

LOCAL UNIQUENESS OF MULTI-PEAK SOLUTIONS TO A CLASS OF KIRCHHOFF EQUATIONS

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Abstract. In this paper, we consider the nonlocal Kirchhoff problem

$$-\left(\epsilon^2 a + \epsilon b \int_{\mathbf{R}^3} |\nabla u|^2\right) \Delta u + V(x)u = u^p, \quad u > 0, \quad u \in H^1(\mathbf{R}^3),$$

where $a, b > 0$, $1 < p < 5$ are constants, $\epsilon > 0$ is a parameter. Under some assumptions on $V(x)$, we show the local uniqueness of positive multi-peak solutions by using the local Pohozaev identity.

1. Introduction and main results

Let $a, b > 0$ and $1 < p < 5$. Consider the singularly perturbed Kirchhoff problem

$$(1.1) \quad -\left(\epsilon^2 a + \epsilon b \int_{\mathbf{R}^3} |\nabla u|^2\right) \Delta u + V(x)u = u^p, \quad u > 0, \quad u \in H^1(\mathbf{R}^3),$$

where $\epsilon > 0$ is a parameter, $V: \mathbf{R}^3 \rightarrow \mathbf{R}$ is a bounded continuous function. Under some mild conditions on V , Luo, Peng, Wang and the last named author of the present paper [22] proved the existence of multi-peak solutions to (1.1). As a continuation of the work [22], this paper is devoted to establish a local uniqueness result for the multi-peak solutions obtained there. For physical background for equation (1.1), the readers are referred to Luo et al. [22] and the references therein.

To be precise, we first give the definition of k -peak solutions of Eq. (1.1) as usual.

Definition 1.1. Let $k \in \mathbf{N}$, $b_j \in \mathbf{R}^3$, $1 \leq j \leq k$. We say that $u_\epsilon \in H^1(\mathbf{R}^3)$ is a k -peak solution of (1.1) concentrated at $\{b_1, b_2, \dots, b_k\}$, if

(i) u_ϵ has k local maximum points $x_\epsilon^j \in \mathbf{R}^3$, $j = 1, 2, \dots, k$, satisfying

$$x_\epsilon^j \rightarrow b_j$$

as $\epsilon \rightarrow 0$ for each j ;

(ii) For any given $\tau > 0$, there exists $R \gg 1$, such that

$$|u_\epsilon(x)| \leq \tau \quad \text{for } x \in \mathbf{R}^3 \setminus \bigcup_{j=1}^k B_{R\epsilon}(x_\epsilon^j);$$

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(iii) There exists $C > 0$ such that

$$\int_{\mathbf{R}^3} (\epsilon^2 a |\nabla u_\epsilon|^2 + u_\epsilon^2) \leq C\epsilon^3.$$

In the simplest case $a = 1$, $b = 0$, equation (1.1) is reduced to the following perturbed Schrödinger equation:

$$(1.2) \quad -\epsilon^2 \Delta u + V(x)u = u^p, \quad u > 0 \text{ in } \mathbf{R}^3,$$

where $1 < p$ is subcritical. There are extensive results on the existence of solutions to equations of type (1.2) for $\epsilon > 0$ sufficiently small in the literature. We only mention for instance e.g. Floer and Weinstein [8], Oh [23, 24], del Pino and Felmer [4, 5], Gui [12], Rabinowitz [25] and the references therein.

One of our motivations of this work is from local uniqueness results for Schrödinger equations (1.2). By local uniqueness, it means that if $u_\epsilon^1, u_\epsilon^2$ are two k -peak solutions of equation (1.2) concentrating at the same k points, then $u_\epsilon^1 \equiv u_\epsilon^2$ for ϵ sufficiently small. The first result in this respect seems to be given by Ghanem [10], where the uniqueness of solutions concentrating at one point for Dirichlet problems with critical nonlinearity on bounded domains is proved. By calculating the number of single-peak solutions, Grossi [11] proved that there is only one solution concentrating at any nondegenerate critical point of $V(x)$. In fact, the uniqueness of single-bump(single-peak) solutions concentrating at some degenerate critical point of $V(x)$ is true in [11] as well. Cao and Heinz [1] proved the uniqueness of multi-peak solutions to (1.2) which concentrate at nondegenerate critical points of $V(x)$ by using the topological degree, see also [10]. Recently, Deng et al. [6] proved the local uniqueness and periodicity for the solution to the prescribed scalar curvature problem

$$-\Delta u = K(x)u^{\frac{N+2}{N-2}}, \quad u > 0 \text{ in } \mathbf{R}^N$$

by the technique of local Pohozaev identity. Their work inspired Cao, Li and Luo [2] to establish local uniqueness of multi-peak solutions to (1.2) under some mild conditions on the potential function $V(x)$. In particular, $V(x)$ is not required to be nondegenerate at concentration points in [2]. For more local uniqueness results using local Pohozaev identities, see e.g. Guo et al. [14] for critical polyharmonic equations and Guo et al. [13] for fractional Laplacian equations.

Let us now recall briefly some recent progress on the problem (1.1). For the existence of single peak solutions to equation (1.1), we refer to e.g. He and Zou [15], Wang et al. [26], He et al. [16, 17, 18], Figueiredo and Santos Júnior [7], where variational methods have been used. We also mention the work [21] of Li et al., where Lyapunov–Schmidt reduction method is applied. Very few result for multi-peak solutions of problem (1.1) exists in the literature. The first one was obtained by Luo et al. [22], where the authors found the right limiting problem for the first time, and then they established some nondegeneracy result, which allows them to apply Lyapunov–Schmidt reduction method to obtain multi-peak solutions. Another result of multi-peak solutions for Kirchhoff equations with general nonlinearity can be found in the quite recent work [19] of Hu and Shuai.

There are very few local uniqueness results for Kirchhoff equations. To the best of our knowledge, the only result in this respect is given by Li et al. [21], where the authors proved that single peak solutions of equation (1.1) obtained there is locally unique under some mild conditions assumed on the function $V(x)$.

In view of the results mentioned above, a natural problem is to study the local uniqueness of the multi-peak solutions of problem (1.1) established by Luo et al. [22]. To state our main result, it is convenience to introduce the following assumptions and notations.

- (V₁) $V \in L^\infty(\mathbf{R}^3)$ and $0 < \inf_{\mathbf{R}^3} V \leq \sup_{\mathbf{R}^3} V < \infty$;
- (V₂) There exist $m > 1$, $\eta > 0$, $k \in \mathbf{N}$, $b_j = (b_{j,1}, b_{j,2}, b_{j,3}) \in \mathbf{R}^3$, $c_{j,i} \in \mathbf{R}$ with $c_{j,i} \neq 0$ for each $i = 1, 2, 3$ and $j = 1, \dots, k$ such that $V \in C^1(B_\eta(b_j))$ and

$$\begin{cases} V(x) = V(b_j) + \sum_{i=1}^3 c_{j,i}|x_i - b_{j,i}|^m + O(|x - b_j|^{m+1}), & x \in B_\eta(b_j), \\ \frac{\partial V}{\partial x_i} = mc_{j,i}|x_i - b_{j,i}|^{m-2}(x_i - b_{j,i}) + O(|x - b_j|^m), & x \in B_\eta(b_j), \end{cases}$$

where $x = (x_1, x_2, x_3) \in \mathbf{R}^3$.

Denote

$$\begin{aligned} \langle u, v \rangle_\epsilon &= \int_{\mathbf{R}^3} (\epsilon^2 a \nabla u \cdot \nabla v + V(x)uv) \quad \text{and} \quad \|u\|_\epsilon^2 = \langle u, u \rangle_\epsilon, \\ H_\epsilon &= \{u \in H^1(\mathbf{R}^3) : \|u\|_\epsilon \equiv \langle u, u \rangle_\epsilon^{1/2} < \infty\}. \end{aligned}$$

The energy functional corresponding to Eq. (1.1) is

$$(1.3) \quad I_\epsilon(u) = \frac{1}{2} \|u\|_\epsilon^2 + \frac{b\epsilon}{4} \left(\int_{\mathbf{R}^3} |\nabla u|^2 \right)^2 - \frac{1}{p+1} \int_{\mathbf{R}^3} u_+^{p+1}$$

for $u \in H_\epsilon$, where $u_+ = \max\{u, 0\}$.

