EXISTENCE OF WEAK SOLUTIONS FOR THE SCHRÖDINGER EQUATION AND ITS APPLICATION

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Abstract. In this paper, we are concerned with the existence of weak solutions for the Schrödinger equation with sign-changing potential in a smooth cone. For solving the Dirichlet boundary-value problem with respect to the Schrödinger operator, we prove the existence of at least one weak solution using changes of Schrödingerean harmonic measure, the energy estimate method and refined inequality technique. Due to the fact that the nonlinearity is allowed to change sign in our formulation, and the novelty of the boundary conditions, these results are new for discrete and arbitrary time scales. As an application, concentration results are also investigated.

1. Introduction and main results

In this paper, we denote the set $\mathbf{R}_+ \times \Omega$ with the domain Ω on $\mathbf{S}^{n-1}$ by $C_n(\Omega)$, where $\mathbf{S}^{n-1}$ denotes the unit sphere in $\mathbf{R}^n$ ($n \geq 2$). And we call it a cone. As in [7, 17], we also consider the following Schödinger equation

$$(1) \qquad -\triangle u + a(P)u(P) = 0,$$

where $P \in C_n(\Omega)$ and a is a nonnegative radical potential satisfying certain conditions. Fractional Laplacian can be understood as the infinitesimal generator of a stable Lèvy diffusion process, and appears in anomalous diffusions in plasmas, flames propagation and chemical reactions in liquids, population dynamics, geographical fluid dynamics, and American options in finance. Please see [1, 3, 20] for more physical background.

In [19], when a is not a radical potential, using variational method, the authors studied the existences and multiplicity results for the following critical Schödinger equation

$$(2) \qquad (-\triangle)^s u + a(P)u(P) = 0$$

where $P \in C_n(\Omega)$, $0 < s \leq 1$ and the potential a satisfies the following condition

$$(3) \qquad 0 < \inf_{P=(r,\Theta)\in C_n(\Omega)} a(P) < \lim_{r\to\infty} a(P) = a_\infty < \infty.$$

Invoking Ljusternick-schnirelmann category and Morse theory (see [6]), we investigate the existences and multiplicity results for the following problem

$$(4) \qquad \epsilon^{2s}(-\triangle)^s u + a(P)u(P) = f(u),$$

where $P \in C_n(\Omega)$. They required that a has a positive lower bound. In [4], the authors studied (4) via penalization method. They needed a has a positive lower

bound and the following local condition

$$0 < c = \inf_{P \in \Lambda} a(P) < \inf_{P \in \partial \Lambda} a(P) \, ,$$

where Λ is open bounded set of $C_n(\Omega)$. In [11], the author considered the Schödinger equation by Lagrange multiplier method when a has a positive lower bound and

$$\lim_{r \to \infty} a(P) = \infty.$$

When $\epsilon \to 0$, (4) is equal to the following Schödinger equation

$$(-\triangle)^s u + \lambda a(P) v = f(P, u),$$

where $P \in C_n(\Omega)$. However, for

$$f(P, \ u) = V(P)|u|^{p-2}u + W(P)|u|^{q-2}u,$$

where $2 < q < p < 2_s^*$, the case is different. Especially, W is allowed to be sign-changing or negative.

In the present paper, our aim is to investigate, a is sign-changing, λ is large, the existences and concentration results for the Schödinger equation. More precisely, we are devoted to study the following problem

(5) $$(-\triangle)^s u + a(P)u\lambda + ua_0(P) = |u|^{p-2}uV(P) + W(P)|u|^{q-2}u,$$

where $P \in C_n(\Omega)$, λ is large enough and $2 < q < p < 2_s^*$.

It is well known that there exists no results in literatures on (5) if a is sign-changing.

