REAL-VARIABLE CHARACTERIZATIONS
OF MUSIELAK–ORLICZ HARDY SPACES
ON SPACES OF HOMOGENEOUS TYPE

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Abstract. Let $(\mathcal{X}, d, \mu)$ be a space of homogeneous type in the sense of Coifman and Weiss. In this article, the authors establish a complete real-variable theory of Musielak–Orlicz Hardy spaces on $(\mathcal{X}, d, \mu)$. To be precise, the authors first introduce the atomic Musielak–Orlicz Hardy space $H_{\mathrm{at}}^{\varphi}(\mathcal{X})$ and then establish its various maximal function characterizations. The authors also investigate the Littlewood–Paley characterizations of $H_{\mathrm{at}}^{\varphi}(\mathcal{X})$ via Lusin area functions, Littlewood–Paley g-functions and Littlewood–Paley g_{λ}^{*}-functions. The authors further obtain the finite atomic characterization of $H_{\mathrm{at}}^{\varphi}(\mathcal{X})$ and its improved version in case $q < \infty$, and their applications to criteria of the boundedness of sublinear operators from $H_{\mathrm{at}}^{\varphi}(\mathcal{X})$ to a quasi-Banach space, which are also applied to the boundedness of Calderón–Zygmund operators. Moreover, the authors find the dual space of $H_{\mathrm{at}}^{\varphi}(\mathcal{X})$, namely, the Musielak–Orlicz BMO space $\mathrm{BMO}^{\varphi}(\mathcal{X})$, present its several equivalent characterizations, and apply it to establish a new characterization of the set of pointwise multipliers for the space $\mathrm{BMO}(\mathcal{X})$. The main novelty of this article is that, throughout the article, except the last section, μ is not assumed to satisfy the reverse doubling condition.

1. Introduction

The classical real Hardy space $H^p(\mathbf{R}^n)$ on the n-dimensional Euclidean space has proved a nice substitute of the Lebesgue space $L^p(\mathbf{R}^n)$ when $p \in (0, 1]$, because some important operators (for instance, the Riesz transforms) are not bounded on $L^p(\mathbf{R}^n)$ but bounded on $H^p(\mathbf{R}^n)$ when $p \in (0, 1]$. The real-variable theory of $H^p(\mathbf{R}^n)$ was initiated by Stein and Weiss [57] and later extensively developed by Fefferman and Stein [13]. Precisely, Fefferman and Stein [13] established various maximal function characterizations of $H^p(\mathbf{R}^n)$ and found a surprising result on the dual space of $H^1(\mathbf{R}^n)$, which is just $\mathrm{BMO}(\mathbf{R}^n)$, the space of functions with bounded mean oscillations introduced by John and Nirenberg [33]. To better understand the structure

of Hardy spaces, one finds that a core idea is to decompose any element of $H^p(\mathbf{R}^n)$ into a sum of some basic elements with some good properties, which are called atoms (see Coifman [8] for the case $n = 1$ and Latter [39] for the general case $n \in \mathbf{N}$). This paves the way for the extensive study of the boundedness on Hardy spaces of many classical operators appearing in harmonic analysis such as Calderón–Zygmund operators and fractional integrals. For more studies on the real-variable theory of Hardy spaces and its applications to harmonic analysis and partial differential equations, we refer the reader to some fundamental references, for instance, [57, 55, 13, 56, 48].

In the last several decades, the classical real Hardy spaces were generalized to other type Hardy spaces. Among this, Ky [35] extended both the Orlicz–Hardy spaces and the weighted Hardy spaces to new Orlicz type Hardy spaces, Musielak–Orlicz Hardy spaces $H^\varphi(\mathbf{R}^n)$, which prove useful. Particularly, the Musielak–Orlicz Hardy space $H^{\log}(\mathbf{R}^n)$, related to the typical growth function

$$(1.1) \qquad \theta(x, t) := \frac{t}{\log(e + |x|) + \log(e + t)}, \quad \forall\, x \in \mathbf{R}^n, \; \forall\, t \in [0, \infty),$$

plays essential roles in the bilinear decompositions of $\mathrm{BMO}(\mathbf{R}^n) \times H^1(\mathbf{R}^n)$. These bilinear decompositions are important to the optimal endpoint estimate for the div-curl lemma, involving the space $H^{\log}(\mathbf{R}^n)$, and have also relations to an implicit conjecture from [6] (see also [5, 4]). These bilinear decompositions also play striking roles in bilinear or subbilinear decompositions, respectively, for linear or sublinear commutators of Calderón–Zygmund operators from [34]; see [36, 37] (or [59]) for more applications. So far, some literature on the real-variable theory of $H^\varphi(\mathbf{R}^n)$ and its applications appeared (see, for instance, [30, 35, 41, 42, 43, 44, 59, 60]).

To extend some classical results of harmonic analysis over Euclidean spaces to more generalized setting, Coifman and Weiss [9, 10] introduced the following notion of the space of homogeneous type, which is a natural setting for the theory of functions spaces and operators in harmonic analysis. Let us first recall the following notion of spaces of homogeneous type from [9, 10]. Suppose that $\mathcal{X}$ is a non-empty set equipped with a *quasi-metric* d with the quasi-triangle constant $A_0 \in [1, \infty)$, namely, for any $x, y, z \in \mathcal{X}$,

    (i) $d(x, y) = d(y, x)$;
    (ii) $d(x, y) = 0$ if and only if $x = y$;
    (iii) d satisfies the quasi-triangle inequality

$$(1.2) \qquad d(x, y) \leq A_0\left[d(x, z) + d(z, y)\right].$$

The triple $(\mathcal{X}, d, \mu)$ is called a *space of homogeneous type* if μ is a non-negative measure satisfying the *doubling condition*: there exists a positive constant $C_{(\mathcal{X})} \in [1, \infty)$ such that, for any ball $B(x, r) := \{y \in \mathcal{X} : d(x, y) < r\}$ with $(x, r) \in \mathcal{X} \times (0, \infty)$,

$$(1.3) \qquad \mu(B(x, 2r)) \leq C_{(\mathcal{X})}\, \mu(B(x, r)),$$

which further implies that there exists a positive constant $\widetilde{C}_{(\mathcal{X})}$ such that, for any $\lambda \in [1, \infty)$ and $(x, r) \in \mathcal{X} \times (0, \infty)$,

$$(1.4) \qquad \mu(B(x, \lambda r)) \leq \widetilde{C}_{(\mathcal{X})}\lambda^\omega \mu(B(x, r)),$$

where $\omega := \log_2 C_{(\mathcal{X})}$.

Remark 1.1. Two typical examples of spaces of homogeneous type are as follows.

(i) Let $\mathcal{X} := \mathbf{R}^n$, $d(x,y) := \left[\sum_{j=1}^n (x_i - y_i)^2\right]^{1/2}$ for any $x := (x_1,\ldots,x_n)$, $y := (y_1,\ldots,y_n) \in \mathbf{R}^n$, and μ be the n-dimensional Lebesgue measure. This is just the Euclidean space equipped with the n-dimensional Lebesgue measure.

(ii) Let $\mathcal{X} := \mathbf{R}^n$, $d(x,y) := \sum_{j=1}^n |x_i - y_i|^{\alpha_i}$ for any $x := (x_1,\ldots,x_n)$, $y := (y_1,\ldots,y_n) \in \mathbf{R}^n$, where $\alpha_1,\ldots,\alpha_n \in (0,\infty)$ are not all equal, and μ be the n-dimensional Lebesgue measure. This is so called the anisotropic Euclidean space equipped with the n-dimensional Lebesgue measure.

Please see [10, pp. 588–590] for more examples of spaces of homogeneous type.

In this setting, Coifman and Weiss [10] introduced the atomic Hardy space $H_{\mathrm{at}}^{p,q}(\mathcal{X})$ and proved that $H_{\mathrm{at}}^{p,q}(\mathcal{X})$ is independent of the choice of q. Hereafter, $H_{\mathrm{at}}^{p,q}(\mathcal{X})$ is simply denoted by $H_{\mathrm{at}}^p(\mathcal{X})$. Coifman and Weiss [10] also showed that its dual space is the Lipschitz space $\mathrm{Lip}_{1/p-1}(\mathcal{X})$ when $p \in (0,1)$, or the space $\mathrm{BMO}(\mathcal{X})$ when $p = 1$.

However, some important tools in the real-variable theory of $H_{\mathrm{at}}^p(\mathcal{X})$, such as the Calderón reproducing formulae, seem to need additional assumptions. To this end, Han et al. [26] (see also [25]) introduced the RD-space, namely, the space of homogeneous type satisfying the following *reverse doubling condition*: there exist some positive constants a_0, $c_{(\mathcal{X})} \in (1,\infty)$ such that, for any $x \in \mathcal{X}$ and $r \in (0, \mathrm{diam}\,(\mathcal{X})/a_0)$ with $\mathrm{diam}\,(\mathcal{X}) := \sup\{d(x,y)\colon x, y \in \mathcal{X}\}$,

$$(1.5) \qquad\qquad \mu(B(x, a_0 r)) \geq c_{(\mathcal{X})} \mu(B(x,r));$$

see [62] for several equivalent characterizations of the reverse doubling condition (1.5). Later, a real-variable theory of various function spaces and their applications were given on RD-spaces; see, for instance, [25, 26, 61, 62, 21] and the references therein.

Recently, motivated by the remarkable wavelet theory of Auscher and Hytönen in [2], Han et al. [23, 24] established a partial real-variable theory of (product) Hardy spaces on spaces of homogeneous type without having recourse to the assumption (1.5), via using the orthonormal basis of regular wavelets from [2, 3] and investigating its applications to the boundedness of Calderón–Zygmund operators. Meanwhile, Fu et al. [19] confirmed a conjecture raised by Bonami and Bernicot (see Ky [38, p. 809]) that $f \times g$ of $f \in H_{\mathrm{at}}^1(\mathcal{X})$ and $g \in \mathrm{BMO}(\mathcal{X})$ can be written into a sum of two bilinear operators on any metric measure space of homogeneous type without having recourse to the reverse doubling condition (1.5), where several equivalent wavelet characterizations of $H_{\mathrm{at}}^1(\mathcal{X})$ from [17] and a typical Musielak–Orlicz Hardy space $H^{\log}(\mathcal{X})$ play crucial roles. The space $H^{\log}(\mathcal{X})$ is also important to the local version of the above bilinear decomposition in [16]. Moreover, the bilinear decomposition in [19] is also useful to the endpoint boundedness of the (sub-)linear commutator $[b, T]$ of a sublinear operator T and $b \in \mathrm{BMO}(\mathcal{X})$ on Hardy spaces in [46, 47]; see the survey [15] for more details.

More recently, He et al. [28] constructed a corresponding wavelet reproducing formulae without having recourse to the reverse doubling condition (1.5). Based on these wavelet reproducing formulae, He et al. [27] further established a complete real-variable theory of Hardy spaces $H^p(\mathcal{X})$ on spaces of homogeneous type without having recourse to the reverse doubling condition (1.5). On another hand, Hou et al. [31] investigated Musielak–Orlicz BMO-type spaces associated with generalized approximations to the identity on spaces of homogeneous type.

Motivated by [27] and the useful space $H^{\log}(\mathcal{X})$, one naturally expects to develop a complete real-variable theory of Musielak–Orlicz Hardy spaces, including the above

useful space $H^{\log}(\mathcal{X})$. In this article, we positively answer this question on $(\mathcal{X}, d, \mu)$. Precisely, we introduce the atomic Musielak–Orlicz Hardy space $H_{\mathrm{at}}^{\varphi}(\mathcal{X})$ and then investigate their various maximal function characterizations, the Littlewood–Paley characterizations, the finite atomic characterizations and the dual space of $H_{\mathrm{at}}^{\varphi}(\mathcal{X})$. We also consider their applications to the boundedness of sublinear operators, including Calderón–Zygmund operators as concrete instances, from $H_{\mathrm{at}}^{\varphi}(\mathcal{X})$ to a quasi-Banach space, and a characterization of the set of pointwise multipliers for the space $\mathrm{BMO}(\mathcal{X})$. It should be pointed out that the main novelty of this article is that, except Section 10, μ is not assumed to satisfy the reverse doubling condition (1.5).

Throughout this article, but except the last section, we *always assume* that $(\mathcal{X}, d, \mu)$ is a space of homogeneous type, which is non-atomic [namely, $\mu(\{x\}) = 0$ for any $x \in \mathcal{X}$] and satisfies $\mathrm{diam}\,(\mathcal{X}) = \infty$. It is known that $\mathrm{diam}\,(\mathcal{X}) = \infty$ implies that $\mu(\mathcal{X}) = \infty$ (see, for instance, [53, Lemma 5.1] or [2, Lemma 8.1]).

The organization of this article is as follows.

In Section 2, we recall some basic notions, notation and known results used in this article.

In Section 3, by establishing a version of Calderón–Zygmund decompositions on balls and borrowing some ideas from the proof of [43, Theorem 2.5], we obtain the John–Nirenberg inequality of $\mathrm{BMO}^{\varphi}(\mathcal{X})$ (see Theorem 3.1 below). Then we apply this John–Nirenberg inequality to establish several equivalent characterizations for $\mathrm{BMO}^{\varphi}(\mathcal{X})$, which are similar to [43, Theorem 2.7] with some technical modifications.

In Section 4, we first recall the important Calderón reproducing formulae established in [27] and then introduce Musielak–Orlicz Hardy spaces, $H^{+,\varphi}(\mathcal{X})$, $H_{\theta}^{\varphi}(\mathcal{X})$ and $H^{*,\varphi}(\mathcal{X})$ defined, respectively, via radial, nontangential and grand maximal functions. In Subsection 4.2, by the pointwise inequalities for those maximal functions and some ideas from the proof of [27, Theorem 3.5], we show that $H^{+,\varphi}(\mathcal{X})$, $H_{\theta}^{\varphi}(\mathcal{X})$ and $H^{*,\varphi}(\mathcal{X})$ are mutually identical with equivalent quasi-norms (see Theorem 4.12 below).

Section 5 is devoted to the atomic characterizations of $H^{*,\varphi}(X)$ (see Theorem 5.4 below). The proof is divided into three parts. In Subsection 5.1, we show $H_{\mathrm{at}}^{\varphi,q}(\mathcal{X}) \subset H^{*,\varphi}(\mathcal{X})$ via the properties of the growth functions φ and some arguments used in the proof of [35, Theorem 4.2]. Then, in Subsection 5.2, we obtain a version of Calderón–Zygmund decompositions for $H^{*,\varphi}(\mathcal{X})$ (see Proposition 5.8 below), which is a generalization of the corresponding results on $H^{p}(\mathcal{X})$ in [27, Proposition 4.9]. In Subsection 5.3, via this Calderón–Zygmund decompositions for $H^{*,\varphi}(\mathcal{X})$, a technical lemma on the density of $L_{\varphi(\cdot,1)}^{q}(\mathcal{X}) \cap H^{*,\varphi}(\mathcal{X})$ in $H^{*,\varphi}(\mathcal{X})$ and some arguments similar to those used in the proof of [27, Theorem 4.2], we prove that $H^{*,\varphi}(\mathcal{X}) \subset H_{\mathrm{at}}^{\varphi,q}(\mathcal{X})$, namely, we establish the atomic decomposition of $H^{*,\varphi}(\mathcal{X})$.

In Section 6, we study the Littlewood–Paley characterizations of $H_{\mathrm{at}}^{\varphi}(\mathcal{X})$, respectively, in terms of Lusin area functions, Littlewood–Paley g-functions and Littlewood–Paley g_{λ}^{*}-functions. Precisely, in Subsection 6.1, via the vector-valued Fefferman–Stein maximal inequality, we first show the independence of the choices of exp-ATIs on the Musielak–Orlicz Hardy space $H^{\varphi}(\mathcal{X})$ defined by the Lusin area functions (see Theorem 6.3 below). Then we obtain the molecular characterizations of $H^{\varphi}(\mathcal{X})$ in Subsection 6.2 (see Theorem 6.8 below) by borrowing some ideas from the proofs of [10, Theorem C] and [18, Theorem 2.12]. Later, in Subsection 6.3, via introducing the adapted atomic Musielak–Orlicz Hardy spaces $H_{\mathrm{at,A}}^{\varphi,q}(\mathcal{X})$ and proving their equivalences to $H_{\mathrm{at}}^{\varphi,q}(\mathcal{X})$ (see Theorem 6.11 below), and some technical lemmas (see Lemmas 4.9 and 6.13 below), we establish the Lusin area function characterizations

of $H^\varphi(\mathcal{X})$ (see Theorem 6.15 below), where the distribution $(\mathcal{G}_0^\epsilon(\varrho, \vartheta))'$ is replaced by its homogeneous version $(\mathring{\mathcal{G}}_0^\epsilon(\varrho, \vartheta))'$ (see Proposition 6.12 below). Using the main results in the above subsections, in Subsection 6.4, we establish the Littlewood–Paley function characterizations of $H^\varphi(\mathcal{X})$ (see Theorem 6.16 below).

In Section 7, we give a finite atomic characterization of $H^\varphi(\mathcal{X})$ and an improved version in case $q < \infty$. Indeed, in Subsection 7.1, combining some ideas used in the proofs of [35, Theorem 3.4] and [27, Theorem 7.1], under the uniformly dominated convergence condition, we show that $\|\cdot\|_{H_{\mathrm{fin}}^{\varphi,q}(\mathcal{X})}$, with $q \in (q(\varphi), \infty)$, and $\|\cdot\|_{H^{*,\varphi}(\mathcal{X})}$ are equivalent quasi-norms on $H_{\mathrm{fin}}^{\varphi,q}(\mathcal{X})$, $\|\cdot\|_{H_{\mathrm{fin}}^{\varphi,\infty}(\mathcal{X})}$ and $\|\cdot\|_{H_{\mathrm{at}}^{\varphi,\infty}(\mathcal{X})}$ are equivalent quasi-norms on $H_{\mathrm{fin}}^{\varphi,\infty}(\mathcal{X}) \cap \mathrm{UC}(\mathcal{X})$, and $H_{\mathrm{fin}}^{\varphi,\infty}(\mathcal{X}) \cap \mathrm{UC}(\mathcal{X})$ is dense in $H_{\mathrm{at}}^{\varphi,\infty}(\mathcal{X})$, where $\mathrm{UC}(\mathcal{X})$ denotes the *set of uniformly continuous functions on $\mathcal{X}$*. The application of the above finite atomic characterizations of $H^\varphi(\mathcal{X})$ to the criterion for the boundedness of sublinear operators from $H^\varphi(\mathcal{X})$ to a quasi-Banach space is also included in this section. By using some ideas from [7, 45], in Subsection 7.2 for the case $q < \infty$, we remove the uniformly dominated convergence condition of the main results of Subsection 7.1, which essentially improves the corresponding results.

In Section 8, with the help of the finite atomic characterization of $H^\varphi(\mathcal{X})$ in Section 7 and some ideas used in the proof of the corresponding Euclidean case (see, for instance, [35, Theorem 3.2]), we find that the dual space of $H^\varphi(\mathcal{X})$ is just the Musielak–Orlicz BMO space $\mathrm{BMO}^\varphi(\mathcal{X})$ (see Theorem 10.6 below).

In Section 9, applying the criteria of the boundedness of sublinear operators from $H^\varphi(\mathcal{X})$ to a quasi-Banach space established in Section 7, we show the boundedness of Calderón–Zygmund operators on $H^\varphi(\mathcal{X})$ (see Theorem 9.2 below).

In Section 10, via the characterizations of $\mathrm{PWM}(\mathrm{BMO}(\mathcal{X}))$ [namely, the set of pointwise multipliers for $\mathrm{BMO}(\mathcal{X})$] from [52, Example 2.8] in terms of some typical Musielak–Orlicz BMO space $\mathrm{BMO}^{\log}(\mathcal{X})$, we establish another characterization of $\mathrm{PWM}(\mathrm{BMO}(\mathcal{X}))$ (see Theorem 10.6 below) by the typical Musielak–Orlicz Hardy space $H^{\log}(\mathcal{X})$, which is a generalization of [35, Theorem 3.3]. The main difficulty is to find a proper equivalent norm of functions in $\mathrm{BMO}^{\log}(\mathcal{X})$ (see Proposition 10.4 below). We point out that the Ahlfors n-regular condition [see (10.1) below] plays important roles in the above characterization of $\mathrm{PWM}(\mathrm{BMO}(\mathcal{X}))$. It is unclear whether or not the Ahlfors n-regular assumption can be relaxed.

Finally, we list some conventions on notation. Throughout this article, let $\mathbf{N} := \{1, 2, \ldots\}$, $\mathbf{Z}_+ := \{0\} \cup \mathbf{N}$, C and c stand for *positive constants* which are independent of the main parameters, but they may change from line to line. Moreover, we use $C_{(\rho, \alpha, \ldots)}$ and $c_{(\rho, \alpha, \ldots)}$ to denote positive constants depending on the indicated parameters $\rho, \alpha, \ldots$. If, for any real functions f and g, $f \le Cg$, we then write $f \lesssim g$ and, if $f \lesssim g \lesssim f$, we then write $f \sim g$. We also use the following convention: If $f \le Cg$ and $g = h$ or $g \le h$, we then write $f \lesssim g \sim h$ or $f \lesssim g \lesssim h$, *rather than $f \lesssim g = h$ or $f \lesssim g \le h$*. For any subset E of $\mathcal{X}$, we let $\mathbf{1}_E$ denote its *characteristic function* and let $E^{\complement}$ denote its *complementary set* $\mathcal{X} \setminus E$. For any ball B, let x_B and r_B denote, respectively, the center and the radius of B. Furthermore, for any $x, y \in \mathcal{X}$, $r, \rho \in (0, \infty)$ and ball $B := B(x, r)$, let

$$\rho B := B(x, \rho r), \quad V(x, r) := \mu(B(x, r)) =: V_r(x) \quad \text{and} \quad V(x, y) := \mu(B(x, d(x, y))).$$

For any $p \in [1, \infty]$, we use p' to denote its *conjugate index*, namely, $1/p + 1/p' = 1$.

2. Preliminaries

In this section, we recall some basic notions, notation and known results used later in this article.

The following version of the space of test functions on $\mathcal{X}$ was originally introduced by Han et al. [25, Definition 2.2] (see also [26, Definition 2.8]).

Definition 2.1. Let $x_1 \in \mathcal{X}$, $r \in (0, \infty)$, $\varrho \in (0, 1]$ and $\vartheta \in (0, \infty)$. The *space of test functions*, $\mathcal{G}(x_1, r, \varrho, \vartheta)$, is defined to be the set of all measurable functions f on $\mathcal{X}$ satisfying that there exists a non-negative constant C such that

(T1) for any $x \in \mathcal{X}$, $|f(x)| \le C \frac{1}{V_r(x_1) + V(x_1, x)} [\frac{r}{r + d(x_1, x)}]^{\vartheta}$;

(T2) for any $x, y \in \mathcal{X}$ satisfying $d(x, y) \le [r + d(x_1, x)]/(2A_0)$,

$$|f(x) - f(y)| \le C \left[\frac{d(x, y)}{r + d(x_1, x)} \right]^{\varrho} \frac{1}{V_r(x_1) + V(x_1, x)} \left[\frac{r}{r + d(x_1, x)} \right]^{\vartheta},$$

where, for any $x, y \in \mathcal{X}$ and $r \in (0, \infty)$, $V_r(x) := \mu(B(x, r))$ and $V(x, y) := \mu(B(x, d(x, y)))$. Moreover, for any $f \in \mathcal{G}(x_1, r, \varrho, \vartheta)$, define

$$\|f\|_{\mathcal{G}(x_1, r, \varrho, \vartheta)} := \inf \{C : C \text{ satisfies (T1) and (T2)}\}.$$

Fix $x_1 \in \mathcal{X}$. It is obvious that $\mathcal{G}(x_1, 1, \varrho, \vartheta)$ is a Banach space. For any $x \in \mathcal{X}$ and $r \in (0, \infty)$, it is easy to see that $\mathcal{G}(x, r, \varrho, \vartheta) = \mathcal{G}(x_1, 1, \varrho, \vartheta)$ with equivalent norms. For notational simplicity, we denote $\mathcal{G}(x_1, 1, \varrho, \vartheta)$ simply by $\mathcal{G}(\varrho, \vartheta)$. For any given $\varepsilon \in (0, 1]$, let $\mathcal{G}_0^{\varepsilon}(\varrho, \vartheta)$ be the completion of the space $\mathcal{G}(\varepsilon, \varepsilon)$ in $\mathcal{G}(\varrho, \vartheta)$ when $\varrho, \vartheta \in (0, \varepsilon]$. Moreover, if $f \in \mathcal{G}_0^{\varepsilon}(\varrho, \vartheta)$, we then let $\|f\|_{\mathcal{G}_0^{\varepsilon}(\varrho, \vartheta)} := \|f\|_{\mathcal{G}(\varrho, \vartheta)}$. Also, we let $\mathring{\mathcal{G}}_0^{\varepsilon}(\varrho, \vartheta) := \{f \in \mathcal{G}_0^{\varepsilon}(\varrho, \vartheta) : \int_{\mathcal{X}} f(x) \, d\mu(x) = 0\}$.

The *dual spaces* $(\mathcal{G}_0^{\varepsilon}(\varrho, \vartheta))'$ and $(\mathring{\mathcal{G}}_0^{\varepsilon}(\varrho, \vartheta))'$ are defined, respectively, to be the sets of all continuous linear functionals $\mathcal{L}$ from $\mathcal{G}_0^{\varepsilon}(\varrho, \vartheta)$ to $\mathbf{C}$ and from $\mathring{\mathcal{G}}_0^{\varepsilon}(\varrho, \vartheta)$ to $\mathbf{C}$, and both are equipped with the weak-$*$ topology. For any $f \in (\mathcal{G}_0^{\varepsilon}(\varrho, \vartheta))'$ and $\phi \in \mathcal{G}_0^{\varepsilon}(\varrho, \vartheta)$, we use the pairing $\langle f, \phi \rangle$ to denote the action of f on ϕ. In particular, when f is locally integrable, it makes sense to write

$$\langle f, \phi \rangle = \int_{\mathcal{X}} f(x) \phi(x) \, d\mu(x).$$

Then we recall the following notion of Orlicz functions. A function $\phi : [0, \infty) \to [0, \infty)$ is called an *Orlicz function* if it is non-decreasing and $\phi(0) = 0$, $\phi(t) > 0$ if $t \in (0, \infty)$, and $\lim_{t \to \infty} \phi(t) = \infty$.

Now, we recall the notions of uniformly lower (resp., upper) types on Musielak–Orlicz functions from [31].

Definition 2.2. For a given function $\varphi : \mathcal{X} \times [0, \infty) \to [0, \infty)$ such that, for almost every $x \in \mathcal{X}$, $\varphi(x, \cdot)$ is an Orlicz function, φ is said to be of *uniformly lower* (resp., *upper*) *type p* if there exists a positive constant $C_{(p)}$, depending on p, such that, for almost every $x \in \mathcal{X}$, $t \in [0, \infty)$ and $s \in (0, 1)$ [resp., $s \in [1, \infty)$],

$$\varphi(x, st) \le C_{(p)} s^p \varphi(x, t).$$

The function φ is said to be of *uniformly lower* (resp., *upper*) *type* if it is of uniformly lower (resp., upper) type p for some $p \in (0, \infty)$ and let

(2.1) $i(\varphi) := \sup\{p \in (0, \infty) : \varphi \text{ is of uniformly lower type } p\}.$

Now, we recall the following notion of the set of Muckenhoupt weights on $\mathcal{X}$ from [58].

Definition 2.3. A measurable function $w\colon \mathcal{X} \to (0, \infty)$ is said to belong to the *set of Muckenhoupt weights* for some $q \in [1, \infty)$, denoted by $\varphi \in A_q(\mathcal{X})$, if, when $q \in (1, \infty)$,

$$[w]_{A_q(\mathcal{X})} := \sup_{B \subset \mathcal{X}} \frac{1}{[\mu(B)]^q} \int_B w(x)\, d\mu(x) \left\{ \int_B [w(y)]^{-1/(q-1)}\, d\mu(y) \right\}^{q-1} < \infty$$

or

$$[w]_{A_1(\mathcal{X})} := \sup_{B \subset \mathcal{X}} \frac{1}{\mu(B)} \int_B w(x)\, d\mu(x) \left(\operatorname*{ess\,sup}_{y \in B} [w(y)]^{-1} \right) < \infty,$$

where the suprema are taken over all balls $B \subset \mathcal{X}$. Let

$$A_\infty(\mathcal{X}) := \bigcup_{q \in [1, \infty)} A_q(\mathcal{X}).$$

We recall the notions of the uniformly Muckenhoupt condition and the uniformly reverse Hölder condition from [31].

Definition 2.4. A function $\varphi\colon \mathcal{X} \times [0, \infty) \to [0, \infty)$ is said to satisfy the *uniformly Muckenhoupt condition* for some $q \in [1, \infty)$, denoted by $\varphi \in \mathbf{A}_q(\mathcal{X})$, if, when $q \in (1, \infty)$,

$$[\varphi]_{\mathbf{A}_q(\mathcal{X})} := \sup_{t \in (0, \infty)} \sup_{B \subset \mathcal{X}} \frac{1}{[\mu(B)]^q} \int_B \varphi(x, t)\, d\mu(x) \left\{ \int_B [\varphi(y, t)]^{-1/(q-1)}\, d\mu(y) \right\}^{q-1} < \infty$$

or

$$[\varphi]_{\mathbf{A}_1(\mathcal{X})} := \sup_{t \in (0, \infty)} \sup_{B \subset \mathcal{X}} \frac{1}{\mu(B)} \int_B \varphi(x, t)\, d\mu(x) \left(\operatorname*{ess\,sup}_{y \in B} [\varphi(y, t)]^{-1} \right) < \infty,$$

where the first suprema are taken over all $t \in (0, \infty)$ and the second ones over all balls $B \subset \mathcal{X}$.

Let

$$\mathbf{A}_\infty(\mathcal{X}) := \bigcup_{q \in [1, \infty)} \mathbf{A}_q(\mathcal{X}).$$

A function $\varphi\colon \mathcal{X} \times [0, \infty) \to [0, \infty)$ is said to satisfy the *uniformly reverse Hölder condition* for some $q \in (1, \infty]$, denoted by $\varphi \in \mathbf{RH}_q(\mathcal{X})$, if, when $q \in (1, \infty)$,

$$[\varphi]_{\mathbf{RH}_q(\mathcal{X})} := \sup_{t \in (0, \infty)} \sup_{B \subset \mathcal{X}} \left\{ \frac{1}{\mu(B)} \int_B [\varphi(x, t)]^q\, d\mu(x) \right\}^{1/q} \left\{ \frac{1}{\mu(B)} \int_B \varphi(y, t)\, d\mu(y) \right\}^{-1} < \infty$$

or

$$[\varphi]_{\mathbf{RH}_\infty(\mathcal{X})} := \sup_{t \in (0, \infty)} \sup_{B \subset \mathcal{X}} \left\{ \operatorname*{ess\,sup}_{y \in B} \varphi(y, t) \right\} \left\{ \frac{1}{\mu(B)} \int_B \varphi(y, t)\, d\mu(y) \right\}^{-1} < \infty,$$

where the first suprema are taken over all $t \in (0, \infty)$ and the second ones over all balls $B \subset \mathcal{X}$.

The following notion of growth functions on spaces of homogeneous type were taken from [31].

Definition 2.5. A function $\varphi\colon \mathcal{X} \times [0, \infty) \to [0, \infty)$ is called a *growth function* if the following conditions are satisfied:

(i) φ is a *Musielak–Orlicz function*, namely,

    (i)$_1$ the function $\varphi(x, \cdot)\colon [0, \infty) \to [0, \infty)$ is an Orlicz function for almost every $x \in \mathcal{X}$;

$(i)_2$ the function $\varphi(\cdot, t)$ is measurable for any $t \in [0, \infty)$.

(ii) $\varphi \in \mathbf{A}_\infty(\mathcal{X})$.

(iii) φ is of uniformly lower type p for some $p \in (0, 1]$ and of uniformly upper type 1.

In what follows, let M denote the *Hardy–Littlewood maximal operator* defined by setting, for any $f \in L^1_{\mathrm{loc}}(\mathcal{X})$ (the space of all locally integrable functions on $\mathcal{X}$) and $x \in \mathcal{X}$,

$$(2.2) \qquad M(f)(x) := \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f(y)| \, d\mu(y),$$

where the supremum is taken over all balls B of $\mathcal{X}$ containing x.

