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AN EXTENSION PROPERTY OF QUASIMÖBIUS MAPPINGS IN METRIC SPACES

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Abstract. In 1991, Väisälä discussed the extension property of quasisymmetric mappings in Banach spaces. In 2009, Haïssinsky got an extension property of quasisymmetric mappings in metric spaces. The purpose of this paper is to establish an extension property of quasimöbius mappings in metric spaces.

1. Introduction

1.1. Extension of quasisymmetric mappings.

Definition 1.1. Suppose that (X,d) and (Y,d_1) are metric spaces and $\eta\colon [0,+\infty) \to [0,+\infty)$ is a homeomorphism with $\eta(0)=0$. A homeomorphism $f\colon (X,d)\to (Y,d_1)$ is called *η-quasisymmetric*, briefly η-QS, if $d(a,x)\le td(x,b)$ implies

$$d_1(a',x')\le \eta(t)d_1(x',b')$$

for any points a, x, b in X and any number $t\ge 0$. Here and hereafter, the primes always stand for the images of the points under the mappings. For example, $a'=f(a)$.

Quasisymmetric mappings originate from the work of Beurling and Ahlfors [4], who defined them as the boundary values of quasiconformal self-mappings of the upper half-plane onto the real line. The general definition of quasisymmetric mappings, i.e., Definition 1.1, is due to Tukia and Väisälä [10]. Since its appearance, the concept of quasisymmetric mappings has been studied by numerous authors. See, for example, [3, 8] for the properties of this class of mappings. In 1991, Väisälä discussed the extension property of quasisymmetric mappings in the setting of Banach spaces and proved the following result.

Theorem A. [13, Theorem 7.39] *Suppose that $f\colon E\to E_1$ is a homeomorphism and $E=A\cup B$ such that the restrictions $f|_A$ and $f|_B$ are η-quasisymmetric, where both E and E_1 are Banach spaces with dimension at least 2, and A and B are subsets of E. Then f is η_1-quasisymmetric, where η_1 depends only on η.*

In [7], the author considered the extension property of quasisymmetric mappings in metric spaces. The obtained result is as follows.

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Theorem B. [7, Theorem 3.1] *Suppose* $X = X_1 \cup X_2$ *and* $Y = Y_1 \cup Y_2$ *are metric spaces with* $\angle(X_1, X_2) > 0$ *and* $\angle(Y_1, Y_2) > 0$. *Assume that* $X_1 \cap X_2$ *and* $Y_1 \cap Y_2$ *are* τ*-uniformly perfect such that* $\operatorname{diam}(X_1 \cap X_2) \geq q\operatorname{diam}X_1$ *for some* $q \in (0, 1)$, *where* "diam" *means "diameter". If* $f\colon X \to Y$ *is a homeomorphism such that for each* $j \in \{1, 2\}$, $f(X_j) = Y_j$ *and* $f|_{X_j}$ *is* η*-quasisymmetric, then* f *is globally* η_1*-quasisymmetric, where* η_1 *depends only on* η, τ, q, *together with the angles* $\angle(X_1, X_2)$ *and* $\angle(Y_1, Y_2)$.

See Section 2 for the definitions of the angle $\angle(X_1, X_2)$ and the uniform perfectness. We remark that the assumption on the uniform perfectness of $Y_1 \cap Y_2$ in Theorem B is redundant since the concept of uniform perfectness is an invariant under quasisymmetric mappings (cf. [14, Lemma C]).

1.2. Extension of quasimöbius mappings. Let (X, d) be a metric space. Its one-point extension is defined via

$$\widehat{X} = \begin{cases} X, & \text{if } X \text{ is bounded,} \\ X \cup \{\infty\}, & \text{if } X \text{ is unbounded.} \end{cases}$$

Let a, b, c, d be points in $\widehat{X}$ with $a \neq b$ and $c \neq d$. Their cross ratio $r_d(a, b, c, d)$ is defined by the formula

$$r_d(a, b, c, d) = \frac{d(a, c)d(b, d)}{d(a, b)d(c, d)}.$$

If $a = c$ or $b = d$, we set $r_d = 0$. If some of these points is ∞, then we omit the factors containing ∞. For example,

$$r_d(a, b, c, \infty) = \frac{d(a, c)}{d(a, b)}.$$

Definition 1.2. Suppose that (X, d) and (Y, d_1) are metric spaces and $\theta\colon [0, \infty) \to [0, \infty)$ is a homeomorphism with $\theta(0) = 0$. A homeomorphism $f\colon (\widehat{X}, d) \to (\widehat{Y}, d_1)$ is called θ*-quasimöbius,* briefly θ-QM, if $r_d(a, b, c, d) \leq t$ implies

$$r_{d_1}(a', b', c', d') \leq \theta(t)$$

for all points a, b, c, d in $\widehat{X}$ and any number $t \geq 0$.

QM mappings were introduced by Väisälä in 1985 [11]. We know that every QS mapping is QM [11, Theorem 3.2] and every QM mapping between two bounded metric spaces is QS [11, p. 222]. Moreover, if a QM mapping fixes ∞, then it is QS [11, Theorem 3.10]. The reader is referred to [11, 12, 13] etc for more properties concerning these two classes of mappings. Also, it has been well known that the class of QM mappings has played an important role in the study of QC mappings (which is the abbreviation of quasiconformal mappings), QS mappings and their relationships (cf. [2, 9, 11, 14] etc.).

The main aim of this paper is to study the extension property of QM mappings. Our result is an analogue of Theorem B for QM mappings, which is as follows.

Theorem 1.1. *Suppose that* (X, d) *and* (Y, d_1) *are metric spaces and the following conditions are satisfied:*

(1) $X = X_1 \cup X_2$ *and* $Y = Y_1 \cup Y_2$ *with* $\angle(X_1, X_2) > 0$ *and* $\angle(Y_1, Y_2) > 0$;
(2) $X_1 \cap X_2$ *is* τ*-uniformly perfect with* $\tau \in (0, 1)$;
(3) *There is a constant* $q \in (0, 1]$ *such that*

$$\operatorname{diam}(X_1 \cap X_2) \geq q \min\{\operatorname{diam}(X_1), \operatorname{diam}(X_2)\}$$

and

$$\operatorname{diam}(Y_1 \cap Y_2) \geq q \min\{\operatorname{diam}(Y_1), \operatorname{diam}(Y_2)\};$$

(4) *$f\colon X \to Y$ is a homeomorphism such that for each $i \in \{1,2\}$, $f(X_i) = Y_i$.*

Then the restrictions $f|_{X_i}$ ($i \in \{1,2\}$) are θ-quasimöbius if and only if f is θ_1-quasimöbius, where θ and θ_1 depend on each other, and τ, q, together with the angles $\angle(X_1, X_2)$ and $\angle(Y_1, Y_2)$.

