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WEIGHTED LOCAL MORREY SPACES

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Abstract. We discuss the boundedness of linear and sublinear operators in two types of weighted local Morrey spaces. One is defined by Natasha Samko in 2008. The other is defined by Yasuo Komori-Furuya and Satoru Shirai in 2009. We characterize the class of weights for which the Hardy–Littlewood maximal operator is bounded. Under a certain integral condition it turns out that the singular integral operators are bounded if and only if the Hardy–Littlewood maximal operator is bounded. As an application of the characterization, the power weight function $|\cdot|^\alpha$ is considered. The condition on α for which the Hardy–Littlewood maximal operator is bounded can be described completely.

1. Introduction

The aim of this paper is to characterize the class of weights for which the Hardy–Littlewood maximal operator M is bounded on the weighted local Morrey space $L\mathcal{M}^p_q(1,w)$ of Samko type and on the weighted local Morrey space $L\mathcal{M}^p_q(w,w)$ of Komori–Shirai type. A similar characterization is obtained for the singular integral operators, the fractional integral operators and the fractional maximal operators. Here and below by a weight we mean a locally integrable function on $\mathbf{R}^n$ which is almost everywhere positive.

We shall consider all cubes in $\mathbf{R}^n$ which have their sides parallel to the coordinate axes. We denote by $\mathcal{Q}$ the family of all such cubes. For a cube $Q \in \mathcal{Q}$ we use $\ell(Q)$ to denote the sides length of Q, $c(Q)$ to denote the center of Q, $|Q|$ to denote the volume of Q and cQ to denote the cube with the same center as Q but with side-length $c\ell(Q)$.

The class A_p with $1 < p < \infty$ is defined to the set of all weights w for which

$$\sup_{Q\in\mathcal{Q}} \frac{1}{|Q|}\int_Q w(x)\,\mathrm{d}x \left(\int_Q w(x)^{-\frac{1}{p-1}}\,\mathrm{d}x\right)^{p-1} < \infty.$$

This class A_p, initiated by Muckenhoupt [42], characterizes the condition for which there exists a constant $C > 0$ such that

$$\int_{\mathbf{R}^n} Mf(x)^p w(x)\,\mathrm{d}x \le C\int_{\mathbf{R}^n} |f(x)|^p w(x)\,\mathrm{d}x$$

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for all measurable functions f, where M is the Hardy–Littlewood maximal operator defined by

$$Mf(x) \equiv \sup_{Q \in \mathcal{Q}} \frac{\chi_Q(x)}{|Q|} \int_Q |f(y)|\,\mathrm{d}y, \quad x \in \mathbf{R}^n.$$

The Hardy–Littlewood maximal operator M plays a fundamental role in harmonic analysis. The Riesz transform, which is given by

$$R_j f(x) \equiv \lim_{\varepsilon \downarrow 0} \int_{\mathbf{R}^n \setminus B(x,\varepsilon)} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\,\mathrm{d}y,$$

is also important in harmonic analysis. Importantly, if $w \in A_p$, then

$$\int_{\mathbf{R}^n} |R_j f(x)|^p w(x)\,\mathrm{d}x \le C \int_{\mathbf{R}^n} |f(x)|^p w(x)\,\mathrm{d}x$$

for all $f \in L^\infty_{\mathrm{c}}$.

Recently, more and more Banach lattices come into play in harmonic analysis. For example, local Morrey spaces play important role to describe the real interpolation of weighted Lebesgue spaces, which covers the off-range of the Stein–Weiss interpolation theorem [3].

Let $1 \le q \le p < \infty$. The *local Morrey norm* $\|\cdot\|_{L\mathcal{M}^p_q}$ is given by

$$\|f\|_{L\mathcal{M}^p_q} \equiv \sup_{R>0} |[-R,R]^n|^{\frac{1}{p}-\frac{1}{q}} \|f\chi_{[-R,R]^n}\|_{L^q}$$

for a measurable function f. The *local Morrey space* $L\mathcal{M}^p_q$ is the set of all the measurable functions f for which the norm $\|f\|_{L\mathcal{M}^p_q}$ is finite. Following the notation in the works [4, 10, 49], we define weighted local Morrey spaces as follows: For a measurable function f and the weights u and w, we write

$$\|f\|_{L\mathcal{M}^p_q(u,w)} \equiv \sup_{R>0} u([-R,R]^n)^{\frac{1}{p}-\frac{1}{q}} \|f\chi_{[-R,R]^n}\|_{L^q(w)}.$$

The *two-weight local Morrey space* $L\mathcal{M}^p_q(u,w)$ is the set of all measurable functions f for which the norm $\|f\|_{L\mathcal{M}^p_q(u,w)}$ is finite. If $u = 1$, then we call $L\mathcal{M}^p_q(1,w)$ the *local Morrey space of Samko type* based on [47, 48] and if $u = w$, then we call $L\mathcal{M}^p_q(w,w)$ the *local Morrey space of Komori–Shirai type* based on [37]. When $p = q$, $L\mathcal{M}^p_p(u,w) = L^p(w)$ and hence $L\mathcal{M}^p_p(1,w) = L\mathcal{M}^p_p(w,w) = L^p(w)$ with coincidence of norms. So, in this case the theory of A_p applies readily.

The weighted local Morrey space $L\mathcal{M}^p_q(u,w)$ is a contrast of the weighted Morrey space $\mathcal{M}^p_q(u,w)$ which consists of all measurable functions f for which the norm

$$\|f\|_{\mathcal{M}^p_q(u,w)} \equiv \sup_{Q \in \mathcal{Q}} u(Q)^{\frac{1}{p}-\frac{1}{q}} \|f\chi_Q\|_{L^p(w)}$$

is finite. If $u(x) = |x|^\beta$ and $w(x) = 1$ or $u(x) = v(x) = |x|^\beta$ with $\beta \in \mathbf{R}$, then we say that $L\mathcal{M}^p_q(u,w)$ is the power weighted local Morrey space and that $\mathcal{M}^p_q(u,w)$ is the power weighted Morrey space.

In this paper, assuming that $1 < q < p < \infty$, we seek a characterization for the Hardy–Littlewood maximal operator M to be bounded mainly on $L\mathcal{M}^p_q(1,w)$ and $L\mathcal{M}^p_q(w,w)$, motivated by the characterization due to Muckenhoupt.

For $\nu \in \mathbf{Z}$ and $m = (m_1, m_2, \ldots, m_n) \in \mathbf{Z}^n$, we define $Q_{\nu m} \equiv \prod_{j=1}^n \left[\frac{m_j}{2^\nu}, \frac{m_j+1}{2^\nu}\right)$. Denote by $\mathcal{D} = \mathcal{D}(\mathbf{R}^n)$ the set of such cubes. The elements in $\mathcal{D}$ are called dyadic cubes. Denote by dist_∞ the ℓ^∞-distance on $\mathbf{R}^n$. We define the *base* $L\mathcal{Q}$ of cubes by

$$L\mathcal{Q} \equiv \{Q \in \mathcal{D} \colon \mathrm{dist}_\infty(\{0\}, Q) = \ell(Q)\}.$$

We notice that the dyadic cubes in $L\mathcal{Q}$ are pairwise disjoint and that

$$\bigcup\{Q\colon Q \in L\mathcal{Q}\} = \mathbf{R}^n \setminus \{0\}.$$

We also notice that the number of dyadic cubes in $L\mathcal{Q}$ with the same side length is $2^n(2^n - 1)$.

We define the weight class $\mathcal{G}$ by the set of all weights u that satisfy the following:

(i) u is doubling and reverse doubling at the origin, that is, there exist $\beta > \alpha > 1$ such that

$$\alpha u([-R, R]^n) \le u([-2R, 2R]^n) \le \beta u([-R, R]^n), \quad R > 0;$$

(ii) u is doubling with respect to $L\mathcal{Q}$ in the following sense: there exists $\gamma > 1$ such that

$$u([-2\ell(Q), 2\ell(Q)]^n) \le \gamma u(Q)$$

for all $Q \in L\mathcal{Q}$.

Our results are besed upon the following structure of weighted local Morrey spaces.

Lemma 1.1. *Let $1 < q < p < \infty$ and let u and w be weights. Assume that $u \in \mathcal{G}$. Then*

$$\|f\|_{L\mathcal{M}^p_q(u,w)} \sim \sup_{Q \in L\mathcal{Q}} u(Q)^{\frac{1}{p}-\frac{1}{q}} \|f\chi_Q\|_{L^q(w)}$$

holds for any measurable function f.

Lemma 1.1 will be proved in Section 2.

Let $1 < q < p < \infty$, and let u and w be weights. For a measurable function g its $L\mathcal{M}^p_q(u,w)$*-associate norm* $\|g\|_{L\mathcal{M}^p_q(u,w)'}$ is defined by

$$\|g\|_{L\mathcal{M}^p_q(u,w)'} \equiv \sup\left\{\|f \cdot g\|_{L^1}\colon f \in L\mathcal{M}^p_q(u,w), \|f\|_{L\mathcal{M}^p_q(u,w)} \le 1\right\}.$$

The space $L\mathcal{M}^p_q(u,w)'$ collects all measurable functions g for which the norm $\|g\|_{L\mathcal{M}^p_q(u,w)'}$ is finite. The space $L\mathcal{M}^p_q(u,w)'$ is called the *Köthe dual* of $L\mathcal{M}^p_q(u,w)$ or the *associated space* of $L\mathcal{M}^p_q(u,w)$. Below, we list our results.

Theorem 1.2. *Let $1 < q < p < \infty$ and let u and w be weights. Assume that $u \in \mathcal{G}$. Then the following are equivalent*:

(1) *The Hardy–Littlewood maximal operator M is bounded on $L\mathcal{M}^p_q(u,w)$;*

(2) *there exists a constant C which is independent of $Q \in L\mathcal{Q}$ such that*

$$\sup_{R\in\mathcal{Q},\, R\subset 2Q} \frac{1}{|R|}\|\chi_R\|_{L^q(w)}\|\chi_R\|_{L^q(w)'} + \sup_{R\in\mathcal{Q},\, R\supset Q} \frac{1}{|R|}\|\chi_Q\|_{L\mathcal{M}^p_q(u,w)}\|\chi_R\|_{L\mathcal{M}^p_q(u,w)'} \le C.$$

As a special case of Theorem 1.2, we obtain the following characterizations of $L\mathcal{M}^p_q(1,w)$ and $L\mathcal{M}^p_q(w,w)$.

Corollary 1.3. *Let $1 < q < p < \infty$ and let w be a weight. Then the following are equivalent*:

(1) *The Hardy–Littlewood maximal operator M is bounded on $L\mathcal{M}^p_q(1,w)$;*

(2) *there exists a constant C which is independent of $Q \in L\mathcal{Q}$ such that*

$$\sup_{R\in\mathcal{Q},\, R\subset 2Q} \frac{1}{|R|}\|\chi_R\|_{L^q(w)}\|\chi_R\|_{L^q(w)'} + \sup_{R\in\mathcal{Q},\, R\supset Q} \frac{1}{|R|}\|\chi_Q\|_{L\mathcal{M}^p_q(1,w)}\|\chi_R\|_{L\mathcal{M}^p_q(1,w)'} \le C.$$

Proposition 1.4. *Let $1 < q < p < \infty$ and let w be a weight. Then the following are equivalent*:

(1) *The Hardy–Littlewood maximal operator M is bounded on $L\mathcal{M}^p_q(w,w)$;*

(2) *$w \in \mathcal{G}$ and there exists a constant C which is independent of $Q \in L\mathcal{Q}$ such that*

$$\sup_{R\in\mathcal{Q},\, R\subset 2Q}\frac{1}{|R|}\|\chi_R\|_{L^q(w)}\|\chi_R\|_{L^q(w)'}+\sup_{R\in\mathcal{Q},\, R\supset Q}\frac{1}{|R|}\|\chi_Q\|_{L\mathcal{M}^p_q(w,w)}\|\chi_R\|_{L\mathcal{M}^p_q(w,w)'}\le C.$$

Our characterization can be applied to singular integral operators including the Riesz transform. A *singular integral operator* is an L^2-bounded linear operator T that comes with a function $K\colon \mathbf{R}^n\times\mathbf{R}^n\to\mathbf{C}$ satisfying the following conditions:

(1) (*Size condition*); For all $x, y \in \mathbf{R}^n$,

$$|K(x,y)|\lesssim |x-y|^{-n}. \tag{1.1}$$

(2) (*Gradient condition*); For all $x, y, z \in \mathbf{R}^n$ satisfying $|x-z|>2|y-z|$,

$$|K(x,z)-K(y,z)|\lesssim |x-y|^{-n-1}|y-z|. \tag{1.2}$$

(3) Let f be an L^2-function. For almost all $x\notin \operatorname{supp}(f)$,

$$Tf(x)=\int_{\mathbf{R}^n}K(x,y)f(y)\,\mathrm{d}y. \tag{1.3}$$

The function K is called the *integral kernel* of T.

