

THE QUASISUPERMINIMIZING CONSTANT FOR THE MINIMUM OF TWO QUASISUPERMINIMIZERS IN \mathbf{R}^n

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Abstract. It was shown in Björn–Björn–Korte [5] that $u := \min\{u_1, u_2\}$ is a \bar{Q} -quasisuperminimizer if u_1 and u_2 are Q -quasisuperminimizers and $\bar{Q} = 2Q^2/(Q+1)$. Moreover, one-dimensional examples therein show that \bar{Q} is close to optimal. In this paper we give similar examples in higher dimensions. The case when u_1 and u_2 have different quasisuperminimizing constants is considered as well.

1. Introduction

Let $\Omega \subset \mathbf{R}^n$ be a nonempty open set and $1 < p < \infty$. A function $u \in W_{\text{loc}}^{1,p}(\Omega)$ is a Q -quasi(super)minimizer in Ω , with $Q \geq 1$, if

$$(1.1) \quad \int_{\varphi \neq 0} |\nabla u|^p dx \leq Q \int_{\varphi \neq 0} |\nabla(u + \varphi)|^p dx$$

for all (nonnegative) $\varphi \in W_0^{1,p}(\Omega)$. Quasiminimizers were introduced by Giaquinta–Giusti [6] as a tool for a unified treatment of variational integrals, elliptic equations and systems, and quasiregular mappings on \mathbf{R}^n .

Quasi(super)minimizers have an interesting theory already in the one-dimensional case, see e.g. [6] and Martio–Sbordone [10]. Kinnunen–Martio [9] showed that one can build a rich potential theory based on quasiminimizers. In particular, they introduced quasisuperharmonic functions, which are related to quasisuperminimizers in a similar way as superharmonic functions are related to supersolutions. See Björn–Björn–Korte [5] for further references.

Kinnunen–Martio [9, Lemmas 3.6 and 3.7] also showed that if u_j is a Q_j -quasisuperminimizer in $\Omega \subset \mathbf{R}^n$ (or in a metric space), $j = 1, 2$, then $u := \min\{u_1, u_2\}$ is a $\min\{Q_1 + Q_2, Q_1 Q_2\}$ -quasisuperminimizer. Björn–Björn–Korte [5] improved upon this result in the following way.

Theorem 1.1. [5, Theorem 1.2] *Let u_j be a Q_j -quasisuperminimizer in $\Omega \subset \mathbf{R}^n$ (or in a metric space), $j = 1, 2$. Then $u := \min\{u_1, u_2\}$ is a \bar{Q} -quasisuperminimizer in Ω , where*

$$(1.2) \quad \bar{Q} = \begin{cases} 1, & \text{if } Q_1 = Q_2 = 1, \\ (Q_1 + Q_2 - 2) \frac{Q_1 Q_2}{Q_1 Q_2 - 1}, & \text{otherwise.} \end{cases}$$

In particular, if $Q_1 = Q_2$, then $\bar{Q} = 2Q_1^2/(Q_1 + 1)$.

It is not known whether \bar{Q} is optimal, but it is the best upper bound known. On the other hand, that u is (in general) not better than a $\max\{Q_1, Q_2\}$ -quasisuperminimizer is rather obvious.

The first examples (and so far the only ones) showing that u is (in general) *not* a $\max\{Q_1, Q_2\}$ -quasisuperminimizer were given in [5]. More precisely, if $1 < Q_1 \leq Q_2$ then there are Q_j -quasisuperminimizers u_j on $(0, 1) \subset \mathbf{R}$ such that $u := \min\{u_1, u_2\}$ is not a Q_2 -quasisuperminimizer. Estimates and concrete examples, showing that the constant \bar{Q} above is not too far from being optimal, were also given in [5]. All examples therein were on $(0, 1) \subset \mathbf{R}$ and our aim in this paper is to obtain similar examples in higher dimensions, i.e. for subsets of \mathbf{R}^n , $n \geq 2$.

The examples in [5] (giving the best lower bounds) were based on power functions $x \mapsto x^\alpha$ and reflections of such functions. For such functions, also in the higher-dimensional case on \mathbf{R}^n , $n \geq 2$, optimal quasi(super)minimizing constants $Q(\alpha, p, n)$ were obtained in Björn–Björn [2], and these formulas for $Q(\alpha, p, n)$ (with $n = 1$) were used in the calculations in [5].

As power-type functions $x \mapsto |x|^\alpha$ only have point singularities, it seems difficult to use them for higher-dimensional analogues of the examples in [5]. Instead we base our examples on log-power functions $\log^\alpha |x|$ and $(-\log |x|)^\alpha$. Since $\log |ex| = 1 - (-\log |x|)$, we are able to scale and translate them and create higher-dimensional examples on annuli, in the spirit of [5]. For this to be possible we need the log-powers to be quasisuperminimizers which requires p to be equal to the conformal dimension n . In particular we obtain the following result.