Throughout this paper, we always make the following assumptions:

(I) $a_0 \in L^\infty(C_n(\Omega))$ and $a_0 := \inf_{x \in \mathrm{R}^N} a_0(P) > 0$.
(II) $a \in C(C_n(\Omega), \ \mathbf{R})$ and a is bounded from below.
(III) The set $\{P \in C_n(\Omega) \colon a(P) < b\}$ has finite measure and is nonempty, where b is a positive constant.
(IV) Define $\Omega = \mathrm{int} \, a^{-1}(0)$. Then it is easy to see that it has smooth boundary and is nonempty. Meanwhile, $\overline{\Omega} = a^{-1}(0)$.
(V) $V(P) \in L^\infty(C_n(\Omega))$, $V_0 := \inf_{P \in C_n(\Omega)} V(P) > 0$, and there exists a positive constant $C(\lambda)$ satisfying $\|V_0\|_{L}\infty \le C(\lambda)$.
(VI) $W(P) \in L^\infty(C_n(\Omega))$. W is sign-changing or negative and

$$\{P \in C_n(\Omega) \colon W(P) \ge 0\}$$

has finite measure.

Theorem 1.1. *If a changes its sign, then (5) has at least a weak solution u_λ.*

Theorem 1.2. *If $a(P) \ge 0$, then (5) has at least a weak solution u_λ. Furthermore, $u_\lambda \to \overline{u}$ in $H^s(C_n(\Omega))$ as $\lambda \to \infty$, where $\overline{u}$ is a weak solution of*

$$\begin{cases} (-\triangle)^s u + a_0(P)u = |u|^{p-2}uV(P) + W(P)|u|^{q-2}u, & in \ \Omega, \\ u = 0, & on \ \partial\Omega. \end{cases}$$

Remark 1.1. *If W satisfies $W(P) \in L^\infty(\mathbf{R})$, $W_0 := \inf_{x \in C_n(\Omega)} W(P) > 0$. Theorems 1.1 and 1.2 remain true. In fact, we only need to modify our proofs mildly.*

Remark 1.2. *a may be coercive or not coercive.*

In [13], $a(P)$ with (V_1)–(IV) is first referred as the steep well potential in the study of nonlinear Schrödinger equation. For more details, the reader is referred to the papers [9, 10, 23, 22], to the book [15] and to the excellent article [18]. Recently,

Properties of weak solutions for the stationary Schrödinger equation with radical potential in a smooth cone and cylinder were studied in [13, 12, 11].

We should point out that we need to reestablish the eigenvalue problem of fractional Schrödinger. And during the research, there are some technical difficulties to recover compact condition. c_λ is uniformly bounded about λ is used to recover compactness, so we need a elaborate estimate since it is different from Schrödinger equations. It is very interesting that the initial thought in our present paper is to study concentration results when V is sign-changing, but we fail since it seems to need $a(P) \geq 0$ in (5) to obtain $\{u_\lambda\}$ is bounded in E.

Denote Lebesgue measure of A by $|A|$. $|\cdot|_p$ is the usual norm of $L^p(C_n(\Omega))$. C and C_i are positive constants, which may be different from line to line. At first, we recall $(-\triangle)^s \colon S \to L^2(C_n(\Omega))$ is defined by (see [24])

$$(-\triangle)^s u(P) = \int_{C_n(\Omega)} \frac{u(P) - u(Q)}{|P - Q|^{n+2s}} \, dQ,$$

where S is be the Schwartz space of rapidly decaying C^∞ functions in $C_n(\Omega)$. The Sobolev space $H^S(C_n(\Omega))$ is defined as

$$H^s(C_n(\Omega)) = \left\{ u \in L^2(C_n(\Omega)) \, \frac{|u(P) - u(Q)|^2}{|P - Q|^{n+2s}} \in L^1(C_n(\Omega) \times C_n(\Omega)) \right\}$$

endowed

$$(u, \, v)_{H^s} = \int_{C_n(\Omega)} \left((-\triangle)^{\frac{s}{2}} u (-\triangle)^{\frac{s}{2}} v + uv \right) dP$$

and norm

$$\|u\|_{H^s}^2 = \int_{C_n(\Omega)} (|(-\triangle)^s 2u|^2 + u^2) \, dP.$$

Let

$$E = \{ u \in H^s(C_n(\Omega)) \colon \int_{C_n(\Omega)} a^+(P) u^2 \, dP < \infty \},$$

be equipped with

$$(u, \, v) = \int_{C_n(\Omega)} \left((-\triangle)^{\frac{s}{2}} u (-\triangle)^{\frac{s}{2}} v + a^+(P) uv \right) dP$$

and

$$\|u\| = (u, \, u)^{\frac{1}{2}}.$$

Fixed $\lambda > 0$,

$$(u, \, v)_\lambda = \int_{1 C_n(\Omega)} \left((-\triangle)^{\frac{s}{2}} u (-\triangle)^{\frac{s}{2}} v + \lambda a^+(P) uv \right) dP, \quad \|u\|_\lambda = (u, \, u)_\lambda^{\frac{1}{2}}$$

Set $E_\lambda = (E, \, \|\cdot\|_\lambda)$. Clearly, $\|u\| \leq \|u\|_\lambda$.