Here and below in this paper we use the following notation: Let $A, B\ge 0$. Then $A\lesssim B$ and $B\gtrsim A$ mean that there exists a constant $C>0$ such that $A\le CB$, where C depends only on the parameters of importance. The symbol $A\sim B$ means that $A\lesssim B$ and $B\lesssim A$ happen simultaneously.

For the singular integral operators, we have the following characterization.

Theorem 1.5. *Assume that $\|\chi_{2^{k+1}Q_0}\|_{\mathrm{L}\mathcal{M}^p_q(u,w)}\ge\alpha\|\chi_{2^kQ_0}\|_{\mathrm{L}\mathcal{M}^p_q(u,w)}$, $Q_0\in \mathrm{L}\mathcal{Q}$ for some $\alpha>1$ independent on $k\in\mathbf{N}$. Let T be a singular integral operator. Let $1<q<p<\infty$ and let u and w be weights. Assume that $u\in\mathcal{G}$ and that there exists a constant C which is independent of $Q\in L\mathcal{Q}$ such that*

$$\sup_{R\in\mathcal{Q},\, R\subset 2Q}\frac{1}{|R|}\|\chi_R\|_{L^q(w)}\|\chi_R\|_{L^q(w)'}+\sup_{R\in\mathcal{Q},\, R\supset Q}\frac{1}{|R|}\|\chi_Q\|_{L\mathcal{M}^p_q(u,w)}\|\chi_R\|_{L\mathcal{M}^p_q(u,w)'}\le C.$$

Then $\|Tf\|_{L\mathcal{M}^p_q(u,w)}\lesssim\|f\|_{L\mathcal{M}^p_q(u,w)}$ for all $f\in L^\infty_{\mathrm{c}}$.

The condition on u and w corresponds to the integral condition considered in [44].

Theorem 1.6. *Let $1<q<p<\infty$ and let w be a weight. Assume that $u\in\mathcal{G}$ and that $\|R_jf\|_{L\mathcal{M}^p_q(u,w)}\lesssim\|f\|_{L\mathcal{M}^p_q(u,w)}$ for all $f\in L^\infty_{\mathrm{c}}$ and for all $j=1,2,\dots,n$. Then there exists a constant C which is independent of $Q\in L\mathcal{Q}$ such that*

$$\sup_{R\in\mathcal{Q},\, R\subset 2Q}\frac{1}{|R|}\|\chi_R\|_{L^q(w)}\|\chi_R\|_{L^q(w)'}+\sup_{R\in\mathcal{Q},\, R\supset Q}\frac{1}{|R|}\|\chi_Q\|_{L\mathcal{M}^p_q(u,w)}\|\chi_R\|_{L\mathcal{M}^p_q(u,w)'}\le C.$$

To investigate weighted Morrey spaces of Samko type, we simply let $u=1$ in Theorems 1.5 and 1.6. Meanwhile, we have investigated the sufficiency of the boundedness of the singular integral operators on weighted Morrey spaces of Komori–Shirai type. However the necessity is somewhat non-trivial. So, we formulate it.

Proposition 1.7. *Let $1<q<p<\infty$, and let w be a doubling weight. Assume in addition that $\|R_jf\|_{L\mathcal{M}^p_q(w,w)}\lesssim\|f\|_{L\mathcal{M}^p_q(w,w)}$ for all $f\in L^\infty_{\mathrm{c}}$ and for all $j=1,2,\dots,n$. Then there exists a constant C which is independent of $Q\in L\mathcal{Q}$ such* *that*

$$\sup_{R\in\mathcal{Q},\, R\subset 2Q} \frac{1}{|R|}\|\chi_R\|_{L^q(w)}\|\chi_R\|_{L^q(w)'} + \sup_{R\in\mathcal{Q},\, R\supset Q} \frac{1}{|R|}\|\chi_Q\|_{L\mathcal{M}^p_q(w,w)}\|\chi_R\|_{L\mathcal{M}^p_q(w,w)'} \le C.$$

We apply our results to special cases.

Proposition 1.8. *Let $1 < q < p < \infty$ and let $w(x) = w_\beta(x) = |x|^\beta$ with $\beta \in \mathbf{R}$. Then the following are equivalent*:

(1) *The maximal operator M is bounded on $L\mathcal{M}^p_q(1,w)$;*
(2) $-\frac{q}{p}n \le \beta < nq\big(1-\frac{1}{p}\big)$.

Proposition 1.9. *Let $1 < q < p < \infty$ and let $w(x) = w_\beta(x) = |x|^\beta$ with $\beta \in \mathbf{R}$. Then the following are equivalent*:

(1) *The maximal operator M is bounded on $L\mathcal{M}^p_q(w,w)$;*
(2) $-n < \beta < n(p-1)$;
(3) $w \in A_p$.

The ranges obtained in Propositions 1.8 and 1.9 are the same as that for weighted Morrey spaces of Komori–Shirai type and Samko type, respectively.

Proposition 1.10. [53] *Let $1 < q < p < \infty$ and let $w(x) = w_\beta(x) = |x|^\beta$ with $\beta \in \mathbf{R}$. Then the following are equivalent*:

(1) *The maximal operator M is bounded on $\mathcal{M}^p_q(1,w)$;*
(2) $-n\dfrac{q}{p} \le \beta < nq\left(1-\dfrac{1}{p}\right)$.

One of the ways to investigate the boundedness of the operators acting on Morrey spaces is to combine the translation and the boundedness of the operators acting on corresponding local Morrey spaces. Propositions 1.8 and 1.10 are significant in that Proposition 1.10 can not be obtained by the translation of Proposition 1.8.

In [33] Iida and the first author obtained a complete characterization of the dual inequality of Stein type in weighted Morrey spaces $\mathcal{M}^p_q(1,w)$ of Samko type. See also [30]. Despite the recent works [44, 46, 53] a complete characterization of the class for which M is bounded on $\mathcal{M}^p_q(1,w)$ or $\mathcal{M}^p_q(w,w)$ is still missing.

Proposition 1.11. *Let $1 < q < p < \infty$ and let $w(x) = w_\beta(x) = |x|^\beta$ with $\beta \in \mathbf{R}$. Then the following are equivalent*:

(1) *The maximal operator M is bounded on $\mathcal{M}^p_q(w,w)$;*
(2) $-n < \beta < n(p-1)$;
(3) $w \in A_p$.

We can consider the weighted norm inequalities for other operators. Let I_α be the *fractional integral operator* given by

$$I_\alpha f(x) \equiv \int_{\mathbf{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,\mathrm{d}y, \quad x \in \mathbf{R}^n, \tag{1.4}$$

for a nonnegative measurable function f. The boundedness of the fractional integral operator can be characterized in a similar way. Let $1 < p < s < \infty$. Recall that the class $A_{p,s}$ of weights is defined to the set of all weights for which there exists a constant $C > 0$ satisfying

$$\|I_\alpha f \cdot w\|_{L^s} \le C\|f\cdot w\|_{L^p}$$

for any nonnegative measurable function f.

To extend this boundedness to Morrey spaces, we can consider two types of boundedness. One is due to Spanne [52] and the other is due to Adams [1]. The following theorem corresponds to the result due to Spanne.

Theorem 1.12. *Let $1 < q < p < \infty$, $1 < t < s < \infty$ and $0 < \alpha < n$ satisfy*

$$\alpha = \frac{n}{p} - \frac{n}{s} = \frac{n}{q} - \frac{n}{t}.$$

Let u and w be weights such that $u \in \mathcal{G}$. Consider the following statements:

(1) *There exists a constant $C > 0$ such that*

$$\|I_\alpha f\|_{L\mathcal{M}^s_t(u,w^t)} \le C\|f\|_{L\mathcal{M}^p_q(u,w^q)}$$

for any nonnegative measurable function f;

(2) *there exists a constant $C > 0$ such that*

$$\|M_\alpha f\|_{L\mathcal{M}^s_t(u,w^t)} \le C\|f\|_{L\mathcal{M}^p_q(u,w^q)}$$

for any nonnegative measurable function f;

(3) *there exists a constant C which is independent of $Q \in L\mathcal{Q}$ such that*

$$\sup_{R\in\mathcal{Q},\, R\subset 2Q} \frac{\ell(R)^\alpha}{|R|}\|\chi_R\|_{L^t(w^t)}\|\chi_R\|_{L^q(w^q)'} \le C$$

and

$$\sup_{R\in\mathcal{Q},\, R\supset Q} \frac{\ell(R)^\alpha}{|R|}\|\chi_Q\|_{L\mathcal{M}^s_t(u,w^t)}\|\chi_R\|_{L\mathcal{M}^p_q(u,w^q)'} \le C.$$

Then;

- (1) *implies* (2) *and* (3).
- (2) *and* (3) *are equivalent.*
- *If there exists $\kappa > 1$ such that $2\|\chi_Q\|_{\mathrm{L}\mathcal{M}^s_t(u,w^t)} \le \kappa\|\chi_{\kappa Q}\|_{\mathrm{L}\mathcal{M}^s_t(u,w^t)}$ for all cubes $Q \in \mathrm{L}\mathcal{Q}$, then* (2) *or* (3) *implies* (1).

In the case of power weighted Morrey spaces of Samko type, we have the following characterization of the boundedness.

Proposition 1.13. *Let $1 < q < p < \infty$ and $1 < t < s < \infty$. Assume that*

$$\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}, \quad \frac{1}{t} = \frac{1}{q} - \frac{\alpha}{n}.$$

Then for the power weight $w(x) = w_\beta(x) = |x|^\beta$ with $\beta \in \mathbf{R}$ the following are equivalent:

(1) *The fractional maximal operator M_α is bounded from $L\mathcal{M}^p_q(1, w^q)$ to $L\mathcal{M}^s_t(1, w^t)$;*

(2) $-\frac{n}{s} \le \beta < \frac{n}{p'}$, *that is, $\beta = \frac{n}{s}$ or $I_\alpha \colon L^p(w^p) \to L^s(w^s)$.*

We can replace M_α by I_α once we exclude the case of $\beta = -\frac{n}{s}$ in (2).

In the case of power weighted Morrey spaces of Komori–Shirai type, we have the following well-known characterization of the boundedness.

Proposition 1.14. *Let $0 < \alpha < n$, $\beta \in \mathbf{R}$, $1 < q < p < \infty$ and $1 < t < s < \infty$. Assume that*

$$\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}, \quad \frac{1}{t} = \frac{1}{s} - \frac{\alpha}{n}.$$

Then for the power weight $w(x) = w_\beta(x) = |x|^\beta$ with $\beta \in \mathbf{R}$ the following are equivalent:

(1) *The fractional integral operator* I_α *is bounded from* $L\mathcal{M}^p_q(w^t, w^q)$ *to* $L\mathcal{M}^s_t(w^t, w^t)$;

(2) $-\dfrac{n}{t} < \beta < \dfrac{ns}{p't}$.

We can replace I_α *by* M_α *once we include the case of* $\beta = -\frac{n}{t}$ *in* (2).

It may be interesting to compare these propositions with the following known results for Lebesgue spaces, the case where $p = q$ and hence $s = t$ in Propositions 1.13 and 1.14.

Proposition 1.15. *Let* $1 < p < s < \infty$, $0 < \alpha < n$ *and* $\beta \in \mathbf{R}$. *Assume* $\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}$. *Then for the power weight* $w(x) = w_\beta(x) = |x|^\beta$ *the following are equivalent*:

(1) *There exists a constant* $C > 0$ *such that*

$$\|I_\alpha f \cdot w_\beta\|_{L^s} \le C\|f \cdot w_\beta\|_{L^p}$$

for all $f \in L^\infty_c$.

(2) $-\dfrac{n}{s} < \beta < \dfrac{n}{p'}$.

We can replace I_α *by* M_α *in the above.*

As for weighted Morrey spaces of Samko type, we have the following conclusion:

Proposition 1.16. [46, Proposition 4.1] *Let* $1 < q < p < \infty$, $0 < \alpha < n$ *and* $1 < t < s < \infty$. *Assume that*

$$\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}, \quad \frac{q}{p} = \frac{t}{s}.$$

Then for the power weight $w(x) = w_\beta(x) = |x|^\beta$ *with* $\beta \in \mathbf{R}$ *the following are equivalent*:

(1) *The fractional maximal operator* M_α *is bounded from* $\mathcal{M}^p_q(1, w^q)$ *to* $\mathcal{M}^s_t(1, w^t)$;

(2) $-\dfrac{n}{s} \le \beta < \dfrac{n}{p'}$.