Now, we list some basic properties of $\mathbf{A}_q(\mathcal{X})$ for any $q \in [1, \infty)$ whose proofs are similar to those of the Euclidean case in [59, Lemma 1.1.3]; the details are omitted here (see [31, Lemmas 2.11 and 2.12] for some details).

Lemma 2.6. *The following conclusions hold true.*

(i) $\mathbf{A}_1(\mathcal{X}) \subset \mathbf{A}_p(\mathcal{X}) \subset \mathbf{A}_q(\mathcal{X})$ *for any* p, q *satisfying* $1 \le p \le q < \infty$.

(ii) $\mathbf{RH}_\infty(\mathcal{X}) \subset \mathbf{RH}_q(\mathcal{X}) \subset \mathbf{RH}_p(\mathcal{X})$ *for any* p, q *satisfying* $1 < p \le q \le \infty$.

(iii) *If* $q \in [1, \infty)$ *and* $\varphi \in \mathbf{A}_q(\mathcal{X})$, *then there exists a positive constant* C *such that, for any ball* $B \subset \mathcal{X}$, *measurable function* f *and* $t \in (0, \infty)$,

$$\left[\frac{1}{\mu(B)} \int_B |f(x)| \, d\mu(x) \right]^q \le C \frac{1}{\varphi(B, t)} \int_B |f(x)|^q \varphi(x, t) \, d\mu(x),$$

here and hereafter, for any measurable set $E \subset \mathcal{X}$ *and* $t \in [0, \infty)$, *let*

$$\varphi(E, t) := \int_E \varphi(x, t) \, d\mu(x).$$

(iv) *If* $\varphi \in \mathbf{A}_q(\mathcal{X})$ *with* $q \in [1, \infty)$, *then there exists a positive constant* C *such that, for any ball* $B \subset \mathcal{X}$, *measurable set* $E \subset B$ *and* $t \in (0, \infty)$,

$$\frac{\varphi(B, t)}{\varphi(E, t)} \le C \left[\frac{\mu(B)}{\mu(E)} \right]^q.$$

(v) *If* $q \in (1, \infty)$ *and* $\varphi \in \mathbf{A}_q(\mathcal{X})$, *then there exists a positive constant* C *such that, for any* $f \in L^1_{\mathrm{loc}}(\mathcal{X})$ *and* $t \in [0, \infty)$,

$$\int_\mathcal{X} [M(f)(x)]^q \varphi(x, t) \, d\mu(x) \le C \int_\mathcal{X} |f(x)|^q \varphi(x, t) \, d\mu(x).$$

We then recall the definition of Musielak–Orlicz spaces from [31, Definition 2.8].

Definition 2.7. Let φ be a Musielak–Orlicz function. The *Musielak–Orlicz space* $L^\varphi(\mathcal{X})$ is defined to be the set of all measurable functions f such that, for some $\lambda \in (0, \infty)$,

$$\int_\mathcal{X} \varphi(x, |f(x)|/\lambda) \, d\mu(x) < \infty,$$

equipped with the *Luxembourg (*also called the *Luxembourg–Nakano) (quasi-)norm*

$$\|f\|_{L^\varphi(\mathcal{X})} := \inf \left\{ \lambda \in (0, \infty) \colon \int_\mathcal{X} \varphi(x, |f(x)|/\lambda) \, d\mu(x) \le 1 \right\}.$$

We further recall some basic properties of $L^\varphi(\mathcal{X})$, which are easy generalizations of the corresponding properties of the Euclidean case in [59, Lemmas 1.1.6 and 1.1.10] to any space of homogeneous type; the details are omitted.

Lemma 2.8. *Let φ be a growth function as in Definition 2.5. Then the following conclusions hold true.*

(i) *The growth function φ is uniformly σ-quasi-subadditive on $\mathcal{X} \times [0, \infty)$, namely, there exists a positive constant C such that, for any $(x, t_j) \in \mathcal{X} \times [0, \infty)$ with $j \in \mathbf{N}$,*

$$\varphi\left(x, \sum_{j=1}^{\infty} t_j\right) \leq C \sum_{j=1}^{\infty} \varphi(x, t_j).$$

(ii) *For any $f \in L^{\varphi}(\mathcal{X}) \setminus \{0\}$,*

$$\int_{\mathcal{X}} \varphi\left(x, \frac{|f(x)|}{\|f\|_{L^{\varphi}(\mathcal{X})}}\right) d\mu(x) = 1.$$

(iii) *For any $\{f_k\}_{k \in \mathbf{N}} \subset L^{\varphi}(\mathcal{X})$, $\lim_{k \to \infty} \|f_k\|_{L^{\varphi}(\mathcal{X})} = 0$ if and only if*

$$\lim_{k \to \infty} \int_{\mathcal{X}} \varphi(x, |f_k(x)|) d\mu(x) = 0.$$

(iii) *For any $(x, t) \in \mathcal{X} \times [0, \infty)$, $\widetilde{\varphi}(x, t) := \int_0^t \frac{\varphi(x, s)}{s} ds$ is a growth function, which is equivalent to φ; moreover, for almost every $x \in \mathcal{X}$, $\widetilde{\varphi}(x, \cdot)$ is continuous and strictly increasing.*

In what follows, for any $f \in L^1_{\mathrm{loc}}(\mathcal{X})$ and any ball $B \subset \mathcal{X}$, let

$$(2.3) \qquad m_B(f) := \frac{1}{\mu(B)} \int_B f(x) \, d\mu(x).$$

Then we generalize the notion of Musielak–Orlicz BMO spaces $\mathrm{BMO}^{\varphi}(\mathbf{R}^n)$ in [35] to any space of homogeneous type as follows.

Definition 2.9. A function $f \in L^1_{\mathrm{loc}}(\mathcal{X})$ is said to belong to $\mathrm{BMO}^{\varphi}(\mathcal{X})$ if

$$\|f\|_{\mathrm{BMO}^{\varphi}(\mathcal{X})} := \sup_B \frac{1}{\|\mathbf{1}_B\|_{L^{\varphi}(\mathcal{X})}} \int_B |f(x) - m_B(f)| \, d\mu(x) < \infty,$$

where the supremum is taken over all balls $B \subset \mathcal{X}$.

Remark 2.10. Fix $x_0 \in \mathcal{X}$. A typical example of the space $\mathrm{BMO}^{\varphi}(\mathcal{X})$ is the space $\mathrm{BMO}^{\log}(\mathcal{X})$ which is just the space $\mathrm{BMO}^{\varphi}(\mathcal{X})$ with

$$(2.4) \qquad \varphi(x, t) := \frac{t}{\log(e + d(x, x_0)) + \log(e + t)}, \quad \forall\, x \in \mathcal{X}, \ \forall\, t \in [0, \infty).$$

We further introduce the following notion of Musielak–Orlicz Campanato spaces $\mathrm{Lip}_{\varphi,q}(\mathcal{X})$, which is a generalization of the corresponding Euclidean case in [43] to any space of homogeneous type.

Definition 2.11. A function $f \in L^1_{\mathrm{loc}}(\mathcal{X})$ is said to belong to the *Musielak–Orlicz Campanato space* $\mathrm{Lip}_{\varphi,q}(\mathcal{X})$ if

$$\|f\|_{\mathrm{Lip}_{\varphi,q}(\mathcal{X})} := \sup_B \frac{1}{\|\mathbf{1}_B\|_{L^{\varphi}(\mathcal{X})}} \left\{ \int_B \left[\frac{|f(x) - m_B(f)|}{\varphi(x, \|\mathbf{1}_B\|_{L^{\varphi}(\mathcal{X})}^{-1})} \right]^q \varphi\left(x, \|\mathbf{1}_B\|_{L^{\varphi}(\mathcal{X})}^{-1}\right) d\mu(x) \right\}^{1/q}$$
$$< \infty,$$

where the supremum is taken over all balls $B \subset \mathcal{X}$.

Remark 2.12. Obviously, $\mathrm{Lip}_{\varphi,1}(\mathcal{X}) = \mathrm{BMO}^{\varphi}(\mathcal{X})$.

3. John–Nirenberg inequality of $\mathrm{BMO}^\varphi(\mathcal{X})$

In this section, we obtain the John–Nirenberg inequality and establish several equivalent characterizations of $\mathrm{BMO}^\varphi(\mathcal{X})$. The following John–Nirenberg inequality for $\mathrm{BMO}^\varphi(\mathcal{X})$ is similar to the corresponding Euclidean case in [43, Theorem 2.5]. For the convenience of the reader, we present some details here.

Theorem 3.1. *Let φ be a growth function as in Definition 2.5 and $f \in \mathrm{BMO}^\varphi(\mathcal{X})$. Then there exist positive constants C_1, C_2 and C_3, independent of f, such that, for any ball $B \subset \mathcal{X}$ and $t \in (0, \infty)$, if $\varphi \in \mathbf{A}_1(\mathcal{X})$, then*

$$\varphi\left(\left\{x \in B : \frac{|f(x) - m_B(f)|}{\varphi(x, \|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1})} > t\right\}, \|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1}\right) \leq C_1 \exp\left\{-\frac{C_2 t}{\|f\|_{\mathrm{Lip}_{\varphi,1}(\mathcal{X})}\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}}\right\}$$

and, if $\varphi \in \mathbf{A}_q(\mathcal{X})$ for some $q \in (1, \infty)$, then

$$\varphi\left(\left\{x \in B : \frac{|f(x) - m_B(f)|}{\varphi(x, \|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1})} > t\right\}, \|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1}\right) \leq C_3 \left[1 + \frac{t}{\|f\|_{\mathrm{Lip}_{\varphi,1}(\mathcal{X})}\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}}\right]^{-q'},$$

where $m_B(f)$ is as in (2.3) and $q' := q/(q-1)$.

To prove Theorem 3.1, we need the following two technical lemmas. The first lemma is an easy generalization of the corresponding Euclidean case in [51, Lemma (3.1)] to any space of homogeneous type; the details are omitted.

Lemma 3.2. *Let $q \in (1, \infty)$ and $\frac{1}{q} + \frac{1}{q'} = 1$. If $w \in A_q(\mathcal{X})$, then there exists a positive constant C such that, for any ball $B \subset \mathcal{X}$ and $t \in (0, \infty)$,*

$$w\left(\{x \in B : w(x) < t\}\right) \leq C\left[t\frac{\mu(B)}{w(B)}\right]^{q'} w(B),$$

here and hereafter, for any non-negative measurable function w and measurable set $E \subset \mathcal{X}$,

$$w(E) := \int_E w(x)\, d\mu(x).$$

The second lemma is a generalization of the weighted version of Calderón–Zygmund decompositions on the Euclidean case in [51, Lemma (3.2)] to any space of homogeneous type.

Lemma 3.3. *Let $w \colon \mathcal{X} \to (0, \infty)$ be a measurable function satisfying the doubling condition, namely, there exists a positive constant $\widetilde{C}_0$ such that, for any ball $B \subset \mathcal{X}$, $w(2B) \leq \widetilde{C}_0 w(B)$ and, for a given ball $B_0 \subset \mathcal{X}$ and $\sigma \in (0, \infty)$, let f be a non-negative function which satisfies that*

$$\frac{1}{w(B_0)}\int_{B_0} f(x)w(x)\, d\mu(x) \leq \sigma.$$

Then there exists a sequence $\{B_k\}_{k \in I}$ (I is some countable index set) of disjoint balls such that, for almost every $x \in B_0 \setminus \bigcup_{k \in I}(4A_0 B_k)$ with A_0 same as in (1.2),

$$(3.1) \qquad\qquad\qquad\qquad f(x) \leq \sigma$$

and, for any $k \in I$,

$$\sigma < \frac{1}{w(B_k)}\int_{B_k \cap B_0} f(x)w(x)\, d\mu(x) \leq \widetilde{C}_0 \sigma.$$

Proof. For a fixed ball B_0 and $\sigma \in (0, \infty)$, let $E := \{x \in \mathcal{X} : f(x)\mathbf{1}_{B_0}(x) > \sigma\} \subset B_0$. Observe that, by the generalized Lebesgue differential theorem (see, for instance, [29, Theorem 1.8]), we know that, for almost every $x \in E$, there exists a ball $B'_x \subset \mathcal{X}$ such that

$$(3.2) \qquad \frac{1}{w(B'_x)} \int_{B'_x} f(y)\mathbf{1}_{B_0}(y)w(y)\,d\mu(y) > \sigma.$$

Let B_x be the largest ball, satisfying (3.2), concentric with B'_x and having the radius of the form $2^m r_{B'_x}$ with $m \in \mathbf{Z}_+$. Then, for $\widetilde{B}_x := 2B_x$,

$$\frac{1}{w(\widetilde{B}_x)} \int_{\widetilde{B}_x} f(y)\mathbf{1}_{B_0}(y)w(y)\,d\mu(y) \le \sigma.$$

From the Vitali–Wiener type covering lemma (see, for instance, [10, Theorem (3.1)]), it follows that there exists a disjoint subsequence $\{B_k\}_{k \in I}$ (I is some countable index set) of $\{B_x\}_{x \in E}$ such that $E \subset \bigcup_{k \in I}(4A_0 B_k)$. Thus, (3.1) follows immediately.

Moreover, by the doubling condition of w, we further have

$$\sigma < \frac{1}{w(B_k)} \int_{B_k \cap B_0} f(y)w(y)\,d\mu(y) \le \frac{w(\widetilde{B}_k)}{w(B_k)} \frac{1}{w(\widetilde{B}_k)} \int_{\widetilde{B}_k} f(y)\mathbf{1}_{B_0}(y)w(y)\,d\mu(y) \le \widetilde{C}_0 \sigma,$$

which completes the proof of Lemma 3.3. $\qquad\qquad\square$

Now, we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. Let $f \in \mathrm{Lip}_{\varphi,1}(\mathcal{X})$ and fix any ball $B_0 \subset \mathcal{X}$. By the homogeneity of the norm of $f \in \mathrm{Lip}_{\varphi,1}(\mathcal{X})$, without loss of generality, we may assume that $\|f\|_{\mathrm{Lip}_{\varphi,1}(\mathcal{X})} = \|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{-1}$; otherwise, we may replace f by

$$\frac{f}{\|f\|_{\mathrm{Lip}_{\varphi,1}(\mathcal{X})}\|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}}.$$

For any $t \in (0, \infty)$ and ball $B \subset B_0$, let

$$\lambda(t, B) := \varphi\left(\left\{x \in B : \frac{|f(x) - m_B(f)|}{\varphi(x, \|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{-1})} > t\right\}, \|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{-1}\right)$$

and

$$(3.3) \qquad \mathcal{F}(t) := \sup_{B \subset B_0} \frac{\lambda(t, B)}{\varphi(B, \|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{-1})}.$$

From $\lambda(t, B) \le \varphi(B, \|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{-1})$, it follows that, for any $t \in (0, \infty)$, $\mathcal{F}(t) \le 1$. By $\|f\|_{\mathrm{Lip}_{\varphi,1}(\mathcal{X})} = \|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{-1}$ and the uniformly upper type 1 property of φ, we obtain

$$(3.4) \qquad \frac{1}{\varphi(B, \|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{-1})} \int_B |f(x) - m_B(f)|\,d\mu(x)$$

$$\le \frac{\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}}{\varphi(B, \|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{-1})\|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}} \le \frac{\widetilde{C}_1\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}}{\varphi(B, \|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1})\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}} = \widetilde{C}_1$$

for the positive constant $\widetilde{C}_1 := \frac{1}{C_{(1)}}$ with $C_{(1)}$ as in Definition 2.2.

Furthermore, let $\sigma \in [\widetilde{C}_1, \infty)$. From Lemma 3.3 with $f(\cdot)$ and $w(\cdot)$ replaced, respectively, by

$$\left[\varphi\left(\cdot, \|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{-1}\right)\right]^{-1} |f(\cdot) - m_B(f)|$$

and $\varphi(\cdot, \|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{-1})$, we deduce that there exist a disjoint sequence $\{B_k\}_{k\in I}$, with I as in Lemma 3.3, of balls and a positive constant $\widetilde{C}_0$ as in Lemma 3.3 such that, for almost every $x \in B \setminus \bigcup_{k\in I}(4A_0 B_k)$,

$$(3.5) \qquad \left[\varphi\left(x, \|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{-1}\right)\right]^{-1} |f(x) - m_B(f)| \le \sigma$$

and, for any $k \in I$,

$$(3.6) \qquad \sigma < \frac{1}{\varphi(B_k, \|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{-1})} \int_{B_k \cap B} |f(x) - m_B(f)| \, d\mu(x) \le \widetilde{C}_0 \sigma.$$

Thus, by this and (3.4), we have

$$(3.7) \qquad \sum_{k\in I} \varphi\left(B_k, \|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{-1}\right) \le \frac{1}{\sigma} \sum_{k\in I} \int_{B_k \cap B} |f(x) - m_B(f)| \, d\mu(x)$$

$$\le \frac{1}{\sigma} \int_B |f(x) - m_B(f)| \, d\mu(x) \le \frac{\widetilde{C}_1}{\sigma} \varphi\left(B, \|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{-1}\right).$$

If $t \in [\sigma, \infty)$, from (3.5) and Lemma 2.6(iv), we then deduce that

$$\lambda(t, B) = \varphi\left(\left\{x \in B : \frac{|f(x) - m_B(f)|}{\varphi(x, \|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{-1})} > t\right\}, \|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{-1}\right)$$

$$\le \sum_{k\in I} \varphi\left(\left\{x \in 4A_0 B_k : \frac{|f(x) - m_B(f)|}{\varphi(x, \|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{-1})} > t\right\}, \|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{-1}\right)$$

$$\lesssim \sum_{k\in I} \varphi\left(\left\{x \in B_k : \frac{|f(x) - m_B(f)|}{\varphi(x, \|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{-1})} > t\right\}, \|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{-1}\right).$$

Thus, for any $t \in [\sigma, \infty)$ and any $s \in [0, t]$, we write

$$(3.8) \qquad \lambda(t, B) \le \sum_{k\in I} \lambda(t - s, B_k)$$

$$+ \sum_{k\in I} \varphi\left(\left\{x \in B_k : \frac{|m_{B_k}(f) - m_B(f)|}{\varphi(x, \|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{-1})} > s\right\}, \|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{-1}\right)$$

$$=: \mathrm{I}_1 + \mathrm{I}_2.$$

For I_1, by (3.3), (3.7) and some arguments similar to those used in the estimation of I_1 in the proof of [43, Theoem 2.5], we then obtain

$$(3.9) \qquad \mathrm{I}_1 \le \frac{\widetilde{C}_1}{\sigma} \mathcal{F}(t - s) \varphi\left(B, \|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{-1}\right).$$

Moreover, from (3.6), we deduce that

$$(3.10) \qquad |m_{B_k}(f) - m_B(f)| \le \frac{1}{\mu(B_k)} \int_{B_k} |f(x) - m_B(f)| \, d\mu(x)$$

$$\le \widetilde{C}_0 \sigma \frac{\varphi(B_k, \|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{-1})}{\mu(B_k)}.$$

We further consider the following two cases.

Case 1. $\varphi \in \mathbf{A}_1(\mathcal{X})$. In this case, we know that there exists a positive constant $\widetilde{C}_2$ such that

$$\frac{\varphi\left(B_k, \|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{-1}\right)}{\mu(B_k)} \leq \widetilde{C}_2 \operatorname*{ess\,inf}_{y \in B_k} \varphi\left(y, \|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{-1}\right),$$

which, together with (3.10), implies that

$$(3.11) \quad \varphi\left(\left\{x \in B_k : \frac{|m_{B_k}(f) - m_B(f)|}{\varphi(x, \|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{-1})} > s\right\}, \|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{-1}\right)$$

$$\leq \varphi\left(\left\{x \in B_k : \frac{\widetilde{C}_0 \widetilde{C}_2 \sigma \operatorname*{ess\,inf}_{y \in B_k} \varphi\left(y, \|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{-1}\right)}{\varphi(x, \|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{-1})} > s\right\}, \|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{-1}\right).$$

Now, we choose $\sigma := 2\widetilde{C}_1$ and $s := 2\widetilde{C}_0\widetilde{C}_1\widetilde{C}_2$. Then we observe that, if $t \in (s, \infty)$, then $\sigma \in (\widetilde{C}_1, t)$ and $s \in (0, t)$ as required.

By (3.8) and (3.11), we conclude that

$$\mathrm{I}_2 \leq \sum_{k \in I} \varphi\left(\left\{x \in B_k : \frac{\operatorname*{ess\,inf}_{y \in B_k} \varphi\left(y, \|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{-1}\right)}{\varphi(x, \|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{-1})} > 1\right\}, \|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{-1}\right) = 0,$$

which, combined with (3.8) and (3.9), further implies that, for any $t \in (s, \infty)$ and $B \subset B_0$,

$$\lambda(t, B) \leq \frac{1}{2}\mathcal{F}(t - s)\varphi\left(B, \|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{-1}\right).$$

Thus, if $t \in (s, \infty)$, then $\mathcal{F}(t) \leq \frac{1}{2}\mathcal{F}(t - s)$.

Moreover, if $m \in \mathbf{N}$ and $t \in (ms, (m-1)s]$, then

$$\mathcal{F}(t) \leq \frac{1}{2}\mathcal{F}(t - s) \leq \cdots \leq \frac{1}{2^m}\mathcal{F}(t - ms).$$

By $\mathcal{F}(t - ms) \leq 1$ and $m \geq \frac{t}{s} - 1$, we conclude that

$$\mathcal{F}(t) \leq 2^{-m} \leq 2^{1-\frac{t}{s}} = 2e^{-\left(\frac{t}{s}\log 2\right)}.$$

Let $C_1 := 2$ and $C_2 := \frac{1}{s}\log 2$. Then we conclude that, for any given $\varphi \in \mathbf{A}_1(\mathcal{X})$ and any $s \in (0, t)$,

$$\varphi\left(\left\{x \in B : \frac{|f(x) - m_B(f)|}{\varphi(x, \|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1})} > t\right\}, \|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1}\right) \leq 2e^{-\left(\frac{1}{s}\log 2\right)t} = C_1 e^{-C_2 t},$$

which implies the desired conclusion of Theorem 3.1 in Case 1.

Case 2. $\varphi \in \mathbf{A}_q(\mathcal{X})$ for some $q \in (1, \infty)$. In this case, by (3.8), (3.10), Lemma 3.2 and (3.7), we know that

$$
\mathrm{I}_2 \leq \sum_{k \in I} \varphi \left(\left\{ x \in B_k : \frac{\widetilde{C}_0 \sigma \varphi \left(B_k, \|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{-1} \right)}{\mu(B_k) \varphi(x, \|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{-1})} > s \right\}, \|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{-1} \right)
$$

$$
= \sum_{k \in I} \varphi \left(\left\{ x \in B_k : \varphi \left(x, \|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{-1} \right) < \frac{\widetilde{C}_0 \sigma \varphi \left(B_k, \|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{-1} \right)}{s \mu(B_k)} \right\}, \|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{-1} \right)
$$

$$
\leq \sum_{k \in I} C_3 \left(\frac{\widetilde{C}_0 \sigma}{s} \right)^{q'} \varphi \left(B_k, \|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{-1} \right) \leq C_3 \left(\frac{\widetilde{C}_0 \sigma}{s} \right)^{q'} \frac{\widetilde{C}_1}{\sigma} \varphi \left(B, \|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{-1} \right),
$$

where C_3 is the same fixed positive constant C as in Lemma 3.2.

From this, (3.8) and (3.9), it follows that, for any $\sigma \in [\widetilde{C}_1, t]$, $s \in (0, t)$ and $B \subset B_0$,

$$
(3.12) \qquad \lambda(t, B) \leq \left[\frac{\widetilde{C}_1}{\sigma} \mathcal{F}(t - s) + C_3 \left(\frac{\widetilde{C}_0 \sigma}{s} \right)^{q'} \frac{\widetilde{C}_1}{\sigma} \right] \varphi \left(B, \|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{-1} \right).
$$

Then, by (3.12) with $\sigma := 4^{q'} \widetilde{C}_1$, $s := \frac{t}{2}$ and $C_0 := \max\{\sigma, \widetilde{C}_1 C_3 (2\widetilde{C}_0)^{q'} \sigma^{q'-1}\}$, we have, for any $t \in (C_0, \infty)$,

$$
(3.13) \qquad \mathcal{F}(t) \leq 4^{-q'} \mathcal{F} \left(\frac{t}{2} \right) + C_0 t^{-q'}.
$$

Now, we show that, if $t \in (C_0, 2C_0]$ and $m \in \mathbf{Z}_+$, then

$$
(3.14) \qquad \mathcal{F}(2^m t) \leq (2C_0)^{q'} (2^m t)^{-q'}.
$$

To this end, we use mathematical induction. Indeed, when $m = 0$, we observe that

$$
\mathcal{F}(2^m t) \leq 1 \leq (2C_0)^{q'} t^{-q'},
$$

which implies (3.14) in this case.

Suppose that (3.14) holds true with m replaced by $m - 1$. Then, from (3.13), we deduce that

$$
\mathcal{F}(2^m t) \leq 4^{-q'} \mathcal{F}(2^{m-1} t) + C_0 (2^m t)^{-q'} \leq 4^{-q'} (2C_0)^{q'} (2^{m-1} t)^{-q'} + C_0 (2^m t)^{-q'}
$$

$$
= (2C_0)^{q'} (2^m t)^{-q'} \left(2^{-q'} + 2^{-q'} C_0^{1-q'} \right),
$$

which, together with the fact that

$$
2^{-q'} + 2^{-q'} C_0^{1-q'} < 2^{-q'} + 2^{-q'} < 1,
$$

implies that (3.14) holds true for m.

Then, by mathematical induction on m, we further conclude that (3.14) holds true for any $m \in \mathbf{Z}_+$. Furthermore, from (3.14), we deduce that, if $t \in (C_0, \infty)$, then

$$
\mathcal{F}(t) \leq (2C_0)^{q'} t^{-q'},
$$

which implies the desired conclusion of Theorem 3.1 in Case 2 and hence completes the proof of Theorem 3.1. $\qquad \square$

In what follows, let

$$
(3.15) \qquad m(\varphi) := \lfloor \omega[q(\varphi)/i(\varphi) - 1] \rfloor,
$$

where, for any $a \in \mathbf{R}$, $\lfloor a \rfloor$ denotes the largest integer which is not bigger than a, ω is as in (1.4), $i(\varphi)$ is as in (2.1) and

$$(3.16) \qquad q(\varphi) := \inf\{q \in [1, \infty) \colon \varphi \in \mathbf{A}_q(\mathcal{X})\}.$$

In the spirit of the corresponding Euclidean case in [43, Theorem 2.7], we obtain the following several equivalent characterizations of $\mathrm{BMO}^{\varphi}(\mathcal{X})$ on $\mathcal{X}$.

Theorem 3.4. *Let φ be a growth function as in Definition 2.5, $q \in [1, [q(\varphi)]')$ and $\epsilon \in (\omega[\frac{q(\varphi)}{i(\varphi)} - 1], \infty)$, where $i(\varphi)$ and $q(\varphi)$ are, respectively, as in (2.1) and (3.16), and $1/q(\varphi) + 1/[q(\varphi)]' = 1$. Then, for any $f \in L^1_{\mathrm{loc}}(\mathcal{X})$, the following statements are mutually equivalent:*

(i) $\|f\|_{\mathrm{BMO}^{\varphi}(\mathcal{X})} < \infty$;
(ii) $\|f\|_{\mathrm{Lip}_{\varphi,q}(\mathcal{X})} < \infty$;

(iii) $\displaystyle \|f\|_{\mathrm{Lip}^{\epsilon}_{\varphi,1}(\mathcal{X})} := \sup_{\{x_0 \in \mathcal{X}, r \in (0,\infty), B := B(x_0,r) \subset \mathcal{X}\}} \frac{r^{\omega+\epsilon}}{\|\mathbf{1}_B\|_{L^{\varphi}(\mathcal{X})}} \int_{\mathcal{X}} \frac{|f(x) - m_B(f)|}{r^{\omega+\epsilon} + [d(x,x_0)]^{\omega+\epsilon}} \, d\mu(x)$
$< \infty$, *where $m_B(f)$ is as in* (2.3).

Moreover, the norms $\|\cdot\|_{\mathrm{BMO}^{\varphi}(\mathcal{X})}$, $\|\cdot\|_{\mathrm{Lip}_{\varphi,q}(\mathcal{X})}$ and $\|\cdot\|_{\mathrm{Lip}^{\epsilon}_{\varphi,1}(\mathcal{X})}$ are mutually equivalent.

Proof. From Theorem 3.1 and some arguments similar to those used in the proof of [43, Theorem 2.7], it follows that "(i) $\Longleftrightarrow$ (ii)" and, for any $f \in L^1_{\mathrm{loc}}(\mathcal{X})$,

$$\|f\|_{\mathrm{Lip}_{\varphi,1}(\mathcal{X})} \sim \|f\|_{\mathrm{Lip}_{\varphi,q}(\mathcal{X})},$$

where the positive equivalence constants are independent of f.