We call $u \in H_\epsilon$ a (weak) solution to equation (1.1) if for any $\psi \in H_\epsilon$, it holds that

$$\langle u, \psi \rangle_\epsilon + b\epsilon \int_{\mathbf{R}^3} |\nabla u|^2 \int_{\mathbf{R}^3} \nabla u \cdot \nabla \psi = \int_{\mathbf{R}^3} u^p \psi.$$

Recall that it was proven in [22, Proposition 2.3] that the limiting problem of Eq. (1.1) is given by the following system

$$(1.4) \quad \begin{cases} -(a + b \sum_{j=1}^k \int |\nabla U^j|^2) \Delta U^i + V(b_i)U^i = (U^i)^p & \text{in } \mathbf{R}^3, \\ U^i > 0 & \text{in } \mathbf{R}^3, \\ U^i \in H^1(\mathbf{R}^3). \end{cases}$$

Hereafter, we denote by (U^1, \dots, U^k) the unique positive radial solutions to the above system (see [22]). Keep in mind that each U^i is also nondegenerate.

Our main result is as follows:

Theorem 1.2. *Assume that $V(x)$ satisfies (V₁)–(V₂). If $u_\epsilon^{(1)}, u_\epsilon^{(2)}$ are two k -peak solutions concentrating at the set of k different points $\{b_1, \dots, b_k\} \subset \mathbf{R}^3$ appeared in the condition (V₂), then for ϵ sufficiently small, we have*

$$u_\epsilon^{(1)} \equiv u_\epsilon^{(2)}$$

and $u_\epsilon^{(j)}$ ($j = 1, 2$) is of the form

$$u_\epsilon^{(j)} = \sum_{i=1}^k U^i((x - x_\epsilon^i)/\epsilon) + \varphi_\epsilon(x)$$

with $x_\epsilon^i \in \mathbf{R}^3$, $\varphi_\epsilon \in H_\epsilon$ satisfying

$$|x_\epsilon^i - b_i| = o(\epsilon), \quad \|\varphi_\epsilon\|_\epsilon = O(\epsilon^{3/2+m})$$

for $i = 1, \dots, k$.

We first explain a characteristic of this multi-peak problem, which is different from the case of Schrödinger equation. To construct multi-peak solutions to the Schrödinger equation (1.2), it is very important to understand the limiting equation as $\epsilon \rightarrow 0$, which is known as the unperturbed Schrödinger equation $-\Delta w + w = w^q$. Denote by w_i the unique (see [20]) positive radial solution to equation

$$-\Delta w_i + V(b_i)w_i = w_i^q \quad \text{in } \mathbf{R}^n.$$

Then, to construct a k -peak solution to Eq. (1.2) concentrated at $\{b_1, \dots, b_k\}$, natural candidates are functions of the form $u_\epsilon = \sum_{i=1}^k w_i((x - x_\epsilon^i)/\epsilon) + \varphi_\epsilon$, where $x_\epsilon^i \rightarrow b_i$ and φ_ϵ should be appropriately chosen such that u_ϵ is indeed a solution to equation (1.2). Accordingly, to construct multi-peak solutions to (1.1), a natural idea is that the building block of the multi-peak solution should be the unique positive radial solution w_i of the unperturbed Kirchhoff equation

$$(1.5) \quad - \left(a + b \int_{\mathbf{R}^3} |\nabla u|^2 \right) \Delta u + V(b_i)u = u^p, \quad u > 0 \text{ in } \mathbf{R}^3.$$

while the fact is that this idea only works for the single-peak solution, as can be seen in Li et al. [21]. That is, there is no multi-peak solutions for (1.1) of the form $u_\epsilon = \sum_{i=1}^k w_i((x - x_\epsilon^i)/\epsilon) + \varphi_\epsilon$, (see [22, Proposition 2.2]). Furthermore, during their proof of that the blow-up point b_j must be critical points of $V(x)$, they found that the right limiting equation of (1.1) is in fact a system of partial differential equations (see [22, Proposition 2.3]).

We mainly use the method in [2] to prove the main result. However, the nonlocal term $(\int_{\mathbf{R}^3} |\nabla u|^2) \Delta u$ causes more technical difficulties than Schrödinger equation (1.2). Moreover, as we consider the multi-peak solution of (1.1), on the one hand, there are many crosses between bubbles, and on the other hand, the limiting equation of (1.1) is a system of partial differential equations (see (1.4)), which is different from the case of single-peak solution and Schrödinger equation (1.2), the difference of these two aspects also makes the estimate more complex.

Now, we give the main idea of the proof of Theorem 1.2. We will follow the idea of Cao, Li and Luo [2]. More precisely, if $u_\epsilon^{(1)}$, $u_\epsilon^{(2)}$ are two distinct solutions as stated in Theorem 1.2, according to Proposition 2.1 in this paper, they have the same form; We firstly get the improved estimate of $|x_\epsilon^i - b_i|$ and $\|\varphi_\epsilon\|_\epsilon$, then for the function $\xi_\epsilon = (u_\epsilon^{(1)} - u_\epsilon^{(2)}) / \|u_\epsilon^{(1)} - u_\epsilon^{(2)}\|_{L^\infty(\mathbf{R}^3)}$, which obviously satisfies $\|\xi_\epsilon\|_{L^\infty(\mathbf{R}^3)} = 1$, we show that $\|\xi_\epsilon\|_{L^\infty(\mathbf{R}^3)} \rightarrow 0$ as $\epsilon \rightarrow 0$, which gives a contradiction, and thus the uniqueness is proved. During the above process, an important tool is a type of local Pohozaev identity (see (2.8) below) from Deng, Lin and Yan [6], which helps us to get the rate of the maximum points x_ϵ^i tending to b_i as $\epsilon \rightarrow 0$. For the estimate of $\|\varphi_\epsilon\|_\epsilon$, we will derive from the nondegeneracy result of Luo et al. [22]. We will need more careful analysis in the procedure due to the presence of the nonlocal term $(\int_{\mathbf{R}^3} |\nabla u|^2) \Delta u$, which leads to the local Pohozaev identity more complicated than the case of the Schrödinger equation (1.2).

Our notations are standard. We write $\int u$ to denote Lebesgue integral of u over \mathbf{R}^3 , unless otherwise stated, and $\|u\|_p = (\int_{\mathbf{R}^3} u^p)^{1/p}$. We use $B_R(x)$ (and $\bar{B}_R(x)$) to denote open (and close) balls in \mathbf{R}^3 centered at x with radius R . We will use C and C_j ($j \in \mathbf{N}$) to denote various positive constants, and $O(t)$, $o(t)$, $o_t(1)$ to mean $|O(t)| \leq C|t|$, $o(t)/t \rightarrow 0$ and $o_t(1) \rightarrow 0$ as $t \rightarrow 0$, respectively.

The paper is organized as follows. In section 2 we give some preliminary estimates. In section 3 we prove the local uniqueness results.

2. Preliminaries

According to [22, Proposition 2.3], the form of k -peak solutions to (1.1) are given by the following:

Proposition 2.1. *Let $k \in \mathbf{N}$, $\{b_1, b_2, \dots, b_k\} \subset \mathbf{R}^3$ and u_ϵ be a k -peak solution of Eq. (1.1) defined as in the Definition 1.1, with local maximum points at x_ϵ^i . Then, for $\epsilon > 0$ sufficiently small, u_ϵ is of the form*

$$(2.1) \quad u_\epsilon(x) = \sum_{i=1}^k U^i((x - x_\epsilon^i)/\epsilon) + \varphi_\epsilon(x)$$

with

$$(2.2) \quad |x_\epsilon^i - b_i| = o(1), \quad \|\varphi_\epsilon\|_\epsilon = o(\epsilon^{\frac{3}{2}}).$$

For simplicity in the below, we introduce the notation

$$w_{\epsilon,y}(x) = w((x - y)/\epsilon).$$

for any function w . The estimates given by (2.2) is not sufficient for later use. Our aim in this section is to prove the following proposition.