Lemma 1.1. $E \hookrightarrow H^S(C_n(\Omega)) \hookrightarrow L^t(C_n(\Omega))$ *for* $t \in [2, 2_{s*}]$ *and* $E \hookrightarrow L^t_{\mathrm{loc}}(C_n(\Omega))$ *for* $t \in [1, 2_s*)$ *is compact.*

Proof. Noting that (III) , it holds that $a^+(P) \not\equiv 0$. Denote

$$A(R) := \{ P = (r, \Theta) \in C_n(\Omega), \, R < r, \, b \leq a^+(P) \},$$
$$B(R) := \{ P = (r, \Theta) \in C_n(\Omega), \, R < r, \, b > a^+(P) \}.$$

Obviously,

$$\int_{A(R)} u^2 \, dP \leq \frac{1}{b} \int_{C_n(\Omega)} u^2 \, dP.$$

Since Hölder inequality and (4) of [14], jointly with $|B(R)| < \infty$, we obtain

$$\int_{B(R)} u^2 \, dP \le C \int_{C_n(\Omega)} |(-\triangle)^{\frac{s}{2}} u|^2 \, dP.$$

Similarly,

$$\int_{C_n(\Omega;(0,R])} u^2 \, dP \le C \int_{C_n(\Omega)} |(-\triangle)^{\frac{s}{2}} u|^2 \, dP.$$

Therefore $E \hookrightarrow H^S(C_n(\Omega))$. It is well known that $H^S(C_n(\Omega)) \hookrightarrow L^p(C_n(\Omega))$ is continuous. The compact embedding follows from Rellich compactness theorem. $\square$

The functional of (5) can be written as follows

(6)
$$I_\lambda(u) = \frac{1}{2} \int_{C_n(\Omega)} (|(-\triangle)^{\frac{s}{2}} u|^2 + \lambda a(P) u^2 + a_0(P) u^2) \, dP$$
$$- \frac{1}{p} \int_{C_n(\Omega)} V(P)|u|^p \, dP - \frac{1}{q} \int_{C_n(\Omega)} W(P)|u|^p \, dP.$$

Clearly, using Lemma 1.1 and Lebesgue dominated theorem, I_λ is C^1 on E. $u \in H^S(C_n(\Omega))$ is a weak solution of (5) if and only if $I'_\lambda(u) = 0$. Since a is sign-changing, the quadric form

$$a_\lambda(u, \, u) \int_{C_n(\Omega)} (|(-\triangle)^{\frac{s}{2}} u|^2 + \lambda a(P) u^2) \, dP$$

is indefinite. Denote

$$D := \{u \in E \colon \operatorname{supp} u \subset a^{-1}([0, \, \infty])\}$$

and

(7)
$$F = \overline{D}^{\|\cdot\|_\lambda} = \overline{\{u \in E \colon \operatorname{supp} u \subset a^{-1}([0, \infty])\}}^{\|\cdot\|_\lambda},$$

which means that F is closure of D under the norm $\|\cdot\|_\lambda$. According to [16, Theorem 12.4], there exists a closed subspace M such that $E_\lambda = F \oplus M$. If $a(P) \ge 0$, then $E = F$.

Consider the bilinear form

$$a_\lambda(u, \, v) = \int_{C_n(\Omega)} ((-\triangle)^{\frac{s}{2}} u(-\triangle)^{\frac{s}{2}} v + \lambda a(P) uv) \, dP,$$

and the eigenvalue problem

(8)
$$(-\triangle)^s u + \lambda a^+(P) u = \alpha \lambda V^-(P) u,$$

where $u \in M$.

