Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 45, 2020, 53–66

A NOTE ON QUASISYMMETRIC HOMEOMORPHISMS

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Abstract. By means of some integral operators and kernel functions, we characterize when a sense preserving homeomorphism h on the unit circle S^1 is quasisymmetric, symmetric or p-integrable asymptotic affine. As an application, we use these results to characterize the pull-back operator, induced by a quasisymmetric homeomorphism on S^1.

1. Introduction and results

Let $\Delta = \{z\colon |z| < 1\}$ be the unit disk in the complex plane $\mathbf{C}$, $\Delta^* = \overline{\mathbf{C}}\backslash\overline{\Delta}$ and $S^1 = \{z\colon |z| = 1\}$. We say that a sense preserving homeomorphism h on the unit circle S^1 is quasisymmetric if there is some $M > 0$ such that

$$\frac{1}{M} \le \left|\frac{h(I_1)}{h(I_2)}\right| \le M$$

for all pairs of adjacent arcs on the unit circle S^1 with the same arc length $|I_1| = |I_2| \le \pi$. Beurling and Ahlfors gave a very important characterization of quasisymmetric homeomorphism (see [BA]).

Theorem 1.1. [BA] *A sense preserving self-homeomorphism h on the unit circle S^1 is quasisymmetric if and only if there exists some quasiconformal homeomorphism of Δ onto itself which has boundary value h.*

In [BA], Beurling and Ahlfors constructed a quasiconformal extension of h, which is called Beurling–Ahlfors extension. There is also another quasiconformal extension, called Douady–Earle extension, of h to the unit disk which is conformally invariant (see [DE]).

Hu and Shen [HS] introduced a integral operator T_h^- which is induced by the following kernel function

$$\phi_h(\zeta, z) = \frac{1}{2\pi i}\int_{S^1} \frac{h(w)}{(1-\zeta w)^2(1-zh(w))}\,dw, \quad (\zeta, z) \in \Delta\times\Delta, \tag{1}$$

where h is a sense preserving homeomorphism on the unit circle S^1. The function ϕ_h is holomorphic and also appeared in [Cui]. The integral operator T_h^- is defined as for any holomorphic function ψ in Δ,

$$T_h^-\psi(\zeta) = \frac{1}{\pi}\iint_\Delta \phi_h(\zeta, z)\psi(\bar z)\,dx\,dy, \quad \zeta\in\Delta. \tag{2}$$

https://doi.org/10.5186/aasfm.2020.4502
2010 Mathematics Subject Classification: Primary 30C62; Secondary 30F60, 32G15.
Key words: Quasisymmetric homeomorphisms, Douady–Earle extension, integral operator, kernel functions, pull-back operator.

Let $p \geq 2$. The Banach space A^p consists of all holomorphic functions ϕ in the unit disk Δ with finite norm

$$\|\phi\|_{A^p} = \left(\frac{1}{\pi}\iint_{\Delta} |\phi(z)|^p(1-|z|^2)^{p-2}\,dx\,dy\right)^{\frac{1}{p}} < \infty. \tag{3}$$

When $p = 2$, A^2 is a Hilbert space with inner product defined as

$$\langle \phi, \psi\rangle = \frac{1}{\pi}\iint_{\Delta} \phi(z)\overline{\psi(z)}\,dx\,dy. \tag{4}$$

We use $SH_0(S^1)$ to denote the set of all sense preserving homeomorphisms h on the unit circle S^1, normalized by

$$\frac{1}{2\pi}\int_{S^1} h(\zeta)|d\zeta| = 0. \tag{5}$$

For $w \in \Delta$, consider the Möbius transformation

$$\Psi_w(\zeta) = \frac{\zeta - w}{1-\overline{w}\zeta}, \quad \zeta \in \Delta. \tag{6}$$

Let $h \in SH_0(S^1)$ and H^w be the Poisson extension of $\Psi_w \circ h$. It is known that H^w is a homeomorphism of $\overline{\Delta}$ onto $\overline{\Delta}$ for fix $w \in \Delta$ (see [Cho]). Let $p \geq 2$, we define kernel function $\mathfrak{H}_{h,p}$ as

$$\mathfrak{H}_{h,p}(w) = \left(\frac{1}{2\pi}\iint_{\Delta} |\overline{\partial} H^w(z)|^p(1-|z|^2)^{p-2}\,dx\,dy\right)^{1/p}, \quad w \in \Delta. \tag{7}$$

The function $\mathfrak{H}_{h,2}$ already appeared in [Cui] and [HS].

We also consider the kernel function $\Phi_{h,p}$, which is defined as

$$\Phi_{h,p}(w) = \left(\frac{1}{2\pi}\iint_{\Delta} |\phi_h(z,w)|^p(1-|z|^2)^{p-2}\,dx\,dy\right)^{1/p}, \quad w \in \Delta, \tag{8}$$

where $p \geq 2$ and $h \in SH_0(S^1)$. The function $\Phi_{h,2}$ has been used to study Teichmüller theory in [HS, SW, TS].

Hu and Shen [HS] proved the following result.

Theorem 1.2. [HS] *Let $h \in SH_0(S^1)$. If h is a quasisymmetric homeomorphism, then the integral operator $T_h^-\colon A^2 \to A^2$ is bounded.*

In this paper, we shall prove that the converse of Theorem 1.2 is also true and therefore obtain a characterization of quasisymmetric homeomorphism. Indeed, we prove the following general case.