Proposition 2.2. *Assume that V satisfies (V_1) and (V_2) . Let $k \in \mathbf{N}$, $\{b_1, \dots, b_k\} \subset \mathbf{R}^3$ and u_ϵ be a k -peak solution of Eq. (1.1) defined as in the Definition 1.1, with local maximum points at x_ϵ^i . Suppose that*

$$u_\epsilon(x) = \sum_{i=1}^k U_{\epsilon,x_\epsilon^i}^i + \varphi_\epsilon(x).$$

Then

$$(2.3) \quad |x_\epsilon^i - b_i| = o(\epsilon)$$

for each i , $1 \leq i \leq k$, and

$$(2.4) \quad \|\varphi_\epsilon\|_\epsilon = O(\epsilon^{3/2+m}).$$

Before proving the above proposition, we first collect some useful facts. Note that since the unique positive solution of equation $-\Delta u + V(b_i)u = u^p$ decays exponentially at infinity (see [9]), we infer that

$$(2.5) \quad \max_{1 \leq i \leq k} (U^i(x) + |\nabla U^i(x)|) = O(e^{-\sigma|x|})$$

for some $\sigma > 0$. Let $0 < \delta < \min\{|b_i - b_j|/4 : i \neq j\}$ and denote

$$D_\delta = \bar{B}_\delta(b_1) \times \dots \times \bar{B}_\delta(b_k).$$

Note that if $(x_\epsilon^1, \dots, x_\epsilon^k) \in D_\delta$, then $|x_\epsilon^i - x_\epsilon^j| \geq |b_i - b_j|/2 \geq 2\delta$ for $i \neq j$, which implies by (2.5) that

$$(2.6) \quad \int \nabla U_{\epsilon,x_\epsilon^i}^i \cdot \nabla U_{\epsilon,x_\epsilon^j}^j + \left(U_{\epsilon,x_\epsilon^i}^i \right)^r \left(U_{\epsilon,x_\epsilon^j}^j \right)^s = O(e^{-\gamma/\epsilon}) \quad \text{if } i \neq j$$

for any given $r, s > 0$, where $\gamma > 0$ is a constant.

We recall an inequality: For any $2 \leq q \leq 6$ there exists a constant $C > 0$ depending only on n, a and q but independent of ϵ , such that

$$(2.7) \quad \|\varphi\|_{L^q(\mathbf{R}^3)} \leq C\epsilon^{\frac{3}{q}-\frac{3}{2}}\|\varphi\|_\epsilon$$

holds for all $\varphi \in H_\epsilon$. For a proof, see e.g. (3.6) of Li et al. [21].

The crucial Pohozaev type identity we will use is as follows:

Proposition 2.3. *Let u be a positive solution to Eq. (1.1). Let Ω be a bounded smooth domain in \mathbf{R}^3 such that $V \in C^1(\Omega)$ holds. Then*

$$(2.8) \quad \int_{\Omega} \frac{\partial V}{\partial x_{\alpha}} u^2 = \left(\epsilon^2 a + \epsilon b \int |\nabla u|^2 \right) \int_{\partial\Omega} \left(|\nabla u|^2 \nu_{\alpha} - 2 \frac{\partial u}{\partial \nu} \frac{\partial u}{\partial x_{\alpha}} \right) \\ + \int_{\partial\Omega} V u^2 \nu_{\alpha} - \frac{2}{p+1} \int_{\partial\Omega} u^{p+1} \nu_{\alpha}$$

for each $\alpha = 1, 2, 3$, where $\nu = (\nu_1, \nu_2, \nu_3)$ is the unit outward normal of $\partial\Omega$.

Proposition 2.3 can be proved by multiplying both sides of equation (1.1) by $\partial_{x_{\alpha}} u$ for each $1 \leq \alpha \leq 3$ and then integrating by parts. We omit the proof, see Cao, Li and Luo [2, Proposition 2.3] for the details.

Proof of Proposition 2.2. For simplicity, write $\varphi = \varphi_{\epsilon}$ in the following. Denote $X_{\epsilon} = (x_{\epsilon}^1, \dots, x_{\epsilon}^k)$, $G_{\epsilon, X_{\epsilon}} = \sum_{i=1}^k U_{\epsilon, x_{\epsilon}^i}^i$. Let

$$\mathcal{L}_{\epsilon} \varphi = - \left(\epsilon^2 a + \epsilon b \int |\nabla G_{\epsilon, X_{\epsilon}}|^2 \right) \Delta \varphi \\ - 2\epsilon b \left(\int \nabla G_{\epsilon, X_{\epsilon}} \cdot \nabla \varphi \right) \Delta G_{\epsilon, X_{\epsilon}} + V(x) \varphi - p G_{\epsilon, X_{\epsilon}}^{p-1} \varphi,$$

for $\varphi \in E_{\epsilon, X_{\epsilon}}$, where

$$E_{\epsilon, X_{\epsilon}} = \left\{ u \in H_{\epsilon} : \langle u, \partial_{x_{\epsilon}^i} U_{\epsilon, x_{\epsilon}^i}^i \rangle = 0 \text{ for } i = 1, \dots, k, j = 1, 2, 3 \right\}.$$

By Proposition 3.4 of [22], there exist $\epsilon, \delta > 0$ small enough, and $\rho > 0$ such that

$$\rho \|\varphi\|_{\epsilon} \leq \|\mathcal{L}_{\epsilon} \varphi\|_{\epsilon}, \quad \varphi \in E_{\epsilon, X_{\epsilon}}.$$

Now we compute $\mathcal{L}_{\epsilon} \varphi$. By the equation (1.4) of U^i , we have

$$- \left(\epsilon^2 a + \epsilon b \sum_{j=1}^k \int |\nabla U_{\epsilon, x_{\epsilon}^j}^j|^2 \right) \Delta U_{\epsilon, x_{\epsilon}^i}^i + V(b_i) U_{\epsilon, x_{\epsilon}^i}^i = (U_{\epsilon, x_{\epsilon}^i}^i)^p \quad \text{in } \mathbf{R}^3.$$

Summing over i on both sides of the equation, we have

$$- \left(\epsilon^2 a + \epsilon b \sum_{j=1}^k \int |\nabla U_{\epsilon, x_{\epsilon}^j}^j|^2 \right) \Delta G_{\epsilon, X_{\epsilon}} + \sum_{i=1}^k V(b_i) U_{\epsilon, x_{\epsilon}^i}^i = \sum_{i=1}^k (U_{\epsilon, x_{\epsilon}^i}^i)^p \quad \text{in } \mathbf{R}^3.$$

As u_{ϵ} is a solution of equation (1.1), we have

$$- \left(\epsilon^2 a + \epsilon b \int |\nabla G_{\epsilon, X_{\epsilon}} + \nabla \varphi|^2 \right) \Delta (G_{\epsilon, X_{\epsilon}} + \varphi) + V(x) (G_{\epsilon, X_{\epsilon}} + \varphi) = (G_{\epsilon, X_{\epsilon}} + \varphi)^p.$$

As a result, φ must satisfy

$$(2.9) \quad \mathcal{L}_{\epsilon} \varphi = \sum_{j=1}^3 r_j + \sum_{j=1}^3 f_j$$

with

$$r_1 = \epsilon b \left(\int |\nabla \varphi|^2 \right) \Delta u_{\epsilon}, \quad r_2 = 2\epsilon b \left(\int \nabla G_{\epsilon, X_{\epsilon}} \cdot \nabla \varphi \right) \Delta \varphi, \\ r_3 = \epsilon b \left(\int |\nabla G_{\epsilon, X_{\epsilon}}|^2 - \sum_{i=1}^k \int |\nabla U_{\epsilon, x_{\epsilon}^i}^i|^2 \right) \Delta G_{\epsilon, X_{\epsilon}},$$

$$\begin{aligned}
f_1 &= \left(\sum_{i=1}^k U_{\epsilon, x_\epsilon^i} + \varphi \right)^p - \left(\sum_{i=1}^k U_{\epsilon, x_\epsilon^i} \right)^p - p \left(\sum_{i=1}^k U_{\epsilon, x_\epsilon^i} \right)^{p-1} \varphi, \\
f_2 &= \left(\sum_{i=1}^k U_{\epsilon, x_\epsilon^i} \right)^p - \sum_{i=1}^k \left(U_{\epsilon, x_\epsilon^i} \right)^p, \\
f_3 &= \sum_{i=1}^k (V(b_i) - V(x)) U_{\epsilon, x_\epsilon^i}.
\end{aligned}$$

Hence¹

$$(2.10) \quad \rho \|\varphi\|_\epsilon^2 \leq \langle \mathcal{L}_\epsilon \varphi, \varphi \rangle = \sum_{j=1}^3 \int r_j \varphi + \sum_{j=1}^3 \int f_j \varphi.$$