Let $A := \operatorname{supp} a^-$. Noting that (III), it holds $|A| < \infty$. It is easy to check that there is $C > 0$, such that for any $u \in M$,

(9)
$$\int_{C_n(\Omega)} (|(-\triangle)^{\frac{s}{2}} u|^2 + \lambda a^+(P) u^2) \, dP \ge C \int_A \lambda a^-(P) u^2 \, dP,$$

(10)
$$W(u) := \int_{C_n(\Omega)} (|(-\triangle)^{\frac{s}{2}} u|^2 + \lambda a^+(P) u^2) \, dP.$$

Lemma 1.2.

$$\alpha_1(\lambda) = \inf_{0 \ne u \in M} \frac{W(u)}{\int_A \lambda a^-(P) u^2 \, dP} > 0$$

is achieved and $\alpha_1(\lambda)$ is the smallest eigenvalue of (8).

Proof. The proof is standard (see [8, p. 74]). The difference is that M is not the whole space E_λ. But M is convex closed set. Let $\{u_k\} \subset M$ be a minimizing sequence. In view of M is convex closed set, there exists $u \in M$ such that $u_k \rightharpoonup u$ in E_λ. Combining with compact embedding Lemma 1.1, similarly to [8], we obtain the desired conclusions. □

Similarly to [8, p. 80] (see also [5, p. 13]), $\lambda > 0$, we can define $\alpha_j(\lambda)$, $j = 2, 3, \ldots$. The following results are essentially known.

Proposition 1. *Each eigen-subspace is finite dimensional.*

Proposition 2. *Fixed $\lambda > 0$, $\alpha_j(\lambda) \xrightarrow{j \to \infty} 0$.*

Proposition 3. *The eigenfunctions for different eigenvalue are orthogonal in E_λ.*

Lemma 1.3. *Fixed j, $a_j(\lambda) \xrightarrow{\lambda \to \infty} 0$.*

Proof. Here we may always assume $j \geq 2$. Let $u_i \in M$ be the corresponding eigenfunctions of $\alpha_i(\lambda)$, $i = 1, 2, \ldots, j - l)$. Denote $V_{j-1} = \mathrm{span}\{u_i, \ i = 1, 2, \ldots, j - 1\}$. $V_{j-1}^\perp$ denotes the orthogonal complement of V_{j-1} in L^2. Let $u \in C_0^\infty(C_n(\Omega))$ such that $\mathrm{supp}\, u \subset \mathrm{supp}\, a^-$, $\mathrm{supp}\, u \cap \mathrm{supp}\, u_i = \emptyset$, $1 \leq i \leq j - 1$. Thus it holds that

$$\alpha_j(\lambda) \leq \frac{\int_{C_n(\Omega)} |\nabla u|^2}{\lambda \int_{C_n(\Omega)} a^-(P)u^2} \to 0 \quad \text{as } \lambda \to \infty. \qquad \square$$

It follows that there exists $\Lambda > 0$ satisfying

$$(11) \qquad \hat{E}_\lambda := \mathrm{span}\{e_j\, \alpha_j(\lambda) \leq 1\}$$

when $\lambda > \Lambda$ from Lemma 1.3 is not empty and $a_\lambda(u, u)$ is negative semidefinite in $\hat{E}_\lambda$, where e_j is the corresponding eigenfunctions of $\alpha_j(\lambda)$. Set $E_\lambda^+ := \mathrm{span}\{e_j\, \alpha_j(\lambda) > 1\}$. In the light of Propositions 1 and 3, $E_\lambda = M \oplus F = \hat{E}_\lambda \oplus E_\lambda^+ \oplus F$ and $\dim E_\lambda < \infty$.

2. Proof of Theorem 1.1

We shall prove that I_λ satisfies linking geometric construction of [21, Theorem 2.1.2].

Lemma 2.1. *For each $\lambda > \Lambda$, there exist $\rho_\lambda > 0$ and $\kappa_\lambda > 0$ such that*

$$I_\lambda(u) \geq \kappa_\lambda,$$

for all $u \in E_\lambda^+ \oplus F$ with $\|u\|_\lambda = \rho_\lambda$.