Theorem 1.2. *Let $p = n \geq 2$ and $1 < Q_1 \leq Q_2$ be given. Then there are functions u_1 and u_2 on $A := \{x \in \mathbf{R}^n : 1/e < |x| < 1\}$ such that u_j is a Q_j -quasisuperminimizer in A , $j = 1, 2$, but $\min\{u_1, u_2\}$ is not a Q_2 -quasiminimizer in A .*

As in [5] we also give lower bounds for the increase in the quasisuperminimizing constant and show that these lower bounds are the same as in the 1-dimensional case considered in [5], see Section 3. In Section 4 we show that one can add dummy variables to these examples, as well as to those in [5]; this is nontrivial and partly relies on results from Björn–Björn [4].

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2. Quasi(sub/super)minimizers

Above we defined what quasiminimizers and quasisuperminimizers are. A function u is a Q -quasisubminimizer if $-u$ is a Q -quasisuperminimizer. Our definition of quasiminimizers (and quasisub- and quasisuperminimizers) is one of several equivalent possibilities, see Proposition 3.2 in A. Björn [1]. In particular, we will use that it is enough to require (1.1) to hold for all (nonnegative) $\varphi \in \text{Lip}_c(\Omega)$, where $\text{Lip}_c(\Omega)$ denotes the space of all Lipschitz functions with compact support in Ω .

When $Q = 1$ we usually drop “quasi” and say (sub/super)minimizer. Being a (sub/super)minimizer is the same as being a (weak) (sub/super)solution of the p -Laplace equation

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0,$$

see Chapter 5 in Heinonen–Kilpeläinen–Martio [8]. The function u is a supersolution of this equation if the left-hand side is nonnegative in a weak sense.

By Giaquinta–Giusti [7, Theorem 4.2], a Q -quasiminimizer can be modified on a set of measure zero so that it becomes locally Hölder continuous in Ω . This continuous Q -quasiminimizer is called a Q -quasiharmonic function, and a p -harmonic function is a continuous minimizer.

If u is a quasisuperminimizer, then $au + b$ is also a quasisuperminimizer whenever $a \geq 0$ and $b \in \mathbf{R}$. Also, u is a Q -quasiminimizer if and only if it is both a Q -quasisubminimizer and a Q -quasisuperminimizer. Quasisuperminimizers are invariant under scaling in the following way.

Lemma 2.1. *Let $\Omega \subset \mathbf{R}^n$ be open and $\tau > 0$. If $u : \Omega \rightarrow \mathbf{R}^n$ is a Q -quasisuperminimizer in Ω , then $v(x) := u(\tau x)$ is a Q -quasisuperminimizer in*

$$\Omega_\tau := \{x : \tau x \in \Omega\}.$$

Proof. Let $\varphi \in W_0^{1,p}(\Omega_\tau)$ be nonnegative. Then $\tilde{\varphi}(x) := \varphi(x/\tau) \in W_0^{1,p}(\Omega)$. Hence, as $u : \Omega \rightarrow \mathbf{R}^n$ is a Q -quasisuperminimizer in Ω ,

$$\begin{aligned} \int_{\varphi \neq 0} |\nabla v|^p dx &= \tau^{p-n} \int_{\tilde{\varphi} \neq 0} |\nabla u|^p dx \leq Q \tau^{p-n} \int_{\tilde{\varphi} \neq 0} |\nabla(u + \tilde{\varphi})|^p dx \\ &= Q \int_{\varphi} |\nabla(v + \varphi)|^p dx. \end{aligned}$$

Thus v is a Q -quasisuperminimizer in Ω_τ . □

The following definition will play a role in Section 3. The finiteness requirement is important for this to be useful, and is always fulfilled when \bar{G} is a compact subset of Ω , by the definition of quasiminimizers.

Definition 2.2. Let u be a Q -quasiminimizer in $\Omega \subset \mathbf{R}^n$ and $G \subset \Omega$ be a nonempty open set. We say that u has the *maximal p -energy allowed by Q on G* if

$$\int_G |\nabla u|^p dx = Q \int_G |\nabla v|^p dx < \infty,$$

where v is the minimizer in G with boundary values $v = u$ on ∂G .

For further discussion on quasi(super)minimizers, as well as references to the literature, we refer to Björn–Björn [2] and Björn–Björn–Korte [5]. We will mainly be interested in radially symmetric functions on \mathbf{R}^n , $n \geq 2$.

For the rest of this section, as well as in most of Section 3, we will only consider the case when $p = n$, the conformal dimension.

Define the annulus

$$A_{r_1,r_2} = \{x \in \mathbf{R}^n : r_1 < |x| < r_2\}, \quad \text{where } 0 \leq r_1 < r_2 \leq \infty.$$

In the conformal case ($p = n$) the logarithm $\log |x|$ is an n -harmonic function, and log-powers are quasiminimizers as we shall see. These log-powers and their optimal quasisuperminimizing constants will be the crucial ingredients in Section 3, when investigating the increase in the quasisuperminimizing constant for the minimum of two quasisuperminimizers. The optimal quasiminimizing and quasisub/superminimizing constants for power functions and log-powers $(-\log |x|)^\alpha$ on punctured unit balls were obtained in Björn–Björn [2]. These are rather easily shown to apply also to the annuli $A_{\gamma,1}$ with $0 < \gamma < 1$, see Theorem 2.5 below. We also need to consider log-powers $\log^\alpha |x| := (\log |x|)^\alpha$ on the annuli $A_{1,\gamma}$, $\gamma > 1$, and their optimal

quasiminimizing and quasisub/superminimizing constants provided by the following result.