Proposition 1.17. [46, Proposition 4.2] *Let* $1 < q < p < \infty$, $0 < \alpha < n$ *and* $1 < t < s < \infty$. *Assume that*

$$\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}, \quad \frac{q}{p} = \frac{t}{s}.$$

Then for the power weight $w(x) = w_\beta(x) = |x|^\beta$ *with* $\beta \in \mathbf{R}$ *the following are equivalent*:

(1) *The fractional integral operator* I_α *is bounded from* $\mathcal{M}^p_q(1, w^q)$ *to* $\mathcal{M}^s_t(1, w^t)$;

(2) $-\dfrac{n}{s} < \beta < \dfrac{n}{p'}$.

Proposition 1.18. *Let* $1 < q < p < \infty$, $0 < \alpha < n$ *and* $1 < t < s < \infty$. *Assume that*

$$\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}, \quad \frac{1}{t} = \frac{1}{q} - \frac{\alpha}{n}, \quad \beta > -\frac{n}{t}.$$

Then for the power weight $w(x) = w_\beta(x) = |x|^\beta$ *with* $\beta \in \mathbf{R}$ *the following are equivalent*:

(1) *The fractional integral operator* I_α *is bounded from* $\mathcal{M}^p_q(w^t, w^q)$ *to* $\mathcal{M}^s_t(w^t, w^t)$;

(2) $-\dfrac{n}{t} < \beta < \dfrac{ns}{p't}$.

We can replace I_α *by* M_α *once we include the case of* $\beta = -\frac{n}{t}$ *in* (2).

We remark that in the case of radial functions, Komori and Sato obtained the following result:

Proposition 1.19. [36] *Let* $1 < p < s < \infty$, $1 < q \le t < \infty$ *and* $0 < \alpha < n$ *satisfy*

$$\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}, \quad \frac{1}{t} = \frac{1}{q} - \frac{\alpha}{n}.$$

Let β *satisfy*

$$\alpha - \frac{n}{q} + \frac{n}{p} < \beta < \frac{n}{q'}.$$

Then for the power weight $w(x) = w_\beta(x) = |x|^\beta$ *with* $\beta \in \mathbf{R}$,

$$\|I_\alpha f \cdot w_\beta\|_{L\mathcal{M}^s_t} \le C\|f \cdot w_\beta\|_{L\mathcal{M}^p_q}$$

for all radial functions $f \in L^\infty_{\rm c}$.

There is a huge amount of literatures dealing with weighted Morrey spaces together with their variants. Many researchers investigated the boundedness properties of the linear operators acting on weighted Morrey spaces. such as sublinear operator [4, 12, 15, 35], singular integral operators [14, 35, 63], commutators [17, 12, 35, 59, 61], pseudo-differential operators [26], the square functions [11], Toeplitz operators [56], the fractional integral operators [12, 28, 31, 32] and fractional integrals associated to operators [51, 54, 55] including the related commutators. Applications to partial differential equations can be found in [8, 19, 50]. Embedding relations together with the envelope are investigated in [22]. A passage to the metric measure spaces is done in [62]. Ye and Wang used the cube testing to get a characterization of a sufficient condition which guarantees the boundedness of the Hardy–Littlewood maximal operator [60]. See [57, 58] for Campanato spaces of Komori–Shirai type.

As for weighted Morrey spaces of Samko type, the boundedness property of the sharp maximal operator, the maximal operator, the singular integral operators, the fractional operarots including the multilinear setting are investigated in [18, 29, 44, 45, 46]. we can find its application to singular integral equations in [41]. Liu considered the boundedness of the pseudo-differential operators in the setting of generalized Morrey spaces [39]. There are many attempts of obtaining a necessary and sufficient condition for the weighted norm inequality. See [34] for a characterization of a sufficient condition which guarantees the boundedness of the Hardy–Littlewood maximal operator.

The two-weighted Morrey spaces of the type $\mathcal{M}^p_q(u, v)$ can be found in [24, 43, 49] including generalized Morrey spaces [5, 9, 12, 13, 20, 21, 25] and their closed subspaces [2].

The remaining part of this paper is organized as follows: In Section 2 we get a characterization of the local Morrey norm $\|\cdot\|_{L\mathcal{M}^p_q}$. Based on the observation in Section 3, we prove Theorem 1.2 in Section 4. Theorems 1.5 and 1.6 are proved in Section 5. Finally, as apply our results for the power weight $w(x) = |x|^\alpha$ in Section 6.

2. Preliminaries

2.1. Structure of weighted local Morrey spaces–the proof of Lemma 1.1. We prove Lemma 1.1.

Proof of Lemma 1.1. The proof consists of two auxiliary equivalences.

We first claim

$$\|f\|_{L\mathcal{M}^p_q(u,w)} \sim \sup_{m\in\mathbf{Z}} (u([-2^{m+1},2^{m+1}]^n)^{\frac{1}{p}-\frac{1}{q}} \|f\chi_{[-2^{m+1},2^{m+1}]^n\setminus[-2^m,2^m]^n}\|_{L^q(w)}. \tag{2.1}$$

It is easy to see that

$$\|f\|_{L\mathcal{M}^p_q(u,w)} \geq \sup_{m\in\mathbf{Z}} u([-2^{m+1},2^{m+1}]^n)^{\frac{1}{p}-\frac{1}{q}} \|f\chi_{[-2^{m+1},2^{m+1}]^n\setminus[-2^m,2^m]^n}\|_{L^q(w)}.$$

To obtain the reverse inequality, we fix $R>0$. Then if we set $j=[1+\log_2 R]$, we have

$$u([-R,R]^n)^{\frac{1}{p}-\frac{1}{q}} \|f\chi_{[-R,R]^n}\|_{L^q(w)} \lesssim u([-2^j,2^j]^n)^{\frac{1}{p}-\frac{1}{q}} \|f\chi_{[-2^j,2^j]^n}\|_{L^q(w)}$$

thanks to the doubling property of u at the origin. Since $q<p$ and u is reverse doubling at the origin, we have

$$u([-2^j,2^j]^n)^{\frac{1}{p}-\frac{1}{q}} \sum_{l=-\infty}^{j-1} u([-2^l,2^l]^n)^{-\frac{1}{p}+\frac{1}{q}} \sim 1$$

and hence

$$\begin{aligned}
&u([-2^j,2^j]^n)^{\frac{1}{p}-\frac{1}{q}} \|f\chi_{[-2^j,2^j]^n}\|_{L^q(w)}\\
&\leq u([-2^j,2^j]^n)^{\frac{1}{p}-\frac{1}{q}} \sum_{l=-\infty}^{j-1} \|f\chi_{[-2^{l+1},2^{l+1}]^n\setminus[-2^l,2^l]^n}\|_{L^q(w)}\\
&\leq u([-2^j,2^j]^n)^{\frac{1}{p}-\frac{1}{q}}\\
&\quad\times \sum_{l=-\infty}^{j-1} u([-2^l,2^l]^n)^{-\frac{1}{p}+\frac{1}{q}} \cdot u([-2^l,2^l]^n)^{\frac{1}{p}-\frac{1}{q}} \|f\chi_{[-2^{l+1},2^{l+1}]^n\setminus[-2^l,2^l]^n}\|_{L^q(w)}\\
&\lesssim u([-2^j,2^j]^n)^{\frac{1}{p}-\frac{1}{q}}\\
&\quad\times \sum_{l=-\infty}^{j-1} u([-2^l,2^l]^n)^{-\frac{1}{p}+\frac{1}{q}} \sup_{m\in\mathbf{Z}} u([-2^{m+1},2^{m+1}]^n)^{\frac{1}{p}-\frac{1}{q}} \|f\chi_{[-2^{m+1},2^{m+1}]^n\setminus[-2^m,2^m]^n}\|_{L^q(w)}\\
&\lesssim \sup_{m\in\mathbf{Z}} u([-2^{m+1},2^{m+1}]^n)^{\frac{1}{p}-\frac{1}{q}} \|f\chi_{[-2^{m+1},2^{m+1}]^n\setminus[-2^m,2^m]^n}\|_{L^q(w)}.
\end{aligned}$$

Thus,

$$\|f\|_{L\mathcal{M}^p_q(u,w)} \lesssim \sup_{m\in\mathbf{Z}} u([-2^{m+1},2^{m+1}]^n)^{\frac{1}{p}-\frac{1}{q}} \|f\chi_{[-2^{m+1},2^{m+1}]^n\setminus[-2^m,2^m]^n}\|_{L^q(w)},$$

which yields (2.1).

Next, we shall verify that

$$\begin{aligned}
&\sup_{m\in\mathbf{Z}} u([-2^{m+1},2^{m+1}]^n)^{\frac{1}{p}-\frac{1}{q}} \|f\chi_{[-2^{m+1},2^{m+1}]^n\setminus[-2^m,2^m]^n}\|_{L^q(w)}\\
&\sim \sup_{Q\in L\mathcal{Q}} u(Q)^{\frac{1}{p}-\frac{1}{q}} \|f\chi_Q\|_{L^q(w)}.
\end{aligned} \tag{2.2}$$

A simple geometric observation shows that

$$\begin{aligned}
&u([-2^{m+1},2^{m+1}]^n)^{\frac{1}{p}-\frac{1}{q}} \|f\chi_{[-2^{m+1},2^{m+1}]^n\setminus[-2^m,2^m]^n}\|_{L^q(w)}\\
&\leq 2^n(2^n-1) \sup_{Q\in L\mathcal{Q},\,\ell(Q)=2^m} u(Q)^{\frac{1}{p}-\frac{1}{q}} \|f\chi_Q\|_{L^q(w)}.
\end{aligned}$$

Thanks to the doubling condition of u (with respect to $L\mathcal{Q}$), we have reverse inequality

$$u(Q)^{\frac{1}{p}-\frac{1}{q}}\|f\chi_Q\|_{L^q(w)} \lesssim u([-2^{m+1},2^{m+1}]^n)^{\frac{1}{p}-\frac{1}{q}}\|f\chi_{[-2^{m+1},2^{m+1}]^n\setminus[-2^m,2^m]^n}\|_{L^q(w)}$$

for all cubes $Q \in L\mathcal{Q}$ with $\ell(Q) = 2^m$. Thus, (2.2) is verified and the proof of the lemma is completed. □

2.2. The Lerner–Hytönen decomposition. To investigate the boundedness property of the singular integral operators in Theorem 1.5, we need the Lerner–Hytönen decomposition.

A collection $\{Q_j^k\}_{k\in\mathbf{N}_0,\, j\in J_k}$ of dyadic cubes is said to be sparse, if the union

$$\Omega_k \equiv \bigcup_{j\in J_k} Q_j^k, \quad k = 1,2,\dots$$

satisfies;

$$\chi_{\Omega_{k+1}} = \sum_{j\in J_{k+1}} \chi_{Q_j^{k+1}} \le \chi_{\Omega_k} = \sum_{j\in J_k} \chi_{Q_j^k} \le 1, \tag{2.3}$$

and

$$2|\Omega_{k+1}\cap Q_j^k| \le |Q_j^k| \quad (j\in J_k).$$

Let Q_0 be a cube and $f\colon Q_0 \to \mathbf{R}$ be a measurable function. Choose $m_f(Q_0)$ so that

$$|\{x\in Q_0\colon f(x) > m_f(Q_0)\}|, \quad |\{x\in Q_0\colon f(x) < m_f(Q_0)\}| \le \frac{1}{2}|Q_0|. \tag{2.4}$$

Note that $m_f(Q_0)$ is not determined uniquely. The quantity $m_f(Q_0)$ is called the median of f over Q_0. The mean oscillation of f over a cube Q of level $\lambda\in(0,1)$ is given by;

$$\omega_\lambda(f;Q) \equiv \inf_{c\in\mathbf{C}}((f-c)\chi_Q)^*(\lambda|Q|),$$

where $*$ denotes the decreasing rearrangement for functions. Hytönen showed that there exists a sparse family $\{Q_j^k\}_{k\in\mathbf{N}\cup\{0\},\, j\in J_k} \subset \mathcal{D}(Q_0)$ such that

$$|f(x) - m_f(Q_0)| \le 2\sum_{k=0}^{\infty}\sum_{j\in J_k}\omega_{2^{-n-2}}(f;Q_j^k)\chi_{Q_j^k}(x)$$

for a.e. $x\in Q_0$ [27]. See also [38].

