,L}$ and the functions

$$v_{n,L} := \min\{v_n|v_n|^{2s}, L^2v_n\} \quad \text{and} \quad w_{n,L} := \min\{v_n|v_n|^s, Lv_n\}.$$

Using the test function $v_{n,L}$, we have

$$(33) \quad \int_{\mathbf{R}^N} \nabla v_n \nabla v_{n,L} + V(\epsilon_n x + \epsilon_n x_n)v_n v_{n,L} = \int_{\mathbf{R}^N} \xi_n v_{n,L}.$$

With an elementary calculation we obtain

$$(34) \quad \int_{\mathbf{R}^N} \nabla v_n \nabla v_{n,L} = (2s + 1) \int_{A_{n,L}} |v_n|^{2s} |\nabla v_n|^2 + L^2 \int_{B_{n,L}} |\nabla v_n|^2$$

and

$$(35) \quad \int_{\mathbf{R}^N} |\nabla w_{n,L}|^2 = (s+1)^2 \int_{A_{n,L}} |v_n|^{2s} |\nabla v_n|^2 + L^2 \int_{B_{n,L}} |\nabla v_n|^2.$$

Combining (33), (34) and (35), we find

$$\int_{\mathbf{R}^N} |\nabla w_{n,L}|^2 \leq (2s+1) \int_{\mathbf{R}^N} \xi_n v_{n,L}.$$

From (12) there exists a constant $C > 0$, independent of L , such that, $|\rho_n| \leq C|v_n|^{2^*-1}$. For $0 < K < L$, we obtain

$$\begin{aligned} \int_{\mathbf{R}^N} |\nabla w_{n,L}|^2 &\leq (2s+1)C \int_{\mathbf{R}^N} |v_n|^{2^*-2} |w_{n,L}|^2 \\ &\leq (2s+1)CK^{2^*-2} \int_{\{v_n \leq K\}} |v_n|^{2(s+1)} + (2s+1)C\eta(K) \left(\int_{\{v_n > K\}} |w_{n,L}|^{2^*} \right)^{\frac{2}{2^*}}, \end{aligned}$$

where $\eta(K) = \left(\int_{\{v_n > K\}} |v_n|^{2^*} \right)^{\frac{2^*-2}{2^*}} \rightarrow 0$ if $K \rightarrow \infty$. Thus,

$$\int_{\mathbf{R}^N} |\nabla w_{n,L}|^2 \leq (2s+1)CK^{2^*-2} \int_{\mathbf{R}^N} |v_n|^{2(s+1)} + (2s+1)C\eta(K)S \int_{\mathbf{R}^N} |\nabla w_{n,L}|^2,$$

where S_0 is the best constant in inequality

$$\left(\int_{\mathbf{R}^N} |u|^{2^*} \right)^{\frac{2}{2^*}} \leq S_0 \int_{\mathbf{R}^N} |\nabla u|^2, \quad \forall u \in D^{1,2}(\mathbf{R}^N).$$

Choosing K , such that, $(2s+1)C_1S_0\eta(K) \leq \frac{1}{2}$, we have

$$\left(\int_{\mathbf{R}^N} |w_{L,n}|^{2^*} \right)^{\frac{2}{2^*}} \leq 2(2s+1)CK^{2^*-2} \int_{\mathbf{R}^N} |v_n|^{2(s+1)}.$$

Since $|w_{n,L}| = |v_n|^{s+1}$ in $A_{n,L}$ and $|w_{n,L}| \leq |v_n|^{s+1}$ in \mathbf{R}^N , thus, if $v_n \in L^{2(s+1)}(\mathbf{R}^N)$, by the dominated convergence theorem, with $L \rightarrow \infty$, we obtain $v_n \in L^{2^*(s+1)}(\mathbf{R}^N)$.

By iteration about s , we have $v_n \in L^p(\mathbf{R}^N)$, for all $p > 1$. From regularity elliptic theory, for p sufficiently large, we get $v_n \in W^{2,p}(\mathbf{R}^N) \hookrightarrow C^{1,\alpha}(\mathbf{R}^N)$, for some $0 < \alpha < 1$. The boundedness of v_n in $H^1(\mathbf{R}^N)$, see Theorem 3.8 (i), implies that $\|v_n\|_{C^{1,\alpha}_{loc}}$ is bounded. From the Schauder estimates, there exists $v \in C(\mathbf{R}^N)$, such that

$$v_n \rightarrow v \quad \text{uniformly on } K,$$

for each compact $K \subset \mathbf{R}^N$. □

Remark 3.10. As a consequence of the proof of the above lemma we have that the solution u_ϵ of $(P_\epsilon)_a$ belongs to $W^{2,\frac{q}{q-1}}(\mathbf{R}^N) \cap C^{1,\alpha}(\mathbf{R}^N)$, for some $\alpha \in (0, 1)$.