Theorem 2.3. *Let $1 < \gamma \leq \infty$, $\alpha > 1 - 1/n$ and $u(x) = \log^\alpha |x|$. Then u is a quasiminimizer in $A_{1,\gamma}$ with*

$$Q_{\alpha,n} = \frac{\alpha^n}{n\alpha - n + 1}$$

being the best quasiminimizing constant. Moreover, u is a Q -quasi(sub/super)minimizer in $A_{1,\gamma}$ as given in Table 1, where Q in Table 1 is the best quasi(sub/super)-minimizing constant. Furthermore, u has the maximal n -energy allowed by $Q_{\alpha,n}$ on $A_{1,\gamma}$ if $\gamma < \infty$.

	quasi-minimizer	quasi-subminimizer	quasi-superminimizer
$1 - \frac{1}{n} < \alpha < 1$	$Q = Q_{\alpha,n}$	$Q = Q_{\alpha,n}$	$Q = 1$
$\alpha = 1$	$Q = 1$	$Q = 1$	$Q = 1$
$\alpha > 1$	$Q = Q_{\alpha,n}$	$Q = 1$	$Q = Q_{\alpha,n}$

Table 1. Optimal quasiminimizing and quasisub/superminimizing constants of $\log^\alpha |x|$ in $A_{1,\gamma}$ and $(-\log |x|)^\alpha$ in $A_{\gamma,1}$, as provided by Theorems 2.3 and 2.5.

The proof is a modification of the proofs of Theorems 7.3 and 7.4 in [2]. For the reader’s convenience we provide the details.

Proof. Let $\varphi(r) = \log^\alpha r$, $1 < r_1 < r_2 < \gamma$, $G = (r_1, r_2)$, $s_1 = \log r_1$, $s_2 = \log r_2$ and $S = s_2/s_1 > 1$. The n -energy of u in A_{r_1,r_2} is given by

$$I_u(A_{r_1,r_2}) := \int_\Omega |\nabla u|^n dx = c_{n-1} \int_{r_1}^{r_2} |\varphi'(r)|^n r^{n-1} dr =: c_{n-1} \hat{I}_\varphi(G),$$

where c_{n-1} is the $(n - 1)$ -dimensional surface area on the sphere \mathbf{S}^{n-1} . Moreover,

$$\begin{aligned} (2.1) \quad \hat{I}_\varphi(G) &= \int_{r_1}^{r_2} \alpha^n \frac{(\log r)^{n\alpha-n}}{r^n} r^{n-1} dr = \int_{s_1}^{s_2} \alpha^n t^{n\alpha-n} dt \\ &= Q_{\alpha,n} (s_2^{n\alpha-n+1} - s_1^{n\alpha-n+1}) = Q_{\alpha,n} s_1^{n\alpha-n+1} (S^{n\alpha-n+1} - 1). \end{aligned}$$

A minimizer is given by $\psi(r) = \log r$, and we have letting $\alpha = 1$ above,

$$\hat{I}_\psi(G) = s_1(S - 1).$$

We want to compare the energy $\hat{I}_\varphi(G)$ with the energy $\hat{I}_\eta(G)$ of the minimizer $\eta = a\psi + b$ having the same boundary values on ∂G as φ . As

$$a = \frac{s_2^\alpha - s_1^\alpha}{s_2 - s_1} = s_1^{\alpha-1} \frac{S^\alpha - 1}{S - 1},$$

their quotient is

$$(2.2) \quad k(S) := \frac{\hat{I}_\varphi(G)}{\hat{I}_\eta(G)} = \frac{\hat{I}_\varphi(G)}{|a|^n \hat{I}_\psi(G)} = Q_{\alpha,n} \frac{S^{n\alpha-n+1} - 1}{S - 1} \left(\frac{S - 1}{S^\alpha - 1} \right)^n,$$

which only depends on S .

Let $s = \sqrt{s_1 s_2}$ and let η_1 and η_2 be the minimizers of the \hat{I} -energy on $G_1 = (e^{s_1}, e^s)$ resp. $G_2 = (e^s, e^{s_2})$ having the same boundary values as φ on ∂G_1 resp. ∂G_2 .