Motivated by this, define the *distributional dyadic maximal operator* $M^{\sharp}_{2^{-n-2}}$ by

$$M^{\sharp,d}_{2^{-n-2}}f(x) \equiv \sup_{Q\in\mathcal{D}(Q_0)} \chi_Q(x)\omega_{2^{-n-2}}(f;Q), \quad x\in\mathbf{R}^n.$$

3. Hardy–Littlewood maximal function – Proof of Theorem 1.2 and Proposition 1.4

3.1. Proof of Theorem 1.2. Assume that (1) holds. One can deduce the condition

$$\frac{1}{|R|}\|\chi_Q\|_{L\mathcal{M}^p_q(u,w)}\|\chi_R\|_{L\mathcal{M}^p_q(u,w)'} \lesssim 1$$

for all dyadic cubes $Q\in L\mathcal{Q}$ and all cubes $R\in\mathcal{Q}$ with $R\supset 2Q$ in a well-known manner: simply use

$$\left(\frac{1}{|R|}\int_R |f(y)|\,\mathrm{d}y\right)\chi_Q \le \inf_{x\in Q} Mf(x).$$

Meanwhile, if $Q \in L\mathcal{Q}$, since we are assuming that M is bounded on $L\mathcal{M}^p_q(u,w)$,

$$\begin{aligned}
&u([-4\ell(Q),4\ell(Q)]^n)^{\frac{1}{p}-\frac{1}{q}}\|\chi_{2Q}M[f\chi_{2Q}]\|_{L^q(w)}\\
&\lesssim \|M[f\chi_{2Q}]\|_{L\mathcal{M}^p_q(u,w)}\\
&\lesssim \|f\chi_{2Q}\|_{L\mathcal{M}^p_q(u,w)}\\
&\sim u([-\ell(Q)/2,\ell(Q)/2]^n)^{\frac{1}{p}-\frac{1}{q}}\|f\chi_{2Q}\|_{L^q(w)}.
\end{aligned}$$

Thus, thanks to the doubling property of u at the origin,

$$\|M[f\chi_{2Q}]\|_{L^q(w)} \lesssim \|f\chi_{2Q}\|_{L^q(w)}.$$

This is equivalent to

$$\sup_{R\in\mathcal{Q},\,R\subset 2Q} \frac{1}{|R|}\|\chi_R\|_{L^q(w)}\|\chi_R\|_{L^q(w)'} \lesssim 1.$$

Let us prove the converse: assume that (2) holds. We have that, for $Q \in L\mathcal{Q}$,

$$\chi_Q(x)Mf(x) \le \chi_Q(x)M[f\chi_{2Q}](x) + C\sup_{R\in\mathcal{Q},\,R\supset 2Q}\frac{\chi_Q(x)}{|R|}\int_R |f(y)|\,\mathrm{d}y.$$

Together with the A_q-property at Q this implies

$$\begin{aligned}
&u(Q)^{\frac{1}{p}-\frac{1}{q}}\|\chi_Q Mf\|_{L^q(w)}\\
&\lesssim u(Q)^{\frac{1}{p}-\frac{1}{q}}\|\chi_Q M[f\chi_{2Q}]\|_{L^q(w)} + u(Q)^{\frac{1}{p}-\frac{1}{q}}\|\chi_Q\|_{L^q(w)}\sup_{R\in\mathcal{Q},\,R\supset 2Q}\frac{1}{|R|}\int_R |f(y)|\,\mathrm{d}y\\
&\lesssim u([-4\ell(Q),4\ell(Q)]^n)^{\frac{1}{p}-\frac{1}{q}}\|f\chi_{2Q}\|_{L^q(w)}\\
&\quad + \|\chi_Q\|_{L\mathcal{M}^p_q(u,w)}\sup_{R\in\mathcal{Q},\,R\supset 2Q}\frac{1}{|R|}\|\chi_R\|_{L\mathcal{M}^p_q(u,w)'}\|f\|_{L\mathcal{M}^p_q(u,w)}\\
&\lesssim \|f\|_{L\mathcal{M}^p_q(u,w)},
\end{aligned}$$

where we have used Lemma 1.1 and the conditions (i) and (ii) of u. Thus, the proof is complete.

3.2. Proof of Proposition 1.4. We need only verify that $w \in \mathcal{G}$, if (1) holds. Once this is verified, then we are in the position of using Theorem 1.2.

For $Q = [-R,R]^n$, $R > 0$, and any measurable set $E \subset Q$, since

$$\frac{|E|}{|Q|}w(Q)^{\frac{1}{p}} = w(Q)^{\frac{1}{p}-\frac{1}{q}}\left\|\frac{|E|}{|Q|}\chi_Q\right\|_{L^q(w)},$$

$L^p(w) \hookrightarrow L\mathcal{M}^p_q(w,w)$ and M is assumed bounded on $L\mathcal{M}^p_q(w,w)$, we have that

$$\frac{|E|}{|Q|}w(Q)^{\frac{1}{p}} \le \|M\chi_E\|_{L\mathcal{M}^p_q(w,w)} \lesssim \|\chi_E\|_{L\mathcal{M}^p_q(w,w)} \le \|\chi_E\|_{L^p(w)} = w(E)^{\frac{1}{p}}.$$

This A_∞-property of w at the origin is more than enough to guarantee that $w \in \mathcal{G}$.

4. Boundedness of the singular integral operators – Proof of Theorems 1.5, 1.6 and Proposition 1.7

4.1. Proof of Theorem 1.5. Let $Q_0 \in L\mathcal{Q}$ be a fixed cube. Form the Lerner–Hytönen decomposition of Tf at Q_0. Then we obtain a sparse family of dyadic cubes

$\{Q_j^k\}_{k\in\mathbf{N}_0,\, j\in J_k}$ in Q_0 satisfying

$$|Tf(x)-m_{Tf}(Q_0)| \le 2\sum_{k=0}^{\infty}\sum_{j\in J_k}\omega_{2^{-n-2}}(Tf;Q_j^k)\chi_{Q_j^k}(x)$$

for almost every $x\in Q_0$. Thus we have

$$\begin{aligned}&u(Q_0)^{\frac{1}{p}-\frac{1}{q}}\|w^{\frac{1}{q}}\chi_{Q_0}Tf\|_{L^q}\\ &\le 2u(Q_0)^{\frac{1}{p}-\frac{1}{q}}\left\|w^{\frac{1}{q}}\sum_{k=0}^{\infty}\sum_{j\in J_k}\omega_{2^{-n-2}}(Tf;Q_j^k)\chi_{Q_j^k}\right\|_{L^q}+u(Q_0)^{\frac{1}{p}-\frac{1}{q}}w(Q_0)^{\frac{1}{q}}|m_{Tf}(Q_0)|.\end{aligned}$$

Let us set $\sigma=w^{-\frac{1}{q-1}}$. To dualize the first term in the right-hand side, we choose a non-negative function $g\in L^{q'}(\sigma)$ and consider

$$\mathrm{I}=\int_{\mathbf{R}^n}\sum_{k=0}^{\infty}\sum_{j\in J_k}\omega(Tf;Q_j^k)\chi_{Q_j^k}(x)g(x)\,dx.$$

Then we have

$$\begin{aligned}\mathrm{I}&=\sum_{k=0}^{\infty}\sum_{j\in J_k}\int_{\mathbf{R}^n}\omega(Tf;Q_j^k)\chi_{Q_j^k}(x)g(x)\,dx\\ &=\sum_{k=0}^{\infty}\sum_{j\in J_k}\omega(Tf;Q_j^k)\int_{Q_j^k}g(x)\,dx\\ &\le\sum_{k=0}^{\infty}\sum_{j\in J_k}\omega(Tf;Q_j^k)|Q_j^k|\inf_{x\in Q_j^k}Mg(x).\end{aligned}$$

Let us set $E_j^k=Q_j^k\setminus\Omega_{k+1}$. Then we have $2|E_j^k|\ge|Q_j^k|$. Since T is weak-$(1,1)$ bounded, we have

$$\mathrm{I}\le 2\sum_{k=0}^{\infty}\sum_{j\in J_k}\inf_{x\in Q_j^k}Mf(x)|E_j^k|\inf_{x\in Q_j^k}Mg(x).$$

Since $\{E_j^k\}_{k\in\mathbf{N},j\in J_k}$ is a disjoint family contained in Q_0, we have

$$\mathrm{I}\le 2\int_{Q_0}Mf(x)Mg(x)\,dx.$$

If we use the $L^{q'}(\sigma)$-boundedness of M in Q_0 and $\|g\|_{L^{q'}(\sigma)}=1$, then we have

$$\left\|w^{\frac{1}{q}}\sum_{k=0}^{\infty}\sum_{j\in J_k}\omega_{2^{-n-2}}(Tf;Q_j^k)\chi_{Q_j^k}\right\|_{L^q}\lesssim\|w^{\frac{1}{q}}Mf\|_{L^q(Q_0)}.$$

Consequently, since M is bounded thanks to Theorem 1.2,

$$\begin{aligned}u(Q_0)^{\frac{1}{p}-\frac{1}{q}}\left\|w^{\frac{1}{q}}\sum_{k=0}^{\infty}\sum_{j\in J_k}\omega_{2^{-n-2}}(Tf;Q_j^k)\chi_{Q_j^k}\right\|_{L^q}&\lesssim u(Q_0)^{\frac{1}{p}-\frac{1}{q}}\|w^{\frac{1}{q}}Mf\|_{L^q(Q_0)}\\ &\lesssim\|Mf\|_{L\mathcal{M}^p_q(u,w)}\\ &\lesssim\|f\|_{L\mathcal{M}^p_q(u,w)}.\end{aligned}$$

Since we are assuming that there exists $\alpha>1$ such that

$$\|\chi_{2^{k+1}Q_0}\|_{L\mathcal{M}^p_q(u,w)}\ge\alpha\|\chi_{2^kQ_0}\|_{L\mathcal{M}^p_q(u,w)}$$

for all $k = 0, 1, 2, \ldots$ and

$$|m_{Tf}(Q_0)| \lesssim \sum_{l=1}^{\infty} \frac{1}{|2^l Q_0|} \int_{2^l Q_0} |f(x)| \,\mathrm{d}x,$$

we have

$$\begin{aligned}|m_{Tf}(Q)| \cdot u(Q_0)^{\frac{1}{p}-\frac{1}{q}} w(Q_0)^{\frac{1}{q}} &\lesssim \sum_{l=1}^{\infty} \alpha^{-l} u(2^l Q_0)^{\frac{1}{p}-\frac{1}{q}} w(2^l Q_0)^{\frac{1}{q}} \frac{1}{|2^l Q_0|} \int_{2^l Q_0} |f(x)| \,\mathrm{d}x \\ &\lesssim \|Mf\|_{L\mathcal{M}^p_q(u,w)} \\ &\lesssim \|f\|_{L\mathcal{M}^p_q(u,w)},\end{aligned}$$

as was to be shown.

4.2. Proof of Theorem 1.6. We say that a sequence $Q_0, Q_1, \ldots, Q_K$ in $L\mathcal{Q}$ is a chain if $\ell(Q_{k-1}) = 2\ell(Q_k)$ and $\overline{Q_{k-1}}$ and $\overline{Q_k}$ intersect at a set of Lebesgue measure zero for all $k = 1, 2, \ldots, K$ and $Q_j \cap Q_j = \emptyset$ if $|j - k| \geq 1$. We need a lemma.

Lemma 4.1. *Let $1 < q < p < \infty$ and let w be a weight. Assume that $u \in \mathcal{G}$ and that $\|R_j f\|_{L\mathcal{M}^p_q(u,w)} \lesssim \|f\|_{L\mathcal{M}^p_q(u,w)}$ for all $f \in L^\infty_{\mathrm{c}}$ and $j = 1, 2, \ldots, n$. Suppose that we are given a chain Q_0, Q_1, Q_2, Q_3 in $L\mathcal{Q}$.*

(1) *The cubes $\overline{2Q_0}$ and $\overline{Q_3}$ do not intersect.*

(2) *For any non-negative $f \in L^\infty_{\mathrm{c}}$ supported on $2Q_0$, we have*

$$\sum_{j=1}^{n} |R_j f(x)| \gtrsim \frac{1}{|Q_0|} \int_{2Q_0} f(y) \,\mathrm{d}y \quad (x \in Q_3).$$

(3) $\|\chi_{2Q_0}\|_{L\mathcal{M}^p_q(u,w)} \sim \|\chi_{Q_3}\|_{L\mathcal{M}^p_q(u,w)}$.

(4) *There exists a constant $C > 0$ independent of $Q_0 \in L\mathcal{Q}$ such that*

$$\frac{1}{|Q_0|} \|\chi_{2Q_0}\|_{L^q(w)} \|\chi_{2Q_0}\|_{L^q(w)'} \leq C.$$

Proof. We suppose that $Q_0 = [2^m, 2^{m+1})^n$ and $Q_3 = [2^{m-3}, 2^{m-2})^n$ for the sake of simplicity.

(1) A geometric observation shows that $\overline{2Q_0} = [2^{m-1}, 5 \cdot 2^{m-1}]^n$ and $\overline{Q_3} = [2^{m-3}, 2^{m-2}]^n$, so that $\overline{2Q_0}$ and $\overline{Q_3}$ do not intersect.

