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REGULARITY OF THE DERIVATIVES OF p-ORTHOTROPIC FUNCTIONS IN THE PLANE FOR $1 < p < 2$

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Abstract. We present a proof of the C^1 regularity of p-orthotropic functions in the plane for $1 < p < 2$, based on the monotonicity of the derivatives. Moreover we achieve an explicit logarithmic modulus of continuity.

1. Introduction

In this work we investigate the regularity of p-orthotropic functions in the plane for $1 < p < 2$. Let $\Omega \subset \mathbf{R}^2$ be an open set. A weak solution of the orthotropic p-Laplace equation (also known as pseudo p-Laplace equation) is a function $u \in W^{1,p}(\Omega)$ such that

$$\sum_{i=1}^{2} \int_\Omega |\partial_i u|^{p-2} \partial_i u \, \partial_i \phi \, \mathrm{d}x = 0 \quad \text{for all} \quad \phi \in W_0^{1,p}(\Omega). \tag{1.1}$$

Equation (1.1) arises as the Euler–Lagrange equation for the functional

$$I_\Omega(v) = \sum_{i=1}^{2} \int_\Omega \frac{|\partial_i v|^p}{p} \, \mathrm{d}x. \tag{1.2}$$

The equation is singular when either one of the derivatives vanishes, and does not fall into the category of equations with p-Laplacian structure. It was proved by Bousquet and Brasco in [1] that weak solutions of (1.1) for $1 < p < \infty$ are $C^1(\Omega)$. A simple proof which gives a logarithmic modulus of continuity for the derivatives is contained in [6] for the case $p \geq 2$. The latter relies on a lemma on the oscillation of monotone functions due to Lebesgue [5] and the fact that derivatives of solutions are monotone (in the sense of Lebesgue). The purpose of this work is to extend this result to the case $1 < p < 2$ employing methods developed in [6]. We obtain the following:

Theorem 1.1. *Let $\Omega \subset \mathbf{R}^2$ and $u \in W^{1,p}(\Omega)$ be a solution of the equation (1.1) for $1 < p < 2$. Fix a ball $B_R \subset\subset \Omega$. Then, for all $j \in \{1, 2\}$ and $B_r \subset\subset B_{R/2}$, we have*

$$\operatorname*{osc}_{B_r} (\partial_j u) \leq C_p \left(\log \left(\frac{R}{r} \right) \right)^{-\frac{1}{2}} \left(\fint_{B_R} |\nabla u|^p \, dx \right)^{\frac{1}{p}}, \tag{1.3}$$

where C_p is a constant depending only on p.

Notation. We indicate balls by $B_r = B_r(a) = \{x \in \mathbf{R}^2 \colon |x - a| < r\}$ and we omit the center when not relevant. Whenever two balls $B_r \subset B_R$ appear in a statement they are implicitly assumed to be concentric. The variable x denotes the

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vector (x_1, x_2) and we denote the partial derivatives of a function f with respect to x_j as $\partial_j f$.

2. Regularization

We will consider a regularized problem by introducing a non degeneracy parameter $\epsilon > 0$. Fix $B_R \subset\subset \Omega \subset \mathbf{R}^2$ and consider the regularized Dirichlet problem

$$\begin{cases} \sum_{i=1}^{2} \int_{B_R} (|\partial_i u^\epsilon|^2 + \epsilon)^{\frac{p-2}{2}} \partial_i u^\epsilon \, \partial_i \phi \, \mathrm{d}x = 0 \\ u^\epsilon - u \in W_0^{1,p}(B_R). \end{cases} \tag{2.1}$$

Note that u^ϵ is the unique minimizer of the regularized functional

$$I^\epsilon_{B_R}(v) = \sum_{i=1}^{2} \int_{B_R} \frac{1}{p} (|\partial_i v|^2 + \epsilon)^{\frac{p}{2}} \, \mathrm{d}x \tag{2.2}$$

among $W^{1,p}(B_R)$ functions v such that $v - u \in W_0^{l,p}(B_R)$. By elliptic regularity theory, the unique solution u^ϵ of (2.1) is smooth in B_R.