Theorem 1.3. *Let $h \in SH_0(S^1)$ and $p \geq 2$.Then the following statements are equivalent.*

(i) *h is a quasisymmetric homeomorphism;*
(ii) *The integral operator $T_h^-\colon A^p \to A^p$ is bounded;*
(iii) $\sup_{w\in\Delta} \mathfrak{H}_{h,p}(w) < \infty$;
(iv) $\sup_{w\in\Delta}(1-|w|^2)\Phi_{h,p}(w) < \infty$.

A sense preserving homeomorphism h is called a symmetric homeomorphism if for any pair of adjacent sub-intervals I_1 and I_2 with $|I_1| = |I_2|$ in S^1, it holds that

$$\frac{|h(I_1)|}{|h(I_2)|} = 1 + o(1), \quad |I_1| = |I_2| \to 0^+. \tag{9}$$

The following result, due to Gardiner and Sullivan [GS], characterizes when a quasisymmetric homeomorphism h is symmetric.

Theorem 1.4. [GS] *A quasisymmetric homeomorphism h is symmetric if and only if h has a quasiconformal extension f to the unit disk so that its complex dilataion $\mu = \partial_{\bar{z}} f / \partial_z f$ satisfies the property that $\mu(z) \to 0$ as $|z| \to 1$.*

In terms of the integral operator T_h^-, kernel functions $\mathfrak{H}_{h,p}$ and $\Phi_{h,p}$, we obtain the following

Theorem 1.5. *Let $p \geq 2$ and h be a quasisymmetric homeomorphism on the unit circle S^1, normalized by (5). Then the following statements are equivalent.*

(I) *h is a symmetric homeomorphism;*
(II) *The integral operator $T_h^- : A^p \to A^p$ is compact;*
(III) $\lim_{|w|\to 1} \mathfrak{H}_{h,p}(w) = 0$;
(IV) $\lim_{|w|\to 1} (1 - |w^2|) \Phi_{h,p}(w) = 0$.

Let $p \geq 2$, the Besov space $B_p(S^1)$ on the unit circle S^1 is the collection of measurable functions f (modulo functions which are constant almost everywhere) for which the norm

$$\|f\|_p = \left(\int_0^{2\pi} \int_0^{2\pi} \frac{|f(e^{it}) - f(e^{i\theta})|^p}{|t - \theta|^2} \, dt \, d\theta \right)^{1/p} \tag{10}$$

is finite (see [Tr]). It is clear that the Besov space $B_p(S^1)$ is a Banach space and $B_p(S^1) \subset B_q(S^1)$ for $p \leq q$. $B_2(S^1)$ is the classic Sobolev space $H^{1/2}$ which consists of all integrable functions $u \in L^1([0, 2\pi])$ on the unit circle with semi-norm

$$\|u\|_p = \left(\sum_{n=-\infty}^{+\infty} |n| |a_n(u)|^2 \right)^{1/2}, \tag{11}$$

where $a_n(u)$ is the n-th Fourier coefficient of u, namely,

$$a_n(u) = \frac{1}{2\pi} \int_0^{2\pi} u(\theta) e^{-in\theta} \, d\theta. \tag{12}$$

Let $p \geq 2$. Recall that a sense preserving homeomorphism h on the unit circle S^1 is p-integrable asymptotic affine homeomorphism if h has a quasiconformal extension f to the unit disk Δ whose complex dilatation μ satisfies

$$\iint_\Delta \frac{|\mu(z)|^p}{(1 - |z|^2)^2} \, dx \, dy < \infty. \tag{13}$$

The 2-integrable asymptotic affine homeomorphism was first introduced by Cui [Cui] and was much investigated in recent years (see [RSS1, RSS2, Shen, TT, STW]). For $p \geq 2$, the p-integrable asymptotic affine homeomorphism was first introduced and investigated by Guo [Guo] (see also [MY, Tang, Ya, HWS, TFS, TS]).

The authors proved the following result, which gives a intrinsic characteriztion of p-integrable asymptotic affine homeomorphism without using quasiconformal extension (see [TS]).

Theorem 1.6. [TS] *Let $p \geq 2$ and h be a quasisymmetric homeomorphism, normalized by (5), on the unit circle S^1.Then h is a p-integrable asymptotic affine homeomorphism if and only if h is absolutely continuous (with respect to the arc-length measure) such that $\log h'$ belongs to $B_p(S^1)$.*

It should be pointed out that the case when $p = 2$ of Theorem 1.6 was proved by Shen in [Shen].

We obtain a new characterization of p-integrable asymptotic affine homeomorphism by means of kernel functions $\mathfrak{H}_{h,p}$ and $\Phi_{h,p}$ in this paper.

Theorem 1.7. *Let $p \geq 2$ and h be a quasisymmetric homeomorphism, normalized by (5), on the unit circle S^1. Then the following statements are equivalent.*

(a) *h is a p-integrable asymptotic affine homeomorphism;*
(b) $\iint_\Delta \frac{\mathfrak{H}_{h,p}(w)^p}{(1-|w|^2)^2}\, du\, dv < \infty$;
(c) $\iint_\Delta \Phi_{h,p}(w)^p (1-|w|^2)^{p-2}\, du\, dv < \infty$.

We point out that the condition (c) in Theorem 1.7 is different from that in Theorem 3.4 in [TS], which is

$$\iint_\Delta \Phi_{h,2}(w)^p (1-|w|^2)^{p-2}\, du\, dv < \infty.$$

The operator T_h^- is also related to the pull-back operator T_h which is defined by

$$T_h(f) = f \circ h, \quad f \in B_p(S^1), \tag{14}$$

where $h \in SH_0(S^1)$. Vodop'yanov proved in [Vo] that the homeomorphism for which T_h is a bounded operator on $B_p(S^1)$ $(p \geq 2)$ is precisely quasisymmetric.