(2) Let $f \in L^\infty_{\mathrm{c}}(2Q_0)$ be a non-negative function. Since

$$-\sum_{j=1}^{n} \frac{x_j - y_j}{|x - y|^{n+1}} \sim \frac{1}{\ell(Q_0)^n}$$

for any $x \in Q_3$ and $y \in 2Q_0$, we have

$$\sum_{j=1}^{n} |R_j f(x)| = -\sum_{j=1}^{n} R_j f(x) = -\sum_{j=1}^{n} \int_{\mathbf{R}^n} \frac{x_j - y_j}{|x - y|^{n+1}} f(y) \, dy \gtrsim \frac{1}{|Q_0|} \int_{2Q_0} f(y) \,\mathrm{d}y.$$

(3) By considering $f = \chi_{2Q_0}$ in (2) and using $\|R_j \chi_{2Q_0}\|_{L\mathcal{M}^p_q(u,w)} \lesssim \|\chi_{2Q_0}\|_{L\mathcal{M}^p_q(u,w)}$, we can show that $\|\chi_{Q_3}\|_{L\mathcal{M}^p_q(u,w)} \gtrsim \|\chi_{2Q_0}\|_{L\mathcal{M}^p_q(u,w)}$. We can swap the role of $2Q_0$ and Q_3 to have $\|\chi_{2Q_0}\|_{L\mathcal{M}^p_q(u,w)} \lesssim \|\chi_{Q_3}\|_{L\mathcal{M}^p_q(u,w)}$ if we go through a similar argument.

(4) From (2) we have

$$\|\chi_{Q_0}\|_{L\mathcal{M}^p_q(u,w)} \frac{1}{|Q_0|} \int_{2Q_0} f(y) \,\mathrm{d}y \lesssim \|f\|_{L\mathcal{M}^p_q(u,w)} \sim u(Q_0)^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q(w)}$$

for any non-negative measurable function f supported on $2Q_0$. It remains to use the duality. $\square$

Going through a similar argument, we have the following corollary.

Corollary 4.2. *Let $1 < q < p < \infty$ and let w be a weight. Assume that $u \in \mathcal{G}$ and that $\|R_j f\|_{L\mathcal{M}^p_q(u,w)} \lesssim \|f\|_{L\mathcal{M}^p_q(u,w)}$ for all $f \in L^\infty_{\rm c}$ and $j = 1, 2, \ldots, n$. Then there exists a constant $C > 0$ independent of $Q \in L\mathcal{Q}$ such that*

$$\sup_{R \in \mathcal{Q},\, R \subset 2Q} \frac{1}{|R|} \|\chi_R\|_{L^q(w)} \|\chi_R\|_{L^q(w)'} \le C.$$

Proof. Let $R \in \mathcal{Q}$ satisfy $R \subset 2Q$. By decomposing R or by expanding R, we can assume that R is a dyadic cube. Then we can choose a sequence $S = S_1, S_2, S_3 = R$ of cubes such that $\ell(R) \sim \ell(S_k) \sim \operatorname{dist}(R, S)$ for $k = 1, 2, 3$, that S_k and S_{k+1} intersect at a point for $k = 1, 2$ and that $2S_1 \cap 2S_3 = \emptyset$. If we replace Q_3 by S and Q_0 by R, respectively in the proof of Lemma 4.1, we can argue as before. $\square$

We move on to the proof of Theorem 1.6. In view of Corollary 4.2, it remains to show that there exists a constant C which is independent of $Q \in L\mathcal{Q}$ such that

$$\sup_{R \in \mathcal{Q},\, R \supset Q} \frac{1}{|R|} \|\chi_Q\|_{L\mathcal{M}^p_q(u,w)} \|\chi_R\|_{L\mathcal{M}^p_q(u,w)'} \le C$$

for all cubes $Q \in L\mathcal{Q}$. By expanding R we can assume that $R \supset 2Q$. By decomposing R suitably, we can further assume that there exists a chain $Q = Q_0, Q_1, Q_2, Q_3$ such that R and $2Q_3$ do not intersect.

Then for any non-negative function $f \in L^\infty$ supported on R, we have

$$\frac{1}{|R|} \int_R f(x)\,\mathrm{d}x \lesssim \sum_{j=1}^n |R_j f(y)|,$$

for all $y \in Q_3$ As a result, assuming that $\|R_j f\|_{L\mathcal{M}^p_q(u,w)} \lesssim \|f\|_{L\mathcal{M}^p_q(u,w)}$, we see

$$\left(\frac{1}{|R|} \int_R f(x)\,\mathrm{d}x\right) \|\chi_{Q_3}\|_{L\mathcal{M}^p_q(u,w)} \lesssim \|f\|_{L\mathcal{M}^p_q(u,w)}.$$

Since $\|\chi_{Q_3}\|_{L\mathcal{M}^p_q(u,w)} \sim \|\chi_Q\|_{L\mathcal{M}^p_q(u,w)}$ thanks to Lemma 4.1(3), we obtain

$$\left(\frac{1}{|R|} \int_R f(x)\,\mathrm{d}x\right) \|\chi_Q\|_{L\mathcal{M}^p_q(u,w)} \lesssim \|f\|_{L\mathcal{M}^p_q(u,w)}.$$

By passing to the Köthe dual, we obtain $\|\chi_R\|_{L\mathcal{M}^p_q(u,w)'} \|\chi_Q\|_{L\mathcal{M}^p_q(u,w)} \lesssim |R|$, as required.

4.3. Proof of Proposition 1.7. We need to prove that $w \in \mathcal{G}$ in particular, w is reverse doubling at the origin. To this end, we observe

$$\|R_j f\|_{L\mathcal{M}^p_q(w,w)} \lesssim \|f\|_{L^p(w)}$$

for all $f \in L^\infty_{\rm c}$. Let Q be a cube centered at the origin and let Q' be a cube such that $|Q| = |Q'|$ and $\sharp(Q \cap Q') = 1$, where $\sharp E$ stands for the cardinality of the set $E \subset \mathbf{R}^n$. Then since w is doubling, we have

$$w(Q)^{\frac{1}{p}} \frac{1}{|Q|} \int_Q |f(y)|\,\mathrm{d}y \sim w(Q')^{\frac{1}{p}} \frac{1}{|Q|} \int_Q |f(y)|\,\mathrm{d}y \lesssim \|f\|_{L^p(w)}$$

for all $f \in L^\infty$ with $\operatorname{supp}(f) \subset Q$. Let R be a cube contained in Q. If we let $f = w^{-\frac{1}{q}} \chi_R$, then we obtain $w \in A_{q+1}$. Thus, we see that $w \in \mathcal{G}$ and we are in the position of using Theorem 1.6.

5. Fractional integral operators – Proof of Theorem 1.12

We can prove that (2) and (3) are equivalent similar to Theorem 1.2. Meanwhile, (1) is clearly stronger than (2). It remains to show that (3) implies (1) under an additional assumption. This is achieved similar to Proposition 1.8 using $M^\sharp_{2^{-n-2}}(I_\alpha f)(x) \lesssim M_\alpha f(x), (x \in \mathbf{R}^n)$.

6. The case of the power weight – Proof of Propositions 1.8 and 1.9, 1.13 and 1.14

Here we consider the case where $w(x) = |x|^\beta$ where $\beta > -n$. Note that $w \in A_\infty$, so that $w \in \mathcal{G}$.

6.1. Proof of Proposition 1.8. Let us assume $-\frac{q}{p}n \le \beta < q\left(1 - \frac{1}{p}\right)n$. It is clear that

$$\sup_{R\in\mathcal{D},\, R\subset Q} \frac{1}{|R|}\|\chi_R\|_{L^q(w)}\|\chi_R\|_{L^q(w)'} \lesssim \sup_{R\in\mathcal{D},\, R\subset Q} \frac{1}{|R|}\|\chi_R w(c_Q)^{\frac{1}{q}}\|_{L^q}\|\chi_R w(c_Q)^{-\frac{1}{q}}\|_{L^{q'}} = 1$$

for all cubes $Q \in L\mathcal{Q}$. Next we will prove

$$\|\chi_Q\|_{L^q(w)}\|\chi_R\|_{L\mathcal{M}^p_q(1,w)'} \le C|Q|^{\frac{1}{q}-\frac{1}{p}}|R|$$

or equivalently

$$\|\chi_Q\|_{L\mathcal{M}^p_q(1,w)}\|\chi_R\|_{L\mathcal{M}^p_q(1,w)'} \lesssim |R|$$

for all cubes $Q \in L\mathcal{Q}$ and R such that $Q \subset R$. To this end, by replacing R with a larger one, say $10R$, we may assume that $R = [-2r, 2r]^n$ is centered at the origin. Write $R^* = [r, 2r]^n$. By the dilation formula for $(L\mathcal{M}^p_q)'$, we have

$$\|\chi_{R^*}(2^l\cdot)w^{-\frac{1}{p}}\|_{(L\mathcal{M}^p_q)'} = 2^{\frac{l\beta}{q}}\|\chi_{R^*}(2^l\cdot)w(2^l\cdot)^{-\frac{1}{q}}\|_{(L\mathcal{M}^p_q)'} = 2^{\frac{l\beta}{q}-\frac{ln}{p'}}\|\chi_{R^*}\cdot w^{-\frac{1}{q}}\|_{(L\mathcal{M}^p_q)'}.$$

Then since $\beta < q\left(1 - \frac{1}{p}\right)n$, we have

$$\begin{aligned}
\|\chi_R\|_{L\mathcal{M}^p_q(1,w)'} &= \|\chi_R w^{-\frac{1}{q}}\|_{(L\mathcal{M}^p_q)'} \le \sum_{l=0}^\infty \|\chi_{2^{-l}R\setminus 2^{-l-1}R} w^{-\frac{1}{q}}\|_{(L\mathcal{M}^p_q)'}\\
&\sim \sum_{l=0}^\infty \|\chi_{R^*}(2^l\cdot)w^{-\frac{1}{q}}\|_{(L\mathcal{M}^p_q)'} \sim \|\chi_{R^*}w^{-\frac{1}{q}}\|_{(L\mathcal{M}^p_q)'}\\
&\sim w(c_{R^*})^{-\frac{1}{q}}\|\chi_{R^*}\|_{(L\mathcal{M}^p_q)'}.
\end{aligned}$$

Thus, since $\frac{q}{p}n + \beta > 0$, thanks to [44, Example 2.3]

$$\begin{aligned}
\|\chi_Q\|_{L^q(w)}\|\chi_R\|_{L\mathcal{M}^p_q(1,w)'} &\lesssim w(c_Q)^{\frac{1}{q}}\|\chi_Q\|_{L^q} w(c_{R^*})^{-\frac{1}{q}}\|\chi_{R^*}\|_{(L\mathcal{M}^p_q)'}\\
&\lesssim \ell(Q)^{\frac{n+\beta}{q}} w(c_{R^*})^{-\frac{1}{q}}|R|^{\frac{1}{p'}} \lesssim \ell(Q)^{\frac{n+\beta}{q}}\ell(R)^{n-\frac{n+\beta}{p}}\\
&\lesssim |Q|^{\frac{1}{q}-\frac{1}{p}}|R|.
\end{aligned}$$

Let us assume that M is bounded on $\mathrm{L}\mathcal{M}^p_q(1, |x|^\beta)$. Then $\chi_{[-1,1]^n} \in L\mathcal{M}^p_q(1, w_\beta)$ and $\frac{\chi_B|\cdot|^{-n}}{\log|\cdot|} \notin L\mathcal{M}^p_q(1, w_\beta)$, where B is a small open ball centered at the origin. We observe that $\chi_{[-1,1]^n} \in L\mathcal{M}^p_q(1, w_\beta)$ if and only if $\beta \ge -\frac{q}{p}n$ and that $\frac{\chi_B|\cdot|^{-n}}{\log|\cdot|} \notin L\mathcal{M}^p_q(1, w_\beta)$ if and only if $\beta < q(1 - \frac{1}{p})n$.

6.2. Proof of Proposition 1.9. According to [6], (2) and (3) are equivalent. If we assume (1), then M is bounded from $L^p(w)$ to $L\mathcal{M}^p_q(w,w)$. As a result,

$$w(Q)^{\frac{1}{p}}\left(\frac{1}{|Q|}\int_Q |f(y)|\,\mathrm{d}y\right) \le C\|f\|_{L^p(w)}$$

for all measurable functions f and all cubes centered at the origin. Thus,

$$\frac{w(Q)}{|Q|}\cdot\left(\frac{\sigma(Q)}{|Q|}\right)^{p-1} \lesssim 1$$

for all cubes centered at the origin, where $\sigma(x) = w(x)^{-\frac{1}{p-1}}$, $x \in \mathbf{R}^n$. As a consequence we have $-n < \beta < n(p-1)$.

Assume $-n < \beta < n(p-1)$. As before, it is clear that

$$\sup_{R\in\mathcal{D},\,R\subset Q}\frac{1}{|R|}\|\chi_R\|_{L^q(w)}\|\chi_R\|_{L^q(w)'} \le C\sup_{R\in\mathcal{D},\,R\subset Q}\frac{1}{|R|}\|\chi_R w(c_Q)^{\frac{1}{q}}\|_{L^q}\|\chi_R w(c_Q)^{-\frac{1}{q}}\|_{L^{q'}} = C$$

for all cubes $Q \in L\mathcal{Q}$.