Now, we show that "(iii) $\Longleftrightarrow$ (i)". It is clear that "(iii) $\Longrightarrow$ (i)" and, for any $f \in L^1_{\mathrm{loc}}(\mathcal{X})$,

$$\|f\|_{\mathrm{Lip}_{\varphi,1}(\mathcal{X})} \lesssim \|f\|_{\mathrm{Lip}^{\epsilon}_{\varphi,1}(\mathcal{X})}.$$

Then we are left to show that "(i) $\Longrightarrow$ (iii)". Fix any $x_0 \in \mathcal{X}$, $r \in (0, \infty)$ and ball $B := B(x_0, r)$. For any $k \in \mathbf{Z}_+ := \{0\} \cup \mathbf{N}$, let $B_k := 2^k B$. Then we write

$$(3.17) \quad \mathrm{I} := \frac{r^{\omega+\epsilon}}{\|\mathbf{1}_B\|_{L^{\varphi}(\mathcal{X})}} \int_{\mathcal{X}} \frac{|f(x) - m_B(f)|}{r^{\omega+\epsilon} + [d(x,x_0)]^{\omega+\epsilon}} \, d\mu(x)$$

$$= \frac{r^{\omega+\epsilon}}{\|\mathbf{1}_B\|_{L^{\varphi}(\mathcal{X})}} \int_B \frac{|f(x) - m_B(f)|}{r^{\omega+\epsilon} + [d(x,x_0)]^{\omega+\epsilon}} \, d\mu(x) + \sum_{k=1}^{\infty} \frac{r^{\omega+\epsilon}}{\|\mathbf{1}_B\|_{L^{\varphi}(\mathcal{X})}} \int_{B_k \setminus B_{k-1}} \cdots$$

$$=: \mathrm{I}_1 + \sum_{k=1}^{\infty} \mathrm{I}_{2,k},$$

where

$$\mathrm{I}_1 := \frac{r^{\omega+\epsilon}}{\|\mathbf{1}_B\|_{L^{\varphi}(\mathcal{X})}} \int_B \frac{|f(x) - m_B(f)|}{r^{\omega+\epsilon} + [d(x,x_0)]^{\omega+\epsilon}} \, d\mu(x)$$

and, for any $k \in \mathbf{N}$,

$$\mathrm{I}_{2,k} := \frac{r^{\omega+\epsilon}}{\|\mathbf{1}_B\|_{L^{\varphi}(\mathcal{X})}} \int_{B_k \setminus B_{k-1}} \frac{|f(x) - m_B(f)|}{r^{\omega+\epsilon} + [d(x,x_0)]^{\omega+\epsilon}} \, d\mu(x).$$

Obviously, we have

$$(3.18) \qquad \mathrm{I}_1 \leq \frac{1}{\|\mathbf{1}_B\|_{L^{\varphi}(\mathcal{X})}} \int_B |f(x) - m_B(f)| \, d\mu(x) \leq \|f\|_{\mathrm{Lip}_{\varphi,1}(\mathcal{X})}.$$

Moreover, observe that $\epsilon \in (\omega[\frac{q(\varphi)}{i(\varphi)} - 1], \infty)$ implies that there exist $p_0 \in (0, i(\varphi))$ and $q_0 \in (q(\varphi), \infty)$ such that $\epsilon > \omega(\frac{q_0}{p_0} - 1)$. Thus, $\varphi \in \mathbf{A}_{q_0}(\mathcal{X})$ and φ is of uniformly lower

type p_0. By some arguments similar to those used in the proof of [43, Theorem 2.7], we conclude that, for any $k \in \mathbf{N}$,

$$(3.19) \qquad \|\mathbf{1}_{B_k}\|_{L^\varphi(\mathcal{X})} \lesssim 2^{k\omega\frac{q_0}{p_0}} \|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})},$$

which, together with (1.4), further implies that, for any $k \in \mathbf{N}$,

$$
\begin{aligned}
(3.20) \qquad \mathrm{I}_{2,k} &\lesssim 2^{-k(\omega+\epsilon)} \frac{1}{\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}} \int_{B_k} |f(x) - m_B(f)| \, d\mu(x) \\
&\lesssim 2^{-k(\omega+\epsilon)} \frac{1}{\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}} \left[\int_{B_k} |f(x) - m_{B_k}(f)| \, d\mu(x) \right. \\
&\quad \left. + \mu(B_k) \, |m_{B_k}(f) - m_B(f)| \right] \\
&\lesssim 2^{-k(\omega+\epsilon)} \frac{1}{\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}} \left[\int_{B_k} |f(x) - m_{B_k}(f)| \, d\mu(x) \right. \\
&\quad \left. + \mu(B_k) \sum_{s=1}^{k} \left| m_{B_s}(f) - m_{B_{s-1}}(f) \right| \right] \\
&\lesssim 2^{-k(\omega+\epsilon)} \left[\frac{1}{\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}} \int_{B_k} |f(x) - m_{B_k}(f)| \, d\mu(x) \right. \\
&\quad \left. + \sum_{s=1}^{k} \frac{\mu(B_k)}{\mu(B_{s-1})} \frac{1}{\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}} \int_{B_{s-1}} |f(x) - m_{B_s}(f)| \, d\mu(x) \right] \\
&\lesssim 2^{-k(\omega+\epsilon)} \left[2^{k\omega\frac{q_0}{p_0}} \|f\|_{\mathrm{Lip}_{\varphi,1}(\mathcal{X})} \right. \\
&\quad \left. + \sum_{s=1}^{k} \frac{\mu(B_k)}{\mu(B_s)} \frac{1}{\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}} \int_{B_s} |f(x) - m_{B_s}(f)| \, d\mu(x) \right] \\
&\lesssim 2^{-k(\omega+\epsilon)} \left[2^{k\omega\frac{q_0}{p_0}} \|f\|_{\mathrm{Lip}_{\varphi,1}(\mathcal{X})} + \sum_{s=1}^{k} 2^{(k-s)\omega} 2^{s\omega\frac{q_0}{p_0}} \|f\|_{\mathrm{Lip}_{\varphi,1}(\mathcal{X})} \right] \\
&\lesssim 2^{-k(\omega+\epsilon)} \left[2^{k\omega\frac{q_0}{p_0}} + 2^{k\omega} \sum_{s=-\infty}^{k} 2^{s\omega(\frac{q_0}{p_0}-1)} \right] \|f\|_{\mathrm{Lip}_{\varphi,1}(\mathcal{X})} \\
&\sim 2^{-k(\omega+\epsilon)} 2^{k\omega\frac{q_0}{p_0}} \|f\|_{\mathrm{Lip}_{\varphi,1}(\mathcal{X})} \sim 2^{-k[\epsilon-\omega(\frac{q_0}{p_0}-1)]} \|f\|_{\mathrm{Lip}_{\varphi,1}(\mathcal{X})}.
\end{aligned}
$$

From (3.17), (3.18), (3.20) and $\epsilon > \omega(\frac{q_0}{p_0} - 1)$, we deduce that

$$\mathrm{I} \lesssim \|f\|_{\mathrm{Lip}_{\varphi,1}(\mathcal{X})} + \sum_{k=1}^{\infty} 2^{-k[\epsilon-\omega(\frac{q_0}{p_0}-1)]} \|f\|_{\mathrm{Lip}_{\varphi,1}(\mathcal{X})} \lesssim \|f\|_{\mathrm{Lip}_{\varphi,1}(\mathcal{X})},$$

which implies that, for any $f \in \mathrm{Lip}_{\varphi,1}(\mathcal{X})$,

$$\|f\|_{\mathrm{Lip}_{\varphi,1}^\epsilon(\mathcal{X})} \lesssim \|f\|_{\mathrm{Lip}_{\varphi,1}(\mathcal{X})}.$$

This finishes the proof of "(i) $\implies$ (iii)" and hence of Theorem 3.4. $\qquad\square$

Remark 3.5. By Theorem 3.4, we know that $\mathrm{Lip}_{\varphi,q}(\mathcal{X})$ is independent of the choice of $q \in [1, [q(\varphi)]')$, with $q(\varphi)$ as in (3.16), in the sense of equivalent norms.

4. Musielak–Orlicz Hardy spaces via various maximal functions

In this section, we introduce Musielak–Orlicz Hardy spaces, $H^{+,\varphi}(\mathcal{X})$, $H_\theta^\varphi(\mathcal{X})$ and $H^{*,\varphi}(\mathcal{X})$, defined, respectively, via radial, nontangential and grand maximal functions and show that $H^{+,\varphi}(\mathcal{X})$, $H_\theta^\varphi(\mathcal{X})$ and $H^{*,\varphi}(\mathcal{X})$ are mutually identical with equivalent quasi-norms.

4.1. Calderón reproducing formulae. In this section, we first recall the important Calderón reproducing formulae established in [28]. Now, we first recall the following system of dyadic cubes given in [32, Theorem 2.2] (see also [2]), which was formulated in [27, Lemma 2.3].

Lemma 4.1. *Fix constants c_0, C_0 and δ such that $0 < c_0 \leq C_0 < \infty$, $\delta \in (0,1)$ and $12A_0^3 C_0 \delta \leq c_0$. Suppose that a set of points, $\{z_\alpha^k : k \in \mathbf{Z},\ \alpha \in \mathscr{A}_k\} \subset \mathcal{X}$ with $\mathscr{A}_k$ ($k \in \mathbf{Z}$) being a countable set of indices, has the following properties: for any $k \in \mathbf{Z}$,*

(i) *$d(z_\alpha^k, z_\beta^k) \geq c_0 \delta^k$ when $\alpha \neq \beta$;*

(ii) *$\min_{\alpha \in \mathscr{A}_k} d(x, z_\alpha^k) \leq C_0 \delta^k$ for any $x \in \mathcal{X}$.*

(The existence of such a set of points is guaranteed by the Zorn lemma.) Then there exists a family of sets, $\{Q_\alpha^k : k \in \mathbf{Z},\ \alpha \in \mathscr{A}_k\}$ (which is called the system of half-open dyadic cubes), satisfying

(iii) *$\mathcal{X} = \bigcup_{\alpha \in \mathscr{A}_k} Q_\alpha^k$, where $\{Q_\alpha^k : \alpha \in \mathscr{A}_k\}$ are mutually disjoint;*

(iv) *if $\ell \geq k$, $\alpha \in \mathscr{A}_k$ and $\beta \in \mathscr{A}_\ell$, then either $Q_\beta^\ell \subset Q_\alpha^k$ or $Q_\alpha^k \cap Q_\beta^\ell = \emptyset$;*

(v) *for any $\alpha \in \mathscr{A}_k$, $B(z_\alpha^k, c_\sharp \delta^k) \subset Q_\alpha^k \subset B(z_\alpha^k, C^\sharp \delta^k) =: B(Q_\alpha^k)$, where $c_\sharp := (3A_0^2)^{-1} c_0$, $C^\sharp := 2A_0 C_0$ and z_α^k is called the "center" of Q_α^k.*

In what follows, for any $k \in \mathbf{Z}$, let

$$(4.1) \qquad \mathcal{X}^k := \{z_\alpha^k\}_{\alpha \in \mathscr{A}_k}, \quad \mathscr{G}_k := \mathscr{A}_{k+1} \setminus \mathscr{A}_k \quad \text{and} \quad \mathcal{Y}^k := \mathcal{X}^{k+1} \setminus \mathcal{X}^k.$$

Then we recall the following notion of the approximation of the identity with exponential decay from [28, Definition 2.7]; see also [27, Definition 2.4].

Definition 4.2. A sequence $\{Q_k\}_{k \in \mathbf{Z}}$ of bounded linear integral operators on $L^2(\mathcal{X})$ is called an *approximation of the identity with exponential decay* (for short, exp-ATI) if there exist constants $C, \nu \in (0, \infty)$, $a \in (0, 1]$ and $\eta \in (0, 1)$ such that, for any $k \in \mathbf{Z}$, the kernel of the operator Q_k, which is still denoted by Q_k, satisfies

(i) (the *identity condition*) $\sum_{k=-\infty}^{\infty} Q_k = I$ in $L^2(\mathcal{X})$, where I is the identity operator on $L^2(\mathcal{X})$;

(ii) (the *size condition*) for any $x, y \in \mathcal{X}$,

$$|Q_k(x, y)| \leq C \frac{1}{\sqrt{V_{\delta^k}(x) V_{\delta^k}(y)}} \exp\left\{-\nu \left[\frac{d(x,y)}{\delta^k}\right]^a\right\}$$

$$\cdot \exp\left\{-\nu \left[\frac{\max\{d(x, \mathcal{Y}^k), d(y, \mathcal{Y}^k)\}}{\delta^k}\right]^a\right\};$$

(iii) (the *regularity condition*) for any $x, \widetilde{x}, y \in \mathcal{X}$ with $d(x, \widetilde{x}) \leq \delta^k$,

$$|Q_k(x, y) - Q_k(\widetilde{x}, y)| + |Q_k(y, x) - Q_k(y, \widetilde{x})|$$

$$\leq C \left[\frac{d(x, \widetilde{x})}{\delta^k}\right]^\eta \frac{1}{\sqrt{V_{\delta^k}(x) V_{\delta^k}(y)}} \exp\left\{-\nu \left[\frac{d(x,y)}{\delta^k}\right]^a\right\}$$

$$\cdot \exp\left\{-\nu \left[\frac{\max\{d(x, \mathcal{Y}^k), d(y, \mathcal{Y}^k)\}}{\delta^k}\right]^a\right\};$$

(iv) (the *second difference regularity condition*) for any x, $\widetilde{x}$, y, $\widetilde{y} \in \mathcal{X}$ with $d(x, \widetilde{x}) \leq \delta^k$ and $d(y, \widetilde{y}) \leq \delta^k$,

$$|[Q_k(x, y) - Q_k(\widetilde{x}, y)] - [Q_k(x, \widetilde{y}) - Q_k(\widetilde{x}, \widetilde{y})]|$$

$$\leq C \left[\frac{d(x, \widetilde{x})}{\delta^k}\right]^\eta \left[\frac{d(y, \widetilde{y})}{\delta^k}\right]^\eta \frac{1}{\sqrt{V_{\delta^k}(x) V_{\delta^k}(y)}} \exp\left\{-\nu \left[\frac{d(x, y)}{\delta^k}\right]^a\right\}$$

$$\cdot \exp\left\{-\nu \left[\frac{\max\{d(x, \mathcal{Y}^k), d(y, \mathcal{Y}^k)\}}{\delta^k}\right]^a\right\};$$

(v) (the *cancellation condition*) for any x, $y \in \mathcal{X}$,

$$\int_{\mathcal{X}} Q_k(x, \widetilde{y}) \, d\mu(\widetilde{y}) = 0 = \int_{\mathcal{X}} Q_k(\widetilde{x}, y) \, d\mu(\widetilde{x}).$$

Via the set of points, $\{z_\alpha^k\}_{k \in \mathbf{Z}, \alpha \in \mathscr{A}_k}$, constructing random dyadic cubes (based on the system of dyadic cubes as in Lemma 4.1) and the spline functions, Auscher and Hytönen [2, Theorem 7.1] constructed the following system $\{\psi_\beta^k\}_{k \in \mathbf{Z}, \beta \in \mathscr{G}_k}$ of wavelet functions on $\mathcal{X}$, which is an orthonormal basis of $L^2(\mathcal{X})$ (see also [28, Theorem 2.6]).

Theorem 4.3. *There exist constants* $a \in (0, 1]$, $\eta \in (0, 1)$, C, $\nu \in (0, \infty)$ *and wavelet functions* $\{\psi_\beta^k\}_{k \in \mathbf{Z}, \beta \in \mathscr{G}_k}$ *satisfying*

(i) *for any* $k \in \mathbf{Z}$, $\beta \in \mathscr{G}_k$ *and* $x \in \mathcal{X}$,

$$\left|\psi_\beta^k(x)\right| \leq C \frac{1}{\sqrt{V(y_\beta^k, \delta^k)}} \exp\left(-\nu \left[\frac{d(y_\beta^k, x)}{\delta^k}\right]^a\right);$$

(ii) *for any* $k \in \mathbf{Z}$, $\beta \in \mathscr{G}_k$ *and* x, $y \in \mathcal{X}$ *such that* $d(x, y) \leq \delta^k$,

$$\left|\psi_\beta^k(x) - \psi_\beta^k(y)\right| \leq C \left[\frac{d(x, y)}{\delta^k}\right]^\eta \frac{1}{\sqrt{V(y_\beta^k, \delta^k)}} \exp\left(-\nu \left[\frac{d(y_\beta^k, x)}{\delta^k}\right]^a\right);$$

(iii) *for any* $k \in \mathbf{Z}$ *and* $\beta \in \mathscr{G}_k$,

$$\int_{\mathcal{X}} \psi_\beta^k(x) \, d\mu(x) = 0.$$

Moreover, the wavelets $\{\psi_\beta^k\}_{k \in \mathbf{Z}, \beta \in \mathscr{G}_k}$ *form an orthonormal basis of* $L^2(\mathcal{X})$ *and an unconditional basis of* $L^p(\mathcal{X})$ *with any given* $p \in (1, \infty)$.

Remark 4.4. (i) Let $\{\psi_\beta^k\}_{k \in \mathbf{Z}, \beta \in \mathscr{G}_k}$ be as in Theorem 4.3. For any $k \in \mathbf{Z}$ and x, $y \in \mathcal{X}$, letting

$$D_k(x, y) := \sum_{\beta \in \mathscr{G}_k} \psi_\beta^k(x) \psi_\beta^k(y),$$

it was shown in [28, p. 291] that the sequence $\{D_k\}_{k \in \mathbf{Z}}$ satisfies all conditions (i) through (v) of Definition 4.2.

(ii) The importance of the index η in Theorem 4.3 (and hence in Definition 4.2) lies not only in the regularity of the wavelet functions but also in the restriction of the index p for all results on Hardy spaces $H^p(\mathcal{X})$; see, for instance, [27, Theorems 3.5, 4.2, 5.10, 6.3 and 7.3].

The following homogeneous continuous Calderón reproducing formula was established in [28, Theorem 4.15].

Theorem 4.5. *Let $\{Q_k\}_{k\in\mathbf{Z}}$ be an exp-ATI and $\varrho, \vartheta \in (0, \eta)$. Then there exists a sequence $\{\widetilde{Q}_k\}_{k\in\mathbf{Z}}$ of bounded linear operators on $L^2(\mathcal{X})$ such that, for any f in $(\mathring{\mathcal{G}}_0^\eta(\varrho, \vartheta))'$ [resp., $L^p(\mathcal{X})$ with any given $p \in (1, \infty)$],*

$$f = \sum_{k=-\infty}^{\infty} \widetilde{Q}_k Q_k f,$$

where the series converges in $(\mathring{\mathcal{G}}_0^\eta(\varrho, \vartheta))'$ [resp., $L^p(\mathcal{X})$ with any given $p \in (1, \infty)$]. Moreover, there exists a positive constant C such that, for any $k \in \mathbf{Z}$, the kernel of $\widetilde{Q}_k$, which is still denoted by $\widetilde{Q}_k$, satisfies the following conditions:

(i) *for any $x, y \in \mathcal{X}$,*

$$\left|\widetilde{Q}_k(x, y)\right| \le C \frac{1}{V_{\delta^k}(x) + V(x, y)} \left[\frac{\delta^k}{\delta^k + d(x, y)}\right]^\vartheta;$$

(ii) *for any $x, \widetilde{x}, y \in \mathcal{X}$ with $d(x, \widetilde{x}) \le (2A_0)^{-1}[\delta^k + d(x, y)]$,*

$$\left|\widetilde{Q}_k(x, y) - \widetilde{Q}_k(\widetilde{x}, y)\right| \le C \left[\frac{d(x, \widetilde{x})}{\delta^k + d(x, y)}\right]^\varrho \frac{1}{V_{\delta^k}(x) + V(x, y)} \left[\frac{\delta^k}{\delta^k + d(x, y)}\right]^\vartheta;$$

(iii) *for any $x \in \mathcal{X}$,*

$$\int_{\mathcal{X}} \widetilde{Q}_k(x, y)\, d\mu(y) = 0 = \int_{\mathcal{X}} \widetilde{Q}_k(y, x)\, d\mu(y).$$

Next we recall the following notion of 1-exp-ATIs from [27, Definition 2.8].

Definition 4.6. A sequence $\{P_k\}_{k\in\mathbf{Z}}$ of bounded linear integral operators on $L^2(\mathcal{X})$ is called an *approximation of the identity with exponential decay and integration 1* (for short, 1-exp-ATI) if $\{P_k\}_{k\in\mathbf{Z}}$ has the following properties:

(i) for any $k \in \mathbf{Z}$, P_k satisfies (ii), (iii) and (iv) of Definition 4.2 but without the exponential decay factor

$$\exp\left\{-\nu\left[\frac{\max\{d(x, \mathcal{Y}^k), d(y, \mathcal{Y}^k)\}}{\delta^k}\right]^a\right\},$$

where $\mathcal{Y}^k$ is as in (4.1);

(ii) for any $k \in \mathbf{Z}$ and $x \in \mathcal{X}$, $\int_{\mathcal{X}} P_k(x, y)\, d\mu(y) = 1 = \int_{\mathcal{X}} P_k(y, x)\, d\mu(y)$;

(iii) for any $k \in \mathbf{Z}$, letting $Q_k := P_k - P_{k-1}$, then $\{Q_k\}_{k\in\mathbf{Z}}$ is an exp-ATI.

Remark 4.7. The existence of the 1-exp-ATI is guaranteed by [2, Lemma 10.1] (see also [27, Remark 2.9]).

4.2. Equivalences of Musielak–Orlicz Hardy spaces via various maximal functions. In this subsection, we show that $H^{+,\varphi}(\mathcal{X})$, $H_\theta^\varphi(\mathcal{X})$ and $H^{*,\varphi}(\mathcal{X})$ mutually coincide with equivalent quasi-norms.

We first introduce the notions of Musielak–Orlicz Hardy spaces defined via various maximal functions, which are generalizations of the corresponding spaces on Hardy spaces $H^p(\mathcal{X})$ in [27, Section 3] [in this case,

(4.2) $\varphi(x, t) := t^p$ for any $(x, t) \in \mathcal{X} \times (0, \infty)$ and any given $p \in (\omega/(\omega + \eta), 1]$

with ω as in (1.4) and η as in Definition 4.2].

Definition 4.8. Let ϱ, $\vartheta \in (0, \eta)$ and $f \in (\mathcal{G}_0^\eta(\varrho, \vartheta))'$ with η as in Definition 4.2. Let $\{P_k\}_{k \in \mathbf{Z}}$ be an 1-exp-ATI. The *radial maximal function* $\mathcal{M}^+(f)$ *of* f is defined by setting

$$\mathcal{M}^+(f)(x) := \sup_{k \in \mathbf{Z}} |P_k f(x)|, \quad \forall x \in \mathcal{X}.$$

The *non-tangential maximal function* $\mathcal{M}_\theta(f)$ *of* f, with *aperture* $\theta \in (0, \infty)$, is defined by setting

$$\mathcal{M}_\theta(f)(x) := \sup_{k \in \mathbf{Z}} \sup_{y \in B(x, \theta \delta^k)} |P_k f(y)|, \quad \forall x \in \mathcal{X}.$$

Moreover, the *grand maximal function* f^* of f is defined by setting, for any $x \in \mathcal{X}$,

$$(4.3) \quad f^*(x) := \sup \left\{ |\langle f, h \rangle| : h \in \mathcal{G}_0^\eta(\varrho, \vartheta), \ \|h\|_{\mathcal{G}(x, r, \varrho, \vartheta)} \leq 1 \text{ for some } r \in (0, \infty) \right\}.$$

Let φ be a growth function as in Definition 2.5. Then the *Musielak–Orlicz Hardy spaces*, $H^{+, \varphi}(\mathcal{X})$, $H_\theta^\varphi(\mathcal{X})$ [with $\theta \in (0, \infty)$] and $H^{*, \varphi}(\mathcal{X})$, are defined, respectively, by setting

$$H^{+, \varphi}(\mathcal{X}) := \left\{ f \in (\mathcal{G}_0^\eta(\varrho, \vartheta))' : \|f\|_{H^{+, \varphi}(\mathcal{X})} := \|\mathcal{M}^+(f)\|_{L^\varphi(\mathcal{X})} < \infty \right\},$$

$$H_\theta^\varphi(\mathcal{X}) := \left\{ f \in (\mathcal{G}_0^\eta(\varrho, \vartheta))' : \|f\|_{H_\theta^\varphi(\mathcal{X})} := \|\mathcal{M}_\theta(f)\|_{L^\varphi(\mathcal{X})} < \infty \right\}$$

and

$$H^{*, \varphi}(\mathcal{X}) := \left\{ f \in (\mathcal{G}_0^\eta(\varrho, \vartheta))' : \|f\|_{H^{*, \varphi}(\mathcal{X})} := \|f^*\|_{L^\varphi(\mathcal{X})} < \infty \right\}.$$

We first give the following property of $H^{*, \varphi}(\mathcal{X})$ on $\mathcal{X}$, which is a generalization of the Euclidean case in [35, Proposition 5.1].

Lemma 4.9. *Let* φ, ϱ *and* ϑ *be as in Definition 4.8 and* η *as in Definition 4.2. Then* $H^{*, \varphi}(\mathcal{X})$ *continuously embeds into* $(\mathcal{G}_0^\eta(\varrho, \vartheta))'$, *namely, there exists a positive constant* C *such that, for any* $f \in H^{*, \varphi}(\mathcal{X})$ *and* $h \in \mathcal{G}_0^\eta(\varrho, \vartheta)$,

$$|\langle f, h \rangle| \leq C \|h\|_{\mathcal{G}_0^\eta(\varrho, \vartheta)} \|f\|_{H^{*, \varphi}(\mathcal{X})}.$$

Proof. Let $h \in \mathcal{G}_0^\eta(\varrho, \vartheta)$. Then it is obvious that, for any $x \in B(x_1, 1)$,

$$\|h\|_{\mathcal{G}_0^\eta(x, 1, \varrho, \vartheta)} \lesssim \|h\|_{\mathcal{G}_0^\eta(x_1, 1, \varrho, \vartheta)} \sim \|h\|_{\mathcal{G}_0^\eta(\varrho, \vartheta)},$$

where $x_1 \in \mathcal{X}$ is as in Definition 2.1. Thus, for any $x \in B(x_1, 1)$,

$$|\langle f, h \rangle| \lesssim \|h\|_{\mathcal{G}_0^\eta(\varrho, \vartheta)} f^*(x),$$

which further implies that

$$|\langle f, h \rangle| \lesssim \|h\|_{\mathcal{G}_0^\eta(\varrho, \vartheta)} \inf_{x \in B(x_1, 1)} f^*(x) \lesssim \|h\|_{\mathcal{G}_0^\eta(\varrho, \vartheta)} \left\| \mathbf{1}_{B(x_1, 1)} \right\|_{L^\varphi(\mathcal{X})}^{-1} \|f\|_{H^{*, \varphi}(\mathcal{X})}.$$

This finishes the proof of Lemma 4.9. $\qquad\qquad\qquad\qquad\qquad\qquad \square$

Remark 4.10. (i) It was shown in [27, (3.1)] that, for any $f \in (\mathcal{G}_0^\eta(\varrho, \vartheta))'$ and $x \in \mathcal{X}$,

$$\mathcal{M}^+ f(x) \leq \mathcal{M}_\theta(f)(x) \lesssim f^*(x),$$

where the implicit positive constant only depends on θ.

(ii) Similarly to the Euclidean case in [35, Propsition 5.2], by Lemma 4.9, we can show that $H^{*, \varphi}(\mathcal{X})$ is complete; the details are omitted.

The following boundedness of the Hardy–Littlewood maximal operator M as in (2.2) on $L^\varphi(\mathcal{X})$ is similar to the Euclidean case in [41, Corollary 2.8] (see also [59, Corollary 2.1.2]); the details are omitted.

Theorem 4.11. *Assume that φ is a Musielak–Orlicz function of uniformly lower type p_φ^- and of uniformly upper type p_φ^+ with $q(\varphi) < p_\varphi^- \le p_\varphi^+ < \infty$, where $q(\varphi)$ is as in (3.16). Then the Hardy–Littlewood maximal function M is bounded on $L^\varphi(\mathcal{X})$ and, moreover, there exists a positive constant C such that, for any $f \in L^\varphi(\mathcal{X})$,*

$$\int_{\mathcal{X}} \varphi(x, M(f)(x))\, d\mu(x) \le C \int_{\mathcal{X}} \varphi(x, |f(x)|)\, d\mu(x).$$

Now, we state the main result of this section, which is a generalization of the corresponding results on Hardy spaces $H^p(\mathcal{X})$ in [27, Theorem 3.5] [in this case, φ is the same as in (4.2)].

Theorem 4.12. *Assume that φ is a growth function as in Definition 2.5 and $\theta \in (0, \infty)$. Then, for any $f \in (\mathcal{G}_0^\eta(\varrho, \vartheta))'$ with $\varrho, \vartheta \in (\omega[\frac{q(\varphi)}{p} - 1], \eta)$ with ω as in (1.4), $q(\varphi)$ as in (3.16), p as in Definition 2.5 and η as in Definition 4.2,*

$$\|f\|_{H^{+,\varphi}(\mathcal{X})} \sim \|f\|_{H_\theta^\varphi(\mathcal{X})} \sim \|f\|_{H^{*,\varphi}(\mathcal{X})},$$

with positive equivalence constants independent of f.

Proof. Let $f \in (\mathcal{G}_0^\eta(\varrho, \vartheta))'$ with $\varrho, \vartheta \in (\omega[\frac{q(\varphi)}{p} - 1], \eta)$. Fix $\theta \in (0, \infty)$. From Remark 4.10(i), it follows that

$$\left\|\mathcal{M}^+(f)\right\|_{L^\varphi(\mathcal{X})} \lesssim \left\|\mathcal{M}_\theta(f)\right\|_{L^\varphi(\mathcal{X})} \lesssim \|f^*\|_{L^\varphi(\mathcal{X})}.$$

Thus, to complete the proof of Theorem 4.12, it suffices to show that, for any $f \in (\mathcal{G}_0^\eta(\varrho, \vartheta))'$ with $\varrho, \vartheta \in (\omega[\frac{q(\varphi)}{p} - 1], \eta)$,

$$(4.4) \qquad \|f^*\|_{L^\varphi(\mathcal{X})} \lesssim \left\|\mathcal{M}^+(f)\right\|_{L^\varphi(\mathcal{X})}.$$

To show (4.4), we first recall the following estimate in [27, (3.5)] that, for any given $r \in (0, \frac{p}{q(\varphi)}) \subset (0, p)$ and, for any $f \in (\mathcal{G}_0^\eta(\varrho, \vartheta))'$ with $\varrho, \vartheta \in (\omega[\frac{q(\varphi)}{p} - 1], \eta)$ and any $x \in \mathcal{X}$,

$$f^*(x) \lesssim \mathcal{M}^+(f)(x) + \left\{M\left(\left[\mathcal{M}^+(f)\right]^r\right)(x)\right\}^{1/r}.$$

By this and the boundedness of M on $L^{\widetilde\varphi}(\mathcal{X})$ with $\widetilde\varphi(x, t) := \varphi(x, t^{1/r})$ for any $x \in \mathcal{X}$ and $t \in (0, \infty)$ (see Theorem 4.11), we conclude that

$$\|f^*\|_{L^\varphi(\mathcal{X})} \lesssim \left\|\mathcal{M}^+(f)\right\|_{L^\varphi(\mathcal{X})} + \left\|M\left(\left[\mathcal{M}^+(f)\right]^r\right)\right\|_{L^{\widetilde\varphi}(\mathcal{X})}^{1/r} \sim \left\|\mathcal{M}^+(f)\right\|_{L^\varphi(\mathcal{X})},$$

where we used the fact that $\widetilde\varphi$ is of uniformly lower type $p/r > q(\varphi)$ when $r \in (0, \frac{p}{q(\varphi)})$. This implies (4.4) and hence finishes the proof of Theorem 4.12. $\qquad\square$

5. Atomic characterizations of $H^{*,\varphi}(\mathcal{X})$

This section is devoted to the atomic characterizations of $H^{*,\varphi}(\mathcal{X})$, which is divided into three parts.

We need the following affiliated spaces, which are generalizations of the corresponding Euclidean case in [35, Definition 2.3] to any space of homogeneous type.

Definition 5.1. For any measurable set E in $\mathcal{X}$, the *space* $L_\varphi^q(E)$ for any given $q \in [1, \infty]$ is defined to be the set of all measurable functions f on $\mathcal{X}$, supported in E, such that

$$
\|f\|_{L_\varphi^q(E)} := \begin{cases} \sup\limits_{t \in (0,\infty)} \left[\dfrac{1}{\varphi(E,t)} \int_\mathcal{X} |f(x)|^q \varphi(x,t)\, d\mu(x) \right]^{1/q} < \infty & \text{when } q \in [1,\infty), \\ \|f\|_{L^\infty(\mathcal{X})} < \infty & \text{when } q = \infty. \end{cases}
$$

We then introduce the notions of (finite) atomic Musielak–Orlicz Hardy spaces on spaces of homogeneous type, which are generalizations of the corresponding Euclidean case in [35, Definition 2.4].

Definition 5.2. A couple (φ, q) is said to be *admissible* if $q \in (q(\varphi), \infty]$ and $m(\varphi) \leq 0$, where $q(\varphi)$ is as in (3.16) and $m(\varphi)$ as in (3.15). A measurable function a is called a (φ, q)-*atom supported in a ball* $B \subset \mathcal{X}$ if the following three conditions hold true:

   (i) supp $a := \{x \in \mathcal{X} : a(x) \neq 0\} \subset B$;
   (ii) $a \in L_\varphi^q(B)$ and $\|a\|_{L_\varphi^q(B)} \leq \|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1}$;
   (iii) $\int_\mathcal{X} a(x)\, d\mu(x) = 0$.

The *atomic Musielak–Orlicz Hardy space* $H_{\mathrm{at}}^{\varphi,q}(\mathcal{X})$ is defined to be the set of all distributions $f \in (\mathcal{G}_0^\eta(\varrho, \vartheta))'$ with

$$
(5.1) \qquad\qquad \varrho, \vartheta \in \left(\omega \left[\frac{q(\varphi)}{p} - 1 \right], \eta \right),
$$

here ω is as in (1.4), p as in Definition 2.5 and η as in Definition 4.2, satisfying that there exists a sequence $\{b_j\}_{j \in \mathbf{N}}$ of multiples of some (φ, q)-atoms supported, respectively, in balls $\{B_j\}_{j \in \mathbf{N}}$, such that

$$
\sum_{j=1}^\infty \varphi\left(B_j, \|b_j\|_{L_\varphi^q(B_j)} \right) < \infty,
$$

and $f = \sum_{j=1}^\infty b_j$ in $(\mathcal{G}_0^\epsilon(\varrho, \vartheta))'$. Moreover, let

$$
\Lambda_q\left(\{b_j\}_{j=1}^\infty \right) := \inf \left\{ \lambda \in (0, \infty) : \sum_{j=1}^\infty \varphi\left(B_j, \frac{\|b_j\|_{L_\varphi^q(B_j)}}{\lambda} \right) \leq 1 \right\}
$$

and

$$
\|f\|_{H_{\mathrm{at}}^{\varphi,q}(\mathcal{X})} := \inf \left\{ \Lambda_q\left(\{b_j\}_{j=1}^\infty \right) : f = \sum_{j=1}^\infty b_j \quad \text{in} \quad (\mathcal{G}_0^\eta(\varrho, \vartheta))' \right\},
$$

where the last infimum is taken over all decompositions of f as above.