4. Proof of Theorem 1.1 in the case $(V_0), (V_1)$ and (V_2)

In this section we assume $(V_0), (V_1)$ and (V_2) ,

$$\Omega := B_{R_\epsilon}(0), \quad m_\epsilon := \max_{x \in \partial B_{R_\epsilon}(0)} u_\epsilon(x) > 0 \quad \text{and} \quad R_\epsilon := \frac{1}{\epsilon},$$

where u_ϵ is a solution of the problem $(P_\epsilon)_a$ given by Theorem 3.8.

Lemma 4.1. *Suppose that (M_1) , (f_1) , (f_2) , (f_3) , (f_4) , (V_0) , (V_1) and (V_2) are satisfied. Then, given $\eta > 0$, there exist $\epsilon^* > 0$ such that*

$$m_\epsilon < \eta, \quad \text{for all } \epsilon \in (0, \epsilon^*).$$

Proof. Proceeding by a contradiction argument, suppose that there exists $\delta_0 > 0$, such that, for each $\eta_n = \frac{1}{n}$, we obtain $\epsilon_n > 0$ small, such that,

$$m_{\epsilon_n} = \max_{x \in \partial B_{\frac{R\epsilon_n}{\epsilon_n}}(0)} u_n(x) \geq \delta_0 > 0 \quad \text{and} \quad \epsilon_n \rightarrow 0,$$

where $u_n := u_{\epsilon_n} \in H^1(\mathbf{R}^N) \cap W^{2, \frac{q}{q-1}}(\mathbf{R}^N)$. By elliptic regularity theory $u_n \in C(\mathbf{R}^N)$, then, there exists $x_n \in \partial B_{\frac{R\epsilon_n}{\epsilon_n}}(0)$, such that, $u_n(x_n) = m_{\epsilon_n}$. Therefore, $u_n(x_n) \geq \delta_0 > 0$.

Now, note that for $v_n(x) := u_n(x + x_n)$ and $\varrho_n(x) := \xi_n(x + x_n)$,

$$\begin{cases} M(\|u_n\|^2)(-\Delta v_n(x) + V(\epsilon_n x + \epsilon_n x_n)v_n(x)) = \varrho_n(x) & \text{a.e. in } \mathbf{R}^N, \\ v_n \in H^1(\mathbf{R}^N) \cap W^{2, \frac{q}{q-1}}(\mathbf{R}^N). \end{cases}$$

The Theorem 3.8 (i), implies that (u_n) is bounded in $H^1(\mathbf{R}^N)$. Therefore, by (12) the sequence $(\xi_n) \subset [g(\epsilon x, u_n), \bar{g}(\epsilon x, u_n)]$ is bounded in $L^{\frac{2^*}{2^*-1}}(\mathbf{R}^N)$.

Thus, up to a subsequence, $v_n \rightharpoonup v$ in $H^1(\mathbf{R}^N)$, $\varrho_n \rightharpoonup \varrho$ in $L^{\frac{2^*}{2^*-1}}(\mathbf{R}^N)$, $M(\|u_n\|^2) \rightarrow m_1$ and by (V_1) , $V(\epsilon_n x_n) \rightarrow \alpha_1$. Taking the limit of $n \rightarrow \infty$, let us deduce that v is a weak solution solution of the problem

$$\begin{cases} m_1(-\Delta v + \alpha_1 v) = \varrho & \text{in } \mathbf{R}^N, \\ v \in H^1(\mathbf{R}^N) \cap W^{2, \frac{q}{q-1}}(\mathbf{R}^N). \end{cases}$$

Using the same argument as in the proof of Lemma 3.9 we obtain $v_n, v \in C(\mathbf{R}^N)$ and a subsequence of v_n which converges uniformly on compact sets of \mathbf{R}^N for v . Then, $v(0) \geq \delta_0 > 0$ and $v \neq 0$.