Next we will establish

$$\|\chi_Q\|_{L^q(w)}\|\chi_R\|_{L\mathcal{M}^p_q(w,w)'} \le Cw(Q)^{\frac{1}{q}-\frac{1}{p}}|R| \tag{6.1}$$

or equivalently

$$\|\chi_Q\|_{L\mathcal{M}^p_q(w,w)}\|\chi_R\|_{L\mathcal{M}^p_q(w,w)'} \lesssim |R|$$

for all cubes $Q \in L\mathcal{Q}$ and R such that $Q \subset R$. To this end, by replacing R with a larger one, say $Q(0, 10\ell(R))$, we may assume that $R = [-2r, 2r]^n$ is centered at the origin. Write $R^* = [r, 2r]^n$. By the dilation formula for $L\mathcal{M}^p_q(w,w)'$, we have

$$\|\chi_{R^*}(2^l\cdot)\|_{L\mathcal{M}^p_q(w,w)'} = 2^{-ln+\frac{l(n+\beta)}{p}}\|\chi_{R^*}\|_{L\mathcal{M}^p_q(w,w)'}.$$

Since w is a power weight

$$\|\chi_{2^{-l}R\setminus 2^{-l-1}R}\|_{L\mathcal{M}^p_q(w,w)'} \sim \|\chi_{R^*}(2^l\cdot)\|_{L\mathcal{M}^p_q(w,w)'}.$$

Thus,

$$\|\chi_R\|_{L\mathcal{M}^p_q(w,w)'} \le \sum_{l=0}^{\infty}\|\chi_{2^{-l}R\setminus 2^{-l-1}R}\|_{L\mathcal{M}^p_q(w,w)'} \lesssim \sum_{l=0}^{\infty}\|\chi_{R^*}(2^l\cdot)\|_{L\mathcal{M}^p_q(w,w)'}.$$

Assuming $\beta < n(p-1)$, we have

$$\begin{aligned}\|\chi_R\|_{L\mathcal{M}^p_q(w,w)'} &\lesssim \sum_{l=0}^{\infty}2^{-ln+\frac{l(n+\beta)}{p}}\|\chi_{R^*}\|_{L\mathcal{M}^p_q(w,w)'} \sim \|\chi_{R^*}\|_{L\mathcal{M}^p_q(w,w)'}\\ &\lesssim w(R^*)^{\frac{1}{q}-\frac{1}{p}}w(c_{R^*})^{-\frac{1}{q}}|R^*|^{\frac{1}{q'}} \lesssim w(R^*)^{-\frac{1}{p}}|R^*|.\end{aligned}$$

As a result, since $n+\beta > 0$, thanks to [44, Example 2.3]

$$\begin{aligned}\|\chi_Q\|_{L^q(w)}\|\chi_R\|_{L\mathcal{M}^p_q(w,w)'} &\lesssim w(Q)^{\frac{1}{q}}w(R^*)^{-\frac{1}{p}}|R^*| \lesssim \ell(Q)^{\frac{n+\beta}{q}}\ell(R)^{n-\frac{n+\beta}{p}}\\ &\lesssim \ell(Q)^{\frac{n+\beta}{q}-\frac{n+\beta}{p}}|R| \lesssim w(Q)^{\frac{1}{q}-\frac{1}{p}}|R|.\end{aligned}$$

Thus (6.1) is proved.

6.3. Proof of Proposition 1.11. (1) is sufficient for (2) and (3) similar to Proposition 1.8. Let Q be a cube. We need to show that

$$w(Q)^{\frac{1}{p}-\frac{1}{q}}\left(\int_Q Mf(x)^q w(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \lesssim \|f\|_{L\mathcal{M}^p_q(w,w)} \tag{6.2}$$

for all cubes Q. If $-n<\beta<n(q-1)$, then we can go through the same argument as [37] which is followed by [12, 40, 56, 59] and so on. So we assume $\beta \ge n(q-1)$. We may assume that Q is centered at the origin or that $0 \notin 4Q$. Let $\varepsilon \in (0, \frac{q}{p}(n(p-1)-\beta))$. We distinguish four cases:

(1) Assume that f is supported on $3Q$ and that Q is centered at the origin. We borrow the idea of [7]. In this case, we have

$$w(Q)=w_\beta(Q) \sim \ell(Q)^{n+\beta}$$

from [44, Example 2.3] and

$$\left(\int_Q Mf(x)^q w(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \lesssim \ell(Q)^{\frac{\beta-n(q-1)+\varepsilon}{q}} \left(\int_Q Mf(x)^q |x|^{n(q-1)-\varepsilon}\,\mathrm{d}x\right)^{\frac{1}{q}},$$

since $\beta \ge n(q-1) > n(q-1)-\varepsilon$. Since $|x|^{n(q-1)-\varepsilon} \in A_q$, we have

$$\left(\int_Q Mf(x)^q |x|^{n(q-1)-\varepsilon}\,\mathrm{d}x\right)^{\frac{1}{q}} \lesssim \left(\int_{3Q} |f(x)|^q |x|^{n(q-1)-\varepsilon}\,\mathrm{d}x\right)^{\frac{1}{q}}.$$

We next decompose $3Q$ dyadically to have

$$\begin{aligned}
&\left(\int_{3Q} |f(x)|^q |x|^{n(q-1)-\varepsilon}\,\mathrm{d}x\right)^{\frac{1}{q}}\\
&\le \sum_{l=0}^{\infty} \left(\int_{3\cdot 2^{-l}Q \setminus 3\cdot 2^{-l-1}Q} |f(x)|^q |x|^{n(q-1)-\varepsilon}\,\mathrm{d}x\right)^{\frac{1}{q}}\\
&\sim \sum_{l=0}^{\infty} (2^{-l}\ell(Q))^{\frac{n(q-1)-\varepsilon-\beta}{q}} \left(\int_{3\cdot 2^{-l}Q \setminus 3\cdot 2^{-l-1}Q} |f(x)|^q |x|^{\beta}\,\mathrm{d}x\right)^{\frac{1}{q}}\\
&\lesssim \sum_{l=0}^{\infty} (2^{-l}\ell(Q))^{\frac{n(q-1)-\varepsilon-\beta}{q}} (2^{-l}\ell(Q))^{\frac{n+\beta}{q}-\frac{n+\beta}{p}} \|f\|_{\mathcal{M}^p_q(1,w_\beta)}.
\end{aligned}$$

Arithmetic shows that

$$\frac{n(q-1)-\varepsilon-\beta}{q}+\frac{n+\beta}{q}-\frac{n+\beta}{p}=\frac{n(p-1)-\beta}{p}-\frac{\varepsilon}{q}>0.$$

Thus, the series in the most right-hand side converges to have

$$\ell(Q)^{\frac{n+\beta}{p}-\frac{n+\beta}{q}} \ell(Q)^{\frac{\beta-n(q-1)+\varepsilon}{q}} \left(\int_{3Q} |f(x)|^q |x|^{n(q-1)-\varepsilon}\,\mathrm{d}x\right)^{\frac{1}{q}} \lesssim \|f\|_{\mathcal{M}^p_q(w,w)}. \tag{6.3}$$

Consequently, we have (6.2).

(2) Assume that f is supported outside $3Q$ and that Q is centered at the origin. In this case, using [44, Example 2.3] again, we have

$$\begin{aligned}
w(Q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q Mf(x)^q w(x)\,\mathrm{d}x\right)^{\frac{1}{q}} &\lesssim \ell(Q)^{\frac{n+\beta}{p}} \sup_{R>\ell(Q)} \frac{1}{|Q(R)|} \int_{Q(R)} |f(y)|\,\mathrm{d}y\\
&\lesssim \sup_{l \in \mathbf{N}} \ell(Q)^{\frac{n+\beta}{p}} \frac{1}{|2^l Q|} \int_{2^l Q \setminus Q} |f(y)|\,\mathrm{d}y.
\end{aligned}$$

Let $l \in \mathbf{N}$ be fixed. We decompose $2^l Q \setminus Q$ dyadically to have

$$\ell(Q)^{\frac{n+\beta}{p}} \frac{1}{|2^l Q|} \int_{2^l Q \setminus Q} |f(y)|\,\mathrm{d}y = \ell(Q)^{\frac{n+\beta}{p}} \sum_{k=1}^{l} \frac{1}{|2^l Q|} \int_{2^k Q \setminus 2^{k-1}Q} |f(y)|\,\mathrm{d}y$$

If we use the Hölder inequality, then we have

$$
\begin{aligned}
&\ell(Q)^{\frac{n+\beta}{p}} \frac{1}{|2^l Q|} \int_{2^l Q \setminus Q} |f(y)| \, \mathrm{d}y \\
&\le \ell(Q)^{\frac{n+\beta}{p}} \sum_{k=1}^{l} 2^{n(k-l)} \left(\frac{1}{|2^k Q|} \int_{2^k Q \setminus 2^{k-1} Q} |f(y)|^q \, \mathrm{d}y \right)^{\frac{1}{q}} \\
&\lesssim \ell(Q)^{\frac{n+\beta}{p} - \frac{\beta}{q}} \sum_{k=1}^{l} 2^{n(k-l) - \frac{\beta}{q} k} \left(\frac{1}{|2^k Q|} \int_{2^k Q \setminus 2^{k-1} Q} |f(y)|^q |y|^\beta \, \mathrm{d}y \right)^{\frac{1}{q}} \\
&\lesssim \sum_{k=1}^{l} 2^{n(k-l) - \frac{n+\beta}{p} k} \|f\|_{\mathcal{M}^p_q(w,w)}.
\end{aligned}
$$

Since $\displaystyle\sum_{k=1}^{l} 2^{n(k-l) - \frac{n+\beta}{p} k} \lesssim 1$, we obtain (6.2).

(3) Assume that f is supported on $3Q$ and that $0 \notin 4Q$. Then we have

$$
\begin{aligned}
&w(Q)^{\frac{1}{p} - \frac{1}{q}} \left(\int_Q Mf(x)^q w(x) \, \mathrm{d}x \right)^{\frac{1}{q}} \\
&\sim (|Q| w(c(Q)))^{\frac{1}{p} - \frac{1}{q}} \left(\int_Q Mf(x)^q w(c(Q)) \, \mathrm{d}x \right)^{\frac{1}{q}} \\
&\lesssim (|Q| w(c(Q)))^{\frac{1}{p} - \frac{1}{q}} \left(\int_{3Q} |f(x)|^q w(c(Q)) \, \mathrm{d}x \right)^{\frac{1}{q}} \\
&\lesssim w(Q)^{\frac{1}{p} - \frac{1}{q}} \left(\int_{3Q} |f(x)|^q w(x) \, \mathrm{d}x \right)^{\frac{1}{q}}.
\end{aligned}
$$

(4) Assume that f is supported outside $3Q$ and that $0 \notin 4Q$. Then we have

$$
\begin{aligned}
&w(Q)^{\frac{1}{p} - \frac{1}{q}} \left(\int_Q Mf(x)^q w(x) \, \mathrm{d}x \right)^{\frac{1}{q}} \\
&\sim (|Q| w(c(Q)))^{\frac{1}{p} - \frac{1}{q}} \left(\int_Q Mf(x)^q w(c(Q)) \, \mathrm{d}x \right)^{\frac{1}{q}} \\
&\sim (|Q| w(c(Q)))^{\frac{1}{p}} \sup_{R:\text{cubes}, R \supset Q} \frac{1}{|R|} \int_{R \setminus 3Q} |f(x)| \, \mathrm{d}x \\
&\sim (|Q| w(c(Q)))^{\frac{1}{p}} \sup_{R:\text{cubes}, R \supset Q, c(Q) = c(R)} \frac{1}{|R|} \int_{R \setminus 3Q} |f(x)| \, \mathrm{d}x.
\end{aligned}
$$

If $0 \notin 8R$, $c(Q) = c(R)$ and $Q \subset R$, then we have

$$
\begin{aligned}
&(|Q| w(c(Q)))^{\frac{1}{p}} \frac{1}{|R|} \int_{R \setminus 3Q} |f(x)| \, \mathrm{d}x \\
&\lesssim (|R| w(c(R)))^{\frac{1}{p}} \frac{1}{w(R)} \int_{R \setminus 3Q} |f(x)| w(x) \, \mathrm{d}x \\
&\lesssim w(R)^{\frac{1}{p}} \left(\frac{1}{w(R)} \int_{R \setminus 3Q} |f(x)|^q w(x) \, \mathrm{d}x \right)^{\frac{1}{q}} \le \|f\|_{L\mathcal{M}^p_q(w,w)}.
\end{aligned}
$$

Suppose instead that $0 \in 8R$, $c(Q) = c(R)$ and $Q \subset R$. Then we have

$$\frac{1}{|R|}\int_{R\setminus 3Q}|f(x)|\,\mathrm{d}x \le \frac{1}{|R|}\left(\int_R |f(x)|^q|x|^{n(q-1)-\varepsilon}\,\mathrm{d}x\right)^{\frac{1}{q}}\left(\int_R |x|^{-n+\frac{\varepsilon}{q-1}}\,\mathrm{d}x\right)^{\frac{1}{q'}}$$
$$\lesssim \frac{1}{|R|}\ell(R)^{\frac{\varepsilon}{q}}\left(\int_R |f(x)|^q|x|^{n(q-1)-\varepsilon}\,\mathrm{d}x\right)^{\frac{1}{q}}.$$

From (6.3) we have

$$\left(\int_R |f(x)|^q|x|^{n(q-1)-\varepsilon}\,\mathrm{d}x\right)^{\frac{1}{q}} \lesssim \ell(R)^{\frac{\beta-n(q-1)-\varepsilon}{q}}w(R)^{-\frac{1}{p}+\frac{1}{q}}\|f\|_{\mathcal{M}^p_q(w,w)}.$$

As a result,

$$w(Q)^{\frac{1}{p}}\frac{1}{|R|}\int_{R\setminus 3Q}|f(x)|\,\mathrm{d}x \lesssim w(R)^{\frac{1}{p}}\ell(R)^{\frac{\beta+n}{q}}w(R)^{-\frac{1}{p}+\frac{1}{q}}\|f\|_{\mathcal{M}^p_q(w,w)} \lesssim \|f\|_{\mathcal{M}^p_q(w,w)}.$$

Putting these results all together, we obtain the desired result.