By the fact that

$$(2.11) \quad \int |\nabla \varphi|^2 = O(\epsilon^{-2} \|\varphi\|_\epsilon^2),$$

we have

$$\begin{aligned}
\int r_1 \varphi &= \epsilon b \left(\int |\nabla \varphi|^2 \right) \int \nabla(G_{\epsilon, X_\epsilon} + \varphi) \cdot \nabla \varphi \\
&= O(\epsilon^{-1} \|\varphi\|_\epsilon^2) \left(\epsilon^{-\frac{1}{2}} \|\varphi\|_\epsilon + \epsilon^{-2} \|\varphi\|_\epsilon^2 \right) = O(\epsilon^{-\frac{3}{2}} \|\varphi\|_\epsilon^3 + \epsilon^{-3} \|\varphi\|_\epsilon^4), \\
\int r_2 \varphi &= 2\epsilon b \left(\int \nabla G_{\epsilon, X_\epsilon} \cdot \nabla \varphi \right) \int |\nabla \varphi|^2 = O(\epsilon \cdot \epsilon^{\frac{1}{2}} \cdot \epsilon^{-1} \|\varphi\|_\epsilon \cdot \epsilon^{-2} \|\varphi\|_\epsilon^2) = O(\epsilon^{-\frac{3}{2}} \|\varphi\|_\epsilon^3)
\end{aligned}$$

and

$$\begin{aligned}
\int r_3 \varphi &= \epsilon b \left(\int |\nabla G_{\epsilon, X_\epsilon}|^2 - \sum_{i=1}^k \int |\nabla U_{\epsilon, x_\epsilon^i}|^2 \right) \int \nabla G_{\epsilon, X_\epsilon} \nabla \varphi \\
&= \epsilon b \int \sum_{i \neq j} \nabla U_{\epsilon, x_\epsilon^i} \nabla U_{\epsilon, x_\epsilon^j} \int \nabla G_{\epsilon, X_\epsilon} \nabla \varphi = O(\epsilon^{-\gamma} \|\varphi\|_\epsilon),
\end{aligned}$$

where $\gamma > 0$ is a constant.

Similar to the proof of (5.14) and (5.15) in [2], we have, for $j = 1$ and $j = 3$,

$$\int f_j \varphi = \epsilon^{\frac{3}{2}} O(\epsilon^m + |x_\epsilon^i - b_i|^m) \|\varphi\|_\epsilon.$$

Since each U^i decays exponentially, we also derive that

$$\int f_2 \varphi = O(\epsilon^\gamma) \|\varphi\|_\epsilon$$

for any $\gamma > 0$. Combining the above estimate with (2.10), we have

$$\|\varphi\|_\epsilon = O(\epsilon^{-\frac{3}{2}} \|\varphi\|_\epsilon^2 + \epsilon^{-3} \|\varphi\|_\epsilon^3) + \epsilon^{\frac{3}{2}} O(\epsilon^m + |x_\epsilon^i - b_i|^m).$$

As a result,

$$(2.12) \quad \|\varphi\|_\epsilon = \epsilon^{\frac{3}{2}} O(\epsilon^m + |x_\epsilon^i - b_i|^m).$$

¹Here we apply an additional property that $\rho \|\varphi\|_\epsilon^2 \leq \langle \mathcal{L}_\epsilon \varphi, \varphi \rangle$ holds for $\varphi \in E_{\epsilon, X_\epsilon}$. As the proof is standard (see e.g. [3]), we omit the details.

Next we prove (2.3) for fix $i \in \{1, \dots, k\}$. It suffices to choose i such that

$$|x_\epsilon^i - b_i| = \max\{|x_\epsilon^j - b_j|: 1 \leq j \leq k\}.$$

Applying the Pohozaev-type identity (2.8) to $u = u_\epsilon$ with $\Omega = B_d(x_\epsilon^i)$, where $0 < d < \delta$ is chosen in such a way (see [2, Lemma 4.5] for details) that

$$(2.13) \quad \int_{\partial B_d(x_\epsilon^i)} (\epsilon^2 |\nabla \varphi|^2 + V(x) |\varphi|^2 + |\varphi|^p) = O\left(\|\varphi\|_\epsilon^2 + \int |\varphi|^p\right)$$

for any $\varphi \in H_\epsilon$. We have

$$(2.14) \quad \int_{B_d(x_\epsilon^i)} \frac{\partial V(x)}{\partial x_\alpha} u_\epsilon^2 = \left(\epsilon^2 a + \epsilon b \int |\nabla u_\epsilon|^2\right) \int_{\partial B_d(x_\epsilon^i)} \left(|\nabla u_\epsilon|^2 \nu_\alpha - 2 \frac{\partial u_\epsilon}{\partial \nu} \frac{\partial u_\epsilon}{\partial x_\alpha}\right) \\ + \int_{\partial B_d(x_\epsilon^i)} V(x) u_\epsilon^2 \nu_\alpha - \frac{2}{p+1} \int_{\partial B_d(x_\epsilon^i)} u_\epsilon^{p+1} \nu_\alpha.$$

We estimate each side of (2.14) as follows. By Proposition 2.1, we know $\|\varphi\|_\epsilon = o(\epsilon^{\frac{3}{2}})$. Thus, a straightforward computation gives

$$(2.15) \quad \|u_\epsilon\|_\epsilon = O(\epsilon^{\frac{3}{2}}).$$

So

$$(2.16) \quad \epsilon^2 a + \epsilon b \int |\nabla u_\epsilon|^2 = O(\epsilon^2).$$

By an elementary inequality, we have

$$|\nabla u_\epsilon|^2 \leq C \sum_{i=1}^k |\nabla U_{\epsilon, x_\epsilon^i}^i|^2 + C |\nabla \varphi|^2.$$

From (2.5), we know

$$\int_{\partial B_d(x_\epsilon^i)} \sum_{i=1}^k |\nabla U_{\epsilon, x_\epsilon^i}^i|^2 = O(e^{-\frac{\lambda}{\epsilon}})$$

holds for some $\lambda > 0$ independent of ϵ . As a result,

$$(2.17) \quad \left(\epsilon^2 a + \epsilon b \int |\nabla u_\epsilon|^2\right) \int_{\partial B_d(x_\epsilon^i)} \left(|\nabla u_\epsilon|^2 \nu_\alpha - 2 \frac{\partial u_\epsilon}{\partial \nu} \frac{\partial u_\epsilon}{\partial x_\alpha}\right) = O(\epsilon^\gamma + \|\varphi\|_\epsilon^2)$$

for any $\gamma > 0$. Using similar arguments and choosing a suitable d if necessary, we also derive

$$\int_{\partial B_d(x_\epsilon^i)} V(x) u_\epsilon^2 \nu_\alpha = O(\epsilon^\gamma + \|\varphi\|_\epsilon^2),$$

since $V(x)$ is bounded. Furthermore, from (2.7) and (2.13), we have

$$\int_{\partial B_d(x_\epsilon^i)} u_\epsilon^{p+1} \nu_\alpha = O(\epsilon^\gamma + (\epsilon^{\frac{3}{p+1} - \frac{3}{2}} \|\varphi\|_\epsilon)^{p+1}) = O(\epsilon^\gamma + \|\varphi\|_\epsilon^2).$$

Combing the above estimate and (2.14), we obtain

$$(2.18) \quad \int_{B_d(x_\epsilon^i)} \frac{\partial V}{\partial x_\alpha} u_\epsilon^2 = O(\epsilon^\gamma + \|\varphi\|_\epsilon^2)$$

for any $\gamma > 0$.