Remark 1.1. By [10, Theorem 2.5], we see that the diameter condition $\operatorname{diam}(X_1 \cap X_2) \geq q \operatorname{diam}(X_1)$ in Theorem B guarantees the one $\operatorname{diam}(Y_1 \cap Y_2) \geq q_1 \operatorname{diam}(Y_1)$, where $q_1 = \frac{1}{2\eta(1/q)}$, because the corresponding mappings are QS. But for QM mappings, this property is no longer valid. This can be seen from Example 4.1 below. Also, we construct three more examples to show that each of the assumptions (1) $\sim$ (3) in Theorem 1.1 cannot be removed. See Examples 4.2 $\sim$ 4.4 below.

We shall prove Theorem 1.1 by applying the inversions introduced by Buckley et al. in [6] or [5]. The proof will be given in Section 3. Some necessary terminologies will be introduced in Section 2, and in Section 4, four examples will be constructed.

2. Preliminaries

2.1. Inversions. Let (X, d) denote a metric space, and let $p \in X$ be a base point. For x, $y \in X_p = X \setminus \{p\}$, let

$$i_p(x,y) = \frac{d(x,y)}{d(x,p)d(y,p)}$$

and

$$d_p(x,y) = \inf\left\{\sum_{i=1}^{k} i_p(x_i, x_{i-1})\colon x = x_0, x_1, \ldots, x_{k-1}, x_k = y \in X_p\right\}.$$

When X is unbounded, for $x \in X_p$, we define

$$i_p(x, \infty) = \frac{1}{d(x,p)}.$$

Then we see that the definition of $d_p(x,y)$ using auxiliary points in X_p is the same as the one using points in $\widehat{X}_p = \widehat{X} \setminus \{p\}$, and so, the distance function d_p on X_p extends to $\widehat{X}_p$.

We call

$$(\operatorname{Inv}_p(X), d_p) = (\widehat{X}_p, d_p)$$

the *inversion* of (X, d) with respect to the base point p. In the following, sometimes, we only use $\operatorname{Inv}_p(X)$ to replace $(\operatorname{Inv}_p(X), d_p)$.

Let us recall the following useful properties concerning the inversions.

Theorem C. [6, Lemma 3.2] *Let (X, d) denote a metric space, and let $p \in X$ be a base point.*

(1) *For all points $x, y \in \operatorname{Inv}_p(X)$,*

$$\frac{1}{4} i_p(x,y) \leq d_p(x,y) \leq i_p(x,y).$$

In particular, d_p is a distance function on $\operatorname{Inv}_p(X)$;

(2) *The identity mapping $\mathrm{id}\colon (X_p, d) \to (X_p, d_p)$ is θ_0-QM, where $\theta_0(t) = 16t$;*

(3) *$(\operatorname{Inv}_p(X), d_p)$ is bounded if and only if p is an isolated point in (X, d).*

2.2. Uniform perfectness and density.

Definition 2.1. A metric space (X, d) is called *τ-uniformly perfect* if there is a constant $\tau \in (0, 1)$ such that for every $x \in X$ and every $r > 0$, $\mathbf{B}(x, r)\backslash\mathbf{B}(x, \tau r) \neq \emptyset$ provided that $X\backslash\mathbf{B}(x, r) \neq \emptyset$, where $\mathbf{B}(x, r)$ denotes the metric ball $\mathbf{B}(x, r) = \{z \in X : d(z, x) < r\}$.

We remark that the τ-uniform perfectness implies the τ_1-uniform perfectness when $0 < \tau_1 \leq \tau < 1$.

Definition 2.2. Suppose that (X, d) is a metric space, a and $b \in X$, and $\{x_i\}_{i\in\mathbf{Z}}$ is a sequence of points in X with $a \neq x_i \neq b$, where $\mathbf{Z}$ denotes the usual integer set.

(1) $\{x_i\}_{i\in\mathbf{Z}}$ is called *a chain* joining a and b if $x_i \to a$ as $i \to -\infty$ and $x_i \to b$ as $i \to +\infty$. Further, if there is a constant $\sigma > 1$ such that for all i,

$$|\log r_d(a, x_i, x_{i+1}, b)| \leq \log \sigma,$$

then $\{x_i\}_{i\in\mathbf{Z}}$ is called *a σ-chain.*

(2) (X, d) is said to be *σ-dense* with $\sigma > 1$ if every pair of points in X can be joined by a σ-chain.

We remark that (1) every σ-dense space does not have any isolated points, and (2) each σ-dense space is σ_1-dense if $\sigma_1 \geq \sigma$.

(In the rest of this paper, we make the following notational convention: Suppose A denotes a condition with data v and B another condition with data v_1. We say that *A implies B quantitatively* if A implies B so that v_1 depends only on v. If A and B imply each other quantitatively, then we say that they are *quantitatively equivalent.*)

The next result means that the σ-density is invariant under QM mappings.

Lemma 2.1. *Let $f : (X, d) \to (Y, d_1)$ be θ-QM between two metric spaces. Then (X, d) is σ-dense if and only if (Y, d_1) is σ_1-dense, quantitatively.*

Proof. Assume that f is θ-QM. Then f^{-1} is θ_1-QM with $\theta_1(t) = 1/\theta^{-1}(1/t)$ (see [11, p. 219]). This fact implies that, to prove this lemma, it suffices to show the necessity. For this, we only need to show that for any u', $v' \in Y$, there exists a $\theta(\sigma)$-chain joining them with $\theta(\sigma) > 1$.

It follows from the density of (X, d) that there exists a σ-chain $\{x_i\}_{i\in\mathbf{Z}}$ joining u and v. Since

$$|\log r_d(u, x_i, x_{i+1}, v)| \leq \log \sigma,$$

we see that

$$|\log r_{d_1}(u', x_i', x_{i+1}', v')| \leq \log \theta(\sigma),$$

and thus, $\{x_i'\}_{i\in\mathbf{Z}}$ is a $\theta(\sigma)$-chain joining u' and v'. Hence the proof of the lemma is complete. □

Theorem D. [14, Lemma E] *Let (X, d) be a metric space. Then the following are quantitatively equivalent:*

(1) *X is uniformly τ-perfect;*
(2) *X is σ-dense.*

The following corollary is a direct consequence of Lemma 2.1 and Theorem D.

Corollary 2.2. *Let $f : (X, d) \to (Y, d_1)$ be θ-QM between two metric spaces. Then (X, d) is uniformly τ-perfect if and only if (Y, d_1) is uniformly τ_1-perfect, quantitatively.*

The following result concerns the removable property of the uniform perfectness, which is useful for the discussions in Section 3.