Fix an index $j \in \{1, 2\}$. Then, replacing ϕ by $\partial_j \phi$ in equation (2.1) and integrating by parts, we find that the derivative $\partial_j u^\epsilon$ satisfies the following equation

$$\sum_{i=1}^{2} \int_{B_R} (\epsilon + |\partial_i u^\epsilon|^2)^{\frac{p-4}{2}} (\epsilon + (p-1)|\partial_i u^\epsilon|^2) \, \partial_i \partial_j u^\epsilon \, \partial_i \phi \, \mathrm{d}x = 0 \tag{2.3}$$

for all $\phi \in C_0^\infty(B_R)$.

We now collect some uniform estimates and convergences (see also [1]).

Lemma 2.1. *Let $u \in W^{1,p}(\Omega)$ be a solution of (1.1) and u^ϵ be a solution of (2.1) for $1 < p < 2$. Then we have*

$$\int_{B_R} |\nabla u^\epsilon|^p \, dx \le C_p \left(\int_{B_R} |\nabla u|^p \, dx + \epsilon^{\frac{p}{2}} R^2 \right) \tag{2.4}$$

where C_p is a constant depending only on p.

Proof. The estimate follows from $I^\epsilon_{B_R}(u^\epsilon) \le I^\epsilon_{B_R}(u)$. □

Proposition 2.2. *Let $u \in W^{1,p}(\Omega)$ be a solution of (1.1) and u^ϵ be a solution of (2.1) for $1 < p < 2$. Then, for all $j \in \{1, 2\}$, we have*

$$\sup_{B_{R/2}} (\epsilon + |\nabla u^\epsilon|^2) \le C_p \left(\fint_{B_R} (\epsilon + |\nabla u^\epsilon|^2)^{\frac{p}{2}} \, dx \right)^{\frac{2}{p}}, \tag{2.5}$$

$$\int_{B_{R/2}} |\nabla \partial_j u^\epsilon|^2 \, dx \le C_p \left(\fint_{B_R} (|\nabla u|^p + \epsilon^{\frac{p}{2}}) \, dx \right)^{\frac{2}{p}}, \tag{2.6}$$

where C_p is a constant depending only on p.

Proof. The proof of the Lipschitz bound can be found in [4] while (2.6) appears in [1]. We provide details for completeness. Note that by a change of variables, the function $u^\epsilon_R(x) = u^\epsilon(x_0 + Rx)$ satisfies the equation

$$\sum_{i=1}^{2} \int_{B_1} (|\partial_i u^\epsilon_R|^2 + R^2 \epsilon)^{\frac{p-2}{2}} \partial_i u^\epsilon_R \partial_i \phi \, \mathrm{d}x = 0 \quad \text{for all} \quad \phi \in W_0^{1,p}(B_1). \tag{2.7}$$

Introduce the notation $w = \epsilon R^2 + |\nabla u_R^\epsilon|^2$ and $a_i(z) = a_i(z_i) = (\epsilon R^2 + |z_i|^2)^{\frac{p-2}{2}} z_i$ so that equation (2.7) rewrites as

$$\sum_{i=1}^{2} \int_{B_1} a_i(\partial_i u_R^\epsilon)\partial_i \phi \, \mathrm{d}x = 0 \quad \text{for all} \quad \phi \in W_0^{1,p}(B_1).$$

For $j \in \{1,2\}$ and $\alpha \geq 0$ take $\phi = \partial_j(\partial_j u_R^\epsilon \, w^{\frac{\alpha}{2}} \xi^2)$ so that $\partial_i \phi = \partial_j(\partial_i \partial_j u_R^\epsilon w^{\frac{\alpha}{2}} \xi^2 + \frac{\alpha}{2} \partial_i w \, w^{\frac{\alpha-2}{2}} \, \partial_j u_R^\epsilon \, \xi^2) + 2\partial_j(\xi \partial_i \xi \, w^{\frac{\alpha}{2}} \, \partial_j u_R^\epsilon)$. Sum in j to get

$$\begin{aligned} A + B := \sum_{i,j=1}^{2} \int_{B_1} & a_i(\partial_i u_R^\epsilon) \partial_j (\partial_i \partial_j u_R^\epsilon w^{\frac{\alpha}{2}} \xi^2 + \frac{\alpha}{2} \partial_i w \, w^{\frac{\alpha-2}{2}} \, \partial_j u_R^\epsilon \, \xi^2) \, \mathrm{d}x \\ & + 2 \sum_{i,j=1}^{2} \int_{B_1} a_i(\partial_i u_R^\epsilon) \partial_j (\xi \partial_i \xi \, w^{\frac{\alpha}{2}} \, \partial_j u_R^\epsilon) \, \mathrm{d}x = 0. \end{aligned}$$