Also let $\tilde{\eta} = \eta_1\chi_{G_1} + \eta_2\chi_{G\setminus G_1}$. Then, as $s/s_1 = s_2/s = \sqrt{S}$, we see that

$$\begin{aligned} \hat{I}_\varphi(G) &= k(S)\hat{I}_{\tilde{\eta}}(G) \leq k(S)\hat{I}_{\tilde{\eta}}(G) = k(S)(\hat{I}_{\eta_1}(G_1) + \hat{I}_{\eta_2}(G_2)) \\ &= k(S)\left(\frac{\hat{I}_\varphi(G_1)}{k(\sqrt{S})} + \frac{\hat{I}_\varphi(G_2)}{k(\sqrt{S})}\right) = \frac{k(S)}{k(\sqrt{S})}\hat{I}_\varphi(G). \end{aligned}$$

As $0 < \hat{I}_\varphi(G) < \infty$, we find that $k(S) \geq k(\sqrt{S})$, and thus

$$(2.3) \quad \sup_{S>1} k(S) = \lim_{S\rightarrow\infty} k(S) = Q_{\alpha,n}.$$

Comparing u with $x \mapsto \eta(|x|)$ shows that the quasiminimizing constant for u cannot be less than $Q_{\alpha,n}$.

To show that $Q_{\alpha,n}$ will do, let ω be a function such that $\omega - \varphi \in \text{Lip}_c((1, \gamma))$. The open set

$$V = \{x \in (1, \gamma) : \omega(x) \neq \varphi(x)\}$$

can be written as a countable (or finite) union of pairwise disjoint intervals $\{I_j\}_j$. We find from (2.2) and (2.3) that

$$(2.4) \quad \hat{I}_\varphi(V) = \sum_j \hat{I}_\varphi(I_j) \leq \sum_j Q_{\alpha,n}\hat{I}_\omega(I_j) = Q_{\alpha,n}\hat{I}_\omega(V).$$

Hence φ is indeed a $Q_{\alpha,n}$ -quasiminimizer for the energy \hat{I} on $(1, \gamma)$.

Next, we turn to u . Let v be such that $v - u \in \text{Lip}_c(A_{1,\gamma})$. Also let

$$\Omega = \{x \in A_{1,\gamma} : v(x) \neq u(x)\}.$$

Using polar coordinates $x = (r, \theta)$, where $r > 1$ and $\theta \in \mathbf{S}^{n-1}$, let

$$V_\theta = \{r : (r, \theta) \in \Omega\} \quad \text{and} \quad v_\theta(r) = v(r, \theta).$$

We then find, applying (2.4) to $G = V_\theta$, that

$$\begin{aligned} I_u(\Omega) &= \int_{\mathbf{S}^{n-1}} \hat{I}_\varphi(V_\theta) d\theta \leq \int_{\mathbf{S}^{n-1}} Q_{\alpha,n}\hat{I}_{v_\theta}(V_\theta) d\theta \\ &= Q_{\alpha,n} \int_\Omega \left| \frac{\partial v}{\partial r} \right|^n dx \leq Q_{\alpha,n} \int_\Omega |\nabla v|^n dx = Q_{\alpha,n}I_v(\Omega), \end{aligned}$$

showing that u is indeed a $Q_{\alpha,n}$ -quasiminimizer in $A_{1,\gamma}$.

It follows directly that the constants in the quasiminimizer column in Table 1 are correct. By Lemma 2.4 below, u is a subminimizer if $\alpha \geq 1$ and a superminimizer if $1 - 1/n < \alpha \leq 1$. As u is a Q -quasiminimizer if and only if it is both a Q -quasisubminimizer and a Q -quasisuperminimizer, it follows that $Q_{\alpha,n}$ is the optimal quasisubminimizing constant when $1 - 1/n < \alpha \leq 1$, and the optimal quasisuperminimizing constant when $\alpha \geq 1$.

Finally, if $\gamma < \infty$, then it follows from (2.1) that

$$\hat{I}_\varphi(A_{1,\gamma}) = Q_{\alpha,n}(\log \gamma)^{n\alpha-n+1}$$

and that the minimizer with the same boundary values has n -energy $(\log \gamma)^{n\alpha-n+1}$, i.e. u has the maximal n -energy allowed by $Q_{\alpha,n}$ on $A_{1,\gamma}$. \square

Lemma 2.4. *Let $u(x) = \log^\alpha |x|$. Then u is a superminimizer in $A_{1,\infty}$ if and only if $0 \leq \alpha \leq 1$. Similarly, u is a subminimizer in $A_{1,\infty}$ if and only if $\alpha \leq 0$ or $\alpha \geq 1$.*

Proof. A straightforward calculation shows that

$$-\operatorname{div}(|\nabla u(x)|^{n-2}\nabla u(x)) = \alpha(1-\alpha)|\alpha|^{n-2}(n-1)(\log|x|)^{(n-1)\alpha-n}|x|^{-n}$$

for $x \in A_{1,\infty}$. The sign of this expression is the same as of $\alpha(1-\alpha)$. The function u is, by definition, a superminimizer if this expression is nonnegative, and a subminimizer if it is nonpositive throughout $A_{1,\infty}$, which thus happens exactly as stated. \square

We also need the following result, which is essentially from Björn–Björn [2].