Next we estimate the left hand side of (2.18). Substitute the form of u_ϵ into the integral, and recall (2.6). We obtain

$$(2.19) \quad \int_{B_d(x_\epsilon^i)} \frac{\partial V}{\partial x_\alpha} u_\epsilon^2 = \int_{B_d(x_\epsilon^i)} \frac{\partial V}{\partial x_\alpha} \left(U_{\epsilon, x_\epsilon^i}^i \right)^2 + 2 \int_{B_d(x_\epsilon^i)} \frac{\partial V}{\partial x_\alpha} U_{\epsilon, x_\epsilon^i}^i \varphi + O(\|\varphi\|_\epsilon^2 + \epsilon^\gamma)$$

for any given constant $\gamma > 0$. By the assumption (V_2) and Hölder's inequality, there holds

$$(2.20) \quad \int_{B_d(x_\epsilon^i)} \frac{\partial V}{\partial x_\alpha} U_{\epsilon, x_\epsilon^i}^i \varphi = \epsilon^{\frac{3}{2}} O(\epsilon^{m-1} + |x_\epsilon^i - b_i|^{m-1}) \|\varphi\|_\epsilon.$$

Applying the assumption (V_2) , we have

$$(2.21) \quad \begin{aligned} & \int_{B_d(x_\epsilon^i)} \frac{\partial V}{\partial x_\alpha} \left(U_{\epsilon, x_\epsilon^i}^i \right)^2 \\ &= m c_{i,\alpha} \int_{B_d(x_\epsilon^i)} |x_\alpha - b_{i,\alpha}|^{m-2} (x_\alpha - b_{i,\alpha}) \left(U_{\epsilon, x_\epsilon^i}^i \right)^2 \\ &+ O \left(\int_{B_d(x_\epsilon^i)} |x_\alpha - b_{i,\alpha}|^m \left(U_{\epsilon, x_\epsilon^i}^i \right)^2 \right) \\ &= m c_{i,\alpha} \epsilon^3 \int_{B_{\frac{d}{\epsilon}}(0)} |\epsilon x_\alpha + x_{\epsilon, \alpha}^i - b_{i,\alpha}|^{m-2} (\epsilon x_\alpha + x_{\epsilon, \alpha}^i - b_{i,\alpha}) (U^i)^2 \\ &+ \epsilon^3 O(\epsilon^m + |x_\epsilon^i - b_i|^m), \end{aligned}$$

where $x_{\epsilon, \alpha}^i$ is the α th exponent of x_ϵ^i .

Therefore, combining (2.18)-(2.21) and (2.12) yields

$$(2.22) \quad \int_{B_{\frac{d}{\epsilon}}(0)} |\epsilon x_\alpha + x_{\epsilon, \alpha}^i - b_{i,\alpha}|^{m-2} (\epsilon x_\alpha + x_{\epsilon, \alpha}^i - b_{i,\alpha}) (U^i(x))^2 = O(\epsilon^m + |x_\epsilon^i - b_i|^m).$$

Note that, for $a, b \in \mathbf{R}$, $m > 1$, $m^* = \min\{m, 2\}$, we have the following inequality

$$(2.23) \quad \left| |a+b|^m - |a|^m - m|a|^{m-2}ab \right| \leq C(|a|^{m-m^*}|b|^{m^*} + b^m)$$

where the constant C is independent of a, b .

Applying (2.23) to $a = \epsilon x_\alpha + x_{\epsilon, \alpha}^i - b_{i,\alpha}$ and $b = -\epsilon x_\alpha$, we have

$$(2.24) \quad \begin{aligned} & \left| |x_{\epsilon, \alpha}^i - b_{i,\alpha}| \left| \int_{B_{\frac{d}{\epsilon}}(0)} |\epsilon x_\alpha + x_{\epsilon, \alpha}^i - b_{i,\alpha}|^{m-2} (\epsilon x_\alpha + x_{\epsilon, \alpha}^i - b_{i,\alpha}) (U^i(x))^2 \right| \right. \\ & \geq \frac{1}{m} |x_{\epsilon, \alpha}^i - b_{i,\alpha}|^m \int_{B_{\frac{d}{\epsilon}}(0)} (U^i(x))^2 \\ & \quad \left. - \frac{C}{m} \int_{B_{\frac{d}{\epsilon}}(0)} (|\epsilon x_\alpha|^m + |\epsilon x_\alpha|^{m^*} |x_{\epsilon, \alpha}^i - b_{i,\alpha}|^{m-m^*}) (U^i(x))^2. \right. \end{aligned}$$

Take $\alpha = i_0 \in \{1, 2, 3\}$ such that $|x_{\epsilon, i_0}^i - b_{i, i_0}| \geq \frac{|x_\epsilon^i - b_i|}{\sqrt{3}}$. Note also that $|x_{\epsilon, i_0}^i - b_{i, i_0}| \leq |x_\epsilon^i - b_i|$. Thus, combining with (2.22) and (2.24), we have

$$\begin{aligned} |x_\epsilon^i - b_i|^m &= |x_\epsilon^i - b_i| O(\epsilon^m + |x_\epsilon^i - b_i|^m) + O(\epsilon^m + |x_\epsilon^i - b_i|^{m-m^*} \epsilon^{m^*}) \\ &= |x_\epsilon^i - b_i| O(\epsilon^m + |x_\epsilon^i - b_i|^m) + O(\epsilon^m) + \frac{1}{2} |x_\epsilon^i - b_i|^m, \end{aligned}$$

which implies

$$|x_\epsilon^i - b_i| = O(\epsilon).$$

Then, from (2.22), we have

$$\int_{B_{\frac{d}{\epsilon}}(0)} \left| x_\alpha + \frac{x_{\epsilon,\alpha}^i - b_{i,\alpha}}{\epsilon} \right|^{m-2} \left(x_\alpha + \frac{x_{\epsilon,\alpha}^i - b_{i,\alpha}}{\epsilon} \right) (U^i(x))^2 = O(\epsilon).$$

From $|x_\epsilon^i - b_i| = O(\epsilon)$, we suppose $\frac{x_\epsilon^i - b_i}{\epsilon} \rightarrow t$ as $\epsilon \rightarrow 0$. Then we have

$$\int_{B_{\frac{d}{\epsilon}}(0)} |x_\alpha + t_\alpha|^{m-2} (x_\alpha + t_\alpha) (U^i(x))^2 = 0,$$

where t_α is the α th component of t for $\alpha = 1, 2, 3$.

As $U^i(x)$ is radially symmetric decreasing, we get $t = 0$. This yields (2.3). Then, combining (2.3) and (2.12) yields (2.4). The proof of Proposition 2.2 is complete. \square

3. Local uniqueness results

In this section, we prove the local uniqueness results. We use a contradiction argument as that of Cao, Li and Luo [2]. Assume $u_\epsilon^{(i)} = \sum_{j=1}^k U_{\epsilon, x_\epsilon^{j(i)}}^j + \varphi^{(i)}$, $i = 1, 2$, are two distinct solutions concentrating on $\{b_1, \dots, b_k\} \subset \mathbf{R}^3$. Set

$$\xi_\epsilon = \frac{u_\epsilon^{(1)} - u_\epsilon^{(2)}}{\|u_\epsilon^{(1)} - u_\epsilon^{(2)}\|_{L^\infty(\mathbf{R}^3)}}$$

and

$$\bar{\xi}_\epsilon(x) = \xi_\epsilon(\epsilon x + x_\epsilon^{i_0(1)})$$

for an arbitrary $i_0 \in \{1, \dots, k\}$. In the below, i_0 will be fixed always. It is clear that

$$\|\bar{\xi}_\epsilon\|_{L^\infty(\mathbf{R}^3)} = 1.$$

To obtain a contradiction, in the rest of this section, we prove

$$(3.1) \quad \|\bar{\xi}_\epsilon\|_{L^\infty(\mathbf{R}^3)} = o(1).$$

To prove (3.1), we will first prove that $|\bar{\xi}_\epsilon| \rightarrow o(1)$ holds locally uniformly, and then prove that it holds at infinity. To this end, we will establish a series of results. First we have

Proposition 3.1. *There holds*

$$\|\xi_\epsilon\|_\epsilon = O(\epsilon^{3/2}).$$

Proof. Since both $u_\epsilon^{(i)}$, $i = 1, 2$, are assumed to be solutions to Eq. (1.1), we obtain

$$(3.2) \quad \begin{aligned} & - \left(\epsilon^2 a + \epsilon b \int |\nabla u_\epsilon^{(1)}|^2 \right) \Delta \xi_\epsilon + V(x) \xi_\epsilon \\ & - \epsilon b \left(\int \nabla(u_\epsilon^{(1)} + u_\epsilon^{(2)}) \cdot \nabla \xi_\epsilon \right) \Delta u_\epsilon^{(2)} = C_\epsilon(x) \xi_\epsilon \end{aligned}$$

and

$$(3.3) \quad \begin{aligned} & - \left(\epsilon^2 a + \epsilon b \int |\nabla u_\epsilon^{(2)}|^2 \right) \Delta \xi_\epsilon + V(x) \xi_\epsilon \\ & - \epsilon b \left(\int \nabla(u_\epsilon^{(1)} + u_\epsilon^{(2)}) \cdot \nabla \xi_\epsilon \right) \Delta u_\epsilon^{(1)} = C_\epsilon(x) \xi_\epsilon, \end{aligned}$$

where

$$C_\epsilon(x) = p \int_0^1 (tu_\epsilon^{(1)}(x) + (1-t)u_\epsilon^{(2)}(x))^{p-1} dx.$$