Theorem 1.8. [Vo] *Let $p \geq 2$ and $h \in SH_0(S^1)$. Then T_h is a bounded operator on $B_p(S^1)$ if and only if h is a quasisymmetric homeomorphism.*

Bourdaud and Sickel [BS] characterized the homeomorphisms for which T_h is a bounded operator on $B_p(S^1)$ for the case when $1 < p < \infty$. For the case when $p = 2$, i.e., the Sobolev space $H^{1/2}$, Nag and Sullivan gave a different proof of Theorem 1.8 in [NS] and proved that the universal Teichmüller space can be embedded in the universal Siegel period matrix space by means of the pull-back operator T_h (see also [TT]). This operator T_h on $H^{1/2}$ has played an important role in the study of Teichmüller theory (see [HS, TT, SW, NS, Pa, Shen, STW, TS]).

By using Theorem 1.3, we shall give the "if" part of Theorem 1.8 a different proof.

We end this introduction section with the organization of the paper. In section 2, we prove Theorem 1.3 by establishing a relationship between the complex dilatation of the Douady–Earle extension of $h \in SH_0(S^1)$ and the integral operator T_h^-. Section 3 is devoted to the proof of Theorems 1.5 and 1.7. We shall use Theorem 1.3 to study the pull-back operator T_h in section 4.

2. Characterizations of quasisymmetric homeomorphisms

In this section, we shall prove Theorem 1.3. Let us begin with some lemmas. The following result will prove very useful in our proof.

Lemma 2.1. [Zhu] *Suppose that (X, μ) is a measure space and $K(x, y)$ is a nonnegative measurable function on $X \times X$, K is the integral operator with kernel $K(x, y)$, that is*

$$K\varphi(z) = \iint_X K(x, y)\varphi(y)\, d\mu(y).$$

Let $1 < p < \infty$ with $1/p + 1/q = 1$. If there exist positive constant C_1 and C_2 and a positive measurable function h on X such that

$$\iint_X K(x,y)h^q(y)\,d\mu(y) \le C_1 h^q(x)$$

for almost every $x \in X$ and

$$\iint_X K(x,y)h^p(x)\,d\mu(x) \le C_2 h^p(y)$$

for almost every $y \in X$, then K is a bounded operator on $L^p(X, d\mu)$ with norm less then or equal to $C_1^{1/q} C_2^{1/p}$.

See [Zhu] for a proof.

We also need the following integral estimates (see [Zhu]).

Lemma 2.2. [Zhu] *Suppose that $z \in \Delta$, $s > 0$ and $t > -1$. Then there exists constant $C > 0$ so that*

$$\frac{1}{C}\frac{1}{(1-|z|^2)^s} \le \iint_\Delta \frac{(1-|w|^2)^t}{|1-z\bar{w}|^{2+t+s}}\,du\,dv \le C\frac{1}{(1-|z|^2)^s}.$$

Let $h \in SH_0(S^1)$ and

$$F(z,w) = \frac{1}{2\pi}\int_{S^1} \frac{h(\zeta)-w}{1-\overline{w}h(\zeta)}\frac{1-|z|^2}{|\zeta - z|^2}|d\zeta|, \tag{15}$$

where $(z,w) \in \Delta \times \Delta$. The Douady–Earle extension $E(h)$ of h is defined as

$$E(h) = \begin{cases} h(z), & \text{for } z \in S^1, \\ w, & \text{where } F(z,w) = 0 \text{ for } z \in \Delta, \end{cases}$$

(see [DE]).

The following result gives an estimate of the complex dilatation of the Douady–Earle extension $E(h)$ of $h \in SH_0(S^1)$ at the origin, which is needed in our proof of Theorem 1.3.

Lemma 2.3. *Let $h \in SH_0(S^1)$ and ν be the complex dilatation of the Douady–Earle extension $E(h)$ of h. Then there exists a positive constant $C_0 > 0$ such that*

$$\frac{|\nu(0)|^p}{(1-|\nu(0)|^2)^{p/2}} \le C_0 \frac{1}{2\pi}\iint_\Delta |\overline{\partial} H(z)|^p (1-|z|^2)^{p-2}\,dx\,dy. \tag{16}$$

Proof. The Fourier coefficient of h are

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int} h(e^{it})\,dt, \quad n = 0, \pm 1, \pm 2, \cdots.$$

It was shown (see [DE, Po, CZ]) that

$$F_z(0,0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-it} h(e^{it})\,dt = a_1, \tag{17}$$

$$F_{\overline{z}}(0,0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{it} h(e^{it})\,dt = a_{-1}, \tag{18}$$

$$F_w(0,0) = 1, \tag{19}$$

$$F_{\overline{w}}(0,0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(e^{it})^2\,dt = b. \tag{20}$$

For simplicity of notations, we write F_z for $F_z(0,0)$, etc. By a straight forward computation, we get

$$\overline{F}_{\overline{z}} + \overline{F}_{\overline{w}}\overline{E(h)}_{\overline{z}} + \overline{F}_w E(h)_{\overline{z}} = 0,$$
$$F_{\overline{z}} + F_{\overline{w}}\overline{E(h)}_{\overline{z}} + F_w E(h)_{\overline{z}} = 0.$$

Therefore, we have

$$\frac{|\nu(0)|^p}{(1-|\nu(0)|^2)^{p/2}} = \frac{|\overline{a_1}b + a_{-1}|^p}{(|a_1|^2 - |a_{-1}|^2)^{p/2}(1-|b|^2)^{p/2}}, \tag{21}$$

where ν is the complex dilataion of $E(h)$. It is known that if h is a sense-preserving homeomorphism on S^1, then there exists a positive constant $\delta > 0$ so that

$$|a_1|^2 - |a_{-1}|^2 = \delta > 0, \quad 1 - |b|^2 \geq \frac{\delta^2}{4}, \tag{22}$$