The *finite atomic Musielak–Orlicz Hardy space* $H_{\mathrm{fin}}^{\varphi,q}(\mathcal{X})$ is defined to be the set of all finite linear combinations f of (φ, q)-atoms, namely, for some $N \in \mathbf{N}$,

$$
f = \sum_{j=1}^N b_j,
$$

where $\{b_j\}_{j=1}^N$ are multiples of some (φ, q)-atoms supported, respectively, in balls $\{B_j\}_{j=1}^N$. Moreover, define

$$\|f\|_{H_{\mathrm{fin}}^{\varphi,q}(\mathcal{X})} := \inf\left\{ \Lambda_q\left(\{b_j\}_{j=1}^N\right) \colon f = \sum_{j=1}^N b_j \right\},$$

where the infimum is taken over all finite decompositions of f as above.

Remark 5.3. It is obvious that $m(\varphi) \le 0$ is equivalent to $\omega q(\varphi) < (\omega + 1)i(\varphi)$. Thus, the real-variable theory of $H_{\mathrm{at}}^{\varphi,q}(\mathcal{X})$ for the case $\omega q(\varphi) \ge (\omega + 1)i(\varphi)$ [or, equivalently, $m(\varphi) > 0$] is failed on $\mathcal{X}$, because the (φ, q)-atoms only have 0-order cancelation.

Now, we are ready to state the main result of this section, which is a generalization of the corresponding results on Hardy spaces $H^p(\mathcal{X})$ in [27, Theorem 4.2] [in this case, φ is the same as in (4.2)].

Theorem 5.4. *Let φ be a growth function, $m(\varphi) \le 0$ with $m(\varphi)$ as in (3.15), $q \in (q(\varphi), \infty]$ and ϱ, ϑ as in (5.1) with $p > \frac{\omega}{\eta+\omega}q(\varphi)$. Then $H^{*,\varphi}(\mathcal{X}) = H_{\mathrm{at}}^{\varphi,q}(\mathcal{X})$ as subspaces of $(\mathcal{G}_0^\eta(\varrho, \vartheta))'$ with equivalent quasi-norms.*

Remark 5.5. Due to Theorem 5.4, $H_{\mathrm{at}}^{\varphi,q}(\mathcal{X})$ is independent of the choice of $q \in (q(\varphi), \infty]$ in the sense of equivalent quasi-norms. In what follows, $H_{\mathrm{at}}^{\varphi,q}(\mathcal{X})$ is simply denoted by $H_{\mathrm{at}}^\varphi(\mathcal{X})$.

5.1. Proof of $\boldsymbol{H_{\mathrm{at}}^{\varphi,q}(\mathcal{X}) \subset H^{*,\varphi}(\mathcal{X})}$. In this subsection, we show that $H_{\mathrm{at}}^{\varphi,q}(\mathcal{X}) \subset H^{*,\varphi}(\mathcal{X})$.

Proof of Theorem 5.4, Part I. In this step, we show $H_{\mathrm{at}}^{\varphi,q}(\mathcal{X}) \subset H^{*,\varphi}(\mathcal{X})$. Let b be a multiple of a (φ, q)-atom supported in a ball $B := B(x_B, r_B)$ with $x_B \in \mathcal{X}$ and $r_B \in (0, \infty)$. We first claim that

$$(5.2) \qquad \int_{\mathcal{X}} \varphi\left(x, b^*(x)\right) d\mu(x) \lesssim \varphi\left(B, \|b\|_{L_\varphi^q(B)}\right).$$

Indeed, let $\widetilde{B} := 2A_0 B$. When $x \in \widetilde{B}$, from the proof of [27, (4.1)], it follows that $b^*(x) \lesssim M(b)(x)$. By Lemma 2.6(iii) and the fact that $\varphi \in \mathbf{A}_q(\mathcal{X})$ for any $q \in (q(\varphi), \infty)$, we conclude that, for any $t \in (0, \infty)$,

$$\|b\|_{L^1(\mathcal{X})} \le \mu(B)\left\{ \frac{1}{\varphi(B,t)} \int_B |b(x)|^q \varphi(x,t)\, d\mu(x) \right\}^{1/q} \le \mu(B)\|b\|_{L_\varphi^q(B)},$$

which, combined with some arguments used in the proof of [27, (4.1)], further implies that, for any $h \in \mathcal{G}_0^\eta(\varrho, \vartheta)$ with $\varrho, \vartheta \in (\omega[\frac{q(\varphi)}{p} - 1], \eta)$, and $x \in (\widetilde{B})^\complement$,

$$|\langle b, h \rangle| \le \left[\frac{r_B}{d(x_B, x)}\right]^\varrho \frac{1}{V(x_B, x)}\|b\|_{L^1(\mathcal{X})} \le \left[\frac{r_B}{d(x_B, x)}\right]^\varrho \frac{\mu(B)}{V(x_B, x)}\|b\|_{L_\varphi^q(B)}.$$

Therefore, taking the supremum over all $h \in \mathcal{G}_0^\eta(\varrho, \vartheta)$ satisfying $\|h\|_{\mathcal{G}(x, r, \varrho, \vartheta)} \le 1$ for some $r \in (0, 1)$, we conclude that, for any $x \in \mathcal{X}$,

$$(5.3) \qquad b^*(x) \lesssim M(b)\mathbf{1}_{\widetilde{B}}(x) + \left[\frac{r_B}{d(x_B, x)}\right]^\varrho \frac{\mu(B)}{V(x_B, x)}\|b\|_{L_\varphi^q(B)}\mathbf{1}_{\widetilde{B}^\complement}(x).$$

Now, we use (5.3) to prove (5.2). To this end, let $q \in (q(\varphi), \infty]$. From the Hölder inequality and the uniformly upper type 1 property of φ, and (iv) and (v) of Lemma 2.6,

we deduce that

$$\int_{\widetilde{B}} \varphi\left(x, b^*(x)\right) d\mu(x)$$

$$\lesssim \int_{\widetilde{B}} \varphi\left(x, M(b)(x)\right) d\mu(x) \lesssim \int_{\widetilde{B}} \left[\frac{M(b)(x)}{\|b\|_{L_\varphi^q(B)}} + 1\right] \varphi\left(x, \|b\|_{L_\varphi^q(B)}\right) d\mu(x)$$

$$\lesssim \varphi\left(\widetilde{B}, \|b\|_{L_\varphi^q(B)}\right)$$

$$+ \frac{1}{\|b\|_{L_\varphi^q(B)}} \left\{\int_{\widetilde{B}} [M(b)(x)]^q \varphi\left(x, \|b\|_{L_\varphi^q(B)}\right) d\mu(x)\right\}^{\frac{1}{q}} \left[\varphi\left(\widetilde{B}, \|b\|_{L_\varphi^q(B)}\right)\right]^{\frac{q-1}{q}}$$

$$\lesssim \left[1 + \frac{1}{\|b\|_{L_\varphi^q(B)}} \|b\|_{L_\varphi^q(B)}\right] \varphi\left(\widetilde{B}, \|b\|_{L_\varphi^q(B)}\right) \lesssim \varphi\left(B, \|b\|_{L_\varphi^q(B)}\right).$$

On another hand, choose $q > q(\varphi)$ such that $\omega[\frac{q}{p} - 1] < \eta$ and $\varrho \in (\omega[\frac{q}{p} - 1], \eta)$. By (5.3), the uniformly lower type $p \in (0, 1]$ property of φ, Lemma 2.6(iv) and $\varrho \in (\omega[\frac{q}{p} - 1], \eta)$, we obtain

$$\int_{\widetilde{B}^\complement} \varphi\left(x, b^*(x)\right) d\mu(x)$$

$$\lesssim \int_{\widetilde{B}^\complement} \varphi\left(x, \left[\frac{r_B}{d(x_B, x)}\right]^\varrho \frac{\mu(B)}{V(x_B, x)} \|b\|_{L_\varphi^q(B)}\right) d\mu(x)$$

$$\lesssim \sum_{k=1}^\infty \int_{2^{k+1}A_0 B \backslash 2^k A_0 B} \varphi\left(x, 2^{-k\varrho} \frac{\mu(B)}{V(x_B, 2^k r_B)} \|b\|_{L_\varphi^q(B)}\right) d\mu(x)$$

$$\lesssim \sum_{k=1}^\infty 2^{-k\varrho p} \left[\frac{\mu(B)}{V(x_B, 2^k r_B)}\right]^p \int_{2^{k+1}A_0 B \backslash 2^k A_0 B} \varphi\left(x, \|b\|_{L_\varphi^q(B)}\right) d\mu(x)$$

$$\lesssim \sum_{k=1}^\infty 2^{-k\varrho p} \left[\frac{\mu(B)}{V(x_B, 2^k r_B)}\right]^p \varphi\left(2^{k+1}A_0 B, \|b\|_{L_\varphi^q(B)}\right)$$

$$\lesssim \sum_{k=1}^\infty 2^{-k\varrho p} \left[\frac{\mu(B)}{V(x_B, 2^k r_B)}\right]^p \left[\frac{\mu(2^{k+1}A_0 B)}{\mu(B)}\right]^q \varphi\left(B, \|b\|_{L_\varphi^q(B)}\right)$$

$$\lesssim \sum_{k=1}^\infty 2^{-k\varrho p} \left[\frac{\mu(2^k B)}{\mu(B)}\right]^{q-p} \varphi\left(B, \|b\|_{L_\varphi^q(B)}\right)$$

$$\lesssim \sum_{k=1}^\infty 2^{-k[\varrho p - \omega(q-p)]} \varphi\left(B, \|b\|_{L_\varphi^q(B)}\right) \lesssim \varphi\left(B, \|b\|_{L_\varphi^q(B)}\right).$$

Thus, we have

$$\int_{\mathcal{X}} \varphi\left(x, b^*(x)\right) d\mu(x) = \int_{\widetilde{B}} \varphi\left(x, b^*(x)\right) d\mu(x) + \int_{\widetilde{B}^\complement} \cdots \lesssim \varphi\left(B, \|b\|_{L_\varphi^q(B)}\right),$$

which completes the proof of the above claim (5.2).

Moreover, for any $f \in H_{\mathrm{at}}^{\varphi,q}(\mathcal{X})$, by Definition 5.2, we know that $f \in (\mathcal{G}_0^\eta(\varrho, \vartheta))'$, with $\varrho, \vartheta \in (\omega[\frac{q(\varphi)}{p} - 1], \eta)$, and there exists a sequence $\{b_j\}_{j=1}^\infty$ of multiples of (φ, q)-atoms such that

$$f = \sum_{j=1}^\infty b_j \quad \text{in } (\mathcal{G}_0^\eta(\varrho, \vartheta))' \quad \text{and} \quad \Lambda_q\left(\{b_j\}_{j=1}^\infty\right) \lesssim \|f\|_{H_{\mathrm{at}}^{\varphi,q}(\mathcal{X})}.$$

Thus, for any $x \in \mathcal{X}$, $f^*(x) \le \sum_{j=1}^\infty (b_j)^*(x)$. From this, Lemma 2.8(i) and the above claim (5.2), it follows that

$$\int_{\mathcal{X}} \varphi\left(x, \frac{f^*(x)}{\Lambda_q(\{b_j\}_{j=1}^\infty)}\right) d\mu(x) \lesssim \sum_{j=1}^\infty \int_{\mathcal{X}} \varphi\left(x, \frac{(b_j)^*(x)}{\Lambda_q(\{b_j\}_{j=1}^\infty)}\right) d\mu(x)$$

$$\lesssim \sum_{j=1}^\infty \varphi\left(B_j, \frac{\|b_j\|_{L_\varphi^q(B)}}{\Lambda_q(\{b_j\}_{j=1}^\infty)}\right) \lesssim 1,$$

which further implies that

$$\|f\|_{H^{*,\varphi}(\mathcal{X})} \sim \|f^*\|_{L^\varphi(\mathcal{X})} \lesssim \Lambda_q\left(\{b_j\}_{j=1}^\infty\right) \lesssim \|f\|_{H_{\mathrm{at}}^{\varphi,q}(\mathcal{X})}.$$

This finishes the proof of $H_{\mathrm{at}}^{\varphi,q}(\mathcal{X}) \subset H^{*,\varphi}(\mathcal{X})$ and hence of the Part I of the proof of Theorem 5.4. $\qquad\square$

5.2. Calderón–Zygmund decompositions of $H^{*,\varphi}(\mathcal{X})$. We obtain a version of Calderón–Zygmund decompositions for $H^{*,\varphi}(\mathcal{X})$ in this subsection.

Suppose that $f \in H_{\mathrm{at}}^{\varphi,q}(\mathcal{X})$. Then, by Definition 5.2, we know that $f \in (\mathcal{G}_0^\eta(\varrho, \vartheta))'$ with ϱ, ϑ as in (5.1). Since the level set $\{x \in \mathcal{X}: f^*(x) > \lambda\}$ with $\lambda \in (0, \infty)$ may not be open even in the case that d is a metric, to obtain the Calderón–Zygmund decomposition of f, we need to use some ideas from [49, Theorem 2] and [21, Remark 2.9] (see also [27, Section 4.2]).

From the proof of [49, Theorem 2], it follows that there exist a constant $\theta \in (0, 1)$ and a metric $d' \sim d^\theta$. For any $x \in \mathcal{X}$ and $r \in (0, \infty)$, define the d'-*ball* $B'(x, r) := \{y \in \mathcal{X}: d'(x, y) < r\}$. Then $(\mathcal{X}, d', \mu)$ is a doubling metric measure space. We also recall the following variant of the space of test functions associated with the metric d' from [27, Definition 4.6].

Definition 5.6. Let $x \in \mathcal{X}$, $r \in (0, \infty)$ and $\widetilde{\varrho}, \widetilde{\vartheta} \in (0, \infty)$. The *space of test functions, denoted by* $G(x, r, \widetilde{\varrho}, \widetilde{\vartheta})$, is defined to be the set of all functions f on $\mathcal{X}$ satisfying that there exists a non-negative constant C such that

(T1)′ for any $y \in \mathcal{X}$, $|f(y)| \le C \frac{1}{\mu(B'(y, r + d'(x, y)))} \left[\frac{r}{r + d'(x, y)}\right]^{\widetilde{\vartheta}}$;

(T2)′ for any $y, \widetilde{y} \in \mathcal{X}$ satisfying $d(y, \widetilde{y}) \le [r + d'(x, y)]/2$,

$$|f(y) - f(\widetilde{y})| \le C \left[\frac{d'(y, \widetilde{y})}{r + d'(x, y)}\right]^{\widetilde{\varrho}} \frac{1}{\mu(B'(y, r + d'(x, y)))} \left[\frac{r}{r + d'(x, y)}\right]^{\widetilde{\vartheta}}.$$

Moreover, for any $f \in G(x, r, \widetilde{\varrho}, \widetilde{\vartheta})$, define

$$\|f\|_{G(x, r, \widetilde{\varrho}, \widetilde{\vartheta})} := \inf\left\{C: C \text{ satisfies } (\mathrm{T1})' \text{ and } (\mathrm{T2})'\right\}.$$

It was shown in [27, Section 4.2] that $\mathcal{G}(x, r, \varrho, \vartheta) = G(x, r^\theta, \varrho/\theta, \vartheta/\theta)$ with equivalent norms. Moreover, for any $\varrho, \vartheta \in (0, \eta)$ and $f \in (\mathcal{G}_0^\eta(\varrho, \vartheta))'$, the *modified grand maximal function* $f^\star$ of f is defined by setting, for any $x \in \mathcal{X}$,

(5.4) $\quad f^\star(x) := \sup\left\{\langle f, h \rangle: h \in \mathcal{G}_0^\eta(\varrho, \vartheta), \|h\|_{G(x, r, \varrho/\theta, \vartheta/\theta)} \le 1 \text{ for some } r \in (0, \infty)\right\}.$

Then we know that $f^\star \sim f^*$ pointwisely on $\mathcal{X}$ with the positive equivalence constants independent of f, where f^* is as in Definition 4.8. For any $\lambda \in (0, \infty)$ and $j \in \mathbf{Z}$, define

$$(5.5) \qquad \Omega_\lambda := \{x \in \mathcal{X} : f^\star(x) > \lambda\} \quad \text{and} \quad \Omega^j := \Omega_{2^j}.$$

It was shown in [21, Remark 2.9(ii)] that, for any $\lambda \in (0, \infty)$, Ω_λ is *open* under the topology induced by d' and hence under the topology induced by d.

Suppose that $\varrho, \vartheta \in (\omega[\frac{q(\varphi)}{p} - 1], \eta)$ and $f \in H_{\mathrm{at}}^{\varphi,q}(\mathcal{X})$. Then $f^\star \in L^\varphi(\mathcal{X})$ and $\mu(\Omega_j) < \infty$ for any $j \in \mathbf{Z}$. Consequently, it was shown in [27, Section 4.2] that, for any $j \in \mathbf{Z}$, there exist $\{x_k^j\}_{k \in I_j}$ with I_j being a countable index set, $\{r_k^j\}_{k \in I_j} \subset (0, \infty)$, $L_0 \in \mathbf{N}$ and a sequence $\{\phi_k^j\}_{k \in I_j}$ of non-negative functions satisfying all conclusions of [27, Propositions 4.4 and 4.5]. For any $j \in \mathbf{Z}$ and $k \in I_j$, the operator Φ_k^j is defined by setting, for any $h \in \mathcal{G}_0^\eta(\varrho, \vartheta)$ and $x \in \mathcal{X}$,

$$\Phi_k^j(h)(x) := \phi_k^j(x) \left[\int_{\mathcal{X}} \phi_k^j(z)\, d\mu(z) \right]^{-1} \int_{\mathcal{X}} [h(x) - h(z)] \phi_k^j(z)\, d\mu(z).$$

By [21, Lemma 4.9], we know that Φ_k^j is bounded on $\mathcal{G}_0^\eta(\varrho, \vartheta)$ with operator norm depending on j and k. Therefore, it makes sense to define a distribution b_k^j on $\mathcal{G}_0^\eta(\varrho, \vartheta)$ by setting, for any $h \in \mathcal{G}_0^\eta(\varrho, \vartheta)$,

$$(5.6) \qquad \left\langle b_k^j, h \right\rangle := \left\langle f, \Phi_k^j(h) \right\rangle.$$

In what follows, for any $j \in \mathbf{Z}$ and $k \in I_j$, let $x_k^j \in \Omega^j$ and $r_k^j \in (0, \infty)$ be defined as in [27, Propositions 4.4 and 4.5] with the open set Ω therein replaced by Ω^j as in (5.5). The following estimate of $(b_k^j)^*$ is taken from [40, Lemma 3.7].

Proposition 5.7. *For any $j \in \mathbf{Z}$ and $k \in I_j$, let b_k^j be defined by (5.6). Then there exists a positive constant C such that, for any $j \in \mathbf{Z}$, $k \in I_j$ and $x \in \mathcal{X}$,*

$$\left(b_k^j\right)^*(x)$$
$$\leq C 2^j \frac{V(x_k^j, r_k^j)}{V(x_k^j, r_k^j) + V(x_k^j, x)} \left[\frac{r_k^j}{r_k^j + d(x_k^j, x)} \right]^\varrho \mathbf{1}_{(B(x_k^j, 16A_0 r_k^j))^\complement}(x) + C f^* \mathbf{1}_{B(x_k^j, 16A_0 r_k^j)}(x).$$

Then we obtain the following Calderón–Zygmund decomposition, which is a generalization of the corresponding results on Hardy spaces $H^p(\mathcal{X})$ in [27, Proposition 4.9] [in this case, φ is the same as in (4.2)].

Proposition 5.8. *Let φ be a growth function as in Definition 2.5. For any $f \in H^{*,\varphi}(\mathcal{X}) \subset (\mathcal{G}_0^\eta(\varrho, \vartheta))'$, with ϱ, ϑ as in (5.1), and, for any $j \in \mathbf{Z}$ and $k \in I_j$, letting b_k^j be defined as in (5.6), then there exists a positive constant C such that, for any $j \in \mathbf{Z}$ and $k \in I_j$,*

$$(5.7) \qquad \int_{\mathcal{X}} \varphi\left(x, \left(b_k^j\right)^*(x)\right) d\mu(x) \leq C \int_{B(x_k^j, 16A_0^4 r_k^j)} \varphi\left(x, f^*(x)\right) d\mu(x);$$

moreover, there exists $b^j \in H^{*,\varphi}(\mathcal{X})$ such that $b^j = \sum_{k \in I_j} b_k^j$ in $H^{*,\varphi}(\mathcal{X})$ and, for any $x \in \mathcal{X}$,

$$(5.8) \qquad \left(b^j\right)^*(x) \leq C 2^j \sum_{k \in I_j} \frac{V(x_k^j, r_k^j)}{V(x_k^j, r_k^j) + V(x_k^j, x)} \left[\frac{r_k^j}{r_k^j + d(x_k^j, x)} \right]^\varrho + C f^*(x) \mathbf{1}_{\Omega^j}(x);$$

if $g^j := f - b^j$ for any $j \in \mathbf{Z}$, then, for any $x \in \mathcal{X}$,

$$(5.9) \quad \left(g^j\right)^*(x) \le C2^j \sum_{k \in I_j} \frac{V(x_k^j, r_k^j)}{V(x_k^j, r_k^j) + V(x_k^j, x)} \left[\frac{r_k^j}{r_k^j + d(x_k^j, x)}\right]^\varrho + Cf^*(x)\mathbf{1}_{(\Omega^j)^\complement}(x).$$

Proof. Since (5.8) and (5.9) can be proved by the same arguments used in the proof of [27, (4.5) and (4.6)], we only need to show (5.7) and that there exists $b^j \in H^{*,\varphi}(\mathcal{X})$ such that $b^j = \sum_{k \in I_j} b_k^j$ in $H^{*,\varphi}(\mathcal{X})$. Indeed, by Proposition 5.7, we write

$$\int_{\mathcal{X}} \varphi\left(x, \left(b_k^j\right)^*(x)\right) d\mu(x)$$

$$\lesssim \int_{\mathcal{X}} \varphi\left(x, 2^j \frac{V(x_k^j, r_k^j)}{V(x_k^j, r_k^j) + V(x_k^j, x)} \left[\frac{r_k^j}{r_k^j + d(x_k^j, x)}\right]^\varrho \mathbf{1}_{(B(x_k^j, 16A_0 r_k^j))^\complement}(x)\right) d\mu(x)$$

$$+ \int_{\mathcal{X}} \varphi\left(x, f^*(x)\mathbf{1}_{B(x_k^j, 16A_0 r_k^j)}(x)\right) d\mu(x) =: \mathrm{I} + \mathrm{II}.$$

By [27, (ii) and (v) of Proposition 4.4], we have, for any $j \in \mathbf{Z}$,

$$(5.10) \quad \Omega^j = \bigcup_{k \in I_j} B\left(x_k^j, 16A_0 r_k^j\right) \text{ and } \left\{B\left(x_k^j, 16A_0 r_k^j\right)\right\}_{k \in I_j} \text{ is finitely overlapped}$$

with the finite positive integer independent of j and k. We immediately observe that

$$\mathrm{II} \sim \int_{B(x_k^j, 16A_0^4 r_k^j)} \varphi\left(x, f^*(x)\right) d\mu(x).$$

For the estimation of I, choose $q > q(\varphi)$ such that $\omega(\frac{q}{p} - 1) < \eta$ and $\varrho \in (\omega[\frac{q}{p} - 1], \eta)$. By $\varrho \in (\omega[\frac{q}{p} - 1], \eta)$ and $f^\star \sim f^*$, we conclude that

$$\mathrm{I} \lesssim \sum_{s=0}^\infty \int_{B(x_k^j, 2^{s+1}16A_0 r_k^j) \setminus B(x_k^j, 2^s 16A_0 r_k^j)} \varphi\left(x, 2^j \frac{V(x_k^j, r_k^j)}{V(x_k^j, 2^s 16A_0^4 r_k^j)} 2^{-s\varrho}\right) d\mu(x)$$

$$\lesssim \sum_{s=0}^\infty 2^{-s\varrho p} \left[\frac{V(x_k^j, r_k^j)}{V(x_k^j, 2^s 16A_0^4 r_k^j)}\right]^p \varphi\left(B(x_k^j, 2^{s+1}16A_0 r_k^j), 2^j\right)$$

$$\lesssim \sum_{s=0}^\infty 2^{-s\varrho p} \left[\frac{V(x_k^j, 2^s 16A_0^4 r_k^j)}{V(x_k^j, r_k^j)}\right]^{q-p} \varphi\left(B(x_k^j, 16A_0 r_k^j), 2^j\right)$$

$$\lesssim \sum_{s=0}^\infty 2^{-s\varrho p} 2^{s\omega(q-p)} \varphi\left(B(x_k^j, 16A_0 r_k^j), 2^j\right) d\mu(x) \lesssim \int_{B(x_k^j, 16A_0^4 r_k^j)} \varphi\left(x, f^*(x)\right) d\mu(x).$$

Combining the estimates of I and II, we then complete the proof of (5.7).

Moreover, from Lemma 2.8(i), (5.7) and (5.10), we deduce that

$$(5.11) \qquad \int_{\mathcal{X}} \varphi\left(x, \left(\sum_{k \in I_j} b_k^j\right)^*(x)\right) d\mu(x)$$

$$\lesssim \int_{\mathcal{X}} \varphi\left(x, \sum_{k \in I_j} \left(b_k^j\right)^*(x)\right) d\mu(x) \lesssim \sum_{k \in I_j} \int_{\mathcal{X}} \varphi\left(x, \left(b_k^j\right)^*(x)\right) d\mu(x)$$

$$\lesssim \sum_{k \in I_j} \int_{B(x_k^j, 16A_0^4 r_k^j)} \varphi(x, f^*(x)) \, d\mu(x) \lesssim \int_{\Omega^j} \varphi(x, f^*(x)) \, d\mu(x) < \infty,$$

which, together with the completeness of $H^{*,\varphi}(\mathcal{X})$, further implies that there exist $\{b^j\}_{j \in I_j} \subset H^{*,\varphi}(\mathcal{X})$ such that $b^j = \sum_{k \in I_j} b_k^j$ in $H^{*,\varphi}(\mathcal{X})$. This finishes the proof of Proposition 5.8. $\qquad\square$

5.3. Atomic characterizations of $H^{*,\varphi}(\mathcal{X})$. In this subsection, we prove that $H^{*,\varphi}(\mathcal{X}) \subset H_{\mathrm{at}}^{\varphi,q}(\mathcal{X})$, namely, the atomic decomposition of $H^{*,\varphi}(\mathcal{X})$. To this end, we first need the following result on the density of $L_{\varphi(\cdot,1)}^q(\mathcal{X}) \cap H^{*,\varphi}(\mathcal{X})$ in $H^{*,\varphi}(\mathcal{X})$, which is a variant of the corresponding Euclidean case in [35, Proposition 5.3].

Lemma 5.9. *For any* $q \in (q(\varphi), \infty)$, $L_{\varphi(\cdot,1)}^q(\mathcal{X}) \cap H^{*,\varphi}(\mathcal{X})$ *is dense in* $H^{*,\varphi}(\mathcal{X})$.

Proof. Let $f \in H^{*,\varphi}(\mathcal{X})$. By the Calderón–Zygmund decomposition of f^* (see Proposition 5.8 with the same notation as therein), we conclude that, for any $j \in \mathbf{Z}$,

$$f = g^j + b^j = g^j + \sum_{k \in I_j} b_k^j.$$

Now, we show that $g^j \in L_{\varphi(\cdot,1)}^q(\mathcal{X}) \cap H^{*,\varphi}(\mathcal{X})$.

Indeed, for any $j \in \mathbf{Z}$, $k \in I_j$ and any $x \in \mathcal{X}$,

$$\frac{V(x_k^j, r_k^j)}{V(x_k^j, r_k^j) + V(x_k^j, x)} \left[\frac{r_k^j}{r_k^j + d(x_k^j, x)} \right]^{\varrho}$$

$$\lesssim \frac{V(x_k^j, r_k^j)}{V(x_k^j, r_k^j + d(x_k^j, x))} \left[\frac{V(x_k^j, r_k^j)}{V(x_k^j, r_k^j + d(x_k^j, x))} \right]^{\frac{\varrho}{\omega}} \sim \left[\frac{V(x_k^j, r_k^j)}{V(x_k^j, r_k^j + d(x_k^j, x))} \right]^{1 + \frac{\varrho}{\omega}}$$

$$\sim \left\{ \frac{1}{V(x_k^j, r_k^j + d(x_k^j, x))} \int_{B(x_k^j, r_k^j + d(x_k^j, x))} \mathbf{1}_{B(x_k^j, r_k^j)}(y) \, d\mu(y) \right\}^{1 + \frac{\varrho}{\omega}}$$

$$\lesssim \left[M \left(\mathbf{1}_{B(x_k^j, r_k^j)} \right)(x) \right]^{1 + \frac{\varrho}{\omega}}.$$

In what follows, let $s_0 := 1 + \frac{\varrho}{\omega}$. From this, (5.9) and the Fefferman–Stein vector-valued maximal inequality (see, for instance, [22, Theorem 1.2]), we deduce that

$$\int_{\mathcal{X}} \left[(g^j)^*(x) \right]^q \varphi(x, 1) \, d\mu(x)$$

$$\lesssim \int_{\mathcal{X}} 2^{jq} \left\{ \sum_{k \in I_j} \frac{V(x_k^j, r_k^j)}{V(x_k^j, r_k^j) + V(x_k^j, x)} \left[\frac{r_k^j}{r_k^j + d(x_k^j, x)} \right]^{\varrho} \right\}^q \varphi(x, 1) \, d\mu(x)$$

$$+ \int_{(\Omega^j)^\complement} [f^*(x)]^q \varphi(x, 1) \, d\mu(x)$$

$$\lesssim 2^{jq} \int_{\mathcal{X}} \left\{ \left[\sum_{k \in I_j} \left\{ M \left(\mathbf{1}_{B(x_k^j, r_k^j)} \right)(x) \right\}^{s_0} \right]^{\frac{1}{s_0}} \right\}^{s_0 q} \varphi(x, 1) \, d\mu(x)$$

$$+ \int_{(\Omega^j)^\complement} [f^*(x)]^q \varphi(x, 1) \, d\mu(x)$$

$$\lesssim 2^{jq} \int_{\mathcal{X}} \left\{ \left[\sum_{k \in I_j} \left\{ \mathbf{1}_{B(x_k^j, r_k^j)}(x) \right\}^{s_0} \right]^{\frac{1}{s_0}} \right\}^{s_0 q} \varphi(x,1) \, d\mu(x) + \int_{(\Omega^j)^\complement} [f^*(x)]^q \, \varphi(x,1) \, d\mu(x)$$

$$\lesssim 2^{jq} \int_{\Omega^j} \varphi(x,1) \, d\mu(x) + \int_{(\Omega^j)^\complement} [f^*(x)]^q \, \varphi(x,1) \, d\mu(x)$$

$$\lesssim 2^{jq} \int_{\Omega^j} \varphi \left(x, \frac{f^*(x)}{2^j} \right) d\mu(x)$$

$$+ \int_{(\Omega^j)^\complement} [f^*(x)]^q \max \left\{ \frac{1}{[f^*(x)]^p}, \frac{1}{f^*(x)} \right\} \varphi \left(x, f^*(x) \right) d\mu(x)$$

$$\lesssim 2^{jq} \max \left\{ \frac{1}{2^{jp}}, \frac{1}{2} \right\} \int_{\Omega^j} \varphi \left(x, f^*(x) \right) d\mu(x)$$

$$+ \max \left\{ 2^{j(q-p)}, 2^{j(q-1)} \right\} \int_{(\Omega^j)^\complement} \varphi \left(x, f^*(x) \right) d\mu(x)$$

$$\sim 2^{jq} \max \left\{ \frac{1}{2^{jp}}, \frac{1}{2} \right\} \int_{\mathcal{X}} \varphi \left(x, f^*(x) \right) d\mu(x) < \infty.$$

Moreover, by (5.11), we know that

$$\int_{\mathcal{X}} \varphi \left(x, \left(g^j - f \right)^* (x) \right) d\mu(x) = \int_{\mathcal{X}} \varphi \left(x, \left(b^j \right)^* (x) \right) d\mu(x)$$

$$\lesssim \int_{\Omega^j} \varphi \left(x, f^*(x) \right) d\mu(x) \to 0$$

as $j \to \infty$. Therefore, by Lemma 2.8(iii), we finally obtain

$$(5.12) \qquad \lim_{j \to \infty} \left\| g^j - f \right\|_{H^{*,\varphi}(\mathcal{X})} = 0,$$

which completes the proof of Lemma 5.9. $\qquad\qquad\square$

Now, we are ready to prove $H^{*,\varphi}(\mathcal{X}) \subset H_{\mathrm{at}}^{\varphi,q}(\mathcal{X})$. In what follows, with all the same notation as in Subsection 5.2, for any $j \in \mathbf{Z}$ and $k \in I_j$, let

$$m_k^j := \frac{1}{\|\phi_k^j\|_{L^1(\mathcal{X})}} \int_{\mathcal{X}} f(y) \phi_k^j(y) \, d\mu(y) \quad \text{and} \quad b_k^j := \left(f - m_k^j \right) \phi_k^j.$$

For any $q \in [1, \infty]$, we also use the *symbol* $L_{\varphi(\cdot,1)}^q(\mathcal{X})$ to denote the weighted Lebesgue space equipped with the following norm: for any $f \in L_{\varphi(\cdot,1)}^q(\mathcal{X})$,

$$\|f\|_{L_{\varphi(\cdot,1)}^q(\mathcal{X})} := \left\{ \int_{\mathcal{X}} |f(x)|^q \varphi(x,1) \, d\mu(x) \right\}^{1/q}$$

with the usual modification made when $q = \infty$.