Lemma 2.3. *Suppose that* (X,d) *is a* τ*-uniformly perfect metric space and* S *denotes a finite sequence of points in* X*. Then* $X\setminus S$ *is* τ'*-uniformly perfect, where* $\tau'=\frac{\tau}{2}$.

Proof. Assume that $S=\{a_1,\dots a_k\}$. To prove this lemma, it suffices to show the following assertion.

Assertion. For any $1\le i\le k$, $X\setminus\{a_1,\dots,a_i\}$ is μ_i-uniformly perfect, where $\mu_i=\big(1-\frac{2^i-1}{2^{i+1}}\big)\tau$.

We start the proof of the assertion with two claims.

Claim 2.1. $X\setminus\{a_1\}$ *is* μ_1*-uniformly perfect, where* $\mu_1=(1-\frac{1}{4})\tau$.

Assume that $x\in X\setminus\{a_1\}$, $r>0$ and $(X\setminus\{a_1\})\setminus\mathbf{B}(x,r)\neq\emptyset$. Then it follows from the uniform perfectness of X that there is w_1 such that

$$w_1\in\mathbf{B}(x,r)\setminus\mathbf{B}(x,\tau r).$$

If $w_1\neq a_1$, then $w_1\in(X\setminus\{a_1\})\cap(\mathbf{B}(x,r)\setminus\mathbf{B}(x,\tau r))\subset(X\setminus\{a_1\})\cap(\mathbf{B}(x,r)\setminus\mathbf{B}(x,(1-\frac{1}{4})\tau r))$. Otherwise, since X is uniformly perfect, we see that a_1 is not an isolated point of X. This implies that there is a sequence $\{w_{1,j}\}_{j=1}^{\infty}\subset X\setminus\{a_1\}$ such that

$$w_{1,j}\to a_1\quad\text{as}\quad j\to\infty.$$

Let

$$\varepsilon_1=\tfrac{1}{2}\min\big\{d(a_1,x)-\big(1-\tfrac{1}{4}\big)\tau r,\ r-d(a_1,x)\big\}.$$

Then $\varepsilon_1>0$. Also, we know that there is a sufficiently large N_1 such that

$$(1-\tfrac{1}{4})\tau r<d(a_1,x)-\varepsilon_1\le d(w_{1,N_1},x)\le d(a_1,x)+\varepsilon_1<r,$$

from which we get

$$w_{1,N_1}\in(X\setminus\{a_1\})\cap(\mathbf{B}(x,r)\setminus\mathbf{B}(x,(1-\tfrac{1}{4})\tau r)).$$

Hence the claim is proved.

Claim 2.2. $X\setminus\{a_1,a_2\}$ *is* μ_2*-uniformly perfect, where* $\mu_2=\big(1-\frac{3}{8}\big)\tau$.

Let $x\in X\setminus\{a_1,a_2\}$ and $r>0$. Assume that $(X\setminus\{a_1,a_2\})\setminus\mathbf{B}(x,r)\neq\emptyset$. It follows from Claim 2.1 that there exists a point w_2 such that

$$w_2\in(X\setminus\{a_1\})\cap(\mathbf{B}(x,r)\setminus\mathbf{B}(x,(1-\tfrac{1}{4})\tau r)).$$

If $w_2\neq a_2$, then $w_2\in(X\setminus\{a_1,a_2\})\cap(\mathbf{B}(x,r)\setminus\mathbf{B}(x,\big(1-\frac{1}{4}\big)\tau r))\subset(X\setminus\{a_1,a_2\})\cap(\mathbf{B}(x,r)\setminus\mathbf{B}(x,\big(1-\frac{3}{8}\big)\tau r))$. Otherwise, there must exist a sequence $\{w_{2,j}\}_{j=1}^{\infty}\subset X\setminus\{a_1,a_2\}$ such that

$$w_{2,j}\to a_2\quad\text{as}\quad j\to\infty.$$

Let

$$\varepsilon_2=\tfrac{1}{2}\min\big\{d(a_2,x)-\big(1-\tfrac{3}{8}\big)\tau r,\ r-d(a_2,x)\big\}.$$

Then $\varepsilon_2>0$, and we also see that there is an integer N_2 such that

$$\big(1-\tfrac{3}{8}\big)\tau r<d(a_2,x)-\varepsilon_2\le d(w_{2,N_2},x)\le d(a_2,x)+\varepsilon_2<r.$$

This implies that

$$w_{2,N_2}\in(X\setminus\{a_1,a_2\})\cap(\mathbf{B}(x,r)\setminus\mathbf{B}(x,\big(1-\tfrac{3}{8}\big)\tau r)),$$

from which the claim follows.

By repeating the discussions as in Claims 2.1 and 2.2, we see that the assertion is true, and hence, the proof of the lemma is complete. □

Remark 2.1. Let $\mathbf{Q}$ be the set of all rational numbers in $\mathbf{R}$ (the real field), and let $X = \mathbf{Q} \cup \{\sqrt{2}, \sqrt{3}\}$. Then we see that X is τ-uniformly perfect for every $\tau \in (0,1)$, but $X \setminus \mathbf{Q} = \{\sqrt{2}, \sqrt{3}\}$ is not μ-uniformly perfect for any $\mu \in (0,1)$. This fact shows that Lemma 2.3 is invalid for the case when the removed set is infinite.

Lemma 2.4. *Suppose that (X,d) is a τ-uniformly perfect metric space and $\operatorname{diam}(X) \geq r$, where r is a positive constant. Then for every $a \in X$, there exists a point $w \in X$ such that*

$$\frac{\tau r}{4} \leq d(w,a) \leq \frac{r}{4}.$$

Proof. Since $\operatorname{diam}(X) \geq r$, for every $a \in X$, there must exist a point $w_0 \in X$ such that

$$d(w_0, a) \geq \frac{r}{3},$$

which guarantees that $X \setminus \mathbf{B}(a, \frac{r}{4}) \neq \emptyset$. By the uniform perfectness of X, we know that there exists a point w such that

$$w \in \mathbf{B}\left(a, \frac{r}{4}\right) \setminus \mathbf{B}\left(a, \frac{\tau r}{4}\right).$$

Hence the proof of the lemma is complete. □

2.3. Angles at seams, weak quasiconvexity and quasiconvexity. Suppose (X,d) is a metric space. Let X_1 and X_2 be two closed subsets of X with $X_1 \cap X_2 \neq \emptyset$. The *seam* is by definition the intersection $X_1 \cap X_2$. Following Agard and Gehring [1], the *angle* $\angle(X_1, X_2)$ between X_1 and X_2 at the seam $X_1 \cap X_2$ is by definition the supremum over all constants $c > 0$ such that for any $(x_1, x_2) \in X_1 \times X_2$,

$$d(x_1, x_2) \geq c \inf_{y \in X_1 \cap X_2} \big\{ d(x_1, y) + d(x_2, y) \big\}.$$