Theorem 2.5. [2, Theorems 7.3 and 7.4] *Let $0 \leq \gamma < 1$, $\alpha > 1 - 1/n$ and $u(x) = (-\log|x|)^\alpha$. Then u is a quasiminimizer in $A_{1,\gamma}$ with*

$$Q_{\alpha,n} = \frac{\alpha^n}{n\alpha - n + 1}$$

being the best quasiminimizing constant. Furthermore, u is a Q -quasi(sub/super)-minimizer in $A_{\gamma,1}$ as given in Table 1, where Q in Table 1 is the best quasi(sub/super)-minimizing constant. Also, u has the maximal n -energy allowed by $Q_{\alpha,n}$ on $A_{\gamma,1}$ if $0 < \gamma < 1$.

Proof. When $\gamma = 0$, the first part follows from Theorem 7.3 in [2]. The proof therein holds equally well when $\gamma > 0$ as S therein still ranges over $1 < S < \infty$, cf. the proof of Theorem 2.3 above.

Similarly, the second part follows from Theorem 7.4 in [2], where again the arguments hold also when $\gamma > 0$. Finally, the last part follows from the formula at the bottom of p. 314 in [2], cf. the end of the proof of Theorem 2.3 above. \square

3. The increase in the quasisuperminimizing constant

In this section we are going to use the log-powers considered in Section 2 to construct higher-dimensional analogues of the examples in Björn–Björn–Korte [5, Section 3], concerning the optimality of (1.2) in Theorem 1.1.

Fix $n \geq 2$. We study quasisuperminimizers on the annulus

$$A := A_{1/e,1} = \{x \in \mathbf{R}^n : 1/e < |x| < 1\}.$$

As before, we let $p = n$ be the conformal dimension.

Example 3.1. Given $Q > 1$, there are exactly two exponents $1 - 1/n < \alpha' < 1 < \alpha$ such that $Q = Q_{\alpha,n} = Q_{\alpha',n}$, where

$$(3.1) \quad Q_{\alpha,n} = \frac{\alpha^n}{n\alpha - n + 1}.$$

This is easily shown by differentiating (3.1) with respect to α and noting that the derivative is negative for $\alpha < 1$ and positive for $\alpha > 1$, and that $Q_\alpha \rightarrow \infty$ as $\alpha \rightarrow 1 - 1/n$ and as $\alpha \rightarrow \infty$.

We let

$$(3.2) \quad u_Q(x) = \log^\alpha|ex| \quad \text{and} \quad \bar{u}_Q(x) = 1 - (-\log|x|)^{\alpha'}.$$

Then $u_Q(y) = \bar{u}_Q(y) = 0$ if $|y| = 1/e$, and $u_Q(y) = \bar{u}_Q(y) = 1$ if $|y| = 1$. By Theorem 2.3 and Lemma 2.1, u_Q is a subminimizer and a Q -quasisuperminimizer in A . The same is true for \bar{u}_Q by Theorem 2.5.

It follows from Theorems 2.3 and 2.5 that u_Q has the maximal n -energy allowed by Q on each annulus $A_{1/e,\gamma}$, $1/e < \gamma \leq 1$, while \bar{u}_Q has the maximal n -energy

allowed by Q on each annulus $A_{\gamma,1}$, $1/e \leq \gamma < 1$. This will be of crucial importance when proving Theorem 1.2, which we will do now.

Proof of Theorem 1.2. Let $1 - 1/n < \alpha_2 < 1 < \alpha_1$ be such that $Q_1 = Q_{\alpha_1,n}$ and $Q_2 = Q_{\alpha_2,n}$ as in (3.1), and let $u_1 := u_{Q_1}$ and $u_2 := \bar{u}_{Q_2}$ be the corresponding quasiminimizers as in (3.2). By Example 3.1, u_j is a subminimizer and a Q_j -quasisuperminimizer in A , $j = 1, 2$.

By Theorem 1.1, the function $u := \min\{u_1, u_2\}$ is a \bar{Q} -quasisuperminimizer in A with the quasisuperminimizing constant \bar{Q} given by (1.2). Let $v = \log |ex|$, which is the minimizer on A with the same boundary values as u , u_1 and u_2 . As u_1 and u_2 are subminimizers they are less than v , which can also be seen directly. Thus $u < v$ in A .

We are going to show that u is not a Q_2 -quasisuperminimizer on A . As $u < v$, to do this it suffices to show that the n -energy

$$I_u := \int_A |\nabla u|^n dx > Q_2 I_v,$$

where

$$(3.3) \quad I_v = \int_A |\nabla v|^n dx = c_{n-1} \int_{1/e}^1 r^{-n} r^{n-1} dr = c_{n-1}.$$