Proof. By the definition of E_λ^+, there is $\delta_\lambda > 0$ satsifying

$$a_\lambda(u, u) \geq \delta_\lambda \|u\|_\lambda^2$$

for $u \in E_\lambda^+$ and

$$a_\lambda(u, u) = \|u\|_\lambda^2,$$

where $u \in F$. Thus for $u = v + w \in E_\lambda^+ \oplus F$, invoking Sobolev embedding, we get

$$I_\lambda(u) = \frac{1}{2}a_\lambda(v, v) + \frac{1}{2}a_\lambda(w, w) + \frac{1}{2}\int_{C_n(\Omega)} a_0(P)u^2\, dP$$

$$- \frac{1}{p}\int_{C_n(\Omega)} V(P)|u|^p\, dP - \frac{1}{q}\int_{C_n(\Omega)} W(P)|u|^q\, dP$$

$$(12) \qquad \geq \frac{1}{2}\min\{\delta_\lambda,\ 1\}\|u\|_\lambda^2 - C_1\|u\|_\lambda^p - C_2\|u\|_\lambda^q.$$

Taking $\epsilon > 0, \rho_\lambda > 0$ and $\kappa_\lambda > 0$ small enough, we have the desired conclusion.   □

By (IV), we take $e_0 \in C_0^\infty(\Omega)$, then $e_0 \in F$.

Lemma 2.2. *For each $\lambda > \Lambda$, there exists $R_\lambda > 0$ such that*

$$\sup_{u \in \partial Q} I_\lambda(u) < \kappa_\lambda,$$

where

$$Q = \{u = v + te_0 : v \in \hat{E}_\lambda,\ t \geq 0,\ \|u\|_\lambda \leq R_\lambda\}.$$

Proof. In view of (V) and (VI) , there exist $C_1 > 0, C_2 > 0$ such that

$$\frac{1}{p}V(P)|u|^p + \frac{1}{q}W(P)|u|^q \geq C_1|u|^p - C_2|t|^q.$$

If $u = v + w \in \hat{E}_\lambda \oplus Re_0$, noting that

$$a_\lambda(u,\ u) = a_\lambda(v,\ v) + a_\lambda(w,\ w) \leq \|u\|_\lambda^2,$$

jointly with all the norms on finite dimensional space are equivalent and $q < p$, we obtain

$$I_\lambda(u) \leq C\|u\|_\lambda^2 + C_3\|u\|_\lambda^q - C_4\|u\|_\lambda^p \xrightarrow{\|u\|_\lambda \to \infty} -\infty.$$

Therefore, there exists $R_\lambda > 0$ such that $I_\lambda(u) \leq 0$ for $u \in \hat{E}_\lambda \oplus Re_0$ with $\|u\|_\lambda = R_{\lambda_\Lambda}$. If $u \in \hat{E}_\lambda$, then we have

$$I_\lambda(u) \leq \frac{1}{2}\int_{C_n(\Omega)} a_0(P)u^2\,dP + C_3\|u\|_\lambda^q - C_4\|u\|_\lambda^p.$$

So we also obtain the conclusion.   □

According to Lemmas 2.1 and 2.2, I_λ has a sequence $\{u_n\} \subset E_\lambda$, i.e.,

$$(13) \qquad\qquad I_\lambda(u_n) \to c,\ (1 + \|u_n\|_\lambda)I_\lambda'(u_n) \to 0 \quad \text{in } E_\lambda^*.$$

Lemma 2.3. *$\{u_n\}$ is bounded (dependent on λ) in E_λ.*

Proof. For n large enough, we have

$$(14)\quad
\begin{aligned}
I_\lambda(u_n) - \frac{1}{q}\langle I_\lambda'(u_n), u_n\rangle &= \left(\frac{1}{2} - \frac{1}{q}\right)\|u_n\|_\lambda^2 - \left(\frac{1}{2} - \frac{1}{q}\right)\int_{C_n(\Omega)} \lambda a^-(P)u_n^2\,dP \\
&\quad + \left(\frac{1}{2} - \frac{1}{q}\right)\int_{C_n(\Omega)} a_0(P)u_n^2\,dP \\
&\quad + \left(\frac{1}{q} - \frac{1}{p}\right)\int_{C_n(\Omega)} V(P)|u_n|^p\,dP = c + 1.
\end{aligned}$$