Definition 2.3. Suppose that (X,d) is a metric space, and X_1 and X_2 are closed subsets of X with $X = X_1 \cup X_2$ and $X_1 \cap X_2 \neq \emptyset$. Let $\mu > 0$. X is called *weakly μ-quasiconvex relative to* (X_1, X_2) if for each pair of points $(x_1, x_2) \in (X_1, X_2)$, there exists a point $z \in X_1 \cap X_2$ such that

$$\min\{d(x_1, z), d(x_2, z)\} \leq \mu d(x_1, x_2).$$

Definition 2.4. A metric space (X,d) is called *ν-quasiconvex* if for any x_1 and $x_2 \in X$, there exists a rectifiable curve γ joining those two points such that

$$\ell(\gamma) \leq \nu d(x_1, x_2),$$

where $\ell(\gamma)$ means the arclength of γ.

The following lemma shows that quasiconvexity implies weak quasiconvexity.

Lemma 2.5. *Suppose that (X,d) is ν-quasiconvex with $X = X_1 \cup X_2$ and $\overline{X}_1 \cap \overline{X}_2 \neq \emptyset$, where the closures are taken in X. Then X is weakly ν-quasiconvex relative to $(\overline{X}_1, \overline{X}_2)$.*

Proof. Let $x_1 \in X_1$ and $x_2 \in X_2$. Then it follows from the assumption of the quasiconvexity of X that there is a curve $\gamma \subset X$ such that

$$\ell(\gamma) \leq \nu d(x_1, x_2).$$

The lemma easily follows since $\min\{d(z,x_1), d(z,x_2)\} \le \ell(\gamma)$ for any $z \in \gamma \cap \overline{X}_1 \cap \overline{X}_2$. □

Our next lemma demonstrates the equivalence between the positive angle and the weak quasiconvexity.

Lemma 2.6. *Suppose that (X,d) is a metric space, and X_1 and X_2 are closed subsets of X with $X = X_1 \cup X_2$ and $X_1 \cap X_2 \ne \emptyset$. Then the following statements are quantitatively equivalent.*

(1) $\angle(X_1, X_2) > 0$;

(2) *X is weakly μ-quasiconvex relative to (X_1, X_2).*

Proof. For the proof of the necessity, let $\angle(X_1, X_2) = c$. Then $c > 0$, and the proof follows from the assertion: For any $x_1 \in X_1$ and $x_2 \in X_2$, there exists a point $u_1 \in X_1 \cap X_2$ such that

$$\min\{d(u_1,x_1), d(u_1,x_2)\} \le \frac{c+1}{2c} d(x_1,x_2). \tag{2.1}$$

To prove this assertion, we only need to consider the case $x_1 \ne x_2$. Since the assumption $c > 0$ implies that

$$c \inf_{y \in X_1 \cap X_2} \{d(y,x_1) + d(y,x_2)\} \le d(x_1,x_2),$$

and because there exists a point $y_1 \in X_1 \cap X_2$ such that

$$d(y_1,x_1) + d(y_1,x_2) \le \inf_{y \in X_1 \cap X_2} \{d(y,x_1) + d(y,x_2)\} + d(x_1,x_2),$$

we know that

$$\min\{d(y_1,x_1), d(y_1,x_2)\} \le \frac{d(y_1,x_1) + d(y_1,x_2)}{2} \le \frac{c+1}{2c} d(x_1,x_2).$$

By letting $u_1 = y_1$, we see that (2.1) is true.

To prove the sufficiency, we only need to show that for any $x_1 \in X_1$ and $x_2 \in X_2$,

$$\frac{1}{2(\mu+1)} \inf_{y \in X_1 \cap X_2} \{d(y,x_1) + d(y,x_2)\} \le d(x_1,x_2). \tag{2.2}$$

Since X is weakly μ-quasiconvex relative to (X_1, X_2), there exists a point $y_0 \in X_1 \cap X_2$ such that

$$\max\{d(y_0,x_1), d(y_0,x_2)\} \le (\mu+1) d(x_1,x_2).$$

Obviously, (2.2) follows from the following inequality:

$$\inf_{y \in X_1 \cap X_2} \{d(y,x_1) + d(y,x_2)\} \le 2\max\{d(y_0,x_1), d(y_0,x_2)\}.$$

Thus the lemma is proved. □

3. Quasimöbius mappings and unions

Lemma 3.1. *Suppose X_1 and X_2 are closed subsets of X with $X = X_1 \cup X_2$ and $X_1 \cap X_2 \ne \emptyset$, where (X,d) is a metric space. If $\operatorname{diam}(X_1 \cap X_2) \ge q \operatorname{diam}(X_1)$ and $X_1 \cap X_2$ is τ-uniformly perfect with $q \in (0,1]$ and $\tau \in (0,1)$, then for any x_{11}, $x_{12} \in X_1$ and $\xi \in X_1 \cap X_2$, there exists a point $\zeta \in X_1 \cap X_2$ such that*

$$\frac{1}{L} d(x_{11}, x_{12}) \le d(\xi, \zeta) \le L d(x_{11}, x_{12}),$$

where $L = \max\{\frac{4}{q}, \frac{1}{\tau}\}$.

Proof. If $\operatorname{diam}(X_1 \cap X_2) < \frac{1}{\tau} d(x_{11}, x_{12})$, since there is a point $\zeta \in X_1 \cap X_2$ such that

$$\tfrac{1}{4}\operatorname{diam}(X_1 \cap X_2) \le d(\zeta, \xi) \le \operatorname{diam}(X_1 \cap X_2),$$

it follows from the assumption $\operatorname{diam}(X_1 \cap X_2) \ge q\operatorname{diam}(X_1)$ of the lemma that

$$\tfrac{q}{4} d(x_{11}, x_{12}) \le d(\zeta, \xi) \le \tfrac{1}{\tau} d(x_{11}, x_{12}). \tag{3.1}$$

Now, we assume that $\operatorname{diam}(X_1 \cap X_2) \ge \frac{1}{\tau} d(x_{11}, x_{12})$. Then Lemma 2.4 guarantees that there exists $\zeta \in X_1 \cap X_2$ such that

$$\tfrac{1}{4} d(x_{11}, x_{12}) \le d(\zeta, \xi) \le \tfrac{1}{4\tau} d(x_{11}, x_{12}). \tag{3.2}$$