Note that $\partial_i w = 2\sum_{j=1}^{2} \partial_i \partial_j u_R^\epsilon \, \partial_j u_R^\epsilon$ and $\partial_i a_i(\partial_i u_R^\epsilon) \geq c_p w^{\frac{p-2}{2}}$ since $1 < p < 2$. Integrate by parts in A. We get $A = A_1 + A_2$ where

$$\begin{aligned} A_1 &:= \sum_{i,j=1}^{2} \int_{B_1} \partial_i a_i(\partial_i u_R^\epsilon)(\partial_i \partial_j u_R^\epsilon)^2 \, w^{\frac{\alpha}{2}} \, \xi^2 \, \mathrm{d}x \geq c_p \sum_{j=1}^{2} \int_{B_1} w^{\frac{p-2+\alpha}{2}} |\nabla \partial_j u_R^\epsilon|^2 \xi^2 \, \mathrm{d}x, \\ A_2 &:= c\alpha \sum_{i,j=1}^{2} \int_{B_1} \partial_i a_i(\partial_i u_R^\epsilon) \partial_i \partial_j u_R^\epsilon \, \partial_j u_R^\epsilon \, \partial_i w \, w^{\frac{\alpha-2}{2}} \, \xi^2 \, \mathrm{d}x \\ &= c\alpha \sum_{i=1}^{2} \int_{B_1} \partial_i a_i(\partial_i u_R^\epsilon)(\partial_i w)^2 w^{\frac{\alpha-2}{2}} \, \xi^2 \, \mathrm{d}x \geq c_p \alpha \int_{B_1} w^{\frac{p-4+\alpha}{2}} |\nabla w|^2 \xi^2 \, \mathrm{d}x. \end{aligned}$$

Now we estimate $B = B_1 + B_2 + B_3$;

$$\begin{aligned} |B_1| &:= \left| \sum_{i,j=1}^{2} \int_{B_1} a_i(\partial_i u_R^\epsilon) w^{\frac{\alpha}{2}} \partial_j u_R^\epsilon \, \partial_j(\xi \partial_i \xi) \, \mathrm{d}x \right| \leq C_p \int_{B_1} w^{\frac{p+\alpha}{2}} (|\nabla \xi|^2 + |\nabla^2 \xi|) \, \mathrm{d}x, \\ |B_2| &:= \left| \frac{\alpha}{2} \sum_{i,j=1}^{2} \int_{B_1} a_i(\partial_i u_R^\epsilon) w^{\frac{\alpha-2}{2}} \partial_j w \, \partial_j u_R^\epsilon \, \xi \, \partial_i \xi \, \mathrm{d}x \right| \leq C\alpha \int_{B_1} w^{\frac{p+\alpha-2}{2}} |\nabla w| \, \xi \, |\nabla \xi| \, \mathrm{d}x \\ &\leq \eta\alpha \int_{B_1} w^{\frac{p-4+\alpha}{2}} |\nabla w|^2 \xi^2 \, \mathrm{d}x + \frac{C\alpha}{\eta} \int_{B_1} |\nabla \xi|^2 \, w^{\frac{p+\alpha}{2}} \, \mathrm{d}x, \\ |B_3| &:= \left| \sum_{i,j=1}^{2} \int_{B_1} a_i(\partial_i u_R^\epsilon) w^{\frac{\alpha}{2}} \partial_j \partial_j u_R^\epsilon \, \xi \, \partial_i \xi \, \mathrm{d}x \right| \leq \sum_{j=1}^{2} \int_{B_1} w^{\frac{p-1+\alpha}{2}} |\nabla \partial_j u_R^\epsilon| \, \xi \, |\nabla \xi| \, \mathrm{d}x \\ &\leq \eta \sum_{j=1}^{2} \int_{B_1} w^{\frac{p-2+\alpha}{2}} |\nabla \partial_j u_R^\epsilon|^2 \xi^2 \, \mathrm{d}x + \frac{C}{\eta} \int_{B_1} |\nabla \xi|^2 \, w^{\frac{p+\alpha}{2}} \, \mathrm{d}x \end{aligned}$$