Thus, combining with (II), we get

$$\|u_n\|_\lambda^2 \leq C_1\lambda \int_{C_n(\Omega)} u_n^2\,dP + C_2.$$

Thus it suffices to prove that $\{u_n\}$ is bounded in $L^2(C_n(\Omega))$. If

$$\int_{C_n(\Omega)} u_n^2\,dP \xrightarrow{n \to \infty} \infty.$$

Denote $v_n = \dfrac{u_n}{|u_n|_2}$, then $|v_n|_2 = 1$. By (14), we have

$$(15) \qquad \|v_n\|_\lambda^2 + \int_{C_n(\Omega)} a_0(P)v_n^2\,dP - \int_{C_n(\Omega)} \lambda a^-(P)v_n^2\,dP \leq \frac{C}{|u_n|_2^2}.$$

Consequently, $\{\|v_n\|_\lambda^2\}$ is bounded and

$$(16) \qquad \int_{C_n(\Omega)} a(P)v_n^2 \, dP \le \frac{C}{|u_n|_2^2} \xrightarrow{n\to\infty} 0.$$

Up to a subsequence we can assume $v_n \rightharpoonup v$ in E_λ. It is easy to obtain $v = 0$ from (15) and Fatou's lemma. Thus, we have

$$\int_{C_n(\Omega)} a(P)v_n^2 \, dP = \int_{\{x:\, a(P)\ge b\}} a(P)v_n^2 \, dP + \int_{\{x:\, a(P)<b\}} a(P)v_n^2 \, dP$$

$$\ge b\left(1 - \int_{\{x:\, a(P)<b\}} a(P)v_n^2 \, dP\right) + o_n(1) \ge b + o_n(1) > 0,$$

which contracts with (16). $\qquad\square$

Next lemma will be used in Lemma 2.5 to prove $(C)_c$ condition.

Lemma 2.4. c_λ *is uniformly bounded about* λ.

Proof. In the light of [21, Theorem 2.1.2], $c_\lambda \in [\kappa_\kappa,\ \sup_{u\in Q} I_\lambda(u)]$. We only need to prove $\sup_{u\in Q} I_\lambda(u)$ has a positive bound from above independent of λ. Set

$$J_\lambda(u) := \frac{1}{2}a_\lambda(u,\ u) + \frac{1}{2}\int_{C_n(\Omega)} a_0(P)u^2 \, dP - C_1 \int_\Omega |u|^q \, dP,$$

where $u \in E_\lambda$.

A simple estimate, $I_\lambda(u) \le J_\lambda(u)$. It follows that there exists $r_\eta > 0$ satisfying

$$\begin{cases} C_1|u|^q \ge \frac{1}{2}\eta u^2, & |u| \ge r_\eta, \\ C_1|u|^q \le \frac{1}{2}\eta u^2, & |u| \le r_\eta, \end{cases}$$

for any $\eta > 0$. Let $u = v + w \in \hat{E}_\lambda \oplus Re_0$. It holds that

$$J_\lambda(u) \le \frac{1}{2}\int_{C_n(\Omega)} |(-\triangle)^s 2w|^2 \, dP + \frac{1}{2}\int_{C_n(\Omega)} a_0(P)u^2 \, dP - \frac{1}{2}\eta \int_\Omega u^2 \, dP$$

$$(17) \qquad + \int_{\{x\in\Omega:\, |u(P)|\le r_\eta\}} \left(\frac{1}{2}\eta u^2 - C_1|u|^q\right) dP$$

$$\le \frac{1}{2}\int_{C_n(\Omega)} |(-\triangle)^s 2w|^2 \, dP + C\|V_0\|_{L^\infty}\|u\|_\lambda^2 - \frac{\eta}{2}\int_\Omega u^2 \, dP + C(\eta).$$

In view of $e_0 \in C_0^\infty(\Omega)$, we get

$$(18) \quad \int_{C_n(\Omega)} |(-\triangle)^{\frac{s}{2}}w|^2 \, dP = a_\lambda(u,\ w) = \int_\Omega u(-\triangle)^s w \, dP \le |(-\triangle)^s w|_{L^2(\Omega)} |u|_{L^2(\Omega)}.$$