(see [Po]). Noting that $|h(\zeta)| = 1$ for $|\zeta| = 1$, we have $|a_1| \leq 1$. Consequently, combining (21) with (22) yields

$$\frac{|\nu(0)|^p}{(1-|\nu(0)|^2)^{p/2}} \leq \frac{\delta^{\frac{3p}{2}}}{2^p}|\overline{a_1}b + a_{-1}|^p \leq \delta^{\frac{3p}{2}}(|a_{-1}|^p + |b|^p). \tag{23}$$

We borrow some ideas from [Cui] to estimate $|a_{-1}|$ and $|b|$. Let H be the Poisson extension of h. Observing that $H(0) = 0$ and using Cauchy–Green formula to the function $zH(z)$, we obtain

$$|a_{-1}| = \left|\frac{1}{2\pi i}\int_{S^1}\frac{\zeta h(\zeta)}{\zeta - 0}d\zeta\right| = \left|\frac{1}{2\pi}\iint_\Delta \overline{\partial}H(z)\,dx\,dy\right|.$$

It follows from the Hölder inequality that there exists a constant $C_1 > 0$ so that

$$\begin{aligned} |a_{-1}|^p &\leq \frac{1}{2\pi}\iint_\Delta |\overline{\partial}H(z)|^p(1-|z|^2)^{p-2}\,dx\,dy\left(\frac{1}{2\pi}\iint_\Delta (1-|z|^2)^{\frac{2-p}{p-1}}\,dx\,dy\right)^{p-1} \\ &\leq C_1\frac{1}{2\pi}\iint_\Delta |\overline{\partial}H(z)|^p(1-|z|^2)^{p-2}\,dx\,dy. \end{aligned} \tag{24}$$

Similarly, using Cauchy–Green formula to the function $H^2(z)$, we deduce that there exists a constant $C_2 > 0$ such that

$$|b| = \left|\frac{1}{2\pi i}\int_{S^1}\frac{h^2(\zeta)}{\zeta - 0}d\zeta\right| = \left|\frac{1}{2\pi}\iint_\Delta \frac{2H(z)}{z}\overline{\partial}H(z)\,dx\,dy\right| \leq C_2\left|\frac{1}{2\pi}\iint_\Delta \overline{\partial}H(z)\,dx\,dy\right|.$$

By using the Hölder inequality agian and arguing similar to (24), we deduce that there exists a constant $C_3 > 0$ so that

$$|b|^p \leq C_3\frac{1}{2\pi}\iint_\Delta |\overline{\partial}H(z)|^p(1-|z|^2)^{p-2}\,dx\,dy. \tag{25}$$

Therefore, it follows from (23), (24) and (25) that there exists a constant $C_4 > 0$ such that

$$\frac{|\nu(0)|^p}{(1-|\nu(0)|^2)^{p/2}} \leq C_4\frac{1}{2\pi}\iint_\Delta |\overline{\partial}H(z)|^p(1-|z|^2)^{p-2}\,dx\,dy. \tag{26}$$

The proof follows. □

We point out that Lemma 2.3 is an extension of Proposition 7 in [CZ], where $p = 2$ and h is assumed to be a quasisymmetric homeomorphism on the unit circle S^1.

Now, we start our proof of Theorem 1.3.

Proof of Theorem 1.3. It is known that

$$\overline{\partial} H^{w_0}(z) = (1-|w_0|^2)\phi_h(\overline{z}, w_0), \tag{27}$$

see [Cui] and [HS]. This shows that (iii) ⇔ (iv). Therefore, it remains to show that (i) ⇒ (ii) ⇒ (iii) ⇒ (i).

We first show that (i) ⇒ (ii). Assume that h is a quasisymmetric homeomorphism on the unit circle S^1, normalized by (5), and f is the Beurling–Ahlfors extension of h into Δ. Let $\zeta \in \Delta$, by a result of [HS], we have

$$T_h^- \psi(\zeta) \;=\; \frac{1}{\pi}\iint_\Delta \frac{\overline{\partial} f(w)\psi(f(w))}{(1-\zeta w)^2}\, du\, dv.$$

Thus,

$$\begin{aligned} \|T_h^-\psi\|_{A^p}^p &= \frac{1}{\pi}\iint_\Delta \left| \frac{1}{\pi}\iint_\Delta \frac{\overline{\partial} f(w)\psi(f(w))}{(1-\zeta w)^2}\, du\, dv\right|^p (1-|\zeta|^2)^{p-2}\, d\xi\, d\eta \\ &\le \frac{1}{\pi^{p+1}}\iint_\Delta \left|\iint_\Delta \frac{(1-|w|^2)^{2-p}|\overline{\partial} f(w)\psi(f(w))|}{|1-\zeta w|^2}(1-|w|^2)^{p-2}\, du\, dv\right|^p \\ &\quad \cdot (1-|\zeta|^2)^{p-2}\, d\xi\, d\eta. \end{aligned} \tag{28}$$

Let $d\mu(w) = (1-|w|^2)^{p-2}\, du\, dv$ and

$$K(\zeta, w) = \frac{(1-|w|^2)^{2-p}}{|1-\zeta w|^2}, \quad (\zeta, w) \in \Delta\times\Delta. \tag{29}$$