where we used $a_i(\partial_i u_R^\epsilon) \leq w^{\frac{p-1}{2}}$ and Young's inequality with a parameter η to be chosen suitably small. We get

$$\begin{aligned} & c_p \sum_{j=1}^{2} \int_{B_1} w^{\frac{p-2+\alpha}{2}} |\nabla \partial_j u_R^\epsilon|^2 \xi^2 \, \mathrm{d}x + c_p \alpha \int_{B_1} w^{\frac{p-4+\alpha}{2}} |\nabla w|^2 \xi^2 \, \mathrm{d}x \\ & \leq C_p(\alpha + 1) \int_{B_1} (|\nabla \xi|^2 + |\nabla^2 \xi|) \, w^{\frac{p+\alpha}{2}} \, \mathrm{d}x. \end{aligned} \tag{2.8}$$

Note that for $\alpha = 0$ we get for all $j \in \{1, 2\}$

$$\int_{B_1} w^{\frac{p-2}{2}} |\nabla \partial_j u_R^\epsilon|^2 \xi^2 \, \mathrm{d}x \le C_p \int_{B_1} (|\nabla \xi|^2 + |\nabla^2 \xi|) \, w^{\frac{p}{2}} \, \mathrm{d}x, \tag{2.9}$$

and since $|\nabla w|^2 \le c \sum_j |\nabla \partial_j u_R^\epsilon|^2 |\nabla u_R^\epsilon|^2$ we have

$$\begin{aligned} \int_{B_1} w^{\frac{p-4}{2}} |\nabla w|^2 \xi^2 \, \mathrm{d}x &\le c \sum_{j=1}^{2} \int_{B_1} w^{\frac{p-4}{2}} |\nabla u_R^\epsilon|^2 |\nabla \partial_j u_R^\epsilon|^2 \xi^2 \, \mathrm{d}x \\ &\le c \sum_{j=1}^{2} \int_{B_1} w^{\frac{p-2}{2}} |\nabla \partial_j u_R^\epsilon|^2 \xi^2 \\ &\le C_p \int_{B_1} (|\nabla \xi|^2 + |\nabla^2 \xi|) \, w^{\frac{p}{2}} \, \mathrm{d}x. \end{aligned} \tag{2.10}$$

Now for $\alpha \ge 1$, (2.8) implies

$$\int_{B_1} w^{\frac{p-4+\alpha}{2}} |\nabla w|^2 \xi^2 \, \mathrm{d}x \le C_p \frac{\alpha+1}{\alpha} \int_{B_1} (|\nabla \xi|^2 + |\nabla^2 \xi|) \, w^{\frac{p+\alpha}{2}} \, \mathrm{d}x \tag{2.11}$$

and combining with (2.10) we get

$$\int_{B_1} |\nabla (w^{\frac{p+\alpha}{4}} \xi)|^2 \, \mathrm{d}x \le C(p+\alpha)^2 \int_{B_1} (|\nabla \xi|^2 + |\nabla^2 \xi|) \, w^{\frac{p+\alpha}{2}} \, \mathrm{d}x$$

for all $\alpha \ge 0$. Using Sobolev's embedding $W_0^{1,2}(B_1) \hookrightarrow L^{2q}(B_1)$ for a fixed $q > 1$ we get

$$\left(\int_{B_1} w^{q \frac{p+\alpha}{2}} \xi^{2q} \, \mathrm{d}x \right)^{\frac{1}{q}} \le C_p (p+\alpha)^2 \int_{B_1} (|\nabla \xi|^2 + |\nabla^2 \xi|) \, w^{\frac{p+\alpha}{2}} \, \mathrm{d}x. \tag{2.12}$$

Now choose a sequence of radii $r_i = 1/2^i + (1 - 1/2^i)\frac{1}{2}$, cut-off functions ξ between r_i and r_{i+1} and $\alpha_i = q^i p - p$ so that $\frac{p+\alpha_i}{2} = \frac{p}{2} q^i$. Using these in (2.12), raising to the power $1/q^i$ and iterating we get for all $i \in \mathbf{N}$

$$\left(\int_{B_{r_{i+1}}} w^{\frac{p}{2} q^{i+1}} \, \mathrm{d}x \right)^{\frac{1}{q^{i+1}}} \le (C_p q^{2i} 2^i)^{\frac{1}{q^i}} \left(\int_{B_{r_i}} w^{\frac{p}{2} q^i} \, \mathrm{d}x \right)^{\frac{1}{q^i}} \le \prod_{j=0}^{i} (C_p q^{2j} 2^j)^{\frac{1}{q^j}} \int_{B_1} w^{\frac{p}{2}} \, \mathrm{d}x.$$