Easily, the lemma follows from (3.1) and (3.2). □

Lemma 3.2. *Suppose that (X, d) is a metric space and the following conditions are satisfied:*

(1) *X_1 and X_2 are closed subsets of X with $X = X_1 \cup X_2$ and $X_1 \cap X_2 \neq \emptyset$;*
(2) *X is weakly μ-quasiconvex relative to (X_1, X_2) with $\mu > 0$;*
(3) *There is a constant $q \in (0, 1]$ such that*

$$\operatorname{diam}(X_1 \cap X_2) \ge q \min\{\operatorname{diam}(X_1), \operatorname{diam}(X_2)\};$$

(4) *$X_1 \cap X_2$ is τ-uniformly perfect with $\tau \in (0, 1)$.*

Then for any $p \in X_1 \cap X_2$, the following hold:

(i) *X_p is weakly μ_1-quasiconvex relative to $((X_1)_p, (X_2)_p)$ with respect to the metric d_p, where $\mu_1 = 16L(\mu + 1)(L + 1)$ and L is the constant from Lemma 3.1;*
(ii) $\operatorname{diam}_p((X_i)_p) = \operatorname{diam}_p((X_1 \cap X_2)_p)$ *for $i \in \{1, 2\}$;*
(iii) *$(X_1 \cap X_2)_p$ is τ_1-uniformly perfect with respect to the metric d_p, where τ_1 depends only on τ.*

Here $\operatorname{diam}_p(M)$ *denotes the diameter of a set M with respect to the metric d_p.*

Proof. By the assumption (4) of the lemma, we see that $X_1 \cap X_2$ has no isolated point. Then Theorem C(3) ensures that all the quantities $\operatorname{diam}_p((X_i)_p)$ $(i \in \{1, 2\})$ and $\operatorname{diam}_p((X_1 \cap X_2)_p)$ are ∞, and so, the statement (ii) of the lemma is true.

Again, the assumption (4) of the lemma along with Lemma 2.3 guarantees that $(X_1 \cap X_2)_p$ is τ'-uniformly perfect, where $\tau' = \frac{\tau}{2}$. Then the statement (iii) of the lemma follows from Theorem C(2), Theorem D and Lemma 2.1.

To complete the proof, it remains to prove the statement (i) of the lemma. We are going to prove that for any $x_1 \in (X_1)_p$ and $x_2 \in (X_2)_p$, there exists $\zeta \in (X_1 \cap X_2)_p$ such that

$$\min\{d_p(x_1, \zeta), d_p(x_2, \zeta)\} \le \mu_1 d_p(x_1, x_2), \tag{3.3}$$

where $\mu_1 = 16L(\mu + 1)(L + 1)$.

If $x_1 \in (X_1 \cap X_2)_p$ (resp. $x_2 \in (X_1 \cap X_2)_p$), by taking $\zeta = x_1$ (resp. $\zeta = x_2$), (3.3) follows.

If $x_1 \in (X_1)_p \setminus ((X_1)_p \cap (X_2)_p)$ and $x_2 \in (X_2)_p \setminus ((X_1)_p \cap (X_2)_p)$, it follows from the assumption (2) of the lemma that there exists $z_0 \in X_1 \cap X_2$ such that

$$\max\{d(x_1, z_0), d(x_2, z_0)\} \le (\mu + 1) d(x_1, x_2). \tag{3.4}$$

Without loss of generality, we may assume that

$$\min\{\operatorname{diam}(X_1), \operatorname{diam}(X_2)\} = \operatorname{diam}(X_1).$$

We divide the discussions into two cases: $d(z_0, p) \le 3d(x_1, z_0)$ and $d(z_0, p) > 3d(x_1, z_0)$.

For the former case, we see from (3.4) that

$$\begin{aligned}\max\{d(x_1,p),d(x_2,p)\} &\le \max\{d(x_1,z_0),d(x_2,z_0)\}+d(z_0,p)\\ &\le 4(\mu+1)d(x_1,x_2).\end{aligned}$$

By replacing x_{11}, x_{12} and ξ in Lemma 3.1 by x_1, p, p, respectively, it follows from Lemma 3.1 that there exists $z_1 \in X_1 \cap X_2$ such that

$$\frac{1}{L}d(x_1,p) \le d(z_1,p) \le Ld(x_1,p), \tag{3.5}$$

and so, (3.5) gives

$$d(z_1,p) \le 4L(\mu+1)d(x_1,x_2).$$

Thus the assumption of this case and (3.4) lead to

$$d(x_2,z_1) \le d(x_2,z_0)+d(z_0,p)+d(z_1,p) \le 4(\mu+1)(L+1)d(x_1,x_2).$$

Thus we deduce from Theorem C(1) and (3.5) that

$$d_p(x_2,z_1) \le \frac{d(x_2,z_1)}{d(x_2,p)d(z_1,p)} \le 16L(\mu+1)(L+1)d_p(x_1,x_2). \tag{3.6}$$

Since $x_1 \neq p$, again, it follows from (3.5) that $z_1 \neq p$. By taking $\zeta = z_1$, (3.3) follows.

For the latter case, i.e., $d(z_0,p) > 3d(x_1,z_0)$, we know that $z_0 \neq p$ and

$$d(x_1,p) \le d(z_0,p)+d(x_1,z_0) \le \tfrac{4}{3}d(z_0,p).$$

Then Theorem C(1) and (3.4) lead to

$$d_p(x_2,z_0) \le \frac{d(x_2,z_0)}{d(x_2,p)d(z_0,p)} \le \frac{16}{3}(\mu+1)d_p(x_1,x_2).$$

By taking $\zeta = z_0$, we know that (3.3) is true. Hence the lemma is proved. □

Proof of Theorem 1.1. The sufficiency is obvious. To prove the necessity, for convenience, let $f_i = f|_{X_i}$ for $i = 1, 2$. Now, we assume that both f_1 and f_2 are θ-QM. Let $p \in X_1 \cap X_2$, and let

$$g = \mathrm{id}_2 \circ f \circ \mathrm{id}_1^{-1} \colon (X_p, d_p) \to (Y_{p'}, d_{1,p'}), \tag{3.7}$$

where $p' = f(p)$, both $\mathrm{id}_1 \colon (X_p, d) \to (X_p, d_p)$ and $\mathrm{id}_2 \colon (Y_{p'}, d_1) \to (Y_{p'}, d_{1,p'})$ are the identity mappings. Also, we use x (resp. y) to denote both x and its image $\mathrm{id}_1(x)$ for any $x \in X_p$ (resp. both y and its image $id_2(y)$ for any $y \in Y_{p'}$). Furthermore, for $i = 1, 2$, let

$$g_i = g|_{(X_i)_p}.$$

Then $g_i(x) = x'$ for all $x \in (X_i)_p$, where $x' = f(x)$.