6.4. Proof of Proposition 1.13. If we assume (1), then we have $\chi_{[-1,1]^n} \in L\mathcal{M}^s_t(1,w^t)$ and $\chi_B|\cdot|^{-n} \notin L\mathcal{M}^p_q(1,w^q)$, or equivalently $\chi_B|\cdot|^{-n+\beta} \notin L\mathcal{M}^p_q$ where B is a small open ball centered at the origin. We observe that $\chi_{[-1,1]^n} \in L\mathcal{M}^s_t(1,w^t)$ if and only if $\beta \ge -\frac{n}{s}$, and that $\chi_B|\cdot|^{-n+\beta} \notin L\mathcal{M}^p_q$, if and only if $\beta - n < -\frac{n}{p}$. If we assume $-\frac{n}{s} \le \beta < \frac{n}{p'}$, then we can argue as we did in the proof of Proposition 1.8, to conclude that $w = w_\beta$ satisfies

$$\sup_{R\in\mathcal{Q},\,R\subset 2Q}\frac{\ell(R)^\beta}{|R|}\|\chi_R\|_{L^t(w^t)}\|\chi_R\|_{L^q(w^q)'}+\sup_{R\in\mathcal{Q},\,R\supset Q}\frac{\ell(R)^\beta}{|R|}\|\chi_Q\|_{L\mathcal{M}^s_t(1,w^t)}\|\chi_R\|_{L\mathcal{M}^p_q(1,w^q)'} \le C$$

for all cubes $Q \in L\mathcal{Q}$.

In fact, if R is a cube such that $R \subset 2Q$, then we have

$$\frac{\ell(R)^\beta}{|R|}\|\chi_R\|_{L^t(w^t)}\|\chi_R\|_{L^q(w^q)'} \sim \frac{\ell(R)^\beta}{|R|}\|\chi_R\|_{L^t}\|\chi_R\|_{(L^q)'} = 1.$$

Meanwhile if R is a cube such that $R \supset Q$, we denote by R^* the cube satisfying $|R^*| = 100^n|R|$ and centered at the origin. Then

$$\frac{\ell(R)^\beta}{|R|}\|\chi_Q\|_{L\mathcal{M}^s_t(1,w^t)}\|\chi_R\|_{L\mathcal{M}^p_q(1,w^q)'} \lesssim \frac{\ell(R^*)^\beta}{|R^*|}|Q|^{\frac{1}{s}}|c(Q)|^\beta\|\chi_{R^*}\|_{L\mathcal{M}^p_q(1,w^q)'}.$$

We note that

$$\|\chi_{R^*}\|_{L\mathcal{M}^p_q(1,w^q)'} = \|w^{-1}\chi_{R^*}\|_{(L\mathcal{M}^p_q)'} \le \|w^{-1}\chi_{R^*}\|_{L^{p'}} \lesssim \ell(R)^{\frac{n}{p'}-\beta},$$

where we used $\beta < \frac{n}{p'}$ to guarantee that $\|w^{-1}\chi_{R^*}\|_{L^{p'}} = \|(w_\beta)^{-1}\chi_{R^*}\|_{L^{p'}}$ is finite. Observe also that $c(Q) \sim \ell(Q)$. As a result, since $-\frac{n}{s} \le \beta$,

$$\frac{\ell(R)^\beta}{|R|}\|\chi_Q\|_{L\mathcal{M}^s_t(1,w^t)}\|\chi_R\|_{L\mathcal{M}^p_q(1,w^q)'} \lesssim \frac{\ell(Q)^{\beta+\frac{n}{s}}}{\ell(R)^{\beta+\frac{n}{s}}} \le 1.$$

We can consider I_α by the use of Theorem 1.12.

6.5. Proof of Proposition 1.14. If we assume (1), then $\chi_{[-1,1]^n} \in L\mathcal{M}^p_q(w^t,w^q)$ and $\frac{\chi_B|\cdot|^{-n}}{\log|\cdot|} \notin L\mathcal{M}^p_q(w^t,w^q)$, where as before B is a small open ball centered at the origin. We observe that $\chi_{[-1,1]^n} \in L\mathcal{M}^p_q(w^t, w^q)$ if and only if $\beta t \geq -n$ and $\frac{\chi_B|\cdot|^{-n}}{\log|\cdot|} \notin L\mathcal{M}^p_q(w^t, w^q)$ if and only if

$$\beta < \frac{ns}{p't}.$$

If I_α is bounded from $L\mathcal{M}^p_q(1, w^q)$ to $L\mathcal{M}^s_t(1, w^t)$, then we can rule out the possibility of $\beta = -\frac{n}{t}$, since $|\cdot|^{-\frac{t\beta+n}{p}+\frac{t\beta-q\beta}{q}} \in L\mathcal{M}^s_t(1, w^t)$. If we assume $-\frac{n}{t} < \beta < \frac{ns}{p't}$, then we can argue as we did in the proof of Proposition 1.9, that $w = w_\beta$ satisfy

$$\begin{aligned}\sup_{R\in\mathcal{Q},\, R\subset 2Q} &\frac{\ell(R)^\beta}{|R|}\|\chi_R\|_{L^t(w^t)}\|\chi_R\|_{L^q(w^q)'}\\ &+ \sup_{R\in\mathcal{Q},\, R\supset Q}\frac{\ell(R)^\beta}{|R|}\|\chi_Q\|_{L\mathcal{M}^s_t(w^t,w^t)}\|\chi_R\|_{L\mathcal{M}^p_q(w^t,w^q)'} \leq C\end{aligned}$$

for all cubes $Q \in L\mathcal{Q}$. In fact,

$$\begin{aligned}\sup_{R\in\mathcal{Q},\, R\subset 2Q} &\frac{\ell(R)^\beta}{|R|}\|\chi_R\|_{L^t(w^t)}\|\chi_R\|_{L^q(w^q)'}\\ &\lesssim \sup_{R\in\mathcal{Q},\, R\subset 2Q}\frac{\ell(R)^\beta w(c_R)}{|R|}\|\chi_R\|_{L^t} w(c_R)^{-1}\|\chi_R\|_{(L^q)'} = 1.\end{aligned}$$

We move on to the proof of

$$\sup_{R\in\mathcal{Q},\, R\supset Q}\frac{\ell(R)^\beta}{|R|}\|\chi_Q\|_{L\mathcal{M}^s_t(w^t,w^t)}\|\chi_R\|_{L\mathcal{M}^p_q(w^t,w^q)'} \lesssim 1.$$

By expanding R suitably, we may assume that R is centered at the origin. Let

$$\lambda = \beta\left(1 + \frac{t}{p} - \frac{t}{q}\right) + \frac{n}{p} - \frac{n}{q} + \varepsilon = \beta\frac{t}{s} + \frac{n}{p} - \frac{n}{q} + \varepsilon.$$

Here $\varepsilon > 0$ is chosen small enough to have $\lambda < \frac{n}{q'}$. This is possible because

$$\frac{n}{q'} - \lambda + \varepsilon = \frac{n}{p'} - \beta\left(1 + \frac{t}{p} - \frac{t}{q}\right) > \left(\frac{ns}{p't} - \beta\right)\left(1 + \frac{t}{p} - \frac{t}{q}\right) > 0.$$

Meanwhile,

$$\|\chi_Q\|_{L\mathcal{M}^s_t(w^t,w^t)} \sim w^t(Q)^{\frac{1}{s}} = \ell(Q)^{\frac{n+\beta t}{s}}$$

and for a non-negative measurable function f

$$\int_R f(x)\,\mathrm{d}x \leq \left(\int_R f(x)^q|x|^{q\lambda}\,\mathrm{d}x\right)^{\frac{1}{q}}\left(\int_R |x|^{-q'\lambda}\,\mathrm{d}x\right)^{\frac{1}{q'}} \lesssim \ell(R)^{\frac{n}{q'}-\lambda}\left(\int_R f(x)^q|x|^{q\lambda}\,\mathrm{d}x\right)^{\frac{1}{q}}.$$

We borrow the idea of [7] again. We decompose

$$\left(\int_R f(x)^q|x|^{q\lambda}\,\mathrm{d}x\right)^{\frac{1}{q}} \leq \sum_{j=1}^{\infty}\left(\int_{2^{1-j}R\setminus 2^{-j}R} f(x)^q|x|^{q\lambda}\,\mathrm{d}x\right)^{\frac{1}{q}}$$

as usual. We note that

$$
\begin{aligned}
&\left(\int_R f(x)^q|x|^{q\lambda}\,\mathrm{d}x\right)^{\frac{1}{q}}\\
&\lesssim \sum_{j=1}^{\infty}((2^{-j}\ell(R))^{n+t\beta})^{\frac{1}{q}-\frac{1}{p}}w^t(2^{-j}R)^{\frac{1}{p}-\frac{1}{q}}\left(\int_{2^{1-j}R\setminus 2^{-j}R} f(x)^q|x|^{q\lambda}\,\mathrm{d}x\right)^{\frac{1}{q}}\\
&\lesssim \sum_{j=1}^{\infty}(2^{-j}\ell(R))^{\varepsilon}w^t(2^{-j}R)^{\frac{1}{p}-\frac{1}{q}}\left(\int_{2^{1-j}R\setminus 2^{-j}R} f(x)^q|x|^{q\lambda+(n+t\beta)\left(1-\frac{q}{p}\right)-q\varepsilon}\,\mathrm{d}x\right)^{\frac{1}{q}}\\
&= \sum_{j=1}^{\infty}(2^{-j}\ell(R))^{\varepsilon}w^t(2^{-j}R)^{\frac{1}{p}-\frac{1}{q}}\left(\int_{2^{1-j}R\setminus 2^{-j}R} f(x)^q|x|^{q\beta}\,\mathrm{d}x\right)^{\frac{1}{q}} \lesssim \ell(R)^{\varepsilon}\|f\|_{\mathcal{M}^p_q(w^t,w^q)}.
\end{aligned}
$$

Thus,

$$
\int_R f(x)\,\mathrm{d}x \lesssim \ell(R)^{\frac{n}{q'}-\lambda+\varepsilon}\|f\|_{\mathcal{M}^p_q(w^t,w^q)}. \tag{6.4}
$$

We note that

$$
\beta-n+\frac{n}{q'}-\lambda+\varepsilon=-\frac{n}{s}-\beta\frac{t}{s}.
$$

Consequently,

$$
\sup_{R\in\mathcal{Q},\,R\supset Q}\frac{\ell(R)^{\beta}}{|R|}\|\chi_Q\|_{L\mathcal{M}^s_t(w^t,w^t)}\|\chi_R\|_{L\mathcal{M}^p_q(w^t,w^q)'} \lesssim \sup_{R\in\mathcal{Q},\,R\supset Q}\ell(Q)^{\frac{n}{s}+\beta\frac{t}{s}}\ell(R)^{-\frac{n}{s}-\beta\frac{t}{s}}=1.
$$

Thus, I_β is bounded from $L\mathcal{M}^p_q(w^t,w^q)$ to $L\mathcal{M}^s_t(w^t,w^t)$ thanks to Theorem 1.12. The proof for M_α is similar.