Observe that $\prod_{i=0}^{\infty} (C_p q^{2i} 2^i)^{\frac{1}{q^i}} = C(p,q) < \infty$ so passing to the limit as $i \to \infty$ we get

$$\sup_{B_{1/2}} w^{\frac{p}{2}} \le C(p,q) \int_{B_1} w^{\frac{p}{2}} \, \mathrm{d}x$$

which, after rescaling, proves (2.5). Now going back to (2.9), choosing a cut-off function between $B_{R/2}$ and B_R and using $1 < p < 2$ we get

$$\int_{B_{R/2}} |\nabla \partial_j u^\epsilon|^2 \, \mathrm{d}x \le C_p \sup_{B_{R/2}} (\epsilon + |\nabla u^\epsilon|^2)^{\frac{2-p}{p}} \fint_{B_R} (\epsilon + |\nabla u^\epsilon|^2)^{\frac{p}{2}} \, \mathrm{d}x.$$

Using (2.5) and (2.4) we obtain (2.6). □

Next we collect some facts about the convergence of u^ϵ to the solution of the degenerate equation. These are sufficient for our purposes.

Proposition 2.3. *Let u^ϵ be the solution of* (2.1) *for $1 < p < 2$ and $u \in W^{1,p}(\Omega)$ the solution of* (1.1). *We have*

- *u^ϵ converges to u locally uniformly in B_R,*
- *∇u^ϵ converges to ∇u in $L^p(B_R)$.*

Proof. From the energy estimate (2.4) we obtain a uniform bound for the L^p norm of ∇u^ϵ. Therefore (up to a subsequence) u^ϵ converges to some $v \in W^{1,p}(B_R)$ weakly in $W^{1,p}(B_R)$ and strongly in $L^p(B_R)$. Note that we have $v - u \in W_0^{1,p}(B_R)$. By weakly lower semicontinuity we get

$$\begin{aligned} I_{B_R}(v) &= \sum_{i=1}^2 \int_{B_R} \frac{|\partial_i v|^p}{p}\,\mathrm{d}x \le \liminf_{\epsilon\to 0} \sum_{i=1}^2 \int_{B_R} \frac{|\partial_i u^\epsilon|^p}{p}\,\mathrm{d}x \\ &\le \liminf_{\epsilon\to 0} \sum_{i=1}^2 \int_{B_R} \frac{1}{p}(|\partial_i u^\epsilon|^2+\epsilon)^{\frac{p}{2}}\,\mathrm{d}x \le \liminf_{\epsilon\to 0} \sum_{i=1}^2 \int_{B_R} \frac{1}{p}(|\partial_i u|^2+\epsilon)^{\frac{p}{2}}\,\mathrm{d}x \\ &= \sum_{i=1}^2 \int_{B_R} \frac{1}{p}|\partial_i u|^p\,\mathrm{d}x = I_{B_R}(u). \end{aligned}$$

Note that in the third inequality we used the minimality of u^ϵ subject to the boundary condition $u^\epsilon - u \in W_0^{1,p}(B_R)$. By uniqueness of the minimizer of I_{B_R} among functions with boundary values u in B_R, we get $v = u$. By the uniform Lipschitz estimate (2.5) and Ascoli–Arzela' theorem we obtain that the convergence is uniform.

Now we show $L^p(B_R)$ convergence of the gradient. Use $\phi = u^\epsilon - u$ as a test function in (2.1), add and subtract the term $(|\partial_i u|^2+\epsilon)^{\frac{p-2}{2}}\partial_i u$ to get

$$\begin{aligned} &\sum_{i=1}^2 \int_{B_R} \left((|\partial_i u^\epsilon|^2+\epsilon)^{\frac{p-2}{2}}\partial_i u^\epsilon - (|\partial_i u|^2+\epsilon)^{\frac{p-2}{2}}\partial_i u\right)(\partial_i u^\epsilon - \partial_i u)\,\mathrm{d}x \\ &= \sum_{i=1}^2 \int_{B_R} (|\partial_i u|^2+\epsilon)^{\frac{p-2}{2}}\partial_i u(\partial_i u - \partial_i u^\epsilon)\,\mathrm{d}x. \end{aligned}$$