Proof of Theorem 5.4, Part II. In this step, we show $H^{*,\varphi}(\mathcal{X}) \subset H_{\mathrm{at}}^{\varphi,q}(\mathcal{X})$. By Lemma 5.9 and a standard density argument (see, for instance, [59, pp. 32–33]), we reduce the proof of this step to showing that, for any $f \in L_{\varphi(\cdot,1)}^q(\mathcal{X}) \cap H^{*,\varphi}(\mathcal{X})$, $f \in H_{\mathrm{at}}^{\varphi,q}(\mathcal{X})$ and

$$\|f\|_{H_{\mathrm{at}}^{\varphi,q}(\mathcal{X})} \lesssim \|f\|_{H^{*,\varphi}(\mathcal{X})}.$$

Indeed, let $f \in L^q_{\varphi(\cdot,1)}(\mathcal{X}) \cap H^{*,\varphi}(\mathcal{X})$. By the Calderón–Zygmund decomposition of f^* (see Proposition 5.8 with the same notation as therein), we conclude that, for any $j \in \mathbf{Z}$,

$$f = g^j + b^j = g^j + \sum_{k \in I_j} b^j_k.$$

For any $j \in \mathbf{N}$, let $h^j := g^{j+1} - g^j = b^j - b^{j+1}$. Then $f - \sum^m_{j=-m} h^j = b^{m+1} + g^{-m}$. From (5.12), it follows that

$$\lim_{j \to \infty} \left\| g^j - f \right\|_{H^{*,\varphi}(\mathcal{X})} = 0.$$

Therefore, $\|(b^{m+1})^*\|_{L^\varphi(\mathcal{X})} = \|g^{m+1} - f\|_{H^{*,\varphi}(\mathcal{X})} \to 0$ as $m \to \infty$. Observe that, by [27, Lemma 4.11(iv)], $g^j \to 0$ uniformly as $j \to -\infty$. Thus,

$$f = \sum_{j=-\infty}^{\infty} \left(g^{j+1} - g^j \right) \quad \text{in } (\mathcal{G}^\eta_0(\varrho, \vartheta))'.$$

Besides, from the proof of $H^{*,p}(\mathcal{X}) \subset H^{p,q}_{\mathrm{at}}(\mathcal{X})$ in [27], we deduce that

$$(5.13) \qquad h^j = b^j - b^{j+1} = \sum_{k \in I_j} \left[b^j_k - \sum_{l \in I_{j+1}} b^{j+1}_l \phi^j_k - L^{j+1}_{k,l} \phi^{j+1}_l \right] =: \sum_{k \in I_j} h^j_k$$

converges in $(\mathcal{G}^\eta_0(\varrho, \vartheta))'$ and almost everywhere on $\mathcal{X}$,

$$(5.14) \qquad \mathrm{supp}\, h^j_k \subset B^j_k := B\left(x^j_k, 16 A^4_0 r^j_k \right) \quad \text{and} \quad \left\| h^j_k \right\|_{L^\infty(\mathcal{X})} \le C_4 2^j,$$

where C_4 is some fixed positive constant. Let

$$(5.15) \qquad \lambda^j_k := C_4 2^j \left\| \mathbf{1}_{B^j_k} \right\|_{L^\varphi(\mathcal{X})} \quad \text{and} \quad a^j_k := \frac{h^j_k}{\lambda^j_k}.$$

Then a^j_k is a (φ, ∞)-atom with $\mathrm{supp}\, a^j_k \subset B^j_k$ and

$$f = \sum_{j=-\infty}^{\infty} \sum_{k \in I_j} h^j_k \quad \text{in } (\mathcal{G}^\eta_0(\varrho, \vartheta))'.$$

Furthermore, by (5.14) and Lemma 2.8(ii), we obtain

$$\sum_{j \in \mathbf{Z}} \sum_{k \in I_j} \varphi \left(B^j_k, \frac{\|h^j_k\|_{L^\infty(\mathcal{X})}}{\|f\|_{H^{*,\varphi}(\mathcal{X})}} \right)$$

$$\lesssim \sum_{j \in \mathbf{Z}} \varphi \left(\Omega^j, \frac{2^j}{\|f\|_{H^{*,\varphi}(\mathcal{X})}} \right) \sim \sum_{j \in \mathbf{Z}} \sum_{\ell=j}^{\infty} \varphi \left(\Omega^\ell \setminus \Omega^{\ell+1}, \frac{2^j}{\|f\|_{H^{*,\varphi}(\mathcal{X})}} \right)$$

$$\lesssim \sum_{\ell \in \mathbf{Z}} \sum_{j=-\infty}^{\ell} 2^{(j-\ell)p} \varphi \left(\Omega^\ell \setminus \Omega^{\ell+1}, \frac{2^\ell}{\|f\|_{H^{*,\varphi}(\mathcal{X})}} \right) \lesssim \sum_{\ell \in \mathbf{Z}} \varphi \left(\Omega^\ell \setminus \Omega^{\ell+1}, \frac{f^*}{\|f\|_{H^{*,\varphi}(\mathcal{X})}} \right)$$

$$\sim \int_{\mathcal{X}} \varphi \left(x, \frac{f^*(x)}{\|f\|_{H^{*,\varphi}(\mathcal{X})}} \right) d\mu(x) \sim 1.$$

Consequently, $\|f\|_{H^{\varphi,q}_{\mathrm{at}}(\mathcal{X})} \lesssim \|f\|_{H^{\varphi,\infty}_{\mathrm{at}}(\mathcal{X})} \lesssim \|f\|_{H^{*,\varphi}(\mathcal{X})}$, which completes the proof of $H^{*,\varphi}(\mathcal{X}) \subset H^{\varphi,q}_{\mathrm{at}}(\mathcal{X})$ and hence of Part II of the proof of Theorem 5.4. So far, the proof of Theorem 5.4 is then completed. $\qquad\square$

6. Littlewood–Paley characterizations of $H_{\mathrm{at}}^{\varphi}(\mathcal{X})$

In this section, we investigate the Littlewood–Paley characterizations of $H_{\mathrm{at}}^{\varphi}(\mathcal{X})$. We first recall the notions of the Lusin area function, the Littlewood–Paley g-function and the Littlewood–Paley g_{λ}^{*}-function from [27, Section 5]. Throughout the whole section, we assume that $q(\varphi) < \frac{p(\eta+\omega)}{\omega}$, which is sharp in the sense that, if $\varphi(x,t) = t^p$ for any $(x,t) \in \mathcal{X} \times [0,\infty)$, then $q(\varphi) = 1$ and hence $q(\varphi) < \frac{p(\eta+\omega)}{\omega}$ if and only if $p > \frac{\omega}{\eta+\omega}$, which returns to the classical case.

Definition 6.1. Let φ be a growth function as in Definition 2.5 and ϱ, ϑ as in (5.1), and let $f \in (\mathring{\mathcal{G}}_0^{\eta}(\varrho,\vartheta))'$ and $\{Q_k\}_{k\in\mathbf{Z}}$ be an exp-ATI.

(i) For any $\theta \in (0,\infty)$, the *Lusin area function of f with aperture θ*, $\mathcal{S}_{\theta}(f)$, is defined by setting, for any $x \in \mathcal{X}$,

$$(6.1) \qquad \mathcal{S}_{\theta}(f)(x) := \left[\sum_{k\in\mathbf{Z}} \int_{B(x,\theta\delta^k)} |Q_k f(y)|^2 \, \frac{d\mu(y)}{V_{\theta\delta^k}(x)} \right]^{\frac{1}{2}}.$$

Particularly, we simply write $\mathcal{S} := \mathcal{S}_1$.

(ii) The *Littlewood–Paley g-function of f*, $g(f)$, is defined by setting, for any $x \in \mathcal{X}$,

$$(6.2) \qquad g(f)(x) := \left[\sum_{k\in\mathbf{Z}} |Q_k f(x)|^2 \right]^{\frac{1}{2}}.$$

(iii) Let $\lambda \in (0,\infty)$. The *Littlewood–Paley g_{λ}^{*}-function of f*, $g_{\lambda}^{*}(f)$, is defined by setting, for any $x \in \mathcal{X}$,

$$(6.3) \qquad g_{\lambda}^{*}(f)(x) := \left\{ \sum_{k\in\mathbf{Z}} \int_{\mathcal{X}} |Q_k f(y)|^2 \left[\frac{\delta^k}{\delta^k + d(x,y)} \right]^{\lambda} \frac{d\mu(y)}{V_{\delta^k}(x) + V_{\delta^k}(y)} \right\}^{\frac{1}{2}}.$$

Then we introduce the notion of Musielak–Orlicz Hardy spaces via Lusin area functions as follows.

Definition 6.2. Let φ be a growth function as in Definition 2.5 and ϱ, ϑ as in (5.1). The *Musielak–Orlicz Hardy space $H^{\varphi}(\mathcal{X})$*, related to the Lusin area function, is defined by setting

$$H^{\varphi}(\mathcal{X}) := \left\{ f \in \left(\mathring{\mathcal{G}}_0^{\eta}(\varrho,\vartheta) \right)' : \|\mathcal{S}(f)\|_{L^{\varphi}(\mathcal{X})} < \infty \right\}$$

and, moreover, for any $f \in H^{\varphi}(\mathcal{X})$, let

$$\|f\|_{H^{\varphi}(\mathcal{X})} := \|\mathcal{S}(f)\|_{L^{\varphi}(\mathcal{X})}.$$

6.1. Independence of exp-ATIs. In this section, we prove that $H^{\varphi}(\mathcal{X})$ is independent of the choices of exp-ATIs. Let $\mathcal{E} := \{E_k\}_{k\in\mathbf{Z}}$ and $\mathcal{Q} := \{Q_k\}_{k\in\mathbf{Z}}$ be two exp-ATIs, we then let $\mathcal{S}_{\mathcal{E}}$ and $\mathcal{S}_{\mathcal{Q}}$ be the Lusin area functions associated with $\mathcal{E}$ and $\mathcal{Q}$, respectively.

Theorem 6.3. *Let φ be a growth function as in Definition 2.5, ϱ, $\vartheta \in (\omega[\frac{q(\varphi)}{p} - 1], \eta)$, $\mathcal{E} := \{E_k\}_{k\in\mathbf{Z}}$ and $\mathcal{Q} := \{Q_k\}_{k\in\mathbf{Z}}$ be two exp-ATIs. Then there exists a positive constant C such that, for any $f \in (\mathring{\mathcal{G}}_0^{\eta}(\varrho,\vartheta))'$,*

$$C^{-1} \|\mathcal{S}_{\mathcal{Q}}(f)\|_{L^{\varphi}(\mathcal{X})} \leq \|\mathcal{S}_{\mathcal{E}}(f)\|_{L^{\varphi}(\mathcal{X})} \leq C \|\mathcal{S}_{\mathcal{Q}}(f)\|_{L^{\varphi}(\mathcal{X})}.$$

In order to prove Theorem 6.3, we need to first establish the Fefferman–Stein vector-valued maximal inequality for $L^\varphi(\mathcal{X})$. We first state the following vector-valued interpolation theorem of Musielak–Orlicz type, which is similar to the corresponding Euclidean case in [41, Theorem 2.9]; the details are omitted.

Recall that the *space $L^\varphi(\ell^r, \mathcal{X})$* is defined to be the set of all measurable functions $\{f_j\}_j$ such that

$$\left[\sum_j |f_j|^r\right]^{1/r} \in L^\varphi(\mathcal{X}),$$

equipped with the (quasi-)norm

$$\left\| \{f_j\}_j \right\|_{L^\varphi(\ell^r, \mathcal{X})} := \left\| \left[\sum_j |f_j|^r\right]^{1/r} \right\|_{L^\varphi(\mathcal{X})}.$$

Theorem 6.4. *Let $p_1, p_2 \in (0, \infty)$, $p_1 < p_2$, $r \in [1, \infty)$ and φ be a Musielak–Orlicz function of uniformly lower type p_φ^- and of uniformly upper type p_φ^+. Suppose that $0 < p_1 < p_\varphi^- < p_\varphi^+ < p_2 < \infty$ and T is a linear operator defined on $L^{p_1}_{\varphi(\cdot,1)}(\mathcal{X}) + L^{p_2}_{\varphi(\cdot,1)}(\mathcal{X})$ satisfying that there exist positive constants C_1 and C_2 such that, for any $i \in \{1, 2\}$ and any sequence $\{f_j\}_j \in L^{p_i}(\ell^r, \mathcal{X})$, $\alpha \in (0, \infty)$ and $t \in (0, \infty)$,*

$$(6.4) \qquad \varphi\left(\left\{x \in \mathcal{X} \colon \left[\sum_j |T(f_j)(x)|^r\right]^{\frac{1}{r}} > \alpha\right\}, t\right)$$

$$\le C_i \alpha^{-p_i} \int_\mathcal{X} \left[\sum_j |f_j(x)|^r\right]^{\frac{p_i}{r}} \varphi(x, t)\, d\mu(x).$$

Then there exists a positive constant C such that, for any sequence $\{f_j\}_j \in L^\varphi(\ell^r, \mathcal{X})$,

$$(6.5) \quad \int_\mathcal{X} \varphi\left(x, \left[\sum_j |T(f_j)(x)|^r\right]^{\frac{1}{r}}\right) d\mu(x) \le C \int_\mathcal{X} \varphi\left(x, \left[\sum_j |f_j(x)|^r\right]^{\frac{1}{r}}\right) d\mu(x).$$

By the Calderón–Zygmund decomposition (see Lemma 3.3) and some arguments similar to those used in the proof of the Euclidean case in [1, Theorme 3.1], we obtain the corresponding result on $\mathcal{X}$ and omit the details here.

Theorem 6.5. *Let $r \in (1, \infty]$.*

(i) *If $p \in [1, \infty)$, then there exists a positive constant $C_{(r,p)}$, depending on r and p, such that $w \in A_p(\mathcal{X})$ if and only if*

$$\omega\left(\left\{x \in \mathcal{X} \colon \left[\sum_j \{M(f_j)(x)\}^r\right]^{\frac{1}{r}} > \alpha\right\}\right) \le C_{(r,p)} t^{-p} \int_\mathcal{X} \left[\sum_j |f_j(x)|^r\right]^{\frac{p}{r}} \omega(x)\, d\mu(x).$$

(ii) *If $p \in (1, \infty)$, then there exists a positive constant $C_{(r,p)}$, depending on r and p, such that $w \in A_p(\mathcal{X})$ if and only if*

$$\int_\mathcal{X} \left[\sum_j \{M(f_j)(x)\}^r\right]^{\frac{p}{r}} \omega(x)\, d\mu(x) \le C_{(r,p)} \int_\mathcal{X} \left[\sum_j |f_j(x)|^r\right]^{\frac{p}{r}} \omega(x)\, d\mu(x).$$

For more study on Muckenhoupt weights on spaces of homogeneous type, we refer the reader to [58, Chapter 1].

Combining Theorems 6.4 and 6.5, we obtain the following vector-valued Fefferman–Stein maximal inequality on spaces of homogeneous type, which is a generalization of the corresponding Euclidean case in [41, Theorem 2.10]; the details are omitted.

Theorem 6.6. *Let* $r \in (1, \infty]$, φ *be a Musielak–Orlicz function of uniformly lower type* p_φ^- *and of uniformly upper type* p_φ^+, *and* $\varphi \in \mathbf{A}_q(\mathcal{X})$. *If* $q(\varphi) < p_\varphi^- < p_\varphi^+ < \infty$, *then there exists a positive constant* C *such that, for any* $\{f_j\}_{j\in\mathbf{Z}} \in L^\varphi(\ell^r, \mathcal{X})$,

$$(6.6) \quad \int_\mathcal{X} \varphi\left(x, \left[\sum_j \{M(f_j)(x)\}^r\right]^{\frac{1}{r}}\right) d\mu(x) \leq C \int_\mathcal{X} \varphi\left(x, \left[\sum_j |f_j(x)|^r\right]^{\frac{1}{r}}\right) d\mu(x).$$

Now, we are ready to show Theorem 6.3.

Proof of Theorem 6.3. By symmetry, we only need to show $\|\mathcal{S}_\mathcal{E}(f)\|_{L^\varphi(\mathcal{X})} \lesssim \|\mathcal{S}_\mathcal{Q}(f)\|_{L^\varphi(\mathcal{X})}$. For any $k \in \mathbf{Z}$, $f \in (\mathring{\mathcal{G}}_0^\eta(\varrho, \vartheta))'$ with ϱ and ϑ as in this theorem, and $x \in \mathcal{X}$, let

$$m_k(f)(x) := \left[\frac{1}{V_{\delta^k}(x)} \int_{B(x,\delta^k)} |Q_k f(y)|^2 \, d\mu(y)\right]^{\frac{1}{2}}.$$

Choose $\widetilde{\varrho}$ and r such that $r \in (\omega/[\omega + \widetilde{\varrho}], p/q(\varphi))$. Then, by the proof of [27, Theorem 5.1], we conclude that, for any $x \in \mathcal{X}$,

$$[\mathcal{S}_\mathcal{E}(f)(x)]^2 \lesssim \sum_{k\in\mathbf{Z}} \{\mathcal{M}([m_k(f)]^r)(x)\}^{\frac{2}{r}},$$

which, combined with Theorem 6.6, further implies that

$$(6.7) \quad \|\mathcal{S}_\mathcal{E}(f)\|_{L^\varphi(\mathcal{X})} \lesssim \left\|\left[\sum_{k\in\mathbf{Z}} \{\mathcal{M}([m_k(f)]^r)\}^{\frac{2}{r}}\right]^{\frac{r}{2}}\right\|_{L^{\widetilde{\varphi}}(\mathcal{X})}^{\frac{1}{r}}$$

$$\lesssim \left\|\left\{\sum_{k\in\mathbf{Z}} [m_k(f)]^2\right\}^{\frac{1}{2}}\right\|_{L^\varphi(\mathcal{X})} \sim \|\mathcal{S}_\mathcal{Q}(f)\|_{L^\varphi(\mathcal{X})},$$

where, for any $x \in \mathcal{X}$ and $t \in (0, \infty)$, $\widetilde{\varphi}(x, t) := \varphi(x, t^{1/r})$ is of uniformly lower type $p/r \in (q(\varphi), \infty)$. This finishes the proof of Theorem 6.3. $\square$

6.2. Molecular characterizations of $H_{\mathrm{at}}^\varphi(\mathcal{X})$. In this subsection, we establish the molecular characterization of $H_{\mathrm{at}}^\varphi(\mathcal{X})$. To this end, we first introduce the following notion of molecular Musielak–Orlicz Hardy spaces on $\mathcal{X}$, which is a generalization of the corresponding Euclidean case in [30] (see also [59]).

Definition 6.7. Let φ be a growth function as in Definition 2.5, $q \in (1, \infty)$, $s \in \mathbf{Z}_+$ and $\epsilon \in (0, \infty)$. A measurable function α is called a (φ, q, ϵ)-*molecule*, related to a ball B, if the following two conditions hold true:

(i) for any $j \in \mathbf{Z}_+$,

$$\|\alpha\|_{L^q(U_j(B))} \leq \delta^{j\epsilon} \left[\mu\left(\delta^{-j}B\right)\right]^{1/q} \|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1},$$

where $U_0(B) := B$ and $U_j(B) := \delta^{-j}B \setminus \delta^{-j+1}B$ for any $j \in \mathbf{N}$;

(ii) $\int_\mathcal{X} \alpha(x) \, d\mu(x) = 0$.

The *molecular Musielak–Orlicz Hardy space* $H_{\mathrm{mol}}^{\varphi,q,\epsilon}(\mathcal{X})$ is defined to be the set of all $f \in (\mathcal{G}_0^\eta(\varrho,\vartheta))'$ with $\varrho, \vartheta \in (\omega[\frac{q(\varphi)}{p}-1],\eta)$ satisfying that there exist a sequence of (φ,q,ϵ)-molecules, $\{\alpha_j\}_{j\in\mathbf{N}}$, related, respectively, to balls $\{B_j\}_{j\in\mathbf{N}}$, and $\{\lambda_j\}_{j\in\mathbf{N}} \subset \mathbf{C}$ such that

$$\sum_{j=1}^\infty \varphi\left(B_j, \frac{|\lambda_j|}{\|\mathbf{1}_{B_j}\|_{L^\varphi(\mathcal{X})}}\right) < \infty$$

and $f = \sum_{j=1}^\infty \lambda_j\alpha_j$ in $(\mathcal{G}_0^\eta(\varrho,\vartheta))'$. Moreover, let

$$(6.8) \qquad \Lambda_q\left(\{\lambda_j\alpha_j\}_{j=1}^\infty\right) := \inf\left\{\lambda \in (0,\infty): \sum_{j=1}^\infty \varphi\left(B_j, \frac{|\lambda_j|}{\lambda\|\mathbf{1}_{B_j}\|_{L^\varphi(\mathcal{X})}}\right) \le 1\right\}$$

and then define

$$\|f\|_{H_{\mathrm{mol}}^{\varphi,q,\epsilon}(\mathcal{X})} := \inf\left\{\Lambda_q\left(\{\lambda_j\alpha_j\}_{j=1}^\infty\right) : f = \sum_{j=1}^\infty \lambda_j\alpha_j \quad \text{in} \quad (\mathring{\mathcal{G}}_0^\eta(\varrho,\vartheta))'\right\},$$

where the last infimum is taken over all decompositions of f as above.

The following molecular characterization of $H_{\mathrm{at}}^\varphi(\mathcal{X})$ is a generalization of the Euclidean case in [30, Theorem 4.13] to any space of homogeneous type (see also [59, Theorem 3.2.10]).

Theorem 6.8. *Let φ be a growth function as in Definition 2.5, $\epsilon \in (\max\{\omega, \omega\frac{q(\varphi)}{i(\varphi)}\},\infty)$, $q \in (q(\varphi)[r(\varphi)]',\infty)$ and $m(\varphi) \le 0$, where $q(\varphi)$, $i(\varphi)$ and $m(\varphi)$ are, respectively, as in (3.16), (2.1) and (3.15), and*

$$(6.9) \qquad\qquad r(\varphi) := \sup\{q \in (1,\infty]: \varphi \in \mathbf{RH}_q(\mathcal{X})\},$$

where $\frac{1}{r(\varphi)} + \frac{1}{[r(\varphi)]'} = 1$. Then $H_{\mathrm{mol}}^{\varphi,q,\epsilon}(\mathcal{X}) = H_{\mathrm{at}}^\varphi(\mathcal{X})$ with equivalent quasi-norms.

Proof. Let φ be a growth function, $\epsilon \in (\max\{\omega, \omega q(\varphi)/i(\varphi)\},\infty)$, $q \in (q(\varphi)[r(\varphi)]',\infty)$ and $m(\varphi) \le 0$. Then (φ,q) is admissible as in Definition 5.2.

First, we easily observe that any (φ,∞)-atom is a (φ,q,ϵ)-molecule. Thus, by Remark 5.5, we have

$$H_{\mathrm{at}}^\varphi(\mathcal{X}) = H_{\mathrm{at}}^{\varphi,\infty}(\mathcal{X}) \subset H_{\mathrm{mol}}^{\varphi,q,\epsilon}(\mathcal{X})$$

and, for any $f \in H_{\mathrm{at}}^\varphi(\mathcal{X})$,

$$\|f\|_{H_{\mathrm{mol}}^{\varphi,q,\epsilon}(\mathcal{X})} \lesssim \|f\|_{H_{\mathrm{at}}^{\varphi,\infty}(\mathcal{X})} \sim \|f\|_{H_{\mathrm{at}}^\varphi(\mathcal{X})}.$$

Now, we show that $H_{\mathrm{mol}}^{\varphi,q,\epsilon}(\mathcal{X}) \subset H_{\mathrm{at}}^{\varphi,q}(\mathcal{X})$. Indeed, let m be any fixed (φ,q,ϵ)-molecule related to a ball $B := B(x_B, r_B)$ with $x_B \in \mathcal{X}$ and $r_B \in (0,\infty)$. It suffices to prove that m is a countable linear combination of $(\varphi,\widetilde{q})$-atoms for some $\widetilde{q} \in (q(\varphi),\infty)$ which is determined later, and (φ,∞)-atoms.

To show this, write $R_0 := B$ and, for any $k \in \mathbf{N}$, $R_k := \delta^{-k}B \setminus \delta^{-k+1}B$, and then let

$$M_k := m\mathbf{1}_{R_k} - \frac{\mathbf{1}_{\delta^{-k}B}}{\mu(\delta^{-k}B)}\int_\mathcal{X} m(y)\mathbf{1}_{R_k}(y)\,d\mu(y)$$

and

$$m_k := \frac{\mathbf{1}_{\delta^{-k}B}}{\mu(\delta^{-k}B)}\int_\mathcal{X} m(y)\mathbf{1}_{R_k}(y)\,d\mu(x).$$

Then

$$(6.10) \qquad m = \sum_{k=0}^{\infty} M_k + \sum_{k=0}^{\infty} m_k.$$

We first deal with $\sum_{k=0}^{\infty} M_k$. Fix $k \in \mathbf{Z}_+$. Now, we claim that M_k is a multiple of a $(\varphi, \widetilde{q})$-atom for some $\widetilde{q} \in (q(\varphi), \infty)$ which is determined later. Indeed, it is obvious that $\int_{\mathcal{X}} M_k(y) \, d\mu(y) = 0$ and $\operatorname{supp} M_k \subset \delta^{-k}B$. By $\epsilon > \omega > \frac{\omega}{q}$ and the fact that

$$(6.11) \qquad \|M_k\|_{L^q(\mathcal{X})} \lesssim \|m \mathbf{1}_{R_k}\|_{L^q(\mathcal{X})} \lesssim \delta^{k\epsilon} \left[\mu \left(\delta^{-k}B \right) \right]^{1/q} \|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1},$$

we conclude that

$$
\begin{aligned}
\left\| \sum_{k=0}^{\infty} M_k \right\|_{L^q(\mathcal{X})}
&\leq \sum_{k=0}^{\infty} \|M_k\|_{L^q(\mathcal{X})} \lesssim \sum_{k=0}^{\infty} \delta^{k\epsilon} \left[\mu \left(\delta^{-k}B \right) \right]^{1/q} \|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1} \\
&\lesssim \sum_{k=0}^{\infty} \delta^{k\epsilon} \delta^{-k\frac{\omega}{q}} [\mu(B)]^{1/q} \|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1} \sim \sum_{k=0}^{\infty} \delta^{k(\epsilon - \frac{\omega}{q})} [\mu(B)]^{1/q} \|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1} \\
&\lesssim [\mu(B)]^{1/q} \|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1} < \infty,
\end{aligned}
$$

which further implies that $\sum_{k=0}^{\infty} M_k$ converges in $L^q(\mathcal{X})$ and hence in $(\mathcal{G}_0^\eta(\varrho, \vartheta))'$ with $\varrho, \vartheta \in (\omega[\frac{q(\varphi)}{p} - 1], \eta)$.

Since $q > q(\varphi)[r(\varphi)]'$, it follows that there exists $\widetilde{q} \in (q(\varphi), \infty)$ such that $q > \widetilde{q}[r(\varphi)]'$, which implies that $\varphi \in \mathbf{RH}_{(\frac{q}{\widetilde{q}})'}(\mathcal{X})$. By this, the Hölder inequality and (6.11), we find that, for any $t \in (0, \infty)$,

$$
\begin{aligned}
&\left\{ \frac{1}{\varphi(\delta^{-k}B, t)} \int_{\delta^{-k}B} [M_k(y)]^{\widetilde{q}} \, \varphi(y, t) \, d\mu(y) \right\}^{\frac{1}{\widetilde{q}}} \\
&\leq \frac{1}{[\varphi(\delta^{-k}B, t)]^{\frac{1}{\widetilde{q}}}} \|M_k\|_{L^q(\mathcal{X})} \left\{ \int_{\delta^{-k}B} [\varphi(y, t)]^{(\frac{q}{\widetilde{q}})'} \, d\mu(y) \right\}^{\frac{1}{\widetilde{q}(\frac{q}{\widetilde{q}})'}} \\
&\lesssim \frac{1}{[\varphi(\delta^{-k}B, t)]^{\frac{1}{\widetilde{q}}}} \delta^{k\epsilon} \left[\mu \left(\delta^{-k}B \right) \right]^{1/q} \|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1} \left[\mu \left(\delta^{-k}B \right) \right]^{(\frac{1}{(\frac{q}{\widetilde{q}})'} - 1)\frac{1}{\widetilde{q}}} [\varphi(\delta^{-k}B, t)]^{\frac{1}{\widetilde{q}}} \\
&\lesssim \delta^{k\epsilon} \|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1} =: \widetilde{C}_2 \delta^{k\epsilon} \|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1},
\end{aligned}
$$

where $\widetilde{C}_2$ is a fixed positive constant. Thus,

$$\|M_k\|_{L_\varphi^{\widetilde{q}}(\mathcal{X})} \leq \widetilde{C}_2 \delta^{k\epsilon} \|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1},$$

which further implies that $\frac{\delta^{-k\epsilon}}{\widetilde{C}_2} M_k \frac{\|\mathbf{1}_B\|}{\|\mathbf{1}_{\delta^{-k}B}\|_{L^\varphi(\mathcal{X})}}$ is a $(\varphi, \widetilde{q})$-atom. This shows the above claim.

Next, we consider $\sum_{k=0}^{\infty} m_k$. For a fixed $k \in \mathbf{Z}_+$, let

$$\mathbf{1}_k := \frac{\mathbf{1}_{\delta^{-k}B}}{\mu(\delta^{-k}B)} \quad \text{and} \quad \widetilde{m}_k := \int_{\mathcal{X}} m(y) \mathbf{1}_{R_k}(y) \, d\mu(y).$$

For any $k \in \mathbf{Z}_+$, let $N_k := \sum_{j=k}^{\infty} \widetilde{m}_j$. We observe that $\int_{\mathcal{X}} m(y)\, d\mu(y) = 0$ and hence $N_0 = 0$. Therefore,

$$
\begin{aligned}
(6.12) \qquad \sum_{k=0}^{\infty} m_k &= \sum_{k=0}^{\infty} \mathbf{1}_k \widetilde{m}_k = \sum_{k=0}^{\infty} \mathbf{1}_k \left(N_k - N_{k+1} \right) \\
&= \sum_{k=0}^{\infty} \left(\mathbf{1}_k - \mathbf{1}_{k-1} \right) N_k + N_0 = \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} \left(\mathbf{1}_k - \mathbf{1}_{k-1} \right) \widetilde{m}_j.
\end{aligned}
$$

For any $k \in \mathbf{Z}_+$ and $j \in \mathbf{Z}_+ \cap [k, \infty)$, let $b_{k,j} := (\mathbf{1}_k - \mathbf{1}_{k-1})\widetilde{m}_j$. We further prove that $b_{k,j}$ is a multiple of a (φ, ∞)-atom. Indeed, it is easy to find that $\operatorname{supp} b_{k,j} \subset \delta^{-k} B$ and $\int_{\mathcal{X}} b_{k,j}(y)\, d\mu(y) = 0$. Also, by the Hölder inequality and (1.4), we have

$$
\begin{aligned}
(6.13) \quad \|b_{k,j}\|_{L^\infty(\mathcal{X})} &\lesssim \frac{1}{\mu(\delta^{-k}B)} \left\| m \mathbf{1}_{R_j} \right\|_{L^1(\mathcal{X})} \lesssim \frac{1}{\mu(\delta^{-k}B)} \left\| m \mathbf{1}_{R_j} \right\|_{L^q(\mathcal{X})} \left[\mu\left(\delta^{-j}B \right) \right]^{\frac{1}{q'}} \\
&\lesssim \frac{1}{\mu(\delta^{-k}B)} \delta^{j\epsilon} \|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1} \left[\mu\left(\delta^{-j}B \right) \right]^{\frac{1}{q}} \left[\mu\left(\delta^{-j}B \right) \right]^{\frac{1}{q'}} \\
&\lesssim \delta^{j\epsilon} \delta^{-(j-k)\omega} \|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1} =: \widetilde{C_3} \delta^{j\epsilon} \delta^{-(j-k)\omega} \|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1},
\end{aligned}
$$

where $\widetilde{C_3}$ is a fixed positive constant.