6.6. Proof of Proposition 1.18. The case of M_α is omitted because this is similar to the case of I_α. If I_α is bounded from $\mathcal{M}^p_q(w^t,w^q)$ to $\mathcal{M}^s_t(w^t,w^t)$, then we have $-\frac{n}{t}<\beta<\frac{ns}{p't}$ as before. Conversely assume $-\frac{n}{t}<\beta<\frac{ns}{p't}$. Let $f\in L^\infty_{\mathrm{c}}$. If $\beta\le 0$, then we can argue as Komori and Shirai did in [37]. Let us assume $\beta>0$. We distinguish four cases as before to show that

$$
w^t(Q)^{\frac{1}{s}-\frac{1}{t}}\left(\int_Q |I_\alpha f(x)|^t|x|^{\beta t}\,\mathrm{d}x\right)^{\frac{1}{t}} \lesssim \|f\|_{\mathcal{M}^p_q(w^t,w^q)}. \tag{6.5}
$$

(1) If Q is a cube centered at 0 and if f is supported on $3Q$, then we have

$$
\begin{aligned}
&w^t(Q)^{\frac{1}{s}-\frac{1}{t}}\left(\int_Q |I_\alpha f(x)|^t|x|^{\beta t}\,\mathrm{d}x\right)^{\frac{1}{t}}\\
&\lesssim w^t(Q)^{\frac{1}{p}-\frac{1}{q}}\ell(Q)^{\beta-\tau}\left(\int_Q |I_\alpha f(x)|^t|x|^{\tau t}\,\mathrm{d}x\right)^{\frac{1}{t}}.
\end{aligned}
$$

Here τ satisfies $\tau\in\left(\frac{t}{s}\beta,\beta\right)$. Since f is supported on $3Q$,

$$
\begin{aligned}
&w^t(Q)^{\frac{1}{s}-\frac{1}{t}}\ell(Q)^{\beta-\tau}\left(\int_Q |I_\alpha f(x)|^t|x|^{\tau t}\,\mathrm{d}x\right)^{\frac{1}{t}}\\
&\lesssim w^t(Q)^{\frac{1}{s}-\frac{1}{t}}\ell(Q)^{\beta-\tau}\left(\int_{3Q} |f(x)|^q|x|^{\tau q}\,\mathrm{d}x\right)^{\frac{1}{q}}.
\end{aligned}
$$

We borrow the idea of [7]. From the definition of the norm,

$$\begin{aligned}
&\left(\int_{3Q} |f(x)|^q |x|^{\tau q}\,\mathrm{d}x\right)^{\frac{1}{q}} \\
&\le \sum_{j=0}^{\infty} \left(\int_{3\cdot 2^{-j}Q\setminus 3\cdot 2^{-j-1}Q} |f(x)|^q |x|^{\tau q}\,\mathrm{d}x\right)^{\frac{1}{q}} \\
&\lesssim \sum_{j=0}^{\infty} (2^{-j}\ell(Q))^{\tau-\beta} \left(\int_{3\cdot 2^{-j}Q\setminus 3\cdot 2^{-j-1}Q} |f(x)|^q |x|^{\beta q}\,\mathrm{d}x\right)^{\frac{1}{q}} \\
&\lesssim \sum_{j=0}^{\infty} (2^{-j}\ell(Q))^{\tau-\beta} w^t(2^{-j}Q)^{\frac{1}{q}-\frac{1}{p}} \|f\|_{\mathcal{M}^p_q(w^t,w^q)}.
\end{aligned}$$

We observe that

$$\tau-\beta+(n+t\beta)\left(\frac{1}{q}-\frac{1}{p}\right)=\tau+n\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{t}{s}\beta \ge n\left(\frac{1}{q}-\frac{1}{p}\right)>0,$$

where we used $\tau > \frac{t}{s}\beta$ for the penultimate inequality. Thus, the series is summable to have (6.5).

(2) If Q is a cube centered at 0 and if f is supported outside $3Q$, then we have

$$w^t(Q)^{\frac{1}{s}-\frac{1}{t}}\left(\int_Q |I_\alpha f(x)|^t |x|^{\beta t}\,\mathrm{d}x\right)^{\frac{1}{t}} \lesssim w^t(Q)^{\frac{1}{s}} \int_{\mathbf{R}^n\setminus 3Q} \frac{|f(y)|}{|y|^{n-\alpha}}\,\mathrm{d}y$$

from the expression of $I_\alpha f(x)$. We note that

$$\begin{aligned}
&\int_{\mathbf{R}^n\setminus 3Q} \frac{|f(y)|}{|y|^{n-\alpha}}\,\mathrm{d}y \\
&\lesssim \sum_{j=1}^{\infty} \frac{1}{\ell(2^jQ)^{n-\alpha}} \int_{2^jQ\setminus 2^{j-1}Q} |f(y)|\,\mathrm{d}y \\
&\lesssim \sum_{j=1}^{\infty} \ell(2^jQ)^{\alpha} \left(\frac{1}{\ell(2^jQ)^n}\int_{2^jQ\setminus 2^{j-1}Q} |f(y)|^q\,\mathrm{d}y\right)^{\frac{1}{q}} \\
&\lesssim \sum_{j=1}^{\infty} \ell(2^jQ)^{\alpha-\beta+\frac{t}{q}\beta} \left(\frac{1}{w^t(2^jQ)}\int_{2^jQ\setminus 2^{j-1}Q} |f(y)|^q |y|^{q\beta}\,\mathrm{d}y\right)^{\frac{1}{q}}.
\end{aligned}$$

Since

$$\frac{1}{q}=\frac{1}{t}+\frac{\alpha}{n}, \quad \frac{1}{p}=\frac{1}{s}+\frac{\alpha}{n},$$

we have

$$\frac{\ell(2^jQ)^{\alpha-\beta+\frac{t}{q}\beta}}{w^t(2^jQ)^{\frac{1}{p}}}=\frac{\ell(2^jQ)^{\alpha+\frac{\alpha\beta t}{n}}}{w^t(2^jQ)^{\frac{1}{p}}}=\frac{\ell(2^jQ)^{\alpha+\frac{\alpha\beta t}{n}}}{w^t(2^jQ)^{\frac{\alpha}{n}}w^t(2^jQ)^{\frac{1}{s}}}\sim w^t(2^jQ)^{-\frac{1}{s}}.$$

Hence we have

$$\int_{\mathbf{R}^n\setminus 3Q} \frac{|f(y)|}{|y|^{n-\alpha}}\,\mathrm{d}y \lesssim w^t(Q)^{-\frac{1}{s}}\|f\|_{\mathcal{M}^p_q(w^t,w^q)}.$$

Putting these estimates all together, we obtain (6.5).

(3) If Q is a cube such that $0 \notin 32Q$ and if f is supported on $3Q$, then by the classical Hardy–Littlewood–Sobolev theorem

$$\begin{aligned} w^t(Q)^{\frac{1}{s}-\frac{1}{t}} \left(\int_Q |I_\alpha f(x)|^t |x|^{\beta t} \, \mathrm{d}x \right)^{\frac{1}{t}} &\lesssim w^t(Q)^{\frac{1}{s}-\frac{1}{t}} |c(Q)|^\beta \left(\int_Q |I_\alpha f(x)|^t \, \mathrm{d}x \right)^{\frac{1}{t}} \\ &\lesssim w^t(Q)^{\frac{1}{s}-\frac{1}{t}} |c(Q)|^\beta \left(\int_{3Q} |f(x)|^q \, \mathrm{d}x \right)^{\frac{1}{q}} \\ &\lesssim w^t(3Q)^{\frac{1}{s}-\frac{1}{t}} \left(\int_{3Q} |f(x)|^q |x|^{\beta q} \, \mathrm{d}x \right)^{\frac{1}{q}} \\ &\lesssim \|f\|_{\mathcal{M}^p_q(w^t, w^q)}. \end{aligned}$$

(4) If Q is a cube such that $0 \notin 32Q$ and if f is supported outside $3Q$, then

$$w^t(Q)^{\frac{1}{s}-\frac{1}{t}} \left(\int_Q |I_\alpha f(x)|^t |x|^{\beta t} \, \mathrm{d}x \right)^{\frac{1}{t}} \lesssim w^t(Q)^{\frac{1}{s}} \int_{\mathbf{R}^n \setminus 3Q} \frac{|f(y)|}{|y - c(Q)|^{n-\alpha}} \, \mathrm{d}y$$

from the integral expression of f. Consequently

$$\begin{aligned} & w^t(Q)^{\frac{1}{s}-\frac{1}{t}} \left(\int_Q |I_\alpha f(x)|^t |x|^{\beta t} \, \mathrm{d}x \right)^{\frac{1}{t}} \\ & \lesssim w^t(Q)^{\frac{1}{s}} \sum_{j=1}^{\infty} \frac{1}{\ell(2^j Q)^{n-\alpha}} \int_{2^{j+1}Q \setminus 2^j Q} |f(y)| \, \mathrm{d}y. \end{aligned}$$

Let $j_0 \geq 6$ be the smallest integer such that $0 \in 2^{j_0-1}Q$. Then since $|y| \sim \ell(2^{j_0}Q)$ for all $y \in 2^{j_0-3}Q$, we have

$$\begin{aligned} & \sum_{j=1}^{j_0-4} \frac{1}{\ell(2^j Q)^{n-\alpha}} \int_{2^{j+1}Q \setminus 2^j Q} |f(y)| \, \mathrm{d}y \\ & \lesssim \sum_{j=1}^{j_0-4} \ell(2^j Q)^\alpha \left(\frac{1}{\ell(2^j Q)^n} \int_{2^{j+1}Q \setminus 2^j Q} |f(y)|^q \, \mathrm{d}y \right)^{\frac{1}{q}} \\ & \lesssim \sum_{j=1}^{j_0-4} \ell(2^j Q)^\alpha \ell(2^{j_0}Q)^{-\beta} \left(\frac{1}{\ell(2^j Q)^n} \int_{2^{j+1}Q \setminus 2^j Q} |f(y)|^q |y|^{q\beta} \, \mathrm{d}y \right)^{\frac{1}{q}} \\ & \lesssim \sum_{j=1}^{j_0-4} \ell(2^j Q)^\alpha \ell(2^{j_0}Q)^{\frac{\beta t}{q}-\beta} w^t(2^j Q)^{-\frac{1}{p}} \|f\|_{\mathcal{M}^p_q(w^t, w^q)}. \end{aligned}$$

We note $\ell(2^j Q)^\alpha w^t(2^j Q)^{-\frac{1}{p}} \sim \ell(2^j Q)^{-\frac{n}{s}} |c(Q)|^{-\frac{\beta t}{p}}$, since

$$\alpha - \frac{n}{p} = -\frac{n}{s}.$$

As a result,

$$\begin{aligned}
&\sum_{j=1}^{j_0-4} \frac{1}{\ell(2^j Q)^{n-\alpha}} \int_{2^{j+1}Q\setminus 2^j Q} |f(y)|\,\mathrm{d}y \\
&\lesssim \ell(2^{j_0}Q)^{-\beta+\frac{\beta t}{q}} \ell(Q)^{-\frac{n}{s}} |c(Q)|^{-\frac{\beta t}{p}} \|f\|_{\mathcal{M}^p_q(w^t,w^q)} \\
&\lesssim \ell(2^{j_0}Q)^{-\beta+\frac{\beta t}{q}-\frac{\beta t}{p}} \ell(Q)^{-\frac{n}{s}} \|f\|_{\mathcal{M}^p_q(w^t,w^q)} \\
&= \ell(2^{j_0}Q)^{-\frac{\beta t}{s}} \ell(Q)^{-\frac{n}{s}} \|f\|_{\mathcal{M}^p_q(w^t,w^q)} \\
&\sim w^t(2^{j_0}Q)^{-\frac{1}{s}} \|f\|_{\mathcal{M}^p_q(w^t,w^q)}.
\end{aligned}$$

Consequently,

$$w^t(Q)^{\frac{1}{s}} \sum_{j=1}^{j_0-4} \frac{1}{\ell(2^j Q)^{n-\alpha}} \int_{2^{j+1}Q\setminus 2^j Q} |f(y)|\,\mathrm{d}y \lesssim \|f\|_{\mathcal{M}^p_q(w^t,w^q)}.$$

We use the idea of [7]. As we did in (6.4), using $\beta < \frac{ns}{p't}$, we have

$$w^t(Q)^{\frac{1}{s}} \sum_{j=j_0-3}^{\infty} \frac{1}{\ell(2^j Q)^{n-\alpha}} \int_{2^{j+1}Q\setminus 2^j Q} |f(y)|\,\mathrm{d}y \lesssim \|f\|_{\mathcal{M}^p_q(w^t,w^q)}.$$

Thus, putting these observations together, we obtain

$$w^t(Q)^{\frac{1}{s}} \sum_{j=1}^{\infty} \frac{1}{\ell(2^j Q)^{n-\alpha}} \int_{2^{j+1}Q\setminus 2^j Q} |f(y)|\,\mathrm{d}y \lesssim \|f\|_{\mathcal{M}^p_q(w^t,w^q)}.$$

All together then, we conclude that I_α is bounded from $\mathcal{M}^p_q(w^t,w^q)$ to $\mathcal{M}^s_t(w^t,w^t)$.

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