Since $\partial_i u - \partial_i u^\epsilon$ converges to 0 weakly in $L^p(B_R)$, the integral in the right hand side converges to 0. We can minorize the integral in the left hand side using the inequality

$$|a-b|^2(\epsilon+|a|^2+|b^2|)^{\frac{p-2}{2}} \le C_p((\epsilon+|a|^2)^{\frac{p-2}{2}}a - (\epsilon+|b|^2)^{\frac{p-2}{2}}b)(a-b)$$

valid for $1 < p < 2$, and obtain that

$$\int_{B_R} \left(\epsilon+|\partial_i u^\epsilon|^2+|\partial_i u|^2\right)^{\frac{p-2}{2}} |\partial_i u^\epsilon - \partial_i u|^2\,\mathrm{d}x \longrightarrow 0 \tag{2.13}$$

as $\epsilon \to 0$, for $i = 1, 2$. Finally by Hölder's inequality

$$\begin{aligned} &\int_{B_R} |\partial_i u^\epsilon - \partial_i u|^p\,\mathrm{d}x \\ &= \int_{B_R} |\partial_i u^\epsilon - \partial_i u|^p \left(\epsilon+|\partial_i u^\epsilon|^2+|\partial_i u|^2\right)^{\frac{p(p-2)}{2}} \left(\epsilon+|\partial_i u^\epsilon|^2+|\partial_i u|^2\right)^{\frac{p(2-p)}{2}}\,\mathrm{d}x \\ &\le \left(\int_{B_R} |\partial_i u^\epsilon - \partial_i u|^2 \left(\epsilon+|\partial_i u^\epsilon|^2+|\partial_i u|^2\right)^{\frac{p-2}{2}}\,\mathrm{d}x\right)^{\frac{p}{2}} \\ &\cdot \left(\int_{B_R} \left(\epsilon+|\partial_i u^\epsilon|^2+|\partial_i u|^2\right)^{\frac{p}{2}}\,\mathrm{d}x\right)^{\frac{2-p}{2}}. \end{aligned}$$

Since the last integral is uniformly bounded in ϵ, using (2.13) we get that $\partial_i u^\epsilon$ converges to $\partial_i u$ in $L^p(B_R)$. □

3. Monotone functions and Lebesgue's lemma

A continuous function $v\colon \Omega \longrightarrow \mathbf{R}$ is monotone (in the sense of Lebesgue) if

$$\max_{\overline{D}} v = \max_{\partial D} v \quad \text{and} \quad \min_{\overline{D}} v = \min_{\partial D} v$$

for all subdomains $D \subset\subset \Omega$. Monotone functions are further discussed in [7].

The next Lemma is due to Lebesgue [5].

Lemma 3.1. *Let $B_R \subset \mathbf{R}^2$ and $v \in C(B_R) \cap W^{1,2}(B_R)$ be monotone in the sense of Lebesgue. Then*

$$(\operatorname*{osc}_{B_r} v)^2 \log\left(\frac{R}{r}\right) \leq \pi \int_{B_R \setminus B_r} |\nabla v(x)|^2 \, dx$$

for every $r < R$.

Proof. Assume v is smooth. Let (η, ζ) be the center of B_R. Let x_1 and x_2 be two points on the circle of radius t, and let $\gamma\colon [0, 2\pi] \longrightarrow \mathbf{R}^2$, $\gamma(s) = (\eta + t\cos(s), \zeta + t\sin(s))$ be a parametrization of the circle such that $\gamma(a) = x_1$ and $\gamma(b) = x_2$. Then we have

$$v(x_1) - v(x_2) = \int_a^b \frac{d}{ds} v(\gamma(s)) \, \mathrm{d}s = \int_a^b \langle \nabla v(\gamma(s)), \gamma'(s) \rangle \, \mathrm{d}s \leq \int_a^b t \, |\nabla v(\gamma(s))| \, \mathrm{d}s.$$

Taking the supremum on angles a and b such that $|a - b| \leq \pi$ and using Hölder's inequality, we get

$$(\operatorname*{osc}_{\partial B_t} v)^2 \leq \pi t^2 \int_0^{2\pi} |\nabla v(\gamma(s))|^2 \, \mathrm{d}s.$$