Let $(\varphi_k,\ \mu_k)$ be the eigenfunctions and eigenvalues of $-\triangle$ in Ω with Dirichlet boundary data. According to [2, Lemmas 3.4 and 3.5], it holds

$$(19) \qquad \|(-\triangle)^s w\|_{L^2(\Omega)} = \left(\sum_{k=1}^\infty a_k^2 \mu_k^s\right)^{\frac{1}{2}} < \infty,$$

where

$$a_k = \int_\Omega w\varphi_k \, dP.$$

In view of $0 < \mu_1 \le \mu_2 \cdots \le \mu_k \cdots$ and $\mu_k \to \infty$ as $k \to \infty$, we get

$$(20) \qquad \|(-\triangle)^{\frac{s}{2}}w\|_{L^2(\Omega)} = \left(\sum_{k=1}^{\infty} a_k^2 \mu_k^{\frac{s}{2}}\right) \frac{1}{2} < \infty.$$

So

$$\|(-\triangle)^s w\|_{L^2(\Omega)} \le C_0 \|(-\triangle)^{\frac{s}{2}}w\|_{L^2(\Omega)},$$

where C_0 is only dependent with e_0.

Noting that (18)–(20) , using Young inequality, it is easy to check

$$(21) \qquad \int_{C_n(\Omega)} |(-\triangle)^{\frac{s}{2}}w|^2 \, dP \le \frac{2}{\eta}C_0^2 \|(-\triangle)^{\frac{s}{2}}w\|_{L^2(\Omega)}^2 + \frac{\eta}{2}\|u\|_{L^2(\Omega)}^2.$$

Let $\eta \ge 4C_0^2$. Then

$$\int_{C_n(\Omega)} |(-\triangle)^{\frac{s}{2}}w|^2 \, \|u\|_{L^2(\Omega)}^2.$$

If $\|V_0\|_{L^\infty} \le \dfrac{2}{CR_\lambda^2}$, according to (17), we have $J_\lambda(u) \le C(\eta) + 1$. $\qquad \square$

Lemma 2.5. *Let M be any positive constant. There is $\Lambda = \Lambda(M) > 0$ such that $(C)_c$ sequence $\{u_n\}$ satisfies $u_n \to u$ in E_λ, where u is a weak critical point of I_λ or I_λ satisfies $(C)_c$ condition, where $c \le M$.*

Proof. In view of Lemma 2.3, if necessary going to a subsequence, $u_n \rightharpoonup u$ in E_λ. Since V is sign-changing, we need to divide into two cases:

Case (i). $I_\lambda(u) < 0$.

Case (ii). $I_\lambda(u) \ge 0$.

If (i) holds, then it is easy to see that u is weak and the proof is completed. If (ii) holds, then we will show that $u_n \to u$ in E_λ. Let $v_n = u_n - u$. By (iii),

$$(22) \qquad \int_{C_n(\Omega)} v_n^2 \, dP = \int_{\{x: \, a(P) \ge b\}}^{v_n^2} dP + \int_{\{x: \, a(P) < b\}}^{v_n} dP \le \frac{1}{\lambda b}\|v_n\|_\lambda^2 + o_n(1).$$

Thus, it follows from interpolation inequality and Sobolev inequalities that

$$(23) \qquad |v_n|_p \le |v_n|_2^\sigma |v_n|_{2_s^*}^{1-\sigma} \le d|v_n|_2^\sigma |(-\triangle)^{\frac{1}{2}}v_n|_2^{1-\sigma} \le d(\lambda b)^{-\frac{\sigma}{2}}\|v_n\|_\lambda + o_n(1),$$

where $0 < \sigma < 1$, d is a constant. We have

$$(24) \qquad I_\lambda(v_n) = I_\lambda(u_n) - I_\lambda(u) + o_n(1), \quad I_\lambda'(v_n) = I_\lambda'(u_n) + o_n(1)$$

from Brezis–Lieb lemma.