Since (u_n) is bounded in $H^1(\mathbf{R}^N)$, M is continuous with $M(t) \geq M(0)$, using similar argument as in [19, proof of the Lemma 5.1] or [1, proof of the Lemma 3.1], we obtain a sequence $(\epsilon_n x_n) \subset \mathbf{R}^N$, such that,

$$\nabla V(\epsilon_n x_n) \rightarrow 0 \quad \text{and} \quad V(\epsilon_n x_n) \rightarrow \alpha_1,$$

for some $\alpha_1 \geq V_0 > 0$. Thus, $(\epsilon_n x_n) \subset \mathbf{R}^N$ is a sequence $(PS)_{\alpha_1}$ with $|\epsilon_n x_n| = R_{\epsilon_n} = \frac{1}{\epsilon_n}$. This contradicts (V_2) and so the lemma is true. \square

Proof of the Theorem 1.1 for (V_0) , (V_1) and (V_2) . Let the constant a of (9). Combining the Theorem 3.8 with the Lemma 4.1, we obtain $\epsilon^* > 0$, such that, for each $\epsilon \in (0, \epsilon^*)$, the problem $(P_\epsilon)_a$ has a positive solution u_ϵ with $\max_{x \in \partial B_{\frac{R\epsilon}{\epsilon}}(0)} u_\epsilon(x) \leq a$. The function $\tilde{u}_\epsilon(x) := 0$ in $B_{\frac{R\epsilon}{\epsilon}}(0)$ and $\tilde{u}_\epsilon(x) := (u_\epsilon - a)_+(x)$ in $B_{\frac{R\epsilon}{\epsilon}}(0)^c$, belongs in $H^1(\mathbf{R}^N)$. Since

$$M(\|u_\epsilon\|^2) \int_{\mathbf{R}^N} [\nabla u_\epsilon \nabla \varphi + V(\epsilon x)u_\epsilon \varphi] = \int_{\mathbf{R}^N} \xi_\epsilon \varphi, \quad \text{for all } \varphi \in H^1(\mathbf{R}^N)$$

and $0 \leq \xi_\epsilon \leq V(\epsilon x)u_\epsilon/k$ in $B_{\frac{R\epsilon}{\epsilon}}(0)^c$, using $\varphi = \tilde{u}_\epsilon$, we have

$$\left(M(0) - \frac{1}{k}\right) \int_{B_{\frac{R\epsilon}{\epsilon}}(0)^c} V(\epsilon x)u_\epsilon(u_\epsilon - a)_+ \leq 0,$$

thus, $u_\epsilon(x) \leq a$ a.e. in $B_{R_\epsilon}(0)^c$. Therefore, u_ϵ is a solution of (6). Moreover, we conclude the proof with the Theorem 3.8. \square

5. Proof of Theorem 1.1 in the case $(V_0), (V_1)$ and (V_3)

In this section we assume $(V_0), (V_1)$ and (V_3) ,

$$\Omega := \Lambda \quad \text{and} \quad m_\epsilon := \max_{x \in \partial\Lambda_\epsilon} u_\epsilon(x) > 0,$$

where u_ϵ is a solution of the problem $(P_\epsilon)_a$ given by Theorem 3.8.

Lemma 5.1. *Suppose that $(M_1), (f_1), (f_2), (f_3), (f_4), (V_0), (V_1)$ and (V_3) are satisfied. Thus, given $\eta > 0$, there exists $\epsilon^* > 0$, such that,*

$$m_\epsilon < \eta, \quad \text{for all } \epsilon \in (0, \epsilon^*).$$

Proof. Suppose that the lemma is not true. Then, repeating the arguments of proof do Lemma 4.1, we can find a sequence $(x_n) \subset \partial\Lambda_{\epsilon_n}$, such that, $\nabla V(\epsilon_n u_n) \rightarrow 0$. Once that $(\epsilon_n x_n) \in \partial\Lambda$ and $V \in C^1(\mathbf{R}^N, \mathbf{R})$, we obtain $x_0 \in \mathbf{R}^N$, such that, $x_0 \in \partial\Lambda$ and $\nabla V(x_0) = 0$. But this contradicts (V_3) . \square

Proof of the Theorem 1.1 for $(V_0), (V_1)$ and (V_3) . Using Lemma 5.1 and arguing similarly as in the proof of the case $(V_0), (V_1)$ and (V_2) we conclude the proof of the theorem. \square

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