Consider the test function $h(w) = (1-|w|^2)^{\frac{3}{2p}-1}$. It follows from Lemma 2.2 that there exists constant $C_1 > 0$ such that

$$\begin{aligned} \iint_\Delta \frac{(1-|w|^2)^{2-p}}{|1-\zeta w|^2} h(w)^q\, d\mu(w) &= \iint_\Delta \frac{(1-|w|^2)^{\frac{3q}{2p}-q}}{|1-\zeta w|^2}\, du\, dv \\ &\le C_1(1-|w|^2)^{\frac{3}{2p}-1} = C_1 h(w)^q. \end{aligned} \tag{30}$$

On the other hand, by Lemma 2.2 again, we deduce that there exists constant $C_2 > 0$ so that

$$\begin{aligned} \iint_\Delta \frac{(1-|w|^2)^{2-p}}{|1-\zeta w|^2} h(\zeta)^p\, d\mu(\zeta) &= \iint_\Delta \frac{(1-|w|^2)^{2-p}(1-|\zeta|^2)^{-1/2}}{|1-\zeta w|^2}\, d\xi\, d\eta \\ &\le C_2(1-|\zeta|^2)^{\frac{3}{2}-p} = C_2 h(\zeta)^p. \end{aligned} \tag{31}$$

Combining (30) with (31) and using Lemma 2.1, we deduce that the following operator

$$K\varphi(\zeta) = \iint_\Delta K(\zeta, w)\varphi(\zeta)\, d\mu(w),$$

is bounded on $L^p(\Delta, d\mu)$. Consequently, it follows from (28) that there exists constant $C_3 > 0$ such that

$$\|T_h^-\psi\|_{A^p}^p \;\le\; \frac{C_3}{\pi^{p+1}}\iint_\Delta |\overline{\partial} f(\zeta)\psi(f(\zeta))|^p(1-|\zeta|^2)^{p-2}\, d\xi\, d\eta. \tag{32}$$

It is well known that the Beurling–Ahlfors extension f is bilipschitz continuous with respect to the hyperbolic metric (see [A, Le]), that is

$$\frac{1}{C_3'}(1-|f(\zeta)|^2) \le (1-|\zeta|^2)J_f^{1/2}(\zeta) \le C_3'(1-|f(\zeta)|^2), \tag{33}$$

where C_3' is a positive constant depending only on the complex dilatation of f and J_f is the Jacobian of f. Let $g = f^{-1}$ and μ be the complex dilatation of g. By (32) and a change of variable, we obtain

$$\begin{aligned}
&\|T_h^- \psi\|_{A^p}^p \\
&\le \frac{C_3}{\pi^{p+1}} \iint_\Delta \frac{|\overline{\partial} f(\zeta)|^p}{(|\partial f(\zeta)|^2 - |\overline{\overline{\partial} f(\zeta)}|^2)^{p/2}} |\psi(f(\zeta))|^p (1-|\zeta|^2)^{p-2} J_f^{p/2}(\zeta)\, d\xi\, d\eta \\
&\le C_4 \frac{1}{\pi} \iint_\Delta \frac{|\mu(w)|^p}{(1-|\mu(w)|^2)^{p/2}} |\psi(w)|^p (1-|w|^2)^{p-2}\, du\, dv \\
&\le C_4 \frac{1}{\pi} \frac{k^p}{(1-k^2)^{p/2}} \|\psi\|_{A^p}^p,
\end{aligned} \tag{34}$$

where $k = \|\mu\|_\infty < 1$ and C_4 is a positive constant depending only on the complex dilatation of f. This shows that (i) $\Rightarrow$ (ii).

We next prove that (ii) $\Rightarrow$ (iii). Let h be a sense-preserving homeomorphism on S^1. Suppose that the operator $T_h^- \colon B_p \to B_p$ is bounded. For $w_0 \in \Delta$, consider the function

$$\psi_{w_0}(\zeta) = \frac{1-|w_0|^2}{(1-w_0\zeta)^2}. \tag{35}$$

By Lemma 2.2, we conclude that there exists a constant $C_5 > 0$ independing of w_0 such that

$$\|\psi_{w_0}\|_{A^p}^p = \frac{1}{\pi} \iint_\Delta \frac{(1-|w_0|^2)^p (1-|\zeta|^2)^{p-2}}{|1-w_0\zeta|^{2p}}\, d\xi\, d\eta \le C_5 < \infty. \tag{36}$$

It was proved in [HS] that

$$T_h^- \psi_{w_0}(\zeta) = (1-|w_0|^2)\phi_h(\zeta, w_0). \tag{37}$$

Noting that $T_h^- \colon B_p \to B_p$ is bounded, combining (36), (37) with (27), we deduce that there exists a constant $C_6 > 0$ so that

$$\mathfrak{H}_{h,p}(w_0)^p = \|T_h^- \psi_{w_0}\|_{A^p}^p \le C_6 \|\psi_{w_0}\|_{A^p}^p \le C_5 C_6 < \infty. \tag{38}$$

This finishes the proof of (ii) $\Rightarrow$ (iii).

Finally, we show that (iii) $\Rightarrow$ (i). Suppose that $h \in SH_0(S^1)$ and $\sup_{w_0 \in \Delta} \mathfrak{H}_{h,p}(w_0)^p < \infty$. By Theorem 1.1, to show that h is quasisymmetric, it is sufficient to show that h has a quasiconformal extension to the unit disk Δ. Indeed, we shall show that the Douady–Earle extension $E(h)$ of h is a quasiconformal mapping of Δ onto Δ.