To finish the proof, we need the following claim.

Claim 3.1. (a) *g_i is θ_1-QS, where $\theta_1(t) = 16\theta(16t)$;*

(b) *g is η_2-QS, where η_2 depends on θ, τ, q, together with the angles $\angle(X_1, X_2)$ and $\angle(Y_1, Y_2)$.*

First, we prove (a). We see from Theorem C(2) that for each $i \in \{1, 2\}$, g_i is θ_1-QM. Without loss generality, we may assume that $i = 1$. To prove this statement, we only need to show that for every triple $\{x, a, b\}$ in $(X_1)_p$,

$$\frac{d_{1,p'}(g_1(x),g_1(a))}{d_{1,p'}(g_1(x),g_1(b))} \le 16\theta\left(16\frac{d_p(x,a)}{d_p(x,b)}\right). \tag{3.8}$$

By the uniform perfectness of $X_1 \cap X_2$, we know that there is a sequence $\{p_n\} \subset (X_1 \cap X_2)_p$ such that $d(p_n, p) \to 0$ as $n \to \infty$. Obviously, $\{p'_n\} \subset (Y_1 \cap Y_2)_{p'}$ and $d_1(p'_n, p') \to 0$ as $n \to \infty$, where $p'_n = f(p_n)$. Since g_1 is θ_1-QM, we have

$$\frac{d_{1,p'}(g_1(x), g_1(a))d_{1,p'}(g_1(b), g_1(p_n))}{d_{1,p'}(g_1(x), g_1(b))d_{1,p'}(g_1(x), g_1(p_n))} \leq 16\theta\left(16\frac{d_p(x,a)d_p(b,p_n)}{d_p(x,b)d_p(x,p_n)}\right). \tag{3.9}$$

Moreover, it follows from Theorem C(1) that all the quantities

$$d_p(x, p_n),\ \ d_p(b, p_n),\ \ d_{1,p'}(g_1(x), g_1(p_n))\ \text{ and }\ d_{1,p'}(g_1(b), g_1(p_n))$$

tend to ∞ as $n \to \infty$. Thus (3.8) follows from (3.9) by letting n tend to ∞.

Next, we shall apply Theorem B to prove (b). For this, we need to check that all the assumptions in Theorem B are satisfied. Since Corollary 2.2 guarantees that $Y_1 \cap Y_2$ is τ_2-uniformly perfect, where τ_2 depends only on θ and τ, we see from Lemmas 2.6 and 3.2, together with the statement (a) of the claim, that all the assumptions in Theorem B are satisfied. By Theorem B, we know that the statement (b) of the claim is true, and thus, the claim is proved.

We are ready to finish the proof of the theorem. Since Claim 3.1(b) implies that $f|_{X_p} = \mathrm{id}_2^{-1} \circ g \circ \mathrm{id}_1$ is θ_2-QM, where θ_2 depends only on η_2, and since p is not an isolated point of X, we see from the homeomorphism of f that f is also θ_2-QM. Now, the proof of the theorem is complete.

4. Some examples

Throughout this section, $\mathbf{C}$ denotes the complex plane and $z = x + iy$ stands for a point in $\mathbf{C}$, where x and $y \in \mathbf{R}$, O denotes the coordinate origin in $\mathbf{C}$, and the metric $d = |\cdot|$ is the usual Euclidean metric.

In this section, our purpose is to construct four examples. The first example shows that the diameter condition in Theorem B is not invariant with respect to QM mappings. The remaining three examples demonstrate that each of the first three assumptions in Theorem 1.1 cannot be removed.

Example 4.1. Let

$$X = X_1 \cup X_2,$$

where $X_1 = (I_1 \cup I_2) \setminus \{O\}$, $X_2 = (I_3 \cup I_4) \setminus \{O\}$,

$$I_1 = \{z \in \mathbf{C}\colon x^2 + y^2 \leq 4,\ x \geq 0,\ y \geq 0\},$$

$$I_2 = \{z \in \mathbf{C}\colon 1 \leq x^2 + y^2 \leq 4,\ x \leq 0,\ y \leq 0\},$$

$$I_3 = \{z \in \mathbf{C}\colon x^2 + y^2 \leq 4,\ x \leq 0,\ y \leq 0\}$$

and

$$I_4 = \{z \in \mathbf{C}\colon 1 \leq x^2 + y^2 \leq 4,\ x \geq 0,\ y \geq 0\}$$

(see Figure 1), and let

$$f\colon X \to Y$$

with $f(z) = \frac{z}{|z|^2}$ and $Y = f(X)$. Then we have the following conclusions.

(1) The homeomorphism f is θ_3-QM, where $\theta_3(t) = 81t$;
(2) $\mathrm{diam}(X_1) = \mathrm{diam}(X_2) = \mathrm{diam}(X_1 \cap X_2) = 4$;
(3) $\mathrm{diam}(f(X_1)) = \mathrm{diam}(f(X_2)) = \infty$, but $\mathrm{diam}(f(X_1) \cap f(X_2)) = 2$.

Proof. The first assertion follows from [11, p. 220], and the rest two assertions are obvious. □

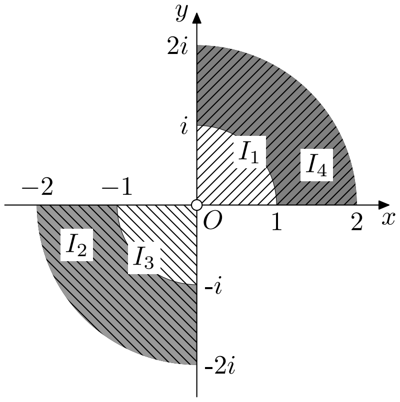

Figure 1. Example 4.1.

The following example shows that the assumption (1) in Theorem 1.1 cannot be removed.

Example 4.2. Let

$$X = X_1 \cup X_2,$$

where $X_1 = J_1 \cup J_3$, $X_2 = J_2 \cup J_3$, $J_1 = \{z \in \mathbf{C}\colon \frac{1}{64} < x^2 + y^2 < 1,\ x > 0,\ y > 0\}$,

$$J_2 = \{z \in \mathbf{C}\colon x^2 + y^2 < 1,\ x > 0,\ y < 0\} \ \text{ and } \ J_3 = \{z \in \mathbf{C} : x^2 + y^2 = 1, x > 0\}$$

(see Figure 2), and let

$$f\colon X \to Y,$$

where $f|_{X_1}(z) = \frac{z}{|z|^2}$, $f|_{X_2}(z) = z$ and $Y = f(X)$. Then the following statements hold.