Adding (3.2) and (3.3) together gives

$$(3.4) \quad \begin{aligned} & - \left(2\epsilon^2 a + \epsilon b \int |\nabla u_\epsilon^{(1)}|^2 + |\nabla u_\epsilon^{(2)}|^2 \right) \Delta \xi_\epsilon + 2V(x)\xi_\epsilon \\ & - \epsilon b \left(\int \nabla(u_\epsilon^{(1)} + u_\epsilon^{(2)}) \cdot \nabla \xi_\epsilon \right) \Delta (u_\epsilon^{(1)} + u_\epsilon^{(2)}) = 2C_\epsilon(x)\xi_\epsilon. \end{aligned}$$

Multiply ξ_ϵ on both sides of (3.4) and integrate over \mathbf{R}^3 . As the terms containing b are positive, we can throw away them. So

$$\|\xi_\epsilon\|_\epsilon^2 \leq \int C_\epsilon \xi_\epsilon^2 dx.$$

As $|\xi_\epsilon| \leq 1$, for $i = 1, 2$, direct computation gives

$$\int \left(U_{\epsilon, x_\epsilon^{j(i)}}^j \right)^{p-1} \xi_\epsilon^2 \leq C\epsilon^3$$

and

$$\begin{aligned} \int (\varphi^{(i)})^{p-1} \xi_\epsilon^2 & \leq \|\varphi^{(i)}\|_{p+1}^{p-1} \|\xi_\epsilon\|_{p+1}^2 \leq C \left(\epsilon^{\frac{3}{p+1} - \frac{3}{2}} \|\varphi^{(i)}\|_\epsilon \right)^{p-1} \left(\epsilon^{\frac{3}{p+1} - \frac{3}{2}} \|\xi_\epsilon\|_\epsilon \right)^2 \\ & = o_\epsilon(1) \|\xi_\epsilon\|_\epsilon^2, \end{aligned}$$

where we have used (2.7) and Proposition 2.1. Hence

$$\begin{aligned} \|\xi_\epsilon\|_\epsilon^2 & \leq \int C_\epsilon \xi_\epsilon^2 dx \leq \int C \left(\sum_{i,j=1}^2 \left(U_{\epsilon, x_\epsilon^{j(i)}}^j \right)^{p-1} + \sum_{i=1}^2 (\varphi^{(i)})^{p-1} \right) \xi_\epsilon^2 dx \\ & = O(\epsilon^3) + o_\epsilon(1) \|\xi_\epsilon\|_\epsilon^2, \end{aligned}$$

which implies the desired estimate. The proof is complete. \square

Next we study the asymptotic behavior of $\bar{\xi}_\epsilon$.

Proposition 3.2. *There exist $d_\beta \in \mathbf{R}$, $\beta = 1, 2, 3$, such that (up to a subsequence)*

$$\bar{\xi}_\epsilon \rightarrow \sum_{\beta=1}^3 d_\beta \frac{\partial U^{i_0}}{\partial x_\beta} \quad \text{in } C_{\text{loc}}^1(\mathbf{R}^3)$$

as $\epsilon \rightarrow 0$. Here i_0 is the index used in the definition of $\bar{\xi}_\epsilon$.

Proof. We will prove that the limiting function of $\bar{\xi}_\epsilon$ belongs to the kernel of a linear operator associated to U^{i_0} . It is straightforward to deduce from (3.2) that $\bar{\xi}_\epsilon$ solves

$$(3.5) \quad \begin{aligned} & - \left(a + \epsilon^{-1}b \int |\nabla u_\epsilon^{(1)}|^2 \right) \Delta \bar{\xi}_\epsilon + V(\epsilon x + x_\epsilon^{i_0(1)}) \bar{\xi}_\epsilon \\ & - \epsilon^{-1}b \left(\int \nabla(u_\epsilon^{(1)} + u_\epsilon^{(2)}) \cdot \nabla \bar{\xi}_\epsilon \right) \Delta u_\epsilon^{(2)}(\epsilon x + x_\epsilon^{i_0(1)}) = C_\epsilon(\epsilon x + x_\epsilon^{i_0(1)}) \bar{\xi}_\epsilon. \end{aligned}$$

Thus, in view of $\|\bar{\xi}_\epsilon\|_{L^\infty(\mathbf{R}^3)} = 1$, the elliptic regularity theory implies that $\bar{\xi}_\epsilon$ is locally uniformly bounded with respect to ϵ in $C_{\text{loc}}^{1,\theta}(\mathbf{R}^3)$ for some $\theta \in (0, 1)$. As a consequence, we assume (up to a subsequence) that

$$\bar{\xi}_\epsilon \rightarrow \bar{\xi} \quad \text{in } C_{\text{loc}}^1(\mathbf{R}^3).$$

We claim that $\bar{\xi} \in \text{Ker } \mathcal{L}^{i_0}$, that is,

$$(3.6) \quad \begin{aligned} & - \left(a + b \sum_{j=1}^k \int |\nabla U^j|^2 \right) \Delta \bar{\xi} - 2b \left(\int \nabla U^{i_0} \cdot \nabla \bar{\xi} \right) \Delta U^{i_0} + V(b_{i_0}) \bar{\xi} \\ & = p(U^{i_0})^{p-1} \bar{\xi}. \end{aligned}$$

Then by the fact that U^{i_0} is nondegenerate, we have $\bar{\xi} = \sum_{\beta=1}^3 d_\beta \frac{\partial U^{i_0}}{\partial x_\beta}$ for some $d_\beta \in \mathbf{R}$ ($\beta = 1, 2, 3$), and thus Proposition 3.2 is proved.

To deduce (3.6), we only need to show that (3.6) is the limiting equation of Eq. (3.5). Observing that

$$\begin{aligned} \epsilon^{-1} \int |\nabla u_\epsilon^{(1)}|^2 &= \epsilon^{-1} \int \left| \sum_{j=1}^k \nabla U^j \left(\frac{x - x_\epsilon^{j(1)}}{\epsilon} \right) + \nabla \varphi^{(1)} \right|^2 \\ &= \epsilon^{-1} \int \left| \sum_{j=1}^k \nabla U^j \left(\frac{x - x_\epsilon^{j(1)}}{\epsilon} \right) \right|^2 + 2\epsilon^{-1} \int \sum_{j=1}^k \nabla U^j \left(\frac{x - x_\epsilon^{j(1)}}{\epsilon} \right) \cdot \nabla \varphi^{(1)} + \epsilon^{-1} \int |\nabla \varphi^{(1)}|^2 \\ &= \epsilon^{-1} \int \sum_{j=1}^k |\nabla U^j \left(\frac{x - x_\epsilon^{j(1)}}{\epsilon} \right)|^2 \\ &\quad + \epsilon^{-1} \sum_{i \neq j} \int \nabla U^j \left(\frac{x - x_\epsilon^{j(1)}}{\epsilon} \right) \cdot \nabla U^i \left(\frac{x - x_\epsilon^{i(1)}}{\epsilon} \right) + O(\epsilon^{-\frac{3}{2}} \|\varphi\|_\epsilon) \\ &= \int \sum_{j=1}^k |\nabla U^j|^2 + \sum_{i \neq j} \int \nabla U^j(z) \cdot \nabla U^i \left(z + \frac{x_\epsilon^{j(1)} - x_\epsilon^{i(1)}}{\epsilon} \right) dz + O(\epsilon^{-\frac{3}{2}} \|\varphi\|_\epsilon) \\ &= \int \sum_{j=1}^k |\nabla U^j|^2 + O(e^{-\frac{2}{\epsilon}}) + O(\epsilon^m), \end{aligned}$$

where we have used (2.5), (2.6) and (2.4), so

$$(3.7) \quad \epsilon^{-1} \int |\nabla u_\epsilon^{(1)}|^2 - \int \sum_{j=1}^k |\nabla U^j|^2 = O(e^{-\frac{2}{\epsilon}}) + O(\epsilon^m) \rightarrow 0$$

as $\epsilon \rightarrow 0$. For convenience, we introduce

$$\bar{u}_\epsilon^{(i)}(x) = u_\epsilon^{(i)}(\epsilon x + x_\epsilon^{i_0(1)}) \quad \text{and} \quad \bar{\varphi}_\epsilon^{(i)} = \varphi_\epsilon^{(i)}(\epsilon x + x_\epsilon^{i_0(1)})$$