It is well known that if h is a sense-preserving homeomorphism on S^1, then $E(h)$ is a homeomorphism of $\overline{\Delta}$ onto $\overline{\Delta}$ (see [DE, Po]). We next estimate the complex dilatation of $E(h)$.

Let $(w_0, z_0) \in \Delta \times \Delta$ with $w_0 = E(h)(z_0)$. Consider the following two Möbius transformations

$$\Psi_{w_0}(\zeta) = \frac{\zeta - w_0}{1-\overline{w_0}\zeta}, \quad \Gamma_{z_0}(\zeta) = \frac{\zeta + z_0}{1+\overline{z_0}\zeta}, \quad \zeta \in \Delta. \tag{39}$$

Let $\mathfrak{h} = \Psi_{w_0} \circ h \circ \Gamma_{z_0}$ and $\mathcal{H}$ be the Poisson extension of $\mathfrak{h}$. Noting that $\mathcal{H} \circ \Gamma_{z_0}^{-1}$ is harmonic, we have $\mathcal{H} \circ \Gamma_{z_0}^{-1} = H^{w_0}$, where H^{w_0} is the Poisson extension of $\Psi_{w_0} \circ h$. Since the Douady–Earle extension is conformal invariant (see [DE]), we have

$$E(\mathfrak{h}) = \Psi_{w_0} \circ E(h) \circ \Gamma_{z_0}.$$

Let μ be the complex dilatation of the converse $E(h)^{-1}$ of $E(h)$. A computation gives

$$\iint_{\Delta} |\overline{\partial}\mathcal{H}(z)|^p(1-|z|^2)^{p-2}\,dx\,dy = \iint_{\Delta} |\overline{\partial}H^{w_0}(z)|^p(1-|z|^2)^{p-2}\,dx\,dy$$

and

$$|\nu_{\mathfrak{h}}(0)| = |\mu(w_0)|,$$

where $\nu_{\mathfrak{h}}$ is the complex dilatation of $E(\mathfrak{h})$. Consequently, applying Lemma 2.3 to the quasisymmetric homeomorphism $\mathfrak{h}$ yields

$$\frac{|\mu(w_0)|^p}{(1-|\mu(w_0)|^2)^{p/2}} \le C_4' \frac{1}{2\pi} \iint_{\Delta} |\overline{\partial}H^{w_0}(z)|^p(1-|z|^2)^{p-2}\,dx\,dy, \tag{40}$$

where C_4' is a positive constant which independs on $w_0 \in \Delta$.

Let ν be the complex dilatation of $E(h)$. Observing that $|\nu(z_0)| = |\mu(w_0)|$, by the condition (iii), we get

$$\sup_{z_0\in\Delta} \frac{|\nu(z_0)|^p}{(1-|\nu(z_0)|^2)^{p/2}} \le C_4' < \infty.$$

On the other hand, it is known that for $z_0 \in \Delta$,

$$|\partial E(h)(z_0)|^2 - |\overline{\partial}E(h)(z_0)|^2 > 0,$$

(see [Po, DE]). Therefore, we conclude that $\|\nu\|_\infty < 1$. This implies $E(h)$ is a quasiconformal mapping of Δ onto Δ. The proof follows. □

3. Characterizations of symmetric and p-integrable asymptotic affine homeomorphisms

In this section, we shall prove Theorem 1.5 and Theorem 1.7, which give some characterizations of symmetric and p-integrable asymptotic affine homeomorphisms.

We first prove Theorem 1.5.

Proof of Theorem 1.5. We first prove that (I) ⇒ (II). Suppose that h is a symmetric homeomorphism. Then h has a quasiconformal extension f to Δ with complex dilatation μ, which is bilipschitz continuous with respect to the hyperbolic metric and satisfies the property that for any $\varepsilon > 0$, there exists a constant $r_0 > 0$ so that $|\mu(z)| < \varepsilon$ for all $|z| > r_0$ (see [GS]). Assume that $\{\psi_n\}_{n=1}^{\infty}$ is a bounded sequence of A_p and converges to zero on any compact subset of Δ. Thus, there exists $N_0 > 0$ so that for all $n > N_0$,

$$\iint_{|w\le r_0} |\psi_n(w)|^p(1-|w|^2)^{p-2}\,du\,dv < \varepsilon.$$

It follows from (34) that for all $n > N_0$,

$$\|T_h^-\psi_n\|_{A^p}^p \le \frac{C_4''k^p}{\pi(1-k^2)^{p/2}}\varepsilon + \frac{C_4''\varepsilon^p}{\pi(1-\varepsilon^2)^{p/2}}\|\psi_n\|_{A_p},$$

where $k = \|\mu\|_\infty$ and $C_4'' > 0$ depends only on k. This implies that $\{T_h^-\psi_n\}_{n=1}^{\infty}$ converges to zero in A_p. Therefore, $T_h^- : A_p \to A_p$ is a compact operator.

We next prove that (II) ⇒ (III). Consider the function ψ_{w_0} as in (35), which tends to zero on any compact subset of Δ as $|w_0| \to 1$. Also, from (36), we have

$\|\psi_{w_0}\|^p_{A_p} < C_5$, where C_5 is independent of w_0. Since $T_h^-: A_p \to A_p$ is compact, we get from the first equality of (38) that

$$\lim_{|w_0|\to 1} \mathfrak{H}_{h,p}(w_0) = \lim_{|w_0|\to 1} \|T_h^- \psi_{w_0}\|_{A^p} = 0.$$

We proceed to show that (III) $\Rightarrow$ (I). Let $E(h)$ be the Douady–Earle extension of h with complex dilatation ν. From (40) and the condition (III), we have

$$\lim_{|w|\to 1} |\mu(w)| = 0,$$

where μ is the complex dilatation of the converse $E(h)^{-1}$ of $E(h)$. Noting that $|\nu(z)| = |\mu(w)|$, where $E(h)(z) = w$, we get

$$\lim_{|z|\to 1} |\nu(z)| = 0.$$

Thus, we conclude from Theorem 1.4 that h is symmetric.