By (6.13), we obtain

$$
\begin{aligned}
\left\| \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} b_{k,j} \right\|_{L^\infty(\mathcal{X})} &\leq \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} \|b_{k,j}\|_{L^\infty(\mathcal{X})} \lesssim \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} \delta^{j\epsilon} \delta^{-(j-k)\omega} \|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1} \\
&\lesssim \|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1} \sum_{k=0}^{\infty} \delta^{k\epsilon} \sum_{j=k}^{\infty} \delta^{(j-k)(\epsilon-\omega)} \lesssim \|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1} < \infty,
\end{aligned}
$$

which further implies that $\sum_{k=0}^{\infty} \sum_{j=k}^{\infty} b_{k,j}$ converges in $L^\infty(\mathcal{X})$ and hence in $(\mathring{\mathcal{G}}_0^\eta(\varrho, \vartheta))'$.

From (6.13) again, it follows that $\frac{\delta^{-j\epsilon}}{\widetilde{C_3}} \delta^{(j-k)\omega} \frac{\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}}{\|\mathbf{1}_{\delta^{-k}B}\|_{L^\varphi(\mathcal{X})}}$ is a (φ, ∞)-atom. Meanwhile, by $\epsilon \in (\omega \frac{q(\varphi)}{i(\varphi)}, \infty)$, we know that there exist $q_0 \in (q(\varphi), \infty)$ and $p_0 \in (0, i(\varphi))$ such that $\epsilon \in (\omega \frac{q_0}{p_0}, \infty)$, which also implies that $\varphi \in \mathbf{A}_{q_0}(\mathcal{X})$ and φ is of uniformly lower type p_0.

From the convergences of both $\sum_{k=0}^{\infty} M_k$ and $\sum_{k=0}^{\infty} \sum_{j=k}^{\infty} b_{k,j}$, (6.12) and (6.10), we deduce that

$$
m = \sum_{k=0}^{\infty} M_k + \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} b_{j,k}
$$

converges in $(\mathcal{G}_0^\eta(\varrho, \vartheta))'$. By these, the above claim and Lemma 2.6(iv), we conclude that, for any $\lambda \in (0, \infty)$,

$$
\begin{aligned}
(6.14) \quad & \sum_{k=0}^{\infty} \varphi \left(\delta^{-k}B, \lambda \|M_k\|_{L_\varphi^{\tilde{q}}(\delta^{-k}B)} \right) + \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} \varphi \left(\delta^{-k}B, \lambda \|b_{k,j}\|_{L^\infty(\mathcal{X})} \right) \\
& \lesssim \sum_{k=0}^{\infty} \varphi \left(\delta^{-k}B, \lambda \delta^{k\epsilon} \|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1} \right) + \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} \varphi \left(\delta^{-k}B, \lambda \delta^{-(j-k)\omega} \delta^{j\epsilon} \|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1} \right)
\end{aligned}
$$

$$\lesssim \sum_{k=0}^{\infty} \delta^{-k\omega q_0} \delta^{kp_0\epsilon} \varphi\left(B, \lambda\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1}\right) + \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} \delta^{-k\omega q_0} \delta^{-(j-k)\omega p_0} \delta^{jp_0\epsilon} \varphi\left(B, \lambda\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1}\right)$$

$$\sim \sum_{k=0}^{\infty} \delta^{kp_0(\epsilon - \omega\frac{q_0}{p_0})} \varphi\left(B, \lambda\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1}\right)$$

$$+ \sum_{k=0}^{\infty} \delta^{kp_0(\epsilon - \omega\frac{q_0}{p_0})} \sum_{j=k}^{\infty} \delta^{(j-k)(\epsilon-\omega)p_0} \varphi\left(B, \lambda\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1}\right)$$

$$\lesssim \varphi\left(B, \lambda\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1}\right).$$

Let $f \in H_{\mathrm{mol}}^{\varphi,q,\epsilon}(\mathcal{X})$. We know that there exist numbers $\{\lambda_j\}_{j=1}^{\infty} \subset \mathbf{C}$ and a sequence $\{m_j\}_{j=1}^{\infty}$ of (φ, q, ϵ)-molecules such that

$$f = \sum_{j=1}^{\infty} \lambda_j m_j \quad \text{in} \quad (\mathcal{G}_0^\eta(\varrho, \vartheta))'$$

and

(6.15) $$\|f\|_{H_{\mathrm{mol}}^{\varphi,q,\epsilon}(\mathcal{X})} \sim \Lambda_q(\{\lambda_j m_j\}_{j=1}^{\infty}).$$

Then, by the above arguments, we find that, for any $j \in \mathbf{N}$, there exist a sequence $\{a_{j,k}\}_{k\in\mathbf{N}}$ of (φ, q)-atoms and numbers $\{\mu_{j,k}\}_{k\in\mathbf{N}} \subset \mathbf{C}$ such that $m_j = \sum_{k=0}^{\infty} \mu_{j,k} a_{j,k}$ in $(\mathcal{G}_0^\eta(\varrho, \vartheta))'$. Thus, we have

$$f = \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} \lambda_j \mu_{j,k} a_{j,k} \quad \text{in} \quad \left(\mathring{\mathcal{G}}_0^\eta(\varrho, \vartheta)\right)',$$

which, together with Lemma 2.8(i), (6.14) and (6.15), further implies that

$$\|f\|_{H_{\mathrm{at}}^\varphi(\mathcal{X})} \lesssim \Lambda_q\left(\{\lambda_j \mu_{j,k} a_{j,k}\}_{j,k=1}^{\infty}\right) \lesssim \Lambda_q\left(\{\lambda_j m_j\}_{j=1}^{\infty}\right) \sim \|f\|_{H_{\mathrm{mol}}^{\varphi,q,\epsilon}(\mathcal{X})}.$$

This finishes the proof of $H_{\mathrm{mol}}^{\varphi,q,\epsilon}(\mathcal{X}) \subset H_{\mathrm{at}}^\varphi(\mathcal{X})$ and hence of Theorem 6.8. $\qquad\square$

6.3. Atomic characterizations of $H^\varphi(\mathcal{X})$. In this section, we aim to obtain the atomic characterizations of $H^\varphi(\mathcal{X})$.

The following notion of homogeneous atomic Musielak–Orlicz Hardy spaces is taken from [21].

Definition 6.9. (i) Let (φ, q) be admissible as in Definition 5.2 and $\varrho, \vartheta \in \left(\omega\left[\frac{q(\varphi)}{p} - 1\right], \eta\right)$. The *homogeneous atomic Musielak–Orlicz Hardy space* $\mathring{H}_{\mathrm{at}}^{\varphi,q}(\mathcal{X})$ is defined in the same way as $H_{\mathrm{at}}^{\varphi,q}(\mathcal{X})$ in Definition 5.2 with $(\mathcal{G}_0^\eta(\varrho, \vartheta))'$ replaced by $(\mathring{\mathcal{G}}_0^\eta(\varrho, \vartheta))'$.

(ii) Let $q \in (1, \infty)$, $s \in \mathbf{Z}_+$ and $\epsilon \in (0, \infty)$. The *homogeneous molecular Musielak–Orlicz Hardy space* $\mathring{H}_{\mathrm{mol}}^{\varphi,q,\epsilon}(\mathcal{X})$ is defined in the same way as $H_{\mathrm{mol}}^{\varphi,q,\epsilon}(\mathcal{X})$ in Definition 6.7 with $(\mathcal{G}_0^\eta(\varrho, \vartheta))'$ replaced by $(\mathring{\mathcal{G}}_0^\eta(\varrho, \vartheta))'$.

Then we introduce notions of adapted atomic Musielak–Orlicz Hardy spaces, which are generalized from the Euclidean case in [18] to any space of homogeneous type.

Definition 6.10. (I) Let (φ, q) be admissible as in Definition 5.2 and $\varrho, \vartheta \in \left(\omega\left[\frac{q(\varphi)}{p} - 1\right], \eta\right)$. A function $a \in L^q(\mathcal{X})$ is called an *adapted* $(\varphi, q)_A$-*atom*

supported in a ball B if Definition 5.2(iii) holds true and, instead of (i) and (ii) of Definition 5.2, a satisfies

(I)$_1$ supp $a \subset B$;

(I)$_2$ $\|a\|_{L^q(\mathcal{X})} \lesssim \dfrac{|B|^{1/q}}{\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}}$.

(II) The *adapted homogeneous atomic Musielak–Orlicz Hardy space* $\mathring{H}_{\mathrm{at,A}}^{\varphi,q}(\mathcal{X})$ is defined to be the set of all $f \in (\mathring{\mathcal{G}}_0^\eta(\varrho,\vartheta))'$ satisfying that there exist a sequence $\{a_j\}_{j\in\mathbf{N}}$ of some $(\varphi,q)_A$-atoms supported, respectively, in balls $\{B_j\}_{j\in\mathbf{N}}$ and $\{\lambda_j\}_{j\in\mathbf{N}} \subset \mathbf{C}$ such that

$$\sum_{j=1}^{\infty} \varphi\left(B_j, \frac{|\lambda_j|}{\|\mathbf{1}_{B_j}\|_{L^\varphi(\mathcal{X})}}\right) < \infty$$

and $f = \sum_{j=1}^{\infty} \lambda_j a_j$ in $(\mathring{\mathcal{G}}_0^\eta(\varrho,\vartheta))'$. Moreover, let

(6.16) $\quad \widetilde{\Lambda}_q\left(\{\lambda_j a_j\}_{j=1}^{\infty}\right) := \inf\left\{\lambda \in (0,\infty): \sum_{j=1}^{\infty} \varphi\left(B_j, \frac{|\lambda_j|}{\lambda\|\mathbf{1}_{B_j}\|_{L^\varphi(\mathcal{X})}}\right) \leq 1\right\}$

and then define

$$\|f\|_{\mathring{H}_{\mathrm{at,A}}^{\varphi,q}(\mathcal{X})} := \inf\left\{\widetilde{\Lambda}_q\left(\{\lambda_j a_j\}_{j=1}^{\infty}\right)\right\},$$

where the last infimum is taken over all decompositions of f as above.

(III) The *adapted atomic Musielak–Orlicz Hardy space* $H_{\mathrm{at,A}}^{\varphi,q}(\mathcal{X})$ is defined similar to $\mathring{H}_{\mathrm{at,A}}^{\varphi,q}(\mathcal{X})$ but with the distribution $(\mathring{\mathcal{G}}_0^\eta(\varrho,\vartheta))'$ replaced by $(\mathcal{G}_0^\eta(\varrho,\vartheta))'$.

Now, we show that $H_{\mathrm{at,A}}^{\varphi,q}(\mathcal{X})$ and $\mathring{H}_{\mathrm{at,A}}^{\varphi,q}(\mathcal{X})$ coincide, respectively, with $H_{\mathrm{at}}^{\varphi,q}(\mathcal{X})$ and $\mathring{H}_{\mathrm{at}}^{\varphi,q}(\mathcal{X})$, which are generalizations of the Euclidean case in [18, Theorem 2.12] to any space of homogeneous type.

Theorem 6.11. *Suppose that* (φ,q) *is admissible as in Definition 5.2, and* $\varrho,\vartheta \in (\omega[\frac{q(\varphi)}{p}-1],\eta)$ *with* $q(\varphi)$ *as in (3.16). Then*

(i) $H_{\mathrm{at,A}}^{\varphi,q}(\mathcal{X}) = H_{\mathrm{at}}^{\varphi,q}(\mathcal{X})$ *with equivalent quasi-norms;*

(ii) $\mathring{H}_{\mathrm{at,A}}^{\varphi,q}(\mathcal{X}) = \mathring{H}_{\mathrm{at}}^{\varphi,q}(\mathcal{X})$ *with equivalent quasi-norms.*

Proof. We only prove (i), because the proof of (ii) is similar. To show (i), we notice that any $(\varphi,\infty)_A$-atom in Definition 6.10 is the same as a (φ,∞)-atom in Definition 5.2 and is also a $(\varphi,q)_A$-atom in Definition 6.10. Thus, by Remark 5.5, we have

$$H_{\mathrm{at}}^{\varphi,q}(\mathcal{X}) = H_{\mathrm{at}}^{\varphi,\infty}(\mathcal{X}) = H_{\mathrm{at,A}}^{\varphi,\infty}(\mathcal{X}) \subset H_{\mathrm{at,A}}^{\varphi,q}(\mathcal{X}).$$

On another hand, observe that any $(\varphi,q)_A$-atom in Definition 6.10 is also a (φ,q,ϵ)-molecule in Definition 6.7, which, combined with Theorem 6.8, further implies that $H_{\mathrm{at,A}}^{\varphi,q}(\mathcal{X}) \subset H_{\mathrm{mol}}^{\varphi,q,\epsilon}(\mathcal{X}) = H_{\mathrm{at}}^{\varphi,q}(\mathcal{X})$. This finishes the proof of Theorem 6.11. $\qquad\square$

The following relationship between $H_{\mathrm{at}}^{\varphi,q}(\mathcal{X})$ and $\mathring{H}_{\mathrm{at}}^{\varphi,q}(\mathcal{X})$ is a generalization of the corresponding results on Hardy spaces $H^p(\mathcal{X})$ in [21, Theorem 5.4] [in this case, φ is the same as in (4.2)].

Proposition 6.12. *Suppose that* (φ,q) *is admissible as in Definition 5.2 and* $\varrho,\vartheta \in (\omega[\frac{q(\varphi)}{p}-1],\eta)$. *Then* $\mathring{H}_{\mathrm{at}}^{\varphi,q}(\mathcal{X}) = H_{\mathrm{at}}^{\varphi,q}(\mathcal{X})$ *with equivalent (quasi-)norms. More precisely, for any* $f \in H_{\mathrm{at}}^{\varphi,q}(\mathcal{X})$, *the restriction of* f *on* $(\mathring{\mathcal{G}}_0^\eta(\varrho,\vartheta))'$ *belongs to*

$\mathring{H}_{\mathrm{at}}^{\varphi,q}(\mathcal{X})$; Conversely, for any $f \in \mathring{H}_{\mathrm{at}}^{\varphi,q}(\mathcal{X})$, there exists a unique $\widetilde{f} \in H_{\mathrm{at}}^{\varphi,q}(\mathcal{X})$ such that $\widetilde{f} = f$ in $(\mathring{\mathcal{G}}_0^\eta(\varrho,\vartheta))'$.

Before proving Proposition 6.12, we need to establish the following auxiliary lemma.

Lemma 6.13. *There exists a positive constant C such that, for any sequence $\{\lambda_j\}_{j\in\mathbf{N}} \subset \mathbf{C}$ and for any sequence $\{a_j\}_{j\in\mathbf{N}}$ of (φ,q)-atoms supported, respectively, in balls $\{B_j\}_{j\in\mathbf{N}}$,*

$$\sum_{j=1}^\infty |\lambda_j| \le C\widetilde{\Lambda}_q\left(\{\lambda_j a_j\}_{j\in\mathbf{N}}\right),$$

where $\widetilde{\Lambda}_q$ is as in (6.16).

Proof. Let $\lambda := \sum_{j=1}^\infty |\lambda_j|$. By the uniformly upper-type 1 property of φ and Lemma 2.8(ii), we have

$$\sum_{j=1}^\infty \int_{B_j} \varphi\left(x, \frac{|\lambda_j|}{\lambda\|\mathbf{1}_{B_j}\|_{L^\varphi(\mathcal{X})}}\right) d\mu(x) \gtrsim \sum_{j=1}^\infty \frac{|\lambda_j|}{\lambda} \int_{B_j} \varphi\left(x, \frac{1}{\|\mathbf{1}_{B_j}\|_{L^\varphi(\mathcal{X})}}\right) d\mu(x)$$

$$\sim \sum_{j=1}^\infty \frac{|\lambda_j|}{\lambda} \sim 1,$$

which implies that

$$\sum_{j=1}^\infty |\lambda_j| \lesssim \widetilde{\Lambda}_q\left(\{\lambda_j a_j\}_{j\in\mathbf{N}}\right).$$

This finishes the proof of Lemma 6.13 □

We are now ready to prove Proposition 6.12.

Proof of Proposition 6.12. By $(\mathcal{G}_0^\eta(\varrho,\vartheta))' \subset (\mathring{\mathcal{G}}_0^\eta(\varrho,\vartheta))'$ with $\varrho, \vartheta \in (\omega[\frac{q(\varphi)}{p}-1], \eta)$, we easily obtain $H_{\mathrm{at}}^{\varphi,q}(\mathcal{X}) \subset \mathring{H}_{\mathrm{at}}^{\varphi,q}(\mathcal{X})$. Thus, it suffices to show that $\mathring{H}_{\mathrm{at}}^{\varphi,q}(\mathcal{X}) \subset H_{\mathrm{at}}^{\varphi,q}(\mathcal{X})$. Indeed, let $f \in \mathring{H}_{\mathrm{at}}^{\varphi,q}(\mathcal{X})$. Then, by Definition 6.9, we know that $f \in (\mathring{\mathcal{G}}_0^\eta(\varrho,\vartheta))'$ and there exist a sequence $\{a_j\}_{j\in\mathbf{N}}$ of (φ,q)-atoms supported, respectively, in balls $\{B_j\}_{j\in\mathbf{N}}$ as in Definition 5.2, and numbers $\{\lambda_j\}_{j\in\mathbf{N}} \subset \mathbf{C}$ such that

$$f = \sum_{j=1}^\infty \lambda_j a_j \quad \text{in } \left(\mathring{\mathcal{G}}_0^\eta(\varrho,\vartheta)\right)'$$

and

$$\widetilde{\Lambda}_q\left(\{\lambda_j a_j\}_{j\in\mathbf{N}}\right) \lesssim \|f\|_{\mathring{H}_{\mathrm{at,A}}^{\varphi,q}(\mathcal{X})},$$

which, combined with Theorem 6.11(ii), implies that

$$(6.17) \qquad\qquad \widetilde{\Lambda}_q\left(\{\lambda_j a_j\}_{j\in\mathbf{N}}\right) \lesssim \|f\|_{\mathring{H}_{\mathrm{at}}^{\varphi,q}(\mathcal{X})}.$$

Now, for any $h \in \mathcal{G}_0^\eta(\varrho,\vartheta)$, define

$$\left\langle \widetilde{f}, h \right\rangle := \sum_{j=1}^\infty \lambda_j \left\langle a_j, h \right\rangle.$$

From Lemmas 4.9 and 6.13, Theorem 5.4 and the fact that $\|a_j\|_{H^{\varphi,q}_{\mathrm{at}}(\mathcal{X})} \lesssim 1$ and (6.17), it follows that

$$\left|\left\langle \widetilde{f}, h \right\rangle\right| \leq \sum_{j=1}^{\infty} |\lambda_j| \, |\langle a_j, h \rangle| \lesssim \sum_{j=1}^{\infty} |\lambda_j| \, \|h\|_{\mathcal{G}^{\eta}_0(\varrho,\vartheta)} \|a_j\|_{H^{*,\varphi}(\mathcal{X})}$$

$$\sim \sum_{j=1}^{\infty} |\lambda_j| \, \|h\|_{\mathcal{G}^{\eta}_0(\varrho,\vartheta)} \|a_j\|_{H^{\varphi,q}_{\mathrm{at}}(\mathcal{X})} \lesssim \sum_{j=1}^{\infty} |\lambda_j| \, \|h\|_{\mathcal{G}^{\eta}_0(\varrho,\vartheta)}$$

$$\lesssim \|h\|_{\mathcal{G}^{\eta}_0(\varrho,\vartheta)} \widetilde{\Lambda}_q\left(\{\lambda_j a_j\}_{j\in\mathbf{N}}\right) \lesssim \|h\|_{\mathcal{G}^{\eta}_0(\varrho,\vartheta)} \|f\|_{\mathring{H}^{\varphi,q}_{\mathrm{at}}(\mathcal{X})}.$$

This shows that $\widetilde{f} \in (\mathcal{G}^{\eta}_0(\varrho,\vartheta))'$ and $\widetilde{f} = \sum_{j=1}^{\infty} \lambda_j a_j$ in $(\mathcal{G}^{\eta}_0(\varrho,\vartheta))'$. Moreover, $\widetilde{f} = f$ in $(\mathring{\mathcal{G}}^{\eta}_0(\varrho,\vartheta))'$, $\widetilde{f} \in H^{*,\varphi}(\mathcal{X})$ and

$$\left\|\widetilde{f}\right\|_{H^{\varphi,q}_{\mathrm{at}}(\mathcal{X})} \lesssim \|f\|_{\mathring{H}^{\varphi,q}_{\mathrm{at}}(\mathcal{X})}.$$

Then we are left to show the uniqueness of the extension $\widetilde{f}$ of f. Indeed, suppose that there exists another extension of f, say, $\widetilde{g} \in H^{\varphi,q}_{\mathrm{at}}(\mathcal{X})$. Then $\widetilde{g} = \widetilde{f}$ on $(\mathring{\mathcal{G}}^{\eta}_0(\varrho,\vartheta))'$, which, together with [21, Lemma 5.2], implies that $\widetilde{f} - \widetilde{g}$ is a constant, denoted by $\widetilde{C}$. If $\widetilde{C} \neq 0$, then it contradicts to the fact that no non-zero constant belongs to $H^{\varphi}_{\theta}(\mathcal{X}) = H^{\varphi,q}_{\mathrm{at}}(\mathcal{X})$. Therefore, $\widetilde{C} = 0$, which further implies that $\widetilde{f} \in H^{\varphi,q}_{\mathrm{at}}(\mathcal{X})$ is the unique extension of f. This finishes the proof of Proposition 6.12. $\qquad\square$

Remark 6.14. Let φ, q and ϵ be as in Theorem 6.8. From the proof of Theorem 6.8, it follows that $\mathring{H}^{\varphi,q,\epsilon}_{\mathrm{mol}}(\mathcal{X}) = \mathring{H}^{\varphi,q}_{\mathrm{at}}(\mathcal{X})$ with equivalent quasi-norms.

Now, we state the main result of this subsection, which is a generalization of the corresponding results on Hardy spaces $H^p(\mathcal{X})$ in [27, Theorem 5.9] [in this case, φ is the same as in (4.2)].

Theorem 6.15. *Assume that (φ, q) is admissible as in Definition 5.2 and $\varrho, \vartheta \in (\omega[\frac{q(\varphi)}{p} - 1], \eta)$ with $q(\varphi) \in [1, \frac{p(\eta+\omega)}{\omega})$ as in (3.16). Then, as subspaces of $(\mathring{\mathcal{G}}^{\eta}_0(\varrho,\vartheta))'$, $H^{\varphi}(\mathcal{X})$ and $\mathring{H}^{\varphi,q}_{\mathrm{at}}(\mathcal{X})$ coincide with equivalent (quasi-)norms.*

Proof. By Theorem 6.11(ii), we have $\mathring{H}^{\varphi,q}_{\mathrm{at}}(\mathcal{X}) = \mathring{H}^{\varphi,q}_{\mathrm{at,A}}(\mathcal{X})$ with equivalent (quasi-)norms. Thus, we first show that $\mathring{H}^{\varphi,q}_{\mathrm{at}}(\mathcal{X}) = \mathring{H}^{\varphi,q}_{\mathrm{at,A}}(\mathcal{X}) \subset H^{\varphi}(\mathcal{X})$. Fix $f \in \mathring{H}^{\varphi,q}_{\mathrm{at,A}}(\mathcal{X})$. Then, by Definition 6.10, we know that there exist sequences $\{\lambda_j\}_{j\in\mathbf{N}} \subset \mathbf{C}$ and $\{a_j\}_{j\in\mathbf{N}}$ of adapted $(\varphi, q)_A$-atoms supported, respectively, in balls $\{B_j\}_{j\in\mathbf{N}} := \{B(x_j, r_j)\}_{j\in\mathbf{N}}$ with $x_j \in \mathcal{X}$ and $r_j \in (0, \infty)$ for any $j \in \mathbf{N}$ as in Definition 6.10, such that $f = \sum_{j=1}^{\infty} \lambda_j a_j$ in $(\mathring{\mathcal{G}}^{\eta}_0(\varrho,\vartheta))'$ with $\varrho, \vartheta \in (\omega[\frac{q(\varphi)}{p} - 1], \eta)$, and

$$\widetilde{\Lambda}_q\left(\{\lambda_j a_j\}_{j\in\mathbf{N}}\right) \lesssim \|f\|_{\mathring{H}^{\varphi,q}_{\mathrm{at,A}}(\mathcal{X})}.$$

Let $\theta \in (0, \infty)$. We aim to show that

(6.18) $$\|S_\theta(f)\|_{L^\varphi(\mathcal{X})} \lesssim \max\left\{\theta^{-\frac{\omega}{2}}, \theta^{(1+q)\omega+\eta}\right\} \|f\|_{\mathring{H}^{\varphi,q}_{\mathrm{at,A}}(\mathcal{X})}.$$

To this end, by the proof of [27, Proposition 5.6], it suffices to prove that, for any $\theta \in [1, \infty)$,

$$\|S_\theta(f)\|_{L^\varphi(\mathcal{X})} \lesssim \theta^{(1+q)\omega+\eta} \|f\|_{\mathring{H}^{\varphi,q}_{\mathrm{at,A}}(\mathcal{X})}.$$

Indeed, we observe that

$$S_\theta(f) \le \sum_{j=1}^\infty S_\theta(\lambda_j a_j) \mathbf{1}_{4A_0^2\theta B_j} + \sum_{j=1}^\infty S_\theta(\lambda_j a_j) \mathbf{1}_{(4A_0^2\theta B_j)^\complement} =: \mathrm{I} + \mathrm{II}.$$

To estimate I, by some standard arguments, we reduce the proof to showing that, for any $(\varphi, q)_A$-atom a supported in a ball $B := B(x_B, r_B)$ with $x_B \in \mathcal{X}$ and $r_B \in (0, \infty)$, and $\lambda \in \mathbf{C}$,

$$\mathrm{A} := \int_{4A_0^2\theta B} \varphi(x, S_\theta(\lambda a)(x)) \, d\mu(x) \lesssim \theta^{(1+q)\omega+\eta} \varphi\left(B, \frac{|\lambda|}{\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}}\right).$$

To this end, since φ is of uniformly upper type 1 and of uniformly lower type p, it follows that

$$\mathrm{A} \lesssim \|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})} \int_{4A_0^2\theta B} \varphi\left(x, \frac{\lambda}{\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}}\right) S_\theta(a)(x) \, d\mu(x)$$

$$+ \|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^p \int_{4A_0^2\theta B} \varphi\left(x, \frac{\lambda}{\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}}\right) [S_\theta(a)(x)]^p \, d\mu(x) =: \mathrm{A}_1 + \mathrm{A}_2.$$

Now, we start to estimate A_1. Choose $q \in [2, \infty)$ such that $q' < r(\varphi)$ with $r(\varphi)$ as in (6.9). Thus, $\varphi \in \mathbf{RH}_{q'}(\mathcal{X})$. Moreover, from the boundedness of S_θ on $L^q(\mathcal{X})$ (see, for instance, the proof of [25, Proposition 2.17]), we deduce that

$$\|S_\theta(a)\|_{L^q(\mathcal{X})} \lesssim \theta^\omega \|a\|_{L^q(\mathcal{X})} \lesssim \theta^\omega \frac{[\mu(B_j)]^{1/q}}{\|\mathbf{1}_{B_j}\|_{L^\varphi(\mathcal{X})}},$$

which, combined with the Hölder inequality, $\varphi \in \mathbf{RH}_{q'}(\mathcal{X})$ and Lemma 2.6(iv), implies that

$$\mathrm{A}_1 \lesssim \|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})} \|S_\theta(a)\|_{L^q(\mathcal{X})} \left\{\int_{4A_0^2\theta B} \left[\varphi\left(x, \frac{|\lambda|}{\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}}\right)\right]^{q'} d\mu(x)\right\}^{1/q'}$$

$$\lesssim \theta^\omega \|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})} \frac{[\mu(B)]^{1/q}}{\|\mathbf{1}_{B_j}\|_{L^\varphi(\mathcal{X})}} [\mu(B)]^{-1/q} \varphi\left(4A_0^2\theta B, \frac{|\lambda|}{\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}}\right)$$

$$\lesssim \theta^{(1+q)\omega} \varphi\left(B, \frac{|\lambda|}{\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}}\right).$$

To deal with A_2, we first observe that $q/p \ge q$ and hence $q' \ge (q/p)'$, which, together with Lemma 2.6(ii), implies that $\mathbf{RH}_{q'}(\mathcal{X}) \subset \mathbf{RH}_{(q/p)'}(\mathcal{X})$. By this, the Hölder inequality and Lemma 2.6(iv), we conclude that

$$\mathrm{A}_2 \lesssim \|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^p \|S_\theta(a)\|_{L^q(\mathcal{X})}^p \left\{\int_{4A_0^2\theta B} \left[\varphi\left(x, \frac{|\lambda|}{\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}}\right)\right]^{(q/p)'} d\mu(x)\right\}^{1/(q/p)'}$$

$$\lesssim \theta^{\omega p} \|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^p \frac{[\mu(B)]^{p/q}}{\|\mathbf{1}_{B_j}\|_{L^\varphi(\mathcal{X})}^p} [\mu(B)]^{-p/q} \varphi\left(4A_0^2\theta B, \frac{|\lambda|}{\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}}\right)$$

$$\lesssim \theta^{(1+q)\omega} \varphi\left(B, \frac{|\lambda|}{\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}}\right),$$

which, combined with the estimate of A_1, further implies that

$$A \lesssim A_1 + A_2 \lesssim \theta^{(1+q)\omega} \varphi\left(B, \frac{|\lambda|}{\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}}\right).$$

Now, we turn to estimate II. From an argument similar to that used in the proof of [27, Proposition 5.6] and (1.4), we deduce that, for any $x \in (4A_0^2\theta B)^\complement$,

$$S_\theta(a)(x) \lesssim \theta^{\omega+\eta} \frac{\mu(B)}{\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}} \left[\frac{r}{d(x,x_B)}\right]^\eta \frac{1}{V(x,x_B)} \lesssim \theta^{\omega+\eta} \frac{1}{\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}} \left[\frac{\mu(B)}{V(x,x_B)}\right]^{\frac{\eta}{\omega}+1}$$

$$\lesssim \theta^{\omega+\eta} \frac{[M(\mathbf{1}_B)(x)]^{\frac{\eta}{\omega}+1}}{\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}}.$$

This, together with Theorem 6.6, further implies that

$$\|\mathrm{II}\|_{L^\varphi(\mathcal{X})} \lesssim \theta^{\omega+\eta} \left\|\sum_{j=1}^\infty |\lambda_j| \frac{[M(\mathbf{1}_{B_j})]^{\frac{\eta}{\omega}+1}}{\|\mathbf{1}_{B_j}\|_{L^\varphi(\mathcal{X})}}\right\|_{L^\varphi(\mathcal{X})}$$

$$\sim \theta^{\omega+\eta} \left\|\sum_{j=1}^\infty \left[M\left(|\lambda_j|^{\frac{\omega}{\omega+\eta}} \frac{\mathbf{1}_{B_j}}{\|\mathbf{1}_{B_j}\|_{L^\varphi(\mathcal{X})}^{\frac{\omega}{\omega+\eta}}}\right)\right]^{\frac{\eta}{\omega}+1}\right\|_{L^\varphi(\mathcal{X})}$$

$$\sim \theta^{\omega+\eta} \left\|\left\{\sum_{j=1}^\infty \left[M\left(|\lambda_j|^{\frac{\omega}{\omega+\eta}} \frac{\mathbf{1}_{B_j}}{\|\mathbf{1}_{B_j}\|_{L^\varphi(\mathcal{X})}^{\frac{\omega}{\omega+\eta}}}\right)\right]^{\frac{\eta}{\omega}+1}\right\}^{\frac{\omega}{\omega+\eta}}\right\|_{L^{\widetilde{\varphi}}(\mathcal{X})}^{\frac{\omega+\eta}{\omega}}$$

$$\lesssim \theta^{\omega+\eta} \left\|\left\{\sum_{j=1}^\infty |\lambda_j| \frac{\mathbf{1}_{B_j}}{\|\mathbf{1}_{B_j}\|_{L^\varphi(\mathcal{X})}}\right\}^{\frac{\omega}{\omega+\eta}}\right\|_{L^{\widetilde{\varphi}}(\mathcal{X})}^{\frac{\omega+\eta}{\omega}} \sim \theta^{\omega+\eta} \left\|\sum_{j=1}^\infty |\lambda_j| \frac{\mathbf{1}_{B_j}}{\|\mathbf{1}_{B_j}\|_{L^\varphi(\mathcal{X})}}\right\|_{L^\varphi(\mathcal{X})}$$

$$\lesssim \theta^{\omega+\eta} \widetilde{\Lambda}_q\left(\{\lambda_j a_j\}_{j\in\mathbf{N}}\right) \lesssim \theta^{\omega+\eta} \|f\|_{\mathring{H}^{\varphi,q}_{\mathrm{at},A}(\mathcal{X})}.$$

From the estimates of I and II, we deduce that

$$\|S_\theta(f)\|_{L^\varphi(\mathcal{X})} \lesssim \|\mathrm{I}\|_{L^\varphi(\mathcal{X})} + \|\mathrm{II}\|_{L^\varphi(\mathcal{X})} \lesssim \theta^{\omega+\eta} \|f\|_{\mathring{H}^{\varphi,q}_{\mathrm{at},A}(\mathcal{X})},$$

which implies (6.18) and hence shows $\mathring{H}^{\varphi,q}_{\mathrm{at}}(\mathcal{X}) \hookrightarrow H^\varphi(\mathcal{X})$. This is the desired conclusion.