For convenience, write $u_j(r) = u_j(x)$ when $r = |x|$. There is a unique number $r_0 \in (1/e, 1)$ such that $u_1(r_0) = u_2(r_0)$ (see below), i.e. such that

$$(3.4) \quad 1 = (-\log r_0)^{\alpha_2} + \log^{\alpha_1} er_0.$$

To see that there is a unique solution, we consider $w = u_2 - u_1$ and note that $w(1/e) = w(1) = 0$. Since $w'(1/e) > 0$ and $w'(1) = \infty$, there is at least one $r \in (1/e, 1)$ such that $w(r) = 0$. Moreover, $w'(r) = 0$ if and only if

$$f(r) := \log(er)(-\log r)^\beta = \left(\frac{\alpha_1}{\alpha_2}\right)^{1/(1-\alpha_1)} > 0, \quad \text{where } \beta = \frac{1 - \alpha_2}{\alpha_1 - 1} > 0.$$

Note that $f'(r) > 0$ if and only if $0 < r < e^{-\beta/(\beta+1)}$ and that $f(r)$ attains its maximum at (and only at) $r = e^{-\beta/(\beta+1)}$. Since $f(1/e) = f(1) = 0$, there are at most two solutions to $w'(r) = 0$, and thus there can be at most one solution to (3.4) which must lie in between the two local extrema of w .

Since u_2 is a subminimizer in A we see that

$$\int_{A_{1/e,r_0}} |\nabla u_1|^n dx > \int_{A_{1/e,r_0}} |\nabla u_2|^n dx,$$

where the strict inequality follows from the uniqueness of solutions to obstacle problems (see e.g. Björn–Björn [3, Theorem 7.2]) and from the fact that u_1 and u_2 differ on a set of positive measure. Hence

$$\int_A |\nabla u|^n dx = \int_{A_{1/e,r_0}} |\nabla u_1|^n dx + \int_{A_{r_0,1}} |\nabla u_2|^n dx > \int_A |\nabla u_2|^n dx = Q_2 I_v,$$

where the last equality follows from the fact that u_2 has the maximal n -energy allowed by Q_2 on A . As $Q_2 \geq Q_1$ this concludes the proof. \square

Theorem 1.2 shows that in general there is an increase in the quasisuperminimizing constant when taking the minimum of two quasiminimizers but does not give any

quantitative estimate of the increase. Next, we are going to analyse the construction in the proof of Theorem 1.2 more carefully to get explicit lower bounds for the increase in the quasisuperminimizing constant.

Given $Q_1, Q_2 > 1$ and $p = n \geq 2$, let $1 - 1/n < \alpha_2 < 1 < \alpha_1$ be such that $Q_1 = Q_{\alpha_1, n}$ and $Q_2 = Q_{\alpha_2, n}$ as in (3.1), and let $u_1 := u_{Q_1}$ and $u_2 := \bar{u}_{Q_2}$ be the corresponding quasiminimizers as in (3.2). Let $r_0 \in (1/e, 1)$ be as in (3.4). Contrary to Theorem 1.2 we here allow for $Q_1 > Q_2$ (the assumption $Q_1 \leq Q_2$ in Theorem 1.2 is only used at the very end of its proof).

It follows from Theorems 2.5 and 2.3 that $u_1 = u_{Q_1}$ has the maximal n -energy allowed by Q_1 on $A_{1/e, r_0}$, while $u_2 = \bar{u}_{Q_2}$ has the maximal n -energy allowed by Q_2 on $A_{r_0, 1}$. Using this, we can calculate the n -energy of $u = \min\{u_1, u_2\}$ as

$$\begin{aligned}
 \int_A |\nabla u|^n dx &= \int_{A_{1/e, r_0}} |\nabla u_1|^n dx + \int_{A_{r_0, 1}} |\nabla u_2|^n dx \\
 &= c_{n-1} (Q_1 (\log er_0)^{n(\alpha_1-1)+1} + Q_2 (-\log r_0)^{n(\alpha_2-1)+1}) \\
 (3.5) \qquad \qquad &=: c_{n-1} \widehat{Q}.
 \end{aligned}$$

Comparing this value with the energy $I_v = c_{n-1}$, given by (3.3), of the minimizer v with the same boundary values as u on A we see that u is not a Q -quasisuperminimizer for any $Q < \widehat{Q}$.

Q	$p = n = 2$	$p = n = 3$	$p = n = 10$	$p = n = 100$	$\bar{Q} = \frac{2Q}{Q+1}$
1.001	1.001480660	1.001480663	1.001480664	1.001480665	1.001500250
1.01	1.014825154	1.014825447	1.014825583	1.014825593	1.015024876
1.125	1.188100103	1.188143910	1.188164386	1.188165836	1.191176471
2	2.619135721	2.621145314	2.622093879	2.622161265	2.666666667
10	17.67321156	17.70495731	17.72058231	17.72170691	18.18181818
100	196.3948537	196.5222958	196.5905036	196.5955633	198.0198020

Table 2. \widehat{Q} for certain values of $p = n$ with $Q_1 = Q_2 = Q$, as well as \bar{Q} from Theorem 1.1.

For specific values of Q_1, Q_2 and $p = n$, one can calculate \widehat{Q} numerically (after first calculating α_1, α_2 and r_0 numerically), which we have done using Maple 18. These results are presented in Table 2, which shows the values of \widehat{Q} for certain values of $p = n$ and with $Q_1 = Q_2 = Q$.