Now diving by t, integrating from r to R, and using polar coordinates we get

$$\int_r^R \frac{(\operatorname{osc}_{\partial B_t} v)^2}{t} \, \mathrm{d}t \leq \pi \int_r^R \int_0^{2\pi} t \, |\nabla v(\gamma(s))|^2 \, \mathrm{d}s \, \mathrm{d}t = \pi \int_{B_R \setminus B_r} |\nabla v(x)|^2 \, \mathrm{d}x.$$

Thanks to the monotonicity of v, for $t \geq r$ we have

$$\operatorname*{osc}_{\partial B_t} v \geq \operatorname*{osc}_{B_t} v \geq \operatorname*{osc}_{B_r} v$$

and we get the result for a smooth function. The general statement follows by approximation. □

The following is credited to [1] (see Lemma 2.14 for the minimum principle).

Lemma 3.2. (Minimum and Maximum principles for the derivatives) *Let u^ϵ be the solution of* (2.1). *Then*

$$\min_{\partial B_r} \partial_j u^\epsilon \leq \partial_j u^\epsilon(x) \leq \max_{\partial B_r} \partial_j u^\epsilon$$

for all $x \in B_r$, $B_r \subset\subset B_R$ and $j = 1, 2$. In particular, $\partial_j u^\epsilon$ is monotone in the sense of Lebesgue.

Proof. We are going to show that given a constant C, if $\partial_j u^\epsilon \leq C$ (resp. $\partial_j u^\epsilon \geq C$) in ∂B_r then $\partial_j u^\epsilon \leq C$ (resp. $\partial_j u^\epsilon \geq C$) in B_r. Let $\phi^\pm = 1_{B_r}(\partial_j u^\epsilon - C)^\pm = 1_{B_r} \max\{\pm(\partial_j u^\epsilon - C), 0\}$ in the equation satisfied by the derivative (2.3). Since u^ϵ is smooth and $\partial_j u^\epsilon \geq C$ (resp. $\partial_j u^\epsilon \leq C$) on ∂B_r we have $\phi^\pm \in W_0^{1,2}(\Omega)$, so they are admissible functions. We get

$$\begin{aligned} 0 &= \sum_{i=1}^{2} \int_{B_r} (\epsilon + |\partial_i u^\epsilon|^2)^{\frac{p-4}{2}} (\epsilon + (p-1)|\partial_i u^\epsilon|^2)\, |\partial_i(\partial_j u^\epsilon - C)^\pm|^2 \,\mathrm{d}x \\ &\geq \epsilon \sum_{i=1}^{2} \int_{B_r} (\epsilon + |\nabla u^\epsilon|^2)^{\frac{p-4}{2}} |\partial_i(\partial_j u^\epsilon - C)^\pm|^2 \,\mathrm{d}x \\ &= \epsilon \int_{B_r} (\epsilon + |\nabla u^\epsilon|^2)^{\frac{p-4}{2}} |\nabla(\partial_j u^\epsilon - C)^\pm|^2 \,\mathrm{d}x. \end{aligned}$$

This implies $(\partial_j u^\epsilon - C)^\pm$ is constant in B_r, and since it is 0 in ∂B_r then $(\partial_j u^\epsilon - C)^\pm = 0$ in B_r. □

4. Proof of the main theorem

Proof of Theorem 1.1. Applying Lemma (3.1) and estimate (2.6) we get for all $r < R/2$

$$(4.1) \qquad (\operatorname*{osc}_{B_r} \partial_j u^\epsilon)^2 \log\left(\frac{R}{r}\right) \leq C \,\|\nabla \partial_j u^\epsilon\|^2_{L^2(B_{R/2})} \leq C \left(\fint_{B_R} |\nabla u|^p \,\mathrm{d}x + \epsilon^{\frac{p}{2}} \right)^{\frac{2}{p}},$$

and hence for all $r < R/2$

$$(4.2) \qquad \operatorname*{osc}_{B_r} \partial_j u^\epsilon \leq C \left(\log\left(\frac{R}{r}\right)\right)^{-\frac{1}{2}} \left(\fint_{B_R} |\nabla u|^p \,\mathrm{d}x + \epsilon^{\frac{p}{2}} \right)^{\frac{1}{p}},$$

where C is a constant independent of ϵ.

Thanks to Proposition (2.3) we can pass to the limit and get (1.3). □

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