Finally, it follows from (27) that (III) $\Leftrightarrow$ (IV). The proof of Theorem 1.5 is completed. □

We next prove Theorem 1.7.

Proof of Theorem 1.7. It follows from (27) that (b) $\Leftrightarrow$ (c). We need only to show that (a) $\Leftrightarrow$ (b). Let h be a p-integrable asymptotic affine homeomorphism on the unit circle S^1. Then h has a quasiconformal extension f to the unit disk Δ with complex dilatation μ, which is bilipschitz continuous with respect to the hyperbolic metric and satisfies

$$\iint_\Delta \frac{|\mu(z)|^p}{(1-|z|^2)^2}\, dx\, dy < \infty, \tag{41}$$

(see [Cui, Tang]). By applying Lemma 2.2, we conclude from (37), (27) and (34) that there exist two positive constants C_1 and C_2 so that

$$\begin{aligned} \iint_\Delta \frac{\mathfrak{H}_{h,p}(w)^p}{(1-|w|^2)^2}\, du\, dv &\le C_1 \frac{1}{\pi} \iint_\Delta \iint_\Delta \frac{|\mu(z)|^p\, |\psi_w(z)|^p}{(1-|\mu(z)|^2)^{p/2}} \frac{(1-|z|^2)^{p-2}}{(1-|w|^2)^2}\, du\, dv\, dx\, dy \\ &\le C_2 \iint_\Delta \frac{|\mu(z)|^p}{(1-|z|^2)^2}\, dx\, dy < \infty, \end{aligned} \tag{42}$$

where ψ_w is defined as in (35). This finishes the proof of (a) $\Rightarrow$ (b).

Conversely, we consider the Douady–Earle extension $E(h)$ of h with complex dilatation ν. From (40), we obtain

$$\iint_\Delta \frac{|\mu(w)|^p}{(1-|w|^2)^2}\, du\, dv < \infty,$$

where μ is the complex dilatation of the converse $E(h)^{-1}$ of $E(h)$. Noting that the set of all p-integrable asymptotic affine homeomorphisms on the unit circle S^1 is a group (see [Tang]), we conclude that

$$\iint_\Delta \frac{|\nu(w)|^p}{(1-|w|^2)^2}\, du\, dv < \infty.$$

This completes the proof of Theorem 1.7. □

Combing Theorem 1.7 and Theorem 1.6 gives the following

Corollary 3.1. *Let $p \geq 2$ and h be a quasisymmetric homeomorphism, normalized by (5), on the unit circle S^1.Then h is absolutely continuous (with respect to the arc-length measure) such that $\log h'$ belongs to $B_p(S^1)$ if and only if*

$$\iint_\Delta \frac{\mathfrak{H}_{h,p}(w)^p}{(1-|w|^2)^2}\,du\,dv < \infty. \tag{43}$$

4. Pull-back operators indecued by quasisymmetric homeomorphisms

In this section, we shall use Theorem 1.3 to prove the "if" part of Theorem 1.8. We first recall some notions. Let $p \geq 2$ and $D_p(\Delta)$ denote the space of all harmonic functions u in the unit disk Δ with semi-norm

$$\|u\|_{D_p} = \left(\frac{1}{\pi}\iint_\Delta (|\partial u(z)| + |\overline{\partial} u(z)|)^p (1-|z|^2)^{p-2}\,dx\,dy\right)^{\frac{1}{p}}. \tag{44}$$

Let H be the Poisson integral operator. It is well known that a integrable function v on the unit circle S^1 belongs to the the Besov space $B_p(S^1)$ if and only if $H(v) \in D_p(\Delta)$ and there is constant $C > 0$ such that for any $v \in B_p(S^1)$,

$$\frac{1}{C}\|v\|_p \leq \|H(v)\|_{D_p} \leq C\|v\|_p$$

(see [Tr], [RS]). We denote by $D_a^p(\Delta)$ be the Banach space of all analytic functions φ in Δ with the semi-norm

$$\|u\|_{D_a^p} = \left(\frac{1}{\pi}\iint_\Delta |\varphi'(z)|^p (1-|z|^2)^{p-2}\,dx\,dy\right)^{\frac{1}{p}}. \tag{45}$$

Then it is clear that $D_p(\Delta) = D_a^p(\Delta) \oplus \overline{D_a^p(\Delta)}$, precisely, for each $u \in D_a^p(\Delta)$, there exists a unique pair of holomorphic functions φ and ψ in $D_a^p(\Delta)$ with $\varphi(0) - u(0) = \psi(0) = 0$ such that $u = \varphi + \overline{\psi}$. Define two operator P^+ and P^- by $P^+u = \varphi$ and $P^-u = \overline{\psi(\overline{z})}$. Let $h \in SH_0(S^1)$, we define two further operators $P_h^+ = P^+ \circ H \circ T_h$ and $P_h^- = P^- \circ H \circ T_h$.