(a) $X = X_1 \cup X_2$ is not weakly μ-quasiconvex relative to (X_1, X_2) for any $\mu > 0$;
(b) Both $X_1 \cap X_2$ and $f(X_1) \cap f(X_2)$ are connected;
(c) $\operatorname{diam}(X_1 \cap X_2) = \operatorname{diam}(X_1) = \operatorname{diam}(X_2) = 2$, $\operatorname{diam}(f(X_1) \cap f(X_2)) = \operatorname{diam}(f(X_2)) = 2$ and $\operatorname{diam}(f(X_1)) = 8\sqrt{2}$;
(d) Both $f|_{X_1}$ and $f|_{X_2}$ are θ_3-QM, where $\theta_3(t) = 81t$;
(e) The homeomorphism f is not θ-QM for any homeomorphism θ.

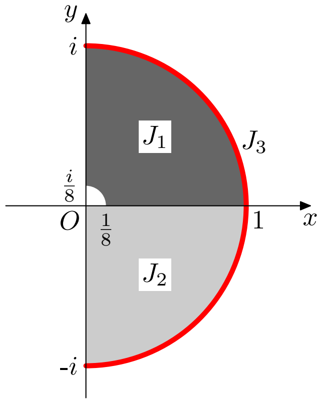

Figure 2. Example 4.2.

Proof. To prove the first statement, let $z_1 = \frac{1}{2} + is$ and $z_2 = \frac{1}{2} - is$, where $0 < s < \frac{1}{2}$. Then we have

$$|z_1 - z_2| = 2s.$$

Moreover, for any $z \in X_1 \cap X_2 = J_3$, we have

$$\min\{|z_1 - z|, |z_2 - z|\} \geq \frac{2 - \sqrt{1+4s^2}}{2}.$$

By letting $s \to 0$, we see that X is not weakly μ-quasiconvex relative to (X_1, X_2) for any $\mu > 0$.

The second and the third statements are obvious, and the forth one easily follows from [11, p. 220]. Now, we remain to show the last statement. For this, let

$$z_3 = \frac{1}{4} - is, \quad z_4 = \frac{1}{2} - is, \quad z_5 = \frac{1}{4} + is \quad \text{and} \quad z_6 = \frac{1}{2} + is,$$

where $0 < s < \frac{1}{2}$. By elementary computations, we obtain

$$r = \frac{|z_3 - z_5||z_4 - z_6|}{|z_3 - z_4||z_5 - z_6|} = 64s^2 \ \text{ and } \ r_1 = \frac{|z_3' - z_5'||z_4' - z_6'|}{|z_3' - z_4'||z_5' - z_6'|} = \frac{2\lambda_1(s)\lambda_2(s)}{\sqrt{(1+16s^2)(1+4s^2)}},$$

where

$$\lambda_1(s) = \sqrt{\left(\frac{15}{4} - 4s^2\right)^2 + (17+16s^2)^2 s^2} \ \text{ and } \ \lambda_2(s) = \sqrt{\left(\frac{3}{2} - 2s^2\right)^2 + (5+4s^2)^2 s^2}.$$

Since $0 < s < \frac{1}{2}$, we see that

$$r_1 \geq \frac{9}{8}\sqrt{10},$$

and thus, the fact $r \to 0$ as $s \to 0$ implies that f is not θ-QM for any homeomorphism θ. □

The purpose of our next example is to illustrate that the assumption (2) in Theorem 1.1 cannot be removed.

Example 4.3. Let

$$X = X_1 \cup X_2,$$

where $X_1 = \{z \in \mathbf{C}\colon x^2 + y^2 \leq 1,\ x \leq 0\} \cup \{p_0\}$, p_0 denotes the point $(1, 0)$ and $X_2 = \{z \in \mathbf{C}\colon x \geq 0,\ y = 0\}$ (see Figure 3), and let

$$f\colon X \to X,$$

where $f|_{X_1}(z) = z$ and $f|_{X_2}(z) = z^2$. Then the following statements hold.

(I) $f(X_i) = X_i$ for each $i = 1, 2$;
(II) $X = X_1 \cup X_2$ is weakly 1-quasiconvex relative to (X_1, X_2);
(III) $X_1 \cap X_2$ is not τ-uniformly perfect for any $\tau \in (0, 1)$;
(IV) $\operatorname{diam}(X_1 \cap X_2) = q \min\{\operatorname{diam}(X_1), \operatorname{diam}(X_2)\} = 1$, where $q = \frac{1}{2}$;
(V) $f|_{X_1}$ and $f|_{X_2}$ are θ_4-QM, where

$$\theta_4(t) = 4t^3 + 8t^2\sqrt{t} + 135t^2 + 8t\sqrt{t} + 90t + 16\sqrt{t};$$

(VI) The homeomorphism f is not θ-QM for any homeomorphism θ.

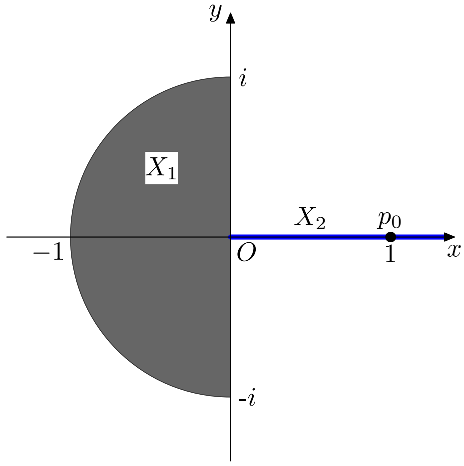

Figure 3. Example 4.3.

Proof. The first statement is obvious. To prove the second statement, it suffices to show that for any $z_1 \in X_1$ and $z_2 \in X_2$, there exists $z^* \in X_1 \cap X_2 (= \{O, p_0\})$ such that

$$\min\{|z_1 - z^*|, |z_2 - z^*|\} \le |z_1 - z_2|. \tag{4.1}$$

If $z_1 \in X_1 \cap X_2$ (resp. $z_2 \in X_1 \cap X_2$), let $z^* = z_1$ (resp. $z^* = z_2$). Then (4.1) is obvious.

If $z_1 \in X_1 \setminus (X_1 \cap X_2)$ and $z_2 \in X_2 \setminus (X_1 \cap X_2)$, then we easily know that the angle formed by the vectors $\overrightarrow{Oz_1}$ and $\overrightarrow{Oz_2}$ is at least $\frac{\pi}{2}$. By letting $z^* = O$, we see that (4.1) is true.