for $i = 1, 2$. Similarly, we have

$$\begin{aligned} & \epsilon^{-1} \int \nabla(u_\epsilon^{(1)} + u_\epsilon^{(2)}) \cdot \nabla \xi_\epsilon - 2 \int \nabla U^{i_0} \cdot \nabla \bar{\xi}_\epsilon \\ &= \int (\nabla \bar{u}_\epsilon^{(1)} + \nabla \bar{u}_\epsilon^{(2)} - 2\nabla U^{i_0}) \cdot \nabla \bar{\xi}_\epsilon \\ &= \int \left(\sum_{j \neq i_0} \nabla U^j \left(z + \frac{x_\epsilon^{i_0(1)} - x_\epsilon^{j(1)}}{\epsilon} \right) + \sum_{j \neq i_0} \nabla U^j \left(z + \frac{x_\epsilon^{i_0(1)} - x_\epsilon^{j(2)}}{\epsilon} \right) \right. \\ &\quad \left. + \nabla U^{i_0}(z) + \nabla U^{i_0} \left(z + \frac{x_\epsilon^{i_0(1)} - x_\epsilon^{i_0(2)}}{\epsilon} \right) - 2\nabla U^{i_0}(z) \right) \cdot \nabla \bar{\xi}_\epsilon + \int (\nabla \bar{\varphi}^{(1)} + \nabla \bar{\varphi}^{(2)}) \cdot \nabla \bar{\xi}_\epsilon \\ &= O(e^{-\frac{2}{\epsilon}} + \epsilon^{m-\frac{1}{2}}) \|\bar{\xi}_\epsilon\|_\epsilon \rightarrow 0 \end{aligned}$$

as $\epsilon \rightarrow 0$, and that, for any $\Phi \in C_0^\infty(\mathbf{R}^3)$,

$$\begin{aligned} & \int \nabla (u_\epsilon^{(2)}(\epsilon x + x_\epsilon^{i_0(1)}) - U^{i_0}) \cdot \nabla \Phi \\ &= \int \nabla \left(U^{i_0} \left(z + \frac{x_\epsilon^{i_0(1)} - x_\epsilon^{i_0(2)}}{\epsilon} \right) + \sum_{j \neq i_0} U^j \left(z + \frac{x_\epsilon^{i_0(1)} - x_\epsilon^{j(2)}}{\epsilon} \right) - U^{i_0}(z) \right) \cdot \nabla \Phi \\ &+ \int \nabla \bar{\varphi}^{(2)} \cdot \nabla \Phi = O(e^{-\frac{2}{\epsilon}} + \epsilon^{m-\frac{1}{2}}) \|\Phi\|_\epsilon \rightarrow 0. \end{aligned}$$

Here, we have used (2.3), which implies

$$\frac{x_\epsilon^{i_0(1)} - x_\epsilon^{i_0(2)}}{\epsilon} = o(1),$$

and used

$$\frac{x_\epsilon^{i_0(1)} - x_\epsilon^{j(i)}}{\epsilon} \rightarrow \infty \quad \text{for } i = 1, 2 \text{ and } j \neq i_0.$$

as $\epsilon \rightarrow 0$. Combining the above two formulas and $\bar{\xi}_\epsilon \rightarrow \bar{\xi}$ in $C_{\text{loc}}^1(\mathbf{R}^3)$, we conclude that

$$\begin{aligned} (3.8) \quad & \epsilon^{-1} b \left(\int \nabla (u_\epsilon^{(1)} + u_\epsilon^{(2)}) \cdot \nabla \xi_\epsilon \right) \Delta u_\epsilon^{(2)}(\epsilon x + x_\epsilon^{i_0(1)}) \\ & \rightarrow 2b \left(\int \nabla U^{i_0} \cdot \nabla \bar{\xi} \right) \Delta U^{i_0} \end{aligned}$$

in $H^{-1}(\mathbf{R}^3)$. Since $V(\epsilon x + x_\epsilon^{i_0(1)}) \rightarrow V(b_{i_0})$ holds locally uniformly, we have

$$V(\epsilon x + x_\epsilon^{i_0(1)}) \bar{\xi}_\epsilon \rightarrow V(b_{i_0}) \bar{\xi}.$$

Also, similar to Cao et. al. [2, proof of (3.11)], we have for any $\Phi \in C_0^\infty(\mathbf{R}^3)$,

$$(3.9) \quad \int C_\epsilon(\epsilon x + x_\epsilon^{i_0(1)}) \bar{\xi}_\epsilon \Phi \rightarrow \int p(U^{i_0})^{p-1} \bar{\xi} \Phi.$$

Finally, combining (3.7) (3.8) (3.9), we obtain (3.6). The proof is complete. \square

Now we prove

Proposition 3.3. *Let d_β be defined as in Proposition 3.2. Then*

$$d_\beta = 0 \quad \text{for } \beta = 1, 2, 3.$$

Proof. We will combine Proposition 2.2 and the Pohozaev-type identity (2.8) to prove that

$$(3.10) \quad d_\beta \int |x_\alpha|^{m-2} x_\alpha U^{i_0}(x) \frac{\partial U^{i_0}}{\partial x_\beta} = 0.$$

Applying (2.8) to $u_\epsilon^{(1)}$ and $u_\epsilon^{(2)}$ with $\Omega = B_d(x_\epsilon^{i_0(1)})$, where d is chosen in the same way as that of Proposition 2.2, which combine with (2.4), (2.13) and Proposition 3.1 implies

$$(3.11) \quad \int_{\partial B_d(x_\epsilon^{i_0(1)})} (\epsilon^2 |\nabla \varphi^{(i)}|^2 + V(x) |\varphi^{(i)}|^2) = O(\|\varphi^{(i)}\|_\epsilon^2) = O(\epsilon^{3+2m})$$

for $i = 1, 2$ and

$$(3.12) \quad \int_{\partial B_d(x_\epsilon^{i_0(1)})} (\epsilon^2 |\nabla \xi_\epsilon|^2 + V(x) |\xi_\epsilon|^2) = O(\epsilon^3).$$

We obtain

$$\begin{aligned}
& \int_{B_d(x_\epsilon^{i_0(1)})} \frac{\partial V}{\partial x_\alpha} (u_\epsilon^{(1)} + u_\epsilon^{(2)}) \xi_\epsilon \\
&= \left(\epsilon^2 a + \epsilon b \int |\nabla u_\epsilon^{(1)}|^2 \right) \int_{\partial B_d(y_\epsilon^{(1)})} (\nabla u_\epsilon^{(1)} + \nabla u_\epsilon^{(2)}) \cdot \nabla \xi_\epsilon \nu_\alpha \\
(3.13) \quad &+ \left(\epsilon^2 a + \epsilon b \int |\nabla u_\epsilon^{(2)}|^2 \right) \int_{\partial B_d(y_\epsilon^{(1)})} \left(\frac{\partial \xi_\epsilon}{\partial x_\alpha} \frac{\partial u_\epsilon^{(2)}}{\partial \nu} - \frac{\partial \xi_\epsilon}{\partial \nu} \frac{\partial u_\epsilon^{(1)}}{\partial x_\alpha} \right) \\
&+ \epsilon b \left(\int \nabla (u_\epsilon^{(1)} + u_\epsilon^{(2)}) \cdot \nabla \xi_\epsilon \right) \int_{\partial B_d(y_\epsilon^{(1)})} (|\nabla u_\epsilon^{(2)}|^2 \nu_\alpha - 2 \partial_\nu u_\epsilon^{(1)} \partial_\alpha u_\epsilon^{(1)}) \\
&+ \int_{\partial B_d(x_\epsilon^{i_0(1)})} V(x) (u_\epsilon^{(1)} + u_\epsilon^{(2)}) \xi_\epsilon \nu_\alpha - 2 \int_{\partial B_d(x_\epsilon^{i_0(1)})} A_\epsilon \xi_\epsilon \nu_\alpha,
\end{aligned}$$