Remark 3.2. The figures for \widehat{Q} in the columns for $p = 2$ and $p = 100$ in Table 2 above are identical to the corresponding columns for \bar{Q} in Table 2 in [5] (there are no columns for $p = 3$ and $p = 10$ therein). This suggests that the increase in quasisuperminimizing constant in the example above is identical to the increase in the 1-dimensional example in [5, pp. 271–272]. This is indeed true, as we shall now show, not only when $Q_1 = Q_2$.

Let as before $p = n \geq 2$ be an integer. (In the example in [5, pp. 271–272], the underlying space is \mathbf{R} , but p can be an arbitrary real number > 1 . To compare it with our construction above we need p to be an integer.)

Let as above $Q_1, Q_2 > 1$ be given, and choose $1 - 1/n < \alpha_2 < 1 < \alpha_1$ such that $Q_1 = Q_{\alpha_1, n}$ and $Q_2 = Q_{\alpha_2, n}$ as in (3.1). With $p = n$ this choice of α_1 and α_2 also satisfies (3.2) in [5]. Next choose $r_0 \in (1/e, 1)$ to be the unique solution of (3.4). Then \widehat{Q} is given by (3.5).

To relate this to \tilde{Q} in [5], we let $x_0 = \log er_0 \in (0, 1)$. It then follows from (3.4) that

$$1 = (1 - x_0)^{\alpha_2} + x_0^{\alpha_1},$$

i.e. x_0 is the unique solution of this equation, which is the same as equation (3.4) in [5]. (That the solution is unique was shown in [5], but also follows from the uniqueness of $r_0 \in (1/e, 1)$.) Using (3.6) in [5] (with $p = n$) we see that

$$(3.6) \quad \tilde{Q} = Q_1 x_0^{n(\alpha_1-1)+1} + Q_2 (1 - x_0)^{n(\alpha_2-1)+1}$$

$$(3.7) \quad = Q_1 (\log er_0)^{n(\alpha_1-1)+1} + Q_2 (-\log r_0)^{n(\alpha_2-1)+1} = \hat{Q}.$$

Q_1	Q_2	$p = 1.2$	$p = 2$	$p = 10$	$p = 100$	\bar{Q}
2	10	10.450759	10.474426	10.477869	10.477946	10.526316
10	2	10.222890	10.293133	10.309651	10.310050	
9	10	16.513457	16.719374	16.762792	16.763819	17.191011
10	9	16.473657	16.689656	16.736154	16.737258	
2	100	100.427051	100.450836	100.454265	100.454342	100.502513
100	2	100.055345	100.111528	100.134063	100.132692	
10	100	107.287586	107.542028	107.596390	107.597679	108.108108
100	10	106.251592	106.758915	106.910025	106.913964	
90	100	185.787954	186.446301	186.634352	186.639194	188.020891
100	90	185.723660	186.399453	186.594523	186.599568	

Table 3. \tilde{Q} (given by (3.6)) for certain values of p , Q_1 and Q_2 , as well as \bar{Q} from Theorem 1.1. When p is an integer these are also the values of \hat{Q} by (3.7).

Remark 3.3. The function \tilde{Q} depends on Q_1 , Q_2 and p , i.e. $\tilde{Q} = \tilde{Q}(Q_1, Q_2, p)$. Given $1 < Q_1 < Q_2$ (and p) it is natural to ask which is larger of $\tilde{Q}(Q_1, Q_2, p)$ and $\tilde{Q}(Q_2, Q_1, p)$. We have calculated some values of $\tilde{Q}(Q_1, Q_2, p)$ using Maple 18, see Table 3. They all indicate that

$$(3.8) \quad \tilde{Q}(Q_1, Q_2, p) > \tilde{Q}(Q_2, Q_1, p) \quad \text{if } 1 < Q_1 < Q_2.$$

Due to the intricate formula (3.6) for \tilde{Q} , involving x_0 , we have not been able to show that this is always the case.

The formula for \tilde{Q} is valid also for nonintegers $p > 1$, and $p = 1.2$ is included in Table 3. However, if p is an integer then $\hat{Q} = \tilde{Q}$, by (3.7), and the same reasoning about the comparison in (3.8) applies to \hat{Q} .

4. Adding dimensions

One way of making higher-dimensional examples from lower-dimensional ones is to add dummy variables. The tensor product $u_1 \otimes u_2(x, y) = u_1(x)u_2(y)$ and tensor sum $u_1 \oplus u_2 = u_1(x) + u_2(y)$ of two harmonic functions is again harmonic, a fact that is well known and easy to prove. The corresponding fact is false for p -harmonic functions, but it was observed in Björn–Björn [4] that the tensor product and sum are quasiminimizers. They moreover showed that the tensor product and sum of quasiminimizers are again quasiminimizers, but typically with an increase in the quasiminimizing constant even if both are 1. However, if one of the quasiminimizers is constant then the increase in the quasiminimizing constant can be avoided, a fact that we shall use. We first recall the following consequence of the results in [4].