A direct computation, there holds that

$$\left(\frac{1}{q} - \frac{1}{p}\right) \int_{C_n(\Omega)} V(P)|v_n|^p \, dP \le I_\lambda(v_n) - \frac{1}{q}\langle I_\lambda'(v_n), \, v_n\rangle$$

$$= c - I_\lambda(u) + o_n(1) \le M + o_n(1).$$

Therefore,

$$(25) \qquad |v_n|_p^p \le \frac{Mpq}{V_0(p-q)} + o_n(1).$$

Noting that condition (VI) and

$$\int_{C_n(\Omega)} \lambda a^-(P)v_n^2 \, dP = o_n(1),$$

it follows from (22)–(25) that

$$o_n(1) = \langle I'_\lambda(v_n),\ v_n \rangle \geq \|v_n\|_\lambda^2 - C_1 |v_n|_p^p + o_n(1)$$

(26)
$$\geq \left(1 - C_1 \left(\frac{Mpq}{V_0(p-q)} \right)^{\frac{p-2}{p}} \frac{d^2}{(\lambda b)^\sigma} \right) \|v_n\|_\lambda^2 + o_n(1).$$

Choosing A large enough, we get $v_n \to 0$ in E_λ. $\qquad\square$

3. Proof of Theorem 1.2

According to our assumptions on Theorem 1.2, we obtain that I_λ has mountain pass geometry. Invoking [14, Theorem 2.2], I_λ has a C_c sequence. Note that $\hat{E}_\lambda = \{0\}$ does not influence the results of Lemmas 2.3 and 2.5. Similarly to Lemmas 2.3 and 2.5, we obtain a weak critical point u_λ for I_λ with $I_\lambda(u_\lambda) \in [\kappa,\ C_0]$ for λ large enough where κ and C_0 are independent of λ.

A direct computation, we get

$$c_{\lambda_n} = I_{\lambda_n}(u_n) - \frac{1}{q}\langle I'_{\lambda_n}(u_n),\ u_n \rangle$$

(27)
$$= \left(\frac{1}{2} - \frac{1}{q} \right) \|u_n\|_{\lambda_n}^2 + \left(\frac{1}{2} - \frac{1}{q} \right) \int_{C_n(\Omega)} a_0(P) u_n^2\, dP$$

$$+ \left(\frac{1}{q} - \frac{1}{p} \right) \int_{1C_n(\Omega)} V(P)|u_n|^p\, dP \geq \left(\frac{1}{2} - \frac{1}{\theta} \right) \|u_n\|_{\lambda_n}^2.$$

It follows from (27) that $\{u_n\}$ is bounded in E. After extracting a subsequence, we may assume $u_n \rightharpoonup \overline{u}$ in E. We have

(28)
$$\int_{C_n(\Omega)} a(P)\overline{u}^2\, dP \leq \liminf_{n\to\infty} \int_{C_n(\Omega)} a(P)\overline{u}_n^2\, dP \leq \liminf_{n\to\infty} \frac{\|u_n\|_{\lambda_n}^2}{\lambda_n} = 0$$

from Fatou's Lemma.

So $\overline{u} = 0$ a.e. in $C_n(\Omega)\backslash a^{-1}(0)$, $\overline{u} \in H_0^s(\Omega)$ by (IV). Since for any $\varphi \in C_0^\infty(\Omega)$, $\langle I'_{\lambda_n}(u_n), \varphi \rangle = 0$, it is easy to check $\overline{u}$ is a weak solution of (5). To complete the proof, we only need to prove $u_n \to \overline{u}$ in E and $\overline{u} \neq 0$. The proof is almost standard and we refer the readers to [14, Section 3.5] with slight modification. Here we sketch the proof for the readers' convenience.

Step 1. Using Lions vanishing [21, Lemma 12.1], we have $u_n \to \overline{u}$ in $L^t(C_n(\Omega))$ for $2 < t < 2_s^*$.

Step 2. By $\langle I'_{\lambda_n}(u_n), u_n \rangle = \langle I'_{\lambda_n}(u_n), \hat{u} \rangle$ we get $u_n \to \overline{u}$ in E.

Step 3. $\overline{u} \neq 0$ is obtained since $u_n \neq 0$ and $u_n \to \overline{u}$ in E.

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