We state the "if" part of Theorem 1.8 as following

Theorem 4.1. *Let $2 \leq p < \infty$ and $h \in SH_0(S^1)$. If h is a quasisymmetric homeomorphism, then $T_h\colon B_p(S^1) \to B_p(S^1)$ is a bounded operator.*

Proof. To prove Theorem 4.1, we need another integral operator, which is defined by the following kernel function

$$\psi_h(\zeta, z) = \frac{1}{2\pi i}\int_{S^1} \frac{h(w)}{(\zeta - w)^2(1 - zh(w))}\,dw, \quad (\zeta, z) \in \Delta \times \Delta. \tag{46}$$

The integral operator T_h^+ is defined as

$$T_h^+\psi(\zeta) = \frac{1}{\pi}\iint_\Delta \psi_h(\zeta, z)\psi(\bar{z})\,dx\,dy, \quad \psi \in A_p, \quad \zeta \in \Delta. \tag{47}$$

Let $\zeta \in \Delta$ and f be the Beurling–Ahlfors extension of h into Δ, by a result of [HS], we have

$$T_h^+\psi(\zeta) = \frac{1}{\pi}\iint_\Delta \frac{\partial f(w)\psi(f(w))}{(1-\zeta\bar{w})^2}\,du\,dv.$$

Thus,

$$\begin{aligned}\|T_h^+\psi\|_{A^p}^p &= \frac{1}{\pi}\iint_\Delta \left|\frac{1}{\pi}\iint_\Delta \frac{\partial f(w)\psi(f(w))}{(1-\zeta\bar{w})^2}\,du\,dv\right|^p (1-|\zeta|^2)^{p-2}\,d\xi\,d\eta \\ &\le \frac{1}{\pi^{p+1}}\iint_\Delta \left|\iint_\Delta \frac{(1-|w|^2)^{2-p}|\partial f(w)\psi(f(w))|}{|1-\zeta\bar{w}|^2}(1-|w|^2)^{p-2}\,du\,dv\right|^p \\ &\quad\cdot (1-|\zeta|^2)^{p-2}\,d\xi\,d\eta.\end{aligned} \tag{48}$$

Let $d\mu(w) = (1-|w|^2)^{p-2}dudv$ and

$$K_1(\zeta, w) = \frac{(1-|w|^2)^{2-p}}{|1-\zeta\bar{w}|^2}, \quad (\zeta, w)\in\Delta\times\Delta. \tag{49}$$

Consider the test function $h(w) = (1-|w|^2)^{\frac{3}{2p}-1}$. By the same method as in Theorem 1.3, we can deduce that the following operator

$$K_1\varphi(\zeta) = \iint_\Delta K_1(\zeta, w)\varphi(\zeta)\,d\mu(w),$$

is bounded on $L^p(\Delta, d\mu)$. Thus, we conclude from (48) that there exists constant $C_1' > 0$ so that

$$\|T_h^+\psi\|_{A^p}^p \le \frac{C_1'}{\pi^{p+1}}\iint_\Delta |\partial f(\zeta)\psi(f(\zeta))|^p(1-|\zeta|^2)^{p-2}\,d\xi\,d\eta. \tag{50}$$

Let J_f be the Jacobian of f and μ the complex dilatation of $g = f^{-1}$. By (50), (33) and a change of variable, we conclude that there exists constant $C_2' > 0$ so that

$$\begin{aligned}&\|T_h^+\psi\|_{A^p}^p \\ &\le \frac{C_1'}{\pi^{p+1}}\iint_\Delta \frac{|\partial f(\zeta)|^p}{(|\partial f(\zeta)|^2 - |\bar{\partial} f(\zeta)|^2)^{p/2}}|\psi(f(\zeta))|^p(1-|\zeta|^2)^{p-2}J_f^{p/2}(\zeta)\,d\xi\,d\eta \\ &\le C_2'\frac{1}{\pi}\iint_\Delta \frac{1}{(1-|\mu(w)|^2)^{p/2}}\,|\psi(w)|^p\,(1-|w|^2)^{p-2}\,du\,dv \\ &\le C_2'\frac{1}{\pi}\frac{1}{(1-k^2)^{p/1}}\|\psi\|_{A^p}^p,\end{aligned} \tag{51}$$

where $k = \|\mu\|_\infty < 1$.

It is cleat that $D\varphi(z) = \varphi'(z)$ determines an isometric isomorphism from $D_a^p(\Delta)$ onto A^p. By the same reasoning as in the proof of Theorem 3.1 in [HS], we can show that on $D_a^p(\Delta)$,

$$D\circ P_h^+ = T_h^+\circ D, \quad D\circ P_h^- = T_h^-\circ D. \tag{52}$$

Thus, we conclude from (34), (51) and (52) that $P_h^+ : D_a^p(\Delta)\to D_a^p(\Delta)$ and $P_h^- : D_a^p(\Delta) \to D_a^p(\Delta)$ are bounded operators.

Let $u\in B_p(S^1)$ and $H(u) = \varphi + \overline{\psi}$. Observe that

$$\begin{aligned}H\circ T_h u(z) &= H\circ T_h\varphi(z) + \overline{H\circ T_h\psi(z)} \\ &= P_h^+\varphi(z) + P_h^-\varphi(\bar{z}) + \overline{P_h^+\psi(z)} + \overline{P_h^-\psi(\bar{z})}.\end{aligned} \tag{53}$$

We conclude from the discussions above that there exist positive constant C_3', C_4', C_5' such that

$$\|T_h u\|_p \le C_3'\|H\circ T_h u\|_{D_p} \le C_4'(\|\varphi\|_{D_a^p} + \|\psi\|_{D_a^p}) \le C_5'\|u\|_p.$$

The proof follows. □

Acknowledgements. This work was supported by National Natural Science Foundation of China (Grant Nos. 11601100) and the foundation of Guizhou Provincial Science and Technology Department (Grant Nos. [2017]7337, [2017]5726).

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Received 23 October 2018 • Accepted 1 February 2019