Let q and ϵ be as in Theorem 6.8. Then we show that $H^\varphi(\mathcal{X}) \subset \mathring{H}^{\varphi,q,\epsilon}_{\mathrm{mol}}(\mathcal{X}) = \mathring{H}^{\varphi,q}_{\mathrm{at}}(\mathcal{X})$. Suppose that $f \in (\mathring{\mathcal{G}}_0^\eta(\varrho,\vartheta))'$ belongs to $H^\varphi(\mathcal{X})$. By Theorem 6.3, without loss of generality, we may choose the exp-ATI $\{D_k\}_{k\in\mathbf{Z}}$ as in Remark 4.4(i). Denote by $\mathcal{D}$ the set of all dyadic cubes. For any $k \in \mathbf{Z}$, define $\Omega_k := \{x \in \mathcal{X}: S(f)(x) > 2^k\}$ and

$$\mathcal{D}_k := \left\{Q \in \mathcal{D}: \mu(Q \cap \Omega_k) > \frac{1}{2}\mu(Q) \text{ and } \mu(Q \cap \Omega_{k+1}) \leq \frac{1}{2}\mu(Q)\right\}.$$

Obviously, for any $Q \in \mathcal{D}$, there exists a unique $k \in \mathbf{Z}$ such that $Q \in \mathcal{D}_k$. Denote by $\{Q^j_{\tau,k}\}_{\tau \in I_{j,k}}$ the set of all maximal cubes in $\mathcal{D}_k$ at level $j \in \mathbf{Z}$, where $I_{j,k} \subset \mathscr{A}_j$. The center of $Q^j_{\tau,k}$ is denoted by $z^j_{\tau,k}$. Then

$$(6.19) \qquad \mathcal{D} = \bigcup_{j,k\in\mathbf{Z}} \bigcup_{\tau\in I_{j,k}} \left\{Q \in \mathcal{D}_k: Q \subset Q^j_{\tau,k}\right\}.$$

Hereafter, we let $D_Q := D_\ell$ when $Q := Q_\beta^{\ell+1}$ for some $\ell \in \mathbf{Z}$ and $\beta \in \mathscr{A}_{\ell+1}$. We now show that

$$(6.20) \qquad f = \sum_{\ell \in \mathbf{Z}} D_\ell D_\ell f = \sum_{k \in \mathbf{Z}} \sum_{j \in \mathbf{Z}} \sum_{\tau \in I_{j,k}} \lambda_{\tau,k}^j b_{\tau,k}^j,$$

where all the equalities hold true in $(\mathring{\mathcal{G}}_0^\eta(\varrho, \vartheta))'$,

$$(6.21) \qquad \lambda_{\tau,k}^j := 2^k \|\mathbf{1}_{Q_{\tau,k}^j}\|_{L^\varphi(\mathcal{X})}$$

and

$$b_{\tau,k}^j(\cdot) := \frac{1}{\lambda_{\tau,k}^j} \sum_{\{Q \in \mathcal{D}_k, \ Q \subset Q_{\tau,k}^j\}} \int_Q D_Q(\cdot, y) D_Q f(y) \, d\mu(y).$$

Before proving (6.20), we first claim that $b_{\tau,k}^j$ is a positive harmless constant multiple of a (φ, q, ϵ_0)-molecule, where ϵ_0 is determined later, related to the ball $B_{\tau,k}^j := B(z_{\tau,k}^j, 4A_0^2 \delta^{j-1})$ as in Definition 6.7. Indeed, for any $Q \in \mathcal{D}_k$ and $Q \in Q_{\tau,k}^j$, assume that $Q := Q_\alpha^{\ell+1}$ for some $\ell \in \mathbf{Z}$ and $\alpha \in \mathscr{A}_{\ell+1}$. By the proof of [27, Proposition 5.7], we know that, for any $y \in Q$,

$$\mu\left(B\left(y, \delta^\ell\right) \cap [Q_{\tau,k}^j \setminus \Omega_{k+1}]\right) \gtrsim V_{\delta^\ell}(y).$$

From this, the Fubini–Tonelli theorem, $V_{\delta^\ell}(x) \sim V_{\delta^\ell}(y)$ for any $x, y \in \mathcal{X}$ with $d(x, y) < \delta^\ell$, and the Hölder inequality, we deduce that, for any $h \in L^{q'}(\mathcal{X})$ with $\|h\|_{L^{q'}(\mathcal{X})} \leq 1$,

$$(6.22) \quad \left|\int_{\mathcal{X}} b_{\tau,k}^j(x) h(x) \, d\mu(x)\right|$$

$$\leq \frac{1}{\lambda_{\tau,k}^j} \sum_{\{Q \in \mathcal{D}_k, \ Q \subset Q_{\tau,k}^j\}} \int_Q |D_Q f(y)| \left|D_Q^* h(y)\right| d\mu(y)$$

$$\lesssim \frac{1}{\lambda_{\tau,k}^j} \sum_{\{Q \in \mathcal{D}_k, \ Q \subset Q_{\tau,k}^j\}} \int_Q \frac{\mu(B(y, \delta^\ell) \cap [Q_{\tau,k}^j \setminus \Omega_{k+1}])}{V_{\delta^\ell}(y)} |D_Q f(y)| \left|D_Q^* h(y)\right| d\mu(y)$$

$$\sim \frac{1}{\lambda_{\tau,k}^j} \sum_{\ell=j-1}^\infty \sum_{\alpha \in \mathscr{A}_{\ell+1}} \sum_{\{Q_\alpha^{\ell+1} \in \mathcal{D}_k, \ Q_\alpha^{\ell+1} \subset Q_{\tau,k}^j\}} \int_{Q_\alpha^{\ell+1}} \frac{\mu(B(y, \delta^\ell) \cap [Q_{\tau,k}^j \setminus \Omega_{k+1}])}{V_{\delta^\ell}(y)}$$
$$\cdot |D_\ell f(y)| |D_\ell^* h(y)| \, d\mu(y)$$

$$\lesssim \frac{1}{\lambda_{\tau,k}^j} \sum_{\ell=j-1}^\infty \int_{Q_{\tau,k}^j} \frac{\mu(B(y, \delta^\ell) \cap [Q_{\tau,k}^j \setminus \Omega_{k+1}])}{V_{\delta^\ell}(y)} |D_\ell f(y)| |D_\ell^* h(y)| \, d\mu(y)$$

$$\sim \frac{1}{\lambda_{\tau,k}^j} \sum_{\ell=j-1}^\infty \int_{Q_{\tau,k}^j} \int_{\mathcal{X}} \mathbf{1}_{B(y, \delta^\ell)}(x) \mathbf{1}_{Q_{\tau,k}^j \setminus \Omega_{k+1}}(x) \frac{d\mu(x)}{V_{\delta^\ell}(x)} |D_\ell f(y)| |D_\ell^* h(y)| \, d\mu(y)$$

$$\sim \frac{1}{\lambda_{\tau,k}^j} \int_{Q_{\tau,k}^j \setminus \Omega_{k+1}} \sum_{\ell=j-1}^\infty \int_{Q_{\tau,k}^j \cap B(x, \delta^\ell)} |D_\ell f(y)| |D_\ell^* h(y)| \frac{d\mu(y)}{V_{\delta^\ell}(x)} \, d\mu(x)$$

$$\lesssim \frac{1}{\lambda_{\tau,k}^j} \int_{Q_{\tau,k}^j \setminus \Omega_{k+1}} \sum_{\ell=j-1}^{\infty} \left[\int_{B(x,\delta^\ell)} |D_\ell f(y)|^2 \frac{d\mu(y)}{V_{\delta^\ell}(x)} \right]^{1/2} \left[\int_{B(x,\delta^\ell)} |D_\ell^* h(y)|^2 \frac{d\mu(y)}{V_{\delta^\ell}(x)} \right]^{1/2} d\mu(x)$$

$$\lesssim \frac{1}{\lambda_{\tau,k}^j} \int_{Q_{\tau,k}^j \setminus \Omega_{k+1}} \left[\sum_{\ell=j-1}^{\infty} \int_{B(x,\delta^\ell)} |D_\ell f(y)|^2 \frac{d\mu(y)}{V_{\delta^\ell}(x)} \right]^{1/2}$$

$$\cdot \left[\sum_{\ell=j-1}^{\infty} \int_{B(x,\delta^\ell)} |D_\ell^* h(y)|^2 \frac{d\mu(y)}{V_{\delta^\ell}(x)} \right]^{1/2} d\mu(x)$$

$$\lesssim \frac{1}{\lambda_{\tau,k}^j} \left\{ \int_{Q_{\tau,k}^j \setminus \Omega_{k+1}} [S(f)(x)]^q \, d\mu(x) \right\}^{1/q} \left\{ \int_{Q_{\tau,k}^j \setminus \Omega_{k+1}} \left[\widetilde{S}(h)(x) \right]^{q'} d\mu(x) \right\}^{1/q'}$$

$$\lesssim \frac{2^k}{\lambda_{\tau,k}^j} \left[\mu\left(Q_{\tau,k}^j\right) \right]^{1/q} \left\| \widetilde{S}(h) \right\|_{L^{q'}(\mathcal{X})},$$

where, for any $x \in \mathcal{X}$,

$$\widetilde{S}(h)(x) := \left[\sum_{\ell=-\infty}^{\infty} \int_{B(x,\delta^\ell)} |D_\ell^* h(y)|^2 \frac{d\mu(y)}{V_{\delta^\ell}(x)} \right]^{1/2}$$

and D_ℓ^* is the adjoint operator on $L^2(\mathcal{X})$ of D_ℓ for any $\ell \in \mathbf{Z}$. By the proof of [27, Theorem 5.10], we conclude that

$$\left\| \widetilde{S}(h) \right\|_{L^{q'}(\mathcal{X})} \lesssim \|\widetilde{g}(h)\|_{L^{q'}(\mathcal{X})},$$

where $\widetilde{g}(h) := [\sum_{\ell \in \mathbf{Z}} |D_\ell^* h|^2]^{1/2}$. Following the arguments used in the proof of [27, Lemma 5.8], we know that $\widetilde{g}$ is bounded on $L^2(\mathcal{X})$, which, combined with the theory of vector-valued Calderón–Zygmund operators (see, for instance, [22]), implies that $\widetilde{g}$ is bounded on $L^{q'}(\mathcal{X})$. Therefore, we have

$$\left| \int_{\mathcal{X}} b_{\tau,k}^j(x) h(x) \, d\mu(x) \right| \lesssim \frac{[\mu(Q_{\tau,k}^j)]^{1/q}}{\|\mathbf{1}_{Q_{\tau,k}^j}\|_{L^\varphi(\mathcal{X})}} \|h\|_{L^{q'}(\mathcal{X})} \lesssim \frac{[\mu(Q_{\tau,k}^j)]^{1/q}}{\|\mathbf{1}_{Q_{\tau,k}^j}\|_{L^\varphi(\mathcal{X})}}.$$

Taking the supremum over all $h \in L^{q'}(\mathcal{X})$ with $\|h\|_{L^{q'}(\mathcal{X})} \leq 1$, we obtain

$$\left\| b_{\tau,k}^j \right\|_{L^q(\mathcal{X})} \lesssim \frac{[\mu(Q_{\tau,k}^j)]^{1/q}}{\|\mathbf{1}_{Q_{\tau,k}^j}\|_{L^\varphi(\mathcal{X})}},$$

which, together with some arguments similar to those used in the proof of [27, Proposition 5.7], further implies that there exists $\epsilon_0 \in (\omega[\frac{q(\varphi)}{p} - 1], \varrho)$ such that

$$(6.23) \qquad \left\| b_{\tau,k}^j \mathbf{1}_{U_m(B_{\tau,k}^j)} \right\|_{L^{q'}(\mathcal{X})} \lesssim \delta^{m\epsilon_0} \left[\frac{\mu(B_{\tau,k}^j)}{\mu(\delta^{-m} B_{\tau,k}^j)} \right]^{1/2} \frac{[\mu(B_{\tau,k}^j)]^{1/q}}{\|\mathbf{1}_{B_{\tau,k}^j}\|_{L^\varphi(\mathcal{X})}}$$

$$\lesssim \delta^{m\epsilon_0} \frac{[\mu(\delta^{-m} B_{\tau,k}^j)]^{1/q}}{\|\mathbf{1}_{B_{\tau,k}^j}\|_{L^\varphi(\mathcal{X})}}.$$

Thus, $b_{\tau,k}^j$ is a positive harmless constant multiple of a (φ, q, ϵ_0)-molecule related to the ball $B_{\tau,k}^j := B(z_{\tau,k}^j, 4A_0^2 \delta^{j-1})$ as in Definition 6.7. This finishes the proof of the above claim.

Moreover, by (6.23), (6.21), Lemma 2.6(iv) and some arguments similar to those used in the proof of [35, Lemma 5.4], we have

$$
\sum_{k\in\mathbf{Z}}\sum_{j\in\mathbf{Z}}\sum_{\tau\in I_{j,k}} \varphi\left(B_{\tau,k}^{j}, \frac{|\lambda_{\tau,k}^{j}|}{\|\mathbf{1}_{B_{\tau,k}^{j}}\|_{L^{\varphi}(\mathcal{X})}\|S(f)\|_{L^{\varphi}(\mathcal{X})}}\right)
$$

$$
\lesssim \sum_{k\in\mathbf{Z}}\sum_{j\in\mathbf{Z}}\sum_{\tau\in I_{j,k}} \varphi\left(Q_{\tau,k}^{j}, \frac{|\lambda_{\tau,k}^{j}|}{\|\mathbf{1}_{Q_{\tau,k}^{j}}\|_{L^{\varphi}(\mathcal{X})}\|S(f)\|_{L^{\varphi}(\mathcal{X})}}\right)
$$

$$
\lesssim \sum_{k\in\mathbf{Z}}\sum_{j\in\mathbf{Z}}\sum_{\tau\in I_{j,k}} \varphi\left(Q_{\tau,k}^{j}, \frac{2^{k}}{\|S(f)\|_{L^{\varphi}(\mathcal{X})}}\right)
$$

$$
\lesssim \sum_{k\in\mathbf{Z}}\sum_{j\in\mathbf{Z}}\sum_{\tau\in I_{j,k}} \varphi\left(Q_{\tau,k}^{j}\cap\Omega_{k}, \frac{2^{k}}{\|S(f)\|_{L^{\varphi}(\mathcal{X})}}\right)\left[\frac{\mu(Q_{\tau,k}^{j})}{\mu(Q_{\tau,k}^{j}\cap\Omega_{k})}\right]^{p_{0}}
$$

$$
\lesssim \sum_{k\in\mathbf{Z}} \varphi\left(\left[\bigcup_{j\in\mathbf{Z}}\bigcup_{\tau\in I_{j,k}} Q_{\tau,k}^{j}\right]\cap\Omega_{k}, \frac{2^{k}}{\|S(f)\|_{L^{\varphi}(\mathcal{X})}}\right)
$$

$$
\lesssim \sum_{k\in\mathbf{Z}} \varphi\left(\Omega_{k}, \frac{2^{k}}{\|S(f)\|_{L^{\varphi}(\mathcal{X})}}\right) \lesssim \int_{\mathcal{X}} \varphi\left(x, \frac{S(f)}{\|S(f)\|_{L^{\varphi}(\mathcal{X})}}\right)d\mu(x) \sim 1,
$$

which implies that

(6.24) $$\Lambda_{q}\left(\{\lambda_{\tau,k}^{j}b_{\tau,k}^{j}\}_{j,\tau,k}\right) \lesssim \|S(f)\|_{L^{\varphi}(\mathcal{X})},$$

where Λ_{q} is as in (6.8).

Then we begin to prove (6.20). To this end, we need to show that

$$
\sum_{k\in\mathbf{Z}}\sum_{j\in\mathbf{Z}}\sum_{\tau\in I_{j,k}} \lambda_{\tau,k}^{j}b_{\tau,k}^{j}
$$

converges in $(\mathcal{G}_{0}^{\eta}(\varrho,\vartheta))'$ and hence in $(\mathring{\mathcal{G}}_{0}^{\eta}(\varrho,\vartheta))'$, because $\mathring{\mathcal{G}}_{0}^{\eta}(\varrho,\vartheta) \subset \mathcal{G}_{0}^{\eta}(\varrho,\vartheta)$. Indeed, let $\{\mathscr{E}_{N}\}_{N\in\mathbf{N}}$ be an increasing sequence of finite subset of $\mathscr{E} := \{(k,j,\tau): k, j \in \mathbf{Z}, \tau \in I_{j,k}\}$ with $\mathscr{E} = \bigcup_{N\in\mathbf{N}}\mathscr{E}_{N}$. For any N, $M \in \mathbf{N}$ with $N > M$, by the above claim, Lemma 4.9, Theorem 6.8 and (6.24), we conclude that, for any $h \in \mathcal{G}_{0}^{\eta}(\varrho,\vartheta)$,

$$
\sum_{(k,j,\tau)\in\mathscr{E}} |\lambda_{\tau,k}^{j}|\,|\langle b_{\tau,k}^{j}, h\rangle| \lesssim \sum_{(k,j,\tau)\in\mathscr{E}} |\lambda_{\tau,k}^{j}|\,\|b_{\tau,k}^{j}\|_{H^{*,\varphi}(\mathcal{X})}\|h\|_{\mathcal{G}_{0}^{\eta}(\varrho,\vartheta)}
$$

$$
\lesssim \sum_{(k,j,\tau)\in\mathscr{E}} |\lambda_{\tau,k}^{j}|\,\|h\|_{\mathcal{G}_{0}^{\eta}(\varrho,\vartheta)} \lesssim \Lambda_{q}\left(\{\lambda_{\tau,k}^{j}b_{\tau,k}^{j}\}_{j,\tau,k}\right)\|h\|_{\mathcal{G}_{0}^{\eta}(\varrho,\vartheta)}
$$

$$
\lesssim \|S(f)\|_{L^{\varphi}(\mathcal{X})}\|h\|_{\mathcal{G}_{0}^{\eta}(\varrho,\vartheta)} < \infty,
$$

which further implies that

$$
\left|\left\langle \sum_{\{(k,j,\tau)\in\mathscr{E}_{N}\backslash\mathscr{E}_{M}\}} \lambda_{\tau,k}^{j}b_{\tau,k}^{j}, h\right\rangle\right| \leq \sum_{\{(k,j,\tau)\in\mathscr{E}_{N}\backslash\mathscr{E}_{M}\}} |\lambda_{\tau,k}^{j}|\,|\langle b_{\tau,k}^{j}, h\rangle| \to 0 \quad\text{as}\quad M \to \infty.
$$

Thus, $\{\sum_{(k,j,\tau)\in\mathscr{E}_{N}} \lambda_{\tau,k}^{j}b_{\tau,k}^{j}\}_{N\in\mathbf{N}}$ is a Cauchy sequence in $(\mathcal{G}_{0}^{\eta}(\varrho,\vartheta))'$, which, combined with the completion of $(\mathcal{G}_{0}^{\eta}(\varrho,\vartheta))'$, implies that $\sum_{(k,j,\tau)\in\mathscr{E}_{N}} \lambda_{\tau,k}^{j}b_{\tau,k}^{j}$ converges to some $\widetilde{f} \in (\mathcal{G}_{0}^{\eta}(\varrho,\vartheta))'$. That is, $\sum_{k\in\mathbf{Z}}\sum_{j\in\mathbf{Z}}\sum_{\tau\in I_{j,k}} \lambda_{\tau,k}^{j}b_{\tau,k}^{j}$ converges in $(\mathcal{G}_{0}^{\eta}(\varrho,\vartheta))'$.

By [24, Corollary 3.5], to prove (6.20), it suffices to show that, for any $\widetilde{k} \in \mathbf{Z}$ and $\widetilde{\beta} \in \mathscr{G}_{\widetilde{k}}$ with $\mathscr{G}_{\widetilde{k}}$ as in (4.1) with k replaced by $\widetilde{k}$,

$$(6.25) \quad \left\langle f, \psi_{\widetilde{\beta}}^{\widetilde{k}} \right\rangle = \left\langle \sum_{\ell \in \mathbf{Z}} D_\ell D_\ell f, \psi_{\widetilde{\beta}}^{\widetilde{k}} \right\rangle = \left\langle \sum_{k \in \mathbf{Z}} \sum_{j \in \mathbf{Z}} \sum_{\tau \in I_{j,k}} \lambda_{\tau,k}^j b_{\tau,k}^j, \psi_{\widetilde{\beta}}^{\widetilde{k}} \right\rangle = \left\langle \widetilde{f}, \psi_{\widetilde{\beta}}^{\widetilde{k}} \right\rangle,$$

where $\{\psi_{\widetilde{\beta}}^{\widetilde{k}}\}_{\widetilde{k} \in \mathbf{Z}, \, \widetilde{\beta} \in \mathscr{G}_{\widetilde{k}}}$ is as in Theorem 4.3.

Let $\{\mathscr{I}_N\}_{N \in \mathbf{N}}$ be an increasing sequence of finite subset of $\mathscr{I} := \{Q \in \mathcal{D}_k, \, Q \subset Q_{\tau,k}^j\}$ with $\mathscr{I} = \bigcup_{N \in \mathbf{N}} \mathscr{I}_N$. From the proof of (6.22), we deduce that, for any $h \in L^2(\mathcal{X})$ with $\|h\|_{L^2(\mathcal{X})} = 1$,

$$\sum_{Q \in \mathscr{I}} |\langle D_Q D_Q f, h \rangle| \lesssim 2^k \left[\mu \left(Q_{\tau,k}^j \right) \right]^{1/q} \left\| \widetilde{S}(h) \right\|_{L^{q'}(\mathcal{X})} \lesssim 2^k \left[\mu \left(Q_{\tau,k}^j \right) \right]^{1/q} < \infty,$$

which further implies that, for any $N, M \in \mathbf{N}$ with $N > M$

$$\left| \left\langle \sum_{Q \in \mathscr{I}_N \setminus \mathscr{I}_M} D_Q D_Q f, h \right\rangle \right| \leq \sum_{Q \in \mathscr{I}_N \setminus \mathscr{I}_M} |\langle D_Q D_Q f, h \rangle| \to 0 \quad \text{as} \quad M \to \infty.$$

Thus, by the completion of $L^2(\mathcal{X})$ and $(L^2(\mathcal{X}))^* = L^2(\mathcal{X})$, where $(L^2(\mathcal{X}))^*$ denotes the dual space of $L^2(\mathcal{X})$, we obtain

$$\sum_{\{Q \in \mathcal{D}_k, \, Q \subset Q_{\tau,k}^j\}} \int_Q D_Q(\cdot, y) D_Q f(y) \, d\mu(y)$$

converges in $L^2(\mathcal{X})$, which, together with $D_{\ell-1}^*(x, y) = D_{\ell-1}(y, x)$ for any $\ell \in \mathbf{Z}$ and $(x, y) \in \mathcal{X} \times \mathcal{X}$, implies that

$$(6.26) \quad \left\langle b_{\tau,k}^j, \psi_{\widetilde{\beta}}^{\widetilde{k}} \right\rangle = \frac{1}{\lambda_{\tau,k}^j} \sum_{\{Q \in \mathcal{D}_k, \, Q \subset Q_{\tau,k}^j\}} \int_Q \left\langle D_Q^*(\cdot, y), \psi_{\widetilde{\beta}}^{\widetilde{k}} \right\rangle D_Q f(y) \, d\mu(y)$$

$$= \frac{1}{\lambda_{\tau,k}^j} \sum_{\ell=j}^\infty \sum_{\beta \in \mathscr{A}_\ell} \sum_{\{Q_\beta^\ell \in \mathcal{D}_k, \, Q_\beta^\ell \subset Q_{\tau,k}^j\}} \int_{Q_\beta^\ell} \left\langle D_{\ell-1}(y, \cdot), \psi_{\widetilde{\beta}}^{\widetilde{k}} \right\rangle D_\ell f(y) \, d\mu(y),$$

where, for any $k \in \mathbf{Z}$, $D_k(\cdot, \cdot)$ is as in Remark 4.4(i) and $D_k^*(\cdot, \cdot)$ denotes the kernel of the adjoint operator D_k^* of D_k on $L^2(\mathcal{X})$.

On another hand, by the orthogonality of $\{\psi_{\widetilde{\beta}}^{\widetilde{k}}\}_{\widetilde{k} \in \mathbf{Z}, \, \widetilde{\beta} \in \mathscr{G}_{\widetilde{k}}}$, we conclude that, for any $y \in \mathcal{X}$,

$$\left\langle D_{\ell-1}(y, \cdot), \psi_{\widetilde{\beta}}^{\widetilde{k}} \right\rangle = \begin{cases} 0, & \text{when } \ell - 1 \neq \widetilde{k}, \\ \psi_{\widetilde{\beta}}^{\widetilde{k}}(y), & \text{when } \ell - 1 = \widetilde{k}, \end{cases}$$

which, combined with (6.26), implies that, if $\widetilde{k} < j-1$, then $\langle b_{\tau,k}^j, \psi_{\widetilde{\beta}}^{\widetilde{k}} \rangle = 0$; if $\widetilde{k} \geq j-1$, then

$$(6.27) \quad \left\langle b_{\tau,k}^j, \psi_{\widetilde{\beta}}^{\widetilde{k}} \right\rangle = \frac{1}{\lambda_{\tau,k}^j} \sum_{\beta \in \mathscr{A}_{\widetilde{k}+1}} \sum_{\{Q_\beta^{\widetilde{k}+1} \in \mathcal{D}_k, \, Q_\beta^{\widetilde{k}+1} \subset Q_{\tau,k}^j\}} \int_{Q_\beta^{\widetilde{k}+1}} \psi_{\widetilde{\beta}}^{\widetilde{k}}(y) D_{\widetilde{k}} f(y) \, d\mu(y).$$

From (6.19), (6.27), the disjointness of $\{Q_{\tau,k}^j\}_{\tau \in I_{j,k}}$ for any fixed $j, k \in \mathbf{Z}$, the disjointness of $\{\mathcal{D}_k\}_{k \in \mathbf{Z}}$, Lemma 4.1(iii) and the orthogonality of $\{\psi_{\widetilde{\beta}}^{\widetilde{k}}\}_{\widetilde{k} \in \mathbf{Z}, \, \widetilde{\beta} \in \mathscr{G}_{\widetilde{k}}}$, it

follows that

$$
\begin{aligned}
\left\langle \widetilde{f}, \psi_{\widetilde{\beta}}^{\widetilde{k}} \right\rangle &= \sum_{k\in\mathbf{Z}} \sum_{j=-\infty}^{\widetilde{k}+1} \sum_{\tau\in I_{j,k}} \sum_{\beta\in\mathscr{A}_{\widetilde{k}+1}} \sum_{\{Q_\beta^{\widetilde{k}+1}\in\mathcal{D}_k,\ Q_\beta^{\widetilde{k}+1}\subset Q_{\tau,k}^j\}} \int_{Q_\beta^{\widetilde{k}+1}} \psi_{\widetilde{\beta}}^{\widetilde{k}}(y) D_{\widetilde{k}} f(y)\, d\mu(y) \\
&= \sum_{\beta\in\mathscr{A}_{\widetilde{k}+1}} \int_{Q_\beta^{\widetilde{k}+1}} \psi_{\widetilde{\beta}}^{\widetilde{k}}(y) D_{\widetilde{k}} f(y)\, d\mu(y) \\
&= \left\langle D_{\widetilde{k}} f, \psi_{\widetilde{\beta}}^{\widetilde{k}} \right\rangle = \sum_{\beta\in\mathscr{G}_k} \left\langle f, \psi_\beta^k \right\rangle \left\langle \psi_\beta^k, \psi_{\widetilde{\beta}}^{\widetilde{k}} \right\rangle = \left\langle f, \psi_{\widetilde{\beta}}^{\widetilde{k}} \right\rangle .
\end{aligned}
$$

This finishes the proof of (6.25) and hence of (6.20).

In the end, by (6.24), Proposition 6.12 and Theorem 6.8, we have

$$
\|f\|_{\mathring{H}_{\mathrm{at}}^{\varphi,q}(\mathcal{X})} \sim \|f\|_{H_{\mathrm{at}}^{\varphi,q}(\mathcal{X})} \sim \|f\|_{H_{\mathrm{mol}}^{\varphi,q,\epsilon}(\mathcal{X})} \lesssim \Lambda_q\left(\left\{\lambda_{\tau,k}^j b_{\tau,k}^j\right\}_{j,\tau,k}\right) \lesssim \|S(f)\|_{L^\varphi(\mathcal{X})}.
$$

This finishes the proof of $H^\varphi(\mathcal{X}) \subset \mathring{H}_{\mathrm{at}}^{\varphi,q}(\mathcal{X})$ and hence of Theorem 6.15. $\qquad\square$

6.4. Littlewood–Paley function characterizations of $H^\varphi(\mathcal{X})$. In this subsection, we establish several characterizations of $H^\varphi(\mathcal{X})$, respectively, in terms of the Lusin area functions with apertures, the Littlewood–Paley g-functions and the Littlewood–Paley g_λ^*-functions, which is a generalization of the corresponding results on Hardy spaces $H^p(\mathcal{X})$ in [27, Theorem 5.10] [in this case, φ is the same as in (4.2)].

Theorem 6.16. *Let φ be a growth function and $\varrho, \vartheta \in (\omega[\frac{q(\varphi)}{p}-1], \eta)$ with $q(\varphi) < \frac{p(\omega+\eta)}{\omega}$ as in (3.16). Assume that $\theta \in (0,\infty)$ and $\lambda \in (\omega[1+\frac{2q(\varphi)}{p}],\infty)$. Then, for any $f \in (\mathring{\mathcal{G}}_0^\eta(\varrho,\vartheta))'$, it holds true that*

$$
(6.28) \qquad \|f\|_{H^\varphi(\mathcal{X})} \sim \|\mathcal{S}_\theta(f)\|_{L^\varphi(\mathcal{X})} \sim \|g_\lambda^*(f)\|_{L^\varphi(\mathcal{X})} \sim \|g(f)\|_{L^\varphi(\mathcal{X})},
$$

provided that any one of (6.28) is finite, with all positive equivalence constants independent of f.

Proof. Let $f \in (\mathring{\mathcal{G}}_0^\eta(\varrho,\vartheta))'$ with $\varrho, \vartheta \in (\omega[\frac{q(\varphi)}{p}-1], \eta)$. Let $S_\theta(f)$, $g_\lambda^*(f)$ and $g(f)$ be defined, respectively, as in (6.1), (6.3) and (6.2) with $\{Q_k\}_{k\in\mathbf{Z}}$ being an exp-ATI, where $\theta \in (0,\infty)$ and $\lambda \in (\omega[1+\frac{2q(\varphi)}{p}],\infty)$.