The third and forth statements are obvious. For the fifth one, obviously, $f|_{X_1}$ is θ_1-QM, where $\theta_5(t) = t$. Hence it is θ_4-QM, where

$$\theta_4(t) = 4t^3 + 8t^2\sqrt{t} + 135t^2 + 8t\sqrt{t} + 90t + 16\sqrt{t}.$$

About $f|_{X_2}$, we first show that $f|_{X_2}$ is η-QS, where

$$\eta(t) = t(t + 2).$$

Let $z_1 = (x_1, 0)$, $z_2 = (x_2, 0)$, $z_3 = (x_3, 0) \in X_2$ with $z_1 \ne z_3$. Then

$$\frac{|f(z_1) - f(z_2)|}{|f(z_1) - f(z_3)|} = \frac{|x_1^2 - x_2^2|}{|x_1^2 - x_3^2|} = \frac{(x_1 + x_2)|z_1 - z_2|}{(x_1 + x_3)|z_1 - z_3|}.$$

Since

$$\frac{x_1 + x_2}{x_1 + x_3} \le 2 + \frac{|z_1 - z_2|}{|z_1 - z_3|},$$

we see that $f|_{X_2}$ is η-QS, where $\eta(t) = t(t + 2)$. Thus the similar reasoning as in the proof of [11, Theorem 3.2] ensures that $f|_{X_2}$ is θ_4-QM.

To finish the proof, it remains to show the last statement. For this, let

$$z_4 = 0, \quad z_5 = s, \quad z_6 = -s \quad \text{and} \quad z_7 = -2s,$$

where $0 < s < 1$. By elementary computations, we obtain

$$r = \frac{|z_4 - z_6||z_5 - z_7|}{|z_4 - z_5||z_6 - z_7|} = 3 \quad \text{and} \quad r_1 = \frac{|z_4' - z_6'||z_5' - z_7'|}{|z_4' - z_5'||z_6' - z_7'|} = \frac{s + 2}{s}.$$

Now, it follows from the fact $r_1 \to +\infty$ as $s \to 0$ that f is not θ-QM for any homeomorphism θ. Hence the proof of the example is complete. □

By the next example, we know that the assumption (3) in Theorem 1.1 cannot be removed.

Example 4.4. Let

$$X = X_1 \cup X_2,$$

where $X_1 = \{z \in \mathbf{C}\colon x^2+y^2 \geq 1,\ x \geq 0,\ y \geq 0\}$, $X_2 = K_1 \cup K_2$, $K_1 = \{z \in \mathbf{C}\colon x^2+y^2 = 1,\ x \geq 0,\ y \geq 0\}$ and $K_2 = \{z \in \mathbf{C}\colon x = 1,\ y \leq 0\}$ (see Figure 4), and let

$$f\colon X \to Y,$$

where $f|_{X_1}(z) = \frac{z}{|z|^2}$, $f|_{X_2}(z) = z$ and $Y = f(X)$. Then the following statements hold.

(i) $X = X_1 \cup X_2$ is weakly 1-quasiconvex relative to (X_1, X_2) and $Y = f(X_1) \cup f(X_2)$ is weakly 1-quasiconvex relative to $(f(X_1), f(X_2))$;
(ii) Both $X_1 \cap X_2$ and $f(X_1) \cap f(X_2)$ are connected;
(iii) $\mathrm{diam}(X_1) = \mathrm{diam}(X_2) = \infty$, $\mathrm{diam}(X_1 \cap X_2) = \sqrt{2}$, and $\mathrm{diam}(f(X_1) \cap f(X_2)) = \min\{\mathrm{diam}(f(X_1)), \mathrm{diam}(f(X_2))\} = \sqrt{2}$;
(iv) Both $f|_{X_1}$ and $f|_{X_2}$ are θ_3-QM, where $\theta_3(t) = 81t$;
(v) f is not θ-QM for any homeomorphism θ.

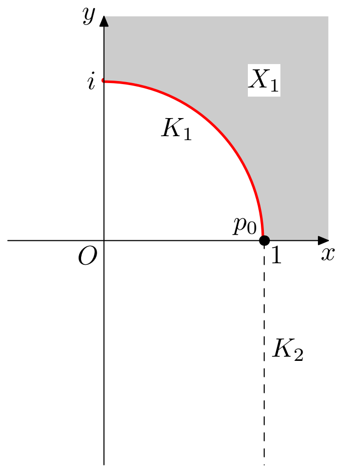

Figure 4. Example 4.4.

Proof. To prove the first statement, we only need to illustrate the relatively weak quasiconvexity of X since the proof of that of Y is similar. To reach this goal, it suffices to show that for any $z_1 \in X_1$ and $z_2 \in X_2$, there exists $z_0 \in X_1 \cap X_2 (= K_1)$ such that

$$\min\{|z_1 - z_0|, |z_2 - z_0|\} \leq |z_1 - z_2|. \tag{4.2}$$

If $z_1 \in K_1$ (resp. $z_2 \in K_1$), let $z_0 = z_1$ (resp. $z_0 = z_2$). Then (4.2) is obvious.

If $z_1 \in X_1 \setminus K_1$ and $z_2 \in X_2 \setminus K_1$, then we easily know that the angle formed by the vectors $\overrightarrow{p_0 z_1}$ and $\overrightarrow{p_0 z_2}$ is at least $\frac{\pi}{2}$ (we recall that p_0 denotes the point $(1,0)$ in $\mathbf{C}$). This fact guarantees that (4.2) holds by letting $z_0 = p_0$.

The second and third statements are obvious, and the forth one follows from [11, p. 220]. To finish the proof, it remains to show the last statement. For this, let

$$z_3 = \frac{1+i}{2}t, \quad z_4 = (1+i)t, \quad z_5 = 1 - 10i \quad \text{and} \quad z_6 = 1 - ti,$$

where $t > 10$ is an integer. By elementary computations, we obtain

$$r = \frac{|z_3 - z_5||z_4 - z_6|}{|z_3 - z_4||z_5 - z_6|} = \frac{\sqrt{(t-2)^2 + (t+20)^2}\sqrt{(t-1)^2 + 4t^2}}{\sqrt{2}t(t-10)}$$

and

$$r_1 = \frac{|z_3' - z_5'||z_4' - z_6'|}{|z_3' - z_4'||z_5' - z_6'|} = \frac{\sqrt{(t-1)^2 + (10t+1)^2}\sqrt{(2t-1)^2 + (2t^2+1)^2}}{\sqrt{2}t(t-10)}.$$

Now, it follows from the fact $r \to \sqrt{5}$ and $r_1 \to +\infty$ as $t \to +\infty$ that f is not θ-QM for any homeomorphism θ. Hence the proof of the example is complete. □

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