Theorem 4.1. *Let u be a Q -quasisuperminimizer in $\Omega \subset \mathbf{R}^n$ and let $I \subset \mathbf{R}$ be an interval. Then $\bar{u} = u \otimes 1$ is a Q -quasisuperminimizer in $\Omega \times I$.*

This result is true also if the first space \mathbf{R}^n is equipped with a so-called p -admissible weight w , see [4]. In particular, by Theorem 3 in [4], $w \otimes 1$ is then a p -admissible weight on \mathbf{R}^{n+1} .

Proof. This is a special case of Theorem 7 in [4], with $u_1 = u$, $u_2 \equiv 1$, $Q_1 = Q$ and $Q_2 = 0$. As mentioned in [4, p. 5196], one is allowed to let $Q_2 = 0$ if u_2 is a constant function. \square

For the purposes in this paper, this is not enough since, typically, the obtained Q is not the optimal quasisuperminimizing constant for \bar{u} even if it is for u . But if I is in addition unbounded then Q is optimal for \bar{u} if it is for u , as we shall now show.

Theorem 4.2. *Let u be a Q -quasisuperminimizer in $\Omega \subset \mathbf{R}^n$, where Q is the optimal quasisuperminimizing constant. Let $I \subset \mathbf{R}$ be an unbounded interval. Then $\bar{u} = u \otimes 1$ is a Q -quasisuperminimizer in $\Omega \times I$, with Q again being optimal.*

Proof. By Theorem 4.1, \bar{u} is a Q -quasisuperminimizer, so it is only the optimality of Q that needs to be shown. If $Q = 1$ there is nothing to prove, so we can assume that $Q > 1$.

Let $0 < \varepsilon < Q$. Since Q is optimal, there is a nonnegative $\varphi \in \text{Lip}_c(\Omega)$ such that

$$\int_{\varphi \neq 0} |\nabla u|^p dx > (Q - \varepsilon) \int_{\varphi \neq 0} |\nabla(u + \varphi)|^p dx,$$

see the beginning of Section 2. As the integral on the left-hand side is positive, also the integral on the right-hand side must be positive, as otherwise u would not be a quasisuperminimizer at all.

Next, let $m > 0$ be given. Since I is unbounded we can find $a \in \mathbf{R}$ so that $[a, a + m + 2] \subset I$. Assume without loss of generality that $a = 0$, and let

$$\bar{\varphi} = \varphi \otimes \varphi_2, \quad \text{where } \varphi_2(t) = \begin{cases} 0, & \text{if } t \leq 0 \text{ or } t \geq m + 2, \\ t, & \text{if } 0 \leq t \leq 1, \\ 1, & \text{if } 1 \leq t \leq m + 1, \\ m + 2 - t, & \text{if } m + 1 \leq t \leq m + 2. \end{cases}$$

Then

$$\int_{\bar{\varphi} \neq 0} |\nabla \bar{u}|^p dx dt \geq m \int_{\varphi \neq 0} |\nabla u|^p dx > (Q - \varepsilon)m \int_{\varphi \neq 0} |\nabla(u + \varphi)|^p dx,$$

while

$$\int_{\bar{\varphi} \neq 0} |\nabla(\bar{u} + \bar{\varphi})|^p dx dt = 2 \int_{\{\varphi \neq 0\} \times (0,1)} |\nabla(\bar{u} + \bar{\varphi})|^p dx dt + m \int_{\varphi \neq 0} |\nabla(u + \varphi)|^p dx.$$

Letting $m \rightarrow \infty$ and then $\varepsilon \rightarrow 0$ shows that Q is optimal, since the last integral is nonzero. \square

It now follows directly from Theorem 4.2 that we can add a dummy variable to the examples constructed in Section 3 and in Björn–Björn–Korte [5, Section 3]. As long as we consider the dummy variable taken over an unbounded interval, we obtain a new example with the same increase in the quasisuperminimizing constant. This can be iterated so that we can add an arbitrary (but finite) number of dummy variables. This way we get higher-dimensional examples on unbounded sets. However, it follows from the following lemma that by taking Cartesian products with large enough

bounded intervals, one can obtain similar bounded examples with an increase in the quasisuperminimizing constant, which is arbitrarily close to the increase in Section 3 resp. [5, Section 3].

Lemma 4.3. *Let $\Omega_1 \subset \Omega_2 \subset \dots \subset \Omega = \bigcup_{j=1}^{\infty} \Omega_j$ be an increasing sequence of open subsets of \mathbf{R}^n . If u is a Q -quasisuperminimizer in Ω_j for every j , then it is also a Q -quasisuperminimizer in Ω .*

Proof. As mentioned at the beginning of Section 2 it is enough to test (1.1) with nonnegative $\varphi \in \text{Lip}_c(\Omega)$. By compactness, $\varphi \in \text{Lip}_c(\Omega_j)$ for some j , and thus (1.1) holds for this particular φ as u is a Q -quasisuperminimizer in Ω_j . \square

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