where $1 \leq \alpha \leq 3$ and

$$A_\epsilon = \int_0^1 (t u_\epsilon^{(1)} + (1-t) u_\epsilon^{(2)})^p.$$

We estimate (3.13) term by term. By (2.16), (3.11) and (3.12), we have

$$\begin{aligned}
& \left(\epsilon^2 a + \epsilon b \int |\nabla u_\epsilon^{(1)}|^2 \right) \int_{\partial B_d(y_\epsilon^{(1)})} |\nabla (u_\epsilon^{(1)} + u_\epsilon^{(2)}) \cdot \nabla \xi_\epsilon \nu_\alpha| \\
& \leq C \left(\epsilon^2 \int_{\partial B_d(y_\epsilon^{(1)})} |\nabla u_\epsilon^{(1)}|^2 \right)^{\frac{1}{2}} \left(\epsilon^2 \int_{\partial B_d(y_\epsilon^{(1)})} |\nabla \xi_\epsilon|^2 \right)^{\frac{1}{2}} \\
& = O(\epsilon^\gamma + \|\varphi_\epsilon\|_\epsilon) O(\epsilon^{\frac{3}{2}}) = O(\epsilon^{3+m})
\end{aligned}$$

by choosing γ sufficiently large. Similarly, we obtain

$$\left(\epsilon^2 a + \epsilon b \int |\nabla u_\epsilon^{(2)}|^2 \right) \int_{\partial B_d(y_\epsilon^{(1)})} \left(\frac{\partial \xi_\epsilon}{\partial x_\alpha} \frac{\partial u_\epsilon^{(2)}}{\partial \nu} - \frac{\partial \xi_\epsilon}{\partial \nu} \frac{\partial u_\epsilon^{(1)}}{\partial x_\alpha} \right) = O(\epsilon^{3+m})$$

and

$$\epsilon b \left(\int \nabla (u_\epsilon^{(1)} + u_\epsilon^{(2)}) \cdot \nabla \xi_\epsilon \right) \int_{\partial B_d(y_\epsilon^{(1)})} (|\nabla u_\epsilon^{(2)}|^2 \nu_\alpha - 2 \partial_\nu u_\epsilon^{(1)} \partial_\alpha u_\epsilon^{(1)}) = O(\epsilon^{3+m}).$$

By (3.11) and (3.12), we have

$$\begin{aligned}
& \int_{\partial B_d(x_\epsilon^{i_0(1)})} V(x) (u_\epsilon^{(1)} + u_\epsilon^{(2)}) \xi_\epsilon \nu_\alpha \\
& \leq C \sum_{i=1}^2 \left(\int_{\partial B_d(x_\epsilon^{i_0(1)})} |u_\epsilon^{(i)}|^2 \right)^{\frac{1}{2}} \left(\int_{\partial B_d(x_\epsilon^{i_0(1)})} |\xi_\epsilon|^2 \right)^{\frac{1}{2}} \\
& = O(\epsilon^{\frac{3}{2}+m} \cdot \epsilon^{\frac{3}{2}}) = O(\epsilon^{3+m}).
\end{aligned}$$

As to A_ϵ , we have

$$A_\epsilon = O(\epsilon^\gamma + \sum_{i=1}^2 |\varphi^{(i)}|^p) \quad \text{on } \partial B_d(x_\epsilon^{i_0(1)})$$

for any given $\gamma > 0$, since U^j , $j = 1, \dots, k$, decay exponentially at infinity. Hence we can deduce that

$$\begin{aligned} \int_{\partial B_d(x_\epsilon^{i_0(1)})} A_\epsilon \xi_\epsilon \nu_\alpha &\leq C\epsilon^\gamma \left(\int_{\partial B_d(x_\epsilon^{i_0(1)})} |\xi_\epsilon|^2 \right)^{\frac{1}{2}} + C \sum_{i=1}^2 \left(\int_{\partial B_d(x_\epsilon^{i_0(1)})} |\varphi^{(i)}|^p \right) |\xi_\epsilon| \\ &\leq C\epsilon^{\gamma+\frac{3}{2}} + O \left(\sum_{i=1}^2 (\epsilon^{\frac{3}{p+1}-\frac{3}{2}} \|\varphi^{(i)}\|_\epsilon)^p (\epsilon^{\frac{3}{p+1}-\frac{3}{2}} \|\xi_\epsilon\|_\epsilon) \right) \\ &= O(\epsilon^{3+pm}), \end{aligned}$$

by choosing $\gamma > \frac{3}{2} + pm$.

Hence by the above estimate, there holds

$$(3.14) \quad \int_{B_d(x_\epsilon^{i_0(1)})} \frac{\partial V}{\partial x_\alpha} (u_\epsilon^{(1)} + u_\epsilon^{(2)}) \xi_\epsilon = O(\epsilon^{3+m}).$$

Next we estimate the left hand side of (3.14). By the assumption (V2), we have

$$\begin{aligned} &\int_{B_d(x_\epsilon^{i_0(1)})} \frac{\partial V}{\partial x_\alpha} (u_\epsilon^{(1)} + u_\epsilon^{(2)}) \xi_\epsilon \\ &= mc_{i_0, \alpha} \int_{B_d(x_\epsilon^{i_0(1)})} |x_\alpha - b_{i_0, \alpha}|^{m-2} (x_\alpha - b_{i_0, \alpha}) (u_\epsilon^{(1)} + u_\epsilon^{(2)}) \xi_\epsilon \\ &\quad + O \left(\int_{B_d(x_\epsilon^{i_0(1)})} |x - b_{i_0}|^m (u_\epsilon^{(1)} + u_\epsilon^{(2)}) \xi_\epsilon \right) =: I_1 + I_2. \end{aligned}$$

Note that

$$I_2 = O \left(\epsilon^{3+m} \int_{B_{\frac{d}{\epsilon}}(0)} \left| x + \frac{x_\epsilon^{i_0(1)} - b_{i_0}}{\epsilon} \right|^m \left(U^{i_0} + U^{i_0} \left(x + \frac{x_\epsilon^{i_0(1)} - x_\epsilon^{i_0(2)}}{\epsilon} \right) + e \right) \bar{\xi}_\epsilon(x) \right),$$

where

$$e = \sum_{i=1}^2 \sum_{j \neq i_0} U^j \left(x + \frac{x_\epsilon^{i_0(1)} - x^{j(i)}}{\epsilon} \right) + \sum_{i=1}^2 \varphi_\epsilon^i(\epsilon x + x_\epsilon^{i_0(1)}).$$

Using the estimates (2.3) and (2.4) and the exponential decay of U^j ($j = 1, \dots, k$), and the fact that $\|\bar{\xi}_\epsilon\|_\infty = 1$ we deduce that

$$I_2 = O(\epsilon^{3+m}).$$

Combining the above estimate with (3.14) yields

$$I_1 = O(\epsilon^{3+m}).$$

By the same arguments as that of I_2 , and using the assumption that $c_{i, \alpha} \neq 0$ for every i, α and using Proposition 3.2, we deduce

$$\begin{aligned} &d_\alpha \int_{B_{\frac{d}{\epsilon}}(0)} |x_\alpha + o_\epsilon(1)|^{m-2} (x_\alpha + o_\epsilon(1)) \left(U^{i_0} + U^{i_0} \left(x + \frac{x_\epsilon^{i_0(1)} - x_\epsilon^{i_0(2)}}{\epsilon} \right) + e \right) \bar{\xi}_\epsilon(x) \\ &= O(\epsilon) \end{aligned}$$

which implies

$$\sum_{\beta=1}^3 d_\beta \int |x_\alpha|^{m-1} x_\alpha U^{i_0}(x) \frac{\partial U^{i_0}}{\partial x_\beta}(x) = 0.$$

Thus $d_\alpha = 0$ for $\alpha = 1, 2, 3$. The proof is complete. \square

Now we can complete the proof of Theorem 1.2.

Proof of Theorem 1.2. Propositions 3.2 and 3.3 imply that

$$\|\bar{\xi}_\epsilon\|_{L^\infty(B_R(0))} = o_\epsilon(1),$$

so we have

$$\|\xi_\epsilon\|_{L^\infty(\cup_{i=1}^k B_{R\epsilon}(x_\epsilon^{i(1)}))} = o_\epsilon(1).$$

In the domain $\mathbf{R}^3 \setminus \cup_{i=1}^k B_{R\epsilon}(x_\epsilon^{i(1)})$, we can apply the same argument as that of [2, Proposition 3.5] to conclude that

$$\|\xi_\epsilon\|_{L^\infty(\mathbf{R}^3 \setminus \cup_{i=1}^k B_{R\epsilon}(x_\epsilon^{i(1)}))} = o_\epsilon(1).$$

Thus $\|\xi_\epsilon\|_{L^\infty(\mathbf{R}^3)} = o_\epsilon(1)$ holds, which is in contradiction with $\|\xi_\epsilon\|_{L^\infty(\mathbf{R}^3)} = 1$. The proof is complete. \square

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