Let (φ,q) be admissible as in Definition 5.2. We first show that $\|S_\theta(f)\|_{L^\varphi(\mathcal{X})} \sim \|f\|_{H^\varphi(\mathcal{X})}$. From (6.18) and Theorem 6.15, we deduce that

$$
\|S_\theta(f)\|_{L^\varphi(\mathcal{X})} \lesssim \|f\|_{\mathring{H}_{\mathrm{at}}^{\varphi,q}(\mathcal{X})} \sim \|f\|_{H^\varphi(\mathcal{X})}.
$$

Conversely, without loss of generality, we may assume that $\|S_\theta(f)\|_{L^\varphi(\mathcal{X})} < \infty$. By the proof of $H^\varphi(\mathcal{X}) \subset \mathring{H}_{\mathrm{at}}^{\varphi,2}(\mathcal{X})$ in Theorem 6.15, we know that $f = \sum_{j=1}^\infty \lambda_j a_j$ in $(\mathring{\mathcal{G}}_0^\eta(\varrho,\vartheta))'$ with a sequence $\{a_j\}_{j=1}^\infty$ of adapted $(\varphi,q)_A$-atoms and numbers $\{\lambda_j\}_{j=1}^\infty \subset \mathbf{C}$ such that

$$
\Lambda_q\left(\{\lambda_j a_j\}_{j\in\mathbf{N}}\right) \lesssim \|S_\theta(f)\|_{L^\varphi(\mathcal{X})},
$$

which, together with Theorem 6.15, further implies that

$$
\|f\|_{H^\varphi(\mathcal{X})} = \|S(f)\|_{L^\varphi(\mathcal{X})} \sim \|f\|_{\mathring{H}_{\mathrm{at}}^{\varphi,q}(\mathcal{X})} \lesssim \Lambda_q\left(\{\lambda_j a_j\}_{j\in\mathbf{N}}\right) \lesssim \|S_\theta(f)\|_{L^\varphi(\mathcal{X})}.
$$

Therefore, we obtain $\|f\|_{H^\varphi(\mathcal{X})} \sim \|S_\theta(f)\|_{L^\varphi(\mathcal{X})}$.

Then we show that $\|S(f)\|_{L^\varphi(\mathcal{X})} \sim \|g_\lambda^*(f)\|_{L^\varphi(\mathcal{X})}$. Notice that

$$S(f) \lesssim g_\lambda^*(f) \lesssim \sum_{j=1}^\infty 2^{\frac{j(\omega-\lambda)}{2}} S_{2^j}(f)$$

(see, for instance, the proof of [27, Theorem 5.10]), which, combined with (6.18) and Theorem 6.15, implies that

$$\|S(f)\|_{L^\varphi(\mathcal{X})}^p \lesssim \|g_\lambda^*(f)\|_{L^\varphi(\mathcal{X})}^p \lesssim \sum_{j=1}^\infty 2^{\frac{j(\omega-\lambda)}{2}p} \|S_{2^j}(f)\|_{L^\varphi(\mathcal{X})}^p$$

$$\lesssim \sum_{j=1}^\infty 2^{\frac{j(\omega-\lambda)}{2}p} \|f\|_{\mathring{H}_{\mathrm{at}}^{\varphi,q}(\mathcal{X})}^p \lesssim \|f\|_{\mathring{H}_{\mathrm{at}}^{\varphi,q}(\mathcal{X})}^p \sim \|S(f)\|_{L^\varphi(\mathcal{X})}^p.$$

This shows that $\|S(f)\|_{L^\varphi(\mathcal{X})}^p \sim \|g_\lambda^*(f)\|_{L^\varphi(\mathcal{X})}^p$.

Now, we prove $\|g(f)\|_{L^\varphi(\mathcal{X})} \sim \|f\|_{H^\varphi(\mathcal{X})}$. Following the proof of (6.18), we easily have

$$\|g(f)\|_{L^\varphi(\mathcal{X})} \lesssim \|f\|_{\mathring{H}_{\mathrm{at}}^{\varphi,q}(\mathcal{X})} \sim \|f\|_{H^\varphi(\mathcal{X})}.$$

To finish the proof of Theorem 6.16, we still need to show that $\|f\|_{H^\varphi(\mathcal{X})} \lesssim \|g(f)\|_{L^\varphi(\mathcal{X})}$. Indeed, by the proof of [27, Theorem 5.10] with the same notation as therein, via choosing r and $\widetilde{\varrho}$ such that $r \in (\frac{\omega}{\omega+\widetilde{\varrho}}, \frac{p}{q(\varphi)})$, we conclude that, for any $x \in \mathcal{X}$,

$$[S(f)(x)]^2 \lesssim \sum_{k'\in\mathbf{Z}} \left[M\left(\sum_{\alpha'\in\mathscr{A}_{k'}} \sum_{m'=1}^{N(k',\alpha')} \left| Q_{k'}f\left(y_{\alpha'}^{k',m'}\right)\right|^r \mathbf{1}_{Q_{\alpha'}^{k',m'}} \right)(x) \right]^{\frac{2}{r}},$$

which, together with Theorem 6.6, further implies that

$$\|f\|_{H^\varphi(\mathcal{X})} = \|[S(f)]^r\|_{L^{\widetilde{\varphi}}(\mathcal{X})}^{\frac{1}{r}}$$

$$\lesssim \left\|\left\{ \sum_{k'\in\mathbf{Z}} \left[M\left(\sum_{\alpha'\in\mathscr{A}_{k'}} \sum_{m'=1}^{N(k',\alpha')} \left| Q_{k'}f\left(y_{\alpha'}^{k',m'}\right)\right|^r \mathbf{1}_{Q_{\alpha'}^{k',m'}} \right) \right]^{\frac{2}{r}} \right\}^{\frac{r}{2}} \right\|_{L^{\widetilde{\varphi}}(\mathcal{X})}^{\frac{1}{r}}$$

$$\lesssim \left\|\left\{ \sum_{k\in\mathbf{Z}} \left[\sum_{\alpha'\in\mathscr{A}_{k'}} \sum_{m'=1}^{N(k',\alpha')} \left| Q_{k'}f\left(y_{\alpha'}^{k',m'}\right)\right|^r \mathbf{1}_{Q_{\alpha'}^{k',m'}} \right]^{\frac{2}{r}} \right\}^{\frac{r}{2}} \right\|_{L^{\widetilde{\varphi}}(\mathcal{X})}^{\frac{1}{r}}$$

$$\sim \left\|\left[\sum_{k'\in\mathbf{Z}} \sum_{\alpha'\in\mathscr{A}_{k'}} \sum_{m'=1}^{N(k',\alpha')} \left| Q_{k'}f\left(y_{\alpha'}^{k',m'}\right)\right|^2 \mathbf{1}_{Q_{\alpha'}^{k',m'}} \right]^{\frac{1}{2}} \right\|_{L^{\varphi}(\mathcal{X})},$$

where $\widetilde{\varphi}(x,t) := \varphi(x,t^{1/r})$ for any $(x,t) \in \mathcal{X} \times [0,\infty)$ is of uniformly lower type $p/r \in (q(\varphi),\infty)$. From this and the arbitrariness of $y_{\alpha'}^{k',m'}$, it follows that

$$\|f\|_{H^\varphi(\mathcal{X})} \lesssim \left\|\left[\sum_{k\in\mathbf{Z}} \sum_{\alpha'\in\mathscr{A}_k} \sum_{m'=1}^{N(k',\alpha')} \inf_{z\in Q_{k'}(y_{\alpha'}^{k',m'})} |Q_{k'}f(z)|^2 \mathbf{1}_{Q_{\alpha'}^{k',m'}} \right]^{\frac{1}{2}} \right\|_{L^\varphi(\mathcal{X})} \lesssim \|g(f)\|_{L^\varphi(\mathcal{X})},$$

which completes the proof of $\|f\|_{H^\varphi(\mathcal{X})} \lesssim \|g(f)\|_{L^\varphi(\mathcal{X})}$ and hence of Theorem 6.16. $\square$

7. Finite atomic characterizations

In this section, we obtain a finite atomic characterization of $H^\varphi(\mathcal{X})$ and also give an improved version in case $q < \infty$.

7.1. Finite atomic characterizations with uniformly locally dominated convergence conditions. In this subsection, we obtain a finite atomic characterization of $H^\varphi(\mathcal{X})$ with uniformly locally dominated convergence conditions.

First, we need to introduce the following uniformly locally dominated convergence condition on φ, which is a generalization of the corresponding Euclidean case in [35] to any space of homogeneous type.

Definition 7.1. A growth function φ is said to satisfy the *uniformly locally dominated convergence condition* if the following holds true: Let $K \subset \mathcal{X}$ be a bounded set and f a measurable function on $\mathcal{X}$ such that $\lim_{m\to\infty} f_m(x) = f(x)$ for almost every $x \in \mathcal{X}$. If there exists a non-negative measurable function g such that $|f_m(x)| \le g(x)$ for almost every $x \in \mathcal{X}$ and

$$\sup_{t \in (0,\infty)} \int_K g(x) \frac{\varphi(x,t)}{\int_K \varphi(y,t)\,d\mu(y)}\,d\mu(x) < \infty,$$

then

$$\lim_{m\to\infty} \sup_{t \in (0,\infty)} \int_K |f_m(x) - f(x)| \frac{\varphi(x,t)}{\int_K \varphi(y,t)\,d\mu(y)}\,d\mu(x) = 0.$$

We first recall some notions and notation from [35, Section 2.4]. Recall that a *quasi-Banach space* $\mathcal{B}$ is a complete vector space equipped with a quasi-norm $\|\cdot\|_{\mathcal{B}}$ which is nonnegative, *non-degenerate* (namely, $\|f\|_{\mathcal{B}} = 0$ if and only if $f = 0$) and homogeneous, and satisfies the *quasi-triangle inequality*, namely, there exists a positive constant $\kappa \in [1,\infty)$ such that, for any $f, g \in \mathcal{B}$, $\|f+g\|_{\mathcal{B}} \le \kappa(\|f\|_{\mathcal{B}} + \|g\|_{\mathcal{B}})$.

Let $\gamma \in (0,1]$ and $m \in \mathbf{N}$. A quasi-Banach space $\mathcal{B}_\gamma$ with the quasi-norm $\|\cdot\|_{\mathcal{B}_\gamma}$ is called a *γ-quasi-Banach space* if there exists a positive constant $\kappa \in [1,\infty)$ such that, for any $m \in \mathbf{N}$ and $\{f_j\}_{j=1}^m \subset \mathcal{B}_\gamma$,

$$\left\| \sum_{j=1}^m f_j \right\|_{\mathcal{B}_\gamma}^\gamma \le \kappa \sum_{j=1}^m \|f_j\|_{\mathcal{B}_\gamma}^\gamma.$$

Similarly to the Euclidean case, we know that, when φ is of uniformly lower type $p \in (0,1]$, $H^\varphi(\mathcal{X})$ and $L^\varphi(\mathcal{X})$ are p-quasi-Banach spaces.

For every given γ-quasi-Banach space $\mathcal{B}_\gamma$ with $\gamma \in (0,1]$ and a linear space $\mathcal{Y}$, an operator T from $\mathcal{Y}$ to $\mathcal{B}_\gamma$ is said to be *$\mathcal{B}_\gamma$-sublinear* if there exists a positive constant $\tilde{\kappa} \in [1,\infty)$ such that

(i) for any $f, g \in \mathcal{Y}$, $\|T(f) - T(g)\|_{\mathcal{B}_\gamma} \le \tilde{\kappa}\|T(f-g)\|_{\mathcal{B}_\gamma}$;
(ii) for any $m \in \mathbf{N}$, $\{\lambda_j\}_{j=1}^m \subset \mathbf{C}$ and $\{f_j\}_{j=1}^m \subset \mathcal{Y}$,

$$\left\| T\left(\sum_{j=1}^m \lambda_j f_j \right) \right\|_{\mathcal{B}_\gamma}^\gamma \le \tilde{\kappa} \sum_{j=1}^m |\lambda_j|^\gamma \|T(f_j)\|_{\mathcal{B}_\gamma}^\gamma.$$

The main result of this section is stated as follows. Let $\mathrm{UC}(\mathcal{X})$ be the *set of all uniformly continuous functions on $\mathcal{X}$*.

Theorem 7.2. *Let φ be a growth function as in Definition 2.5, (φ, q) be admissible as in Definition 5.2 and φ satisfy the uniformly locally dominated convergence condition as in Definition 7.1. Then*

(i) *If $q \in (q(\varphi), \infty)$, then $\|\cdot\|_{H_{\mathrm{fin}}^{\varphi,q}(\mathcal{X})}$ and $\|\cdot\|_{H^\varphi(\mathcal{X})}$ are equivalent quasi-norms on $H_{\mathrm{fin}}^{\varphi,q}(\mathcal{X})$;*

(ii) *$\|\cdot\|_{H_{\mathrm{fin}}^{\varphi,\infty}(\mathcal{X})}$ and $\|\cdot\|_{H^\varphi(\mathcal{X})}$ are equivalent quasi-norms on $H_{\mathrm{fin}}^{\varphi,\infty}(\mathcal{X}) \cap \mathrm{UC}(\mathcal{X})$;*

(iii) *$H_{\mathrm{fin}}^{\varphi,\infty}(\mathcal{X}) \cap \mathrm{UC}(\mathcal{X})$ is a dense subspace of $H^\varphi(\mathcal{X})$.*

Proof. We first show (i). Let $q \in (q(\varphi), \infty)$. By Theorems 6.15 and 5.4, and Proposition 6.12, we know that $H^\varphi(\mathcal{X})$ and $H^{*,\varphi}(\mathcal{X})$ coincide with equivalent quasi-norms and, for any $f \in H^\varphi(\mathcal{X})$,

$$(7.1) \qquad \|f\|_{H^\varphi(\mathcal{X})} \sim \|f\|_{\mathring{H}_{\mathrm{at}}^{\varphi,q}(\mathcal{X})} \sim \|f\|_{H_{\mathrm{at}}^{\varphi,q}(\mathcal{X})} \sim \|f\|_{H^{*,\varphi}(\mathcal{X})}.$$

Thus, to show (i), it suffices to prove that, for any $f \in H_{\mathrm{fin}}^{\varphi,q}(\mathcal{X})$,

$$\|f\|_{H_{\mathrm{fin}}^{\varphi,q}(\mathcal{X})} \lesssim \|f\|_{H^{*,\varphi}(\mathcal{X})}.$$

By the homogeneity of the quasi-norm of $H^{*,\varphi}(\mathcal{X})$, without loss of generality, we may assume that $\|f\|_{H^{*,\varphi}(\mathcal{X})} = 1$.

Obviously, for any $f \in H_{\mathrm{fin}}^{\varphi,q}(\mathcal{X})$, $f \in L_{\varphi(\cdot,1)}^q(\mathcal{X}) \cap H^{*,\varphi}(\mathcal{X})$ and $\int_{\mathcal{X}} f(x)\, d\mu(x) = 0$. By the atomic characterization in Part II of the proof of Theorem 5.4, namely, the proof of $H^{*,\varphi}(\mathcal{X}) \subset H_{\mathrm{at}}^{\varphi,q}(\mathcal{X})$ (with the same notation as therein),

$$f = \sum_{j \in \mathbf{Z}} \sum_{k \in I_j} \lambda_k^j a_k^j = \sum_{j \in \mathbf{Z}} \sum_{k \in I_j} h_k^j = \sum_{j \in \mathbf{Z}} h_j$$

converges both in $(\mathcal{G}_0^\eta(\varrho, \vartheta))'$ and almost everywhere on $\mathcal{X}$. Here and hereafter, h_j, h_k^j, λ_k^j and a_k^j are as in (5.13) and (5.15).

From $f \in H_{\mathrm{fin}}^{\varphi,q}(\mathcal{X})$, we deduce that there exists a ball $B(x_1, R)$ with $x_1 \in \mathcal{X}$ and $R \in (0, \infty)$ such that $\mathrm{supp}\, f \subset B(x_1, R)$. In the remainder of this proof, let $\widetilde{B} := B(x_1, 16 A_0^4 R)$. Now, we claim that, for any $x \in (\widetilde{B})^\complement$,

$$(7.2) \qquad f^\star(x) \le c_1 \|\mathbf{1}_{\widetilde{B}}\|_{L^\varphi(\mathcal{X})}^{-1}$$

for some fixed positive constant c_1 independent of f and x, where $f^\star$ is as in (5.4). Suppose that $\varphi \in (\mathcal{G}_0^\eta(\varrho, \vartheta))'$ with $\|\varphi\|_{\mathcal{G}(x,r,\varrho,\vartheta)} \le 1$ for some $r \in (0, \infty)$. By the proof of [27, (7.1)], we know that, for any $y \in B(x_1, d(x, x_1))$,

$$|\langle f, \varphi \rangle| \lesssim f^*(y),$$

where f^* is as in (4.3), and hence, by (3.19), for any $x \in (\widetilde{B})^\complement$,

$$f^\star(x) \lesssim \inf_{y \in B(x_1, R)} f^*(y) \lesssim \|\mathbf{1}_{B(x_1,R)}\|_{L^\varphi(\mathcal{X})}^{-1} \|f^*\|_{L^\varphi(\mathcal{X})} \sim \|\mathbf{1}_{\widetilde{B}}\|_{L^\varphi(\mathcal{X})}^{-1} \|f\|_{H^{*,\varphi}(\mathcal{X})}$$

$$=: c_1 \|\mathbf{1}_{\widetilde{B}}\|_{L^\varphi(\mathcal{X})}^{-1}$$

for some fixed positive constant c_1 independent of f and x, which completes the proof of the above claim.

Moreover, let

$$(7.3) \qquad \widetilde{j} := \max\left\{ j \in \mathbf{Z} : \ 2^j \le c_1 \|\mathbf{1}_{\widetilde{B}}\|_{L^\varphi(\mathcal{X})}^{-1} \right\}.$$

Then, by the atomic characterization in Part II of the proof of Theorem 5.4, namely, the proof of $H^{*,\varphi}(\mathcal{X}) \subset H^{\varphi,q}_{\mathrm{at}}(\mathcal{X})$, we know that

$$h := \sum_{j \leq \widetilde{j}} \sum_{k \in I_j} h_k^j \quad \text{and} \quad \ell := \sum_{j > \widetilde{j}} \sum_{k \in I_j} h_k^j$$

converge both in $(\mathcal{G}_0^\eta(\varrho, \vartheta))'$ and almost everywhere on $\mathcal{X}$, where $\{h_k^j\}_{j \in \mathbf{Z}, k \in I_j}$ are the same as in (5.13) and, indeed, h_k^j for any $i \in \mathbf{Z}$ and $k \in I_j$ is a multiple of a (φ, ∞)-atom as in Definition 5.2. Thus, $f = h + \ell$ and, by the above claim (7.2), we conclude that, if $j > \widetilde{j}$, then

$$\Omega^j := \left\{ x \in \mathcal{X} : f^\star(x) > 2^j \right\} \subset \widetilde{B},$$

which, combined with $\mathrm{supp}\, f \subset B(x_1, R)$, further implies that $\mathrm{supp}\, h \subset \widetilde{B}$. Now, we show that h is a multiple of a (φ, ∞)-atom as in Definition 5.2. Indeed, we recall some results from Part II of the proof of Theorem 5.4, namely, the proof of $H^{*,\varphi}(\mathcal{X}) \subset H^{\varphi,q}_{\mathrm{at}}(\mathcal{X})$, that

$$(7.4) \qquad \left\| h_k^j \right\|_{L^\infty(\mathcal{X})} \lesssim 2^j, \quad \mathrm{supp}\, h_k^j \subset B_k^j := B\left(x_k^j, 16 A_0^4 r_k^j\right) \subset \Omega^j$$

$$\text{and} \quad \sum_{j \in I_k} \mathbf{1}_{B(x_k^j, 16 A_0^4 r_k^j)} \lesssim 1.$$

Therefore, we have

$$\|h\|_{L^\infty(\mathcal{X})} \leq \sum_{j \leq \widetilde{j}} \|h_j\|_{L^\infty(\mathcal{X})} \lesssim \sum_{j \leq \widetilde{j}} 2^j \lesssim \|\mathbf{1}_{\widetilde{B}}\|_{L^\varphi(\mathcal{X})}^{-1}.$$

Thus, to prove that h is a multiple of a (φ, ∞)-atom, it suffices to show that $\int_{\mathcal{X}} h(x) \, d\mu(x) = 0$. To this end, we first need to prove that f is a multiple of a classical $(1, \widetilde{q})$-atom as in [10, p. 591] for some $\widetilde{q} \in (1, \frac{q}{q(\varphi)})$. Let $\widetilde{q} \in (1, \frac{q}{q(\varphi)})$. Then $\varphi \in \mathbf{A}_{q/\widetilde{q}}(\mathcal{X})$ and hence

$$\left[\frac{1}{V(x_1, R)} \int_{B(x_1, R)} |f(x)|^{\widetilde{q}} \, d\mu(x) \right]^{1/\widetilde{q}}$$

$$\leq \left[\frac{1}{\varphi(B(x_1, R), 1)} \int_{B(x_1, R)} |f(x)|^q \varphi(x, 1) \, d\mu(x) \right]^{1/q} < \infty$$

by Lemma 2.6(iii) when $q \in (q(\varphi), \infty)$. From this and $\int_{\mathcal{X}} f(x) \, d\mu(x) = 0$, it follows that f is a multiple of a classical $(1, \widetilde{q})$-atom and hence $f^* \in L^1(\mathcal{X})$.

Moreover, by (7.4), for any $x \in \mathcal{X}$, we have

$$(7.5) \qquad \sum_{j > \widetilde{j}} \sum_{k \in I_j} \left| h_k^j(x) \right|$$

$$\lesssim \sum_{j > \widetilde{j}} 2^j \sum_{k \in I_j} \mathbf{1}_{B_k^j}(x) \lesssim \sum_{j > \widetilde{j}} 2^j \mathbf{1}_{\Omega^j}(x) \sim \sum_{j > \widetilde{j}} 2^j \sum_{m=j}^\infty \mathbf{1}_{\Omega^m \setminus \Omega^{m+1}}(x)$$

$$\sim \sum_{m=\widetilde{j}+1}^\infty \sum_{j=\widetilde{j}+1}^m 2^j \mathbf{1}_{\Omega^m \setminus \Omega^{m+1}}(x) \sim \sum_{m=\widetilde{j}+1}^\infty 2^m \mathbf{1}_{\Omega^m \setminus \Omega^{m+1}}(x)$$

$$\sim f^*(x) \sum_{m=\widetilde{j}+1}^\infty \mathbf{1}_{\Omega^m \setminus \Omega^{m+1}}(x) \sim f^*(x) \mathbf{1}_{\Omega^{\widetilde{j}+1}}(x) \lesssim f^*(x) \mathbf{1}_{\Omega^j}(x) \in L^1(\mathcal{X}),$$

which, together with the Lebesgue dominated convergence theorem, implies that

$$\int_{\mathcal{X}} \ell(x) \, d\mu(x) = \sum_{j > \widetilde{j}} \sum_{k \in I_j} \int_{\mathcal{X}} h_k^j(x) \, d\mu(x) = 0.$$

Moreover, by this and $\int_{\mathcal{X}} f(x) \, d\mu(x) = 0$, we obtain

$$\int_{\mathcal{X}} h(x) \, d\mu(x) = \int_{\mathcal{X}} f(x) \, d\mu(x) - \int_{\mathcal{X}} \ell(x) \, d\mu(x) = 0,$$

which further implies that h is a multiple of a (φ, ∞)-atom as in Definition 5.2.

Now, we deal with ℓ. For any $N \in \mathbf{N}$, let

$$F_N := \left\{ (k, j) \colon j \in \mathbf{Z}, \ k \in I_j, \ j > \widetilde{j}, \ |k| + |j| \le N \right\} \quad \text{and} \quad \ell_N := \sum_{(k,j) \in F_N} h_k^j.$$

Then $\|\ell_N\|_{H_{\mathrm{fin}}^{\varphi,q}(\mathcal{X})} \le \Lambda(\{h_k^j\}_{(k,j) \in F_N}) \lesssim 1$. We then claim that, for any $\epsilon \in (0, \infty)$, there exists $N \in \mathbf{N}$ such that $\varepsilon^{-1}(\ell - \ell_N)$ is a (φ, q)-atom supported in the ball $\widetilde{B}$ and $\|\ell - \ell_N\|_{H_{\mathrm{fin}}^{\varphi,q}(\mathcal{X})} < \varepsilon$. Indeed, observe that, for any $j > \widetilde{j}$,

$$\operatorname{supp}(\ell - \ell_N) \subset \Omega^j \subset \widetilde{B} \quad \text{and} \quad \int_{\mathcal{X}} [\ell(x) - \ell_N(x)] \, d\mu(x) = 0.$$

To prove the above claim, it suffices to show that $\lim_{N \to \infty} \|\ell - \ell_N\|_{L_{\varphi}^q(\widetilde{B})} = 0$. To this end, by $f \in L_{\varphi}^q(\widetilde{B})$, $f^* \lesssim M(f)$ (see, for instance, [21, Proposition 3.9]) and Lemma 2.6(v), we conclude that $f^* \in L_{\varphi}^q(\widetilde{B})$, which, combined with (7.5) and the uniformly locally dominated convergence condition as in Definition 7.1, further implies that $\lim_{N \to \infty} \|\ell - \ell_N\|_{L_{\varphi}^q(\widetilde{B})} = 0$. This shows the above claim.

Furthermore, for any $\varepsilon \in (0, \infty)$, we choose $N \in \mathbf{N}$ such that $\|\ell_N - \ell\|_{H_{\mathrm{fin}}^{\varphi,q}(\mathcal{X})} < \epsilon$. Thus, $f = h + \ell_N + (\ell - \ell_N)$ is a finite linear combination of both (φ, ∞)-atoms and (φ, q)-atoms and

$$\|f\|_{H_{\mathrm{fin}}^{\varphi,q}(\mathcal{X})} \lesssim \|h\|_{H_{\mathrm{fin}}^{\varphi,q}(\mathcal{X})} + \|\ell_N\|_{H_{\mathrm{fin}}^{\varphi,q}(\mathcal{X})} + \|\ell - \ell_N\|_{H_{\mathrm{fin}}^{\varphi,q}(\mathcal{X})} \lesssim 1,$$

which completes the proof of (i).

To show (ii), by Theorems 6.15 and 5.4, and Proposition 6.12, we know that $H^{\varphi}(\mathcal{X})$ and $H_{\mathrm{at}}^{\varphi,\infty}(\mathcal{X})$ coincide with equivalent quasi-norms and, for any $f \in H^{\varphi}(\mathcal{X})$,

$$(7.6) \qquad \|f\|_{H^{\varphi}(\mathcal{X})} \sim \|f\|_{\dot{H}_{\mathrm{at}}^{\varphi,q}(\mathcal{X})} \sim \|f\|_{H_{\mathrm{at}}^{\varphi,q}(\mathcal{X})} \sim \|f\|_{H_{\mathrm{at}}^{\varphi,\infty}(\mathcal{X})}.$$

Thus, to show (ii), it suffices to prove that, for any $f \in H_{\mathrm{fin}}^{\varphi,\infty}(\mathcal{X}) \cap \mathrm{UC}(\mathcal{X})$,

$$\|f\|_{H_{\mathrm{fin}}^{\varphi,\infty}(\mathcal{X})} \lesssim \|f\|_{H_{\mathrm{at}}^{\varphi,\infty}(\mathcal{X})}.$$

By the homogeneity of the quasi-norm of $H^{*,\varphi}(\mathcal{X})$, without loss of generality, we may assume that $\|f\|_{H_{\mathrm{at}}^{\varphi,\infty}(\mathcal{X})} = 1$. We use some arguments from the proof of [27, Theorem 7.1].

Since $f \in H_{\mathrm{fin}}^{\varphi,\infty}(\mathcal{X}) \cap \mathrm{UC}(\mathcal{X})$, it follows that f is bounded. Thus, there exists a positive integer $\widetilde{\widetilde{j}} > \widetilde{j}$ such that $\Omega^j = \emptyset$ for any integer $j > \widetilde{\widetilde{j}}$. Consequently,

$$\ell = \sum_{\widetilde{j} < j \le \widetilde{\widetilde{j}}} \sum_{k} h_k^j,$$

where ℓ and $\{h_k^j\}_{j\in\mathbf{Z},\,k\in I_j}$ are the same as in the proof of (i). By $f \in \mathrm{UC}(\mathcal{X})$, we obtain, for any $\epsilon \in (0,\infty)$, there exists $\sigma \in (0,\infty)$ such that, if $d(x,y) \leq \sigma$, then $|f(x) - f(y)| < \epsilon$. Write $\ell := \ell_1^\sigma + \ell_2^\sigma$ with

$$\ell_1^\sigma := \sum_{(j,k)\in G_1} h_k^j \quad \text{and} \quad \ell_2^\sigma := \sum_{(j,k)\in G_2} h_k^j,$$

where

$$G_1 := \left\{ (k,j) \colon j \in \mathbf{Z},\ k \in I_j,\ 12A_0^3 r_k^j \geq \sigma,\ \widetilde{j} < j \leq \widetilde{\widetilde{j}} \right\}$$

and

$$G_2 := \left\{ (k,j) \colon j \in \mathbf{Z},\ k \in I_j,\ 12A_0^3 r_k^j < \sigma,\ \widetilde{j} < j \leq \widetilde{\widetilde{j}} \right\}.$$

Notice that Ω^j is bounded for any $j \in (\widetilde{j}, \widetilde{\widetilde{j}}]$, which, together with [27, Proposition 4.4(vi)], implies that ℓ_1^σ is a finite linear combination of (φ,∞)-atoms as in Definition 5.2 and $\|\ell_1^\sigma\|_{H_{\mathrm{fin}}^{\varphi,\infty}(\mathcal{X})} \leq \Lambda_\infty(\{h_k^j\}_{(k,j)\in G_1}) \lesssim 1$.

For ℓ_2^σ, from the proof of [27, Theorem 7.1], we deduce that $\operatorname{supp} \ell_2^\sigma \subset \widetilde{B}$, $\int_{\mathcal{X}} \ell_2^\sigma(x)\,d\mu(x) = 0$ and

$$\|\ell_2^\sigma\|_{L^\infty(\mathcal{X})} \lesssim \epsilon.$$

Thus, ℓ_2^σ is a multiple of a (φ,∞)-atom as in Definition 5.2. This proves that $\|\ell_2^\sigma\|_{H_{\mathrm{fin}}^{\varphi,\infty}(\mathcal{X})} \lesssim 1$ and hence

$$\|f\|_{H_{\mathrm{fin}}^{\varphi,\infty}(\mathcal{X})} \lesssim \|h\|_{H_{\mathrm{fin}}^{\varphi,q}(\mathcal{X})} + \|\ell_1^\sigma\|_{H_{\mathrm{fin}}^{\varphi,\infty}(\mathcal{X})} + \|\ell_2^\sigma\|_{H_{\mathrm{fin}}^{\varphi,q}(\mathcal{X})} \lesssim 1,$$

which completes the proof of (ii).

Finally, to prove (iii), let $q \in (q(\varphi),\infty)$ and a be any (φ,∞)-atom supported in $B(z,r)$ with $z \in \mathcal{X}$ and $r \in (0,\infty)$. By (7.6) and the proof of [27, Theorem 7.1(iii)] (with the same notation as therein), we know that, to show (iii), it suffices to show that

$$\lim_{k\to\infty} \|S_k a - a\|_{L_\varphi^q(B(z,2A_0 r))} = 0.$$

Let $\widetilde{q} \in (1,\infty)$ satisfy $\frac{\widetilde{q}}{q} > [r(\varphi)]' > 1$. Then $(\frac{\widetilde{q}}{q})' < r(\varphi)$ and hence $\varphi \in \mathbf{RH}_{(\frac{\widetilde{q}}{q})'}(\mathcal{X})$. By this and the Hölder inequality, we conclude that, for any $t \in (0,\infty)$,

$$\frac{1}{\varphi(B(z,2A_0 r),t)} \int_{B(z,2A_0 r)} |S_k a(x) - a(x)|^q\,\varphi(x,t)\,d\mu(x)$$

$$\leq \frac{1}{\varphi(B(z,2A_0 r),t)} \left[\int_{B(z,2A_0 r)} |S_k a(x) - a(x)|^{\widetilde{q}}\,d\mu(x) \right]^{\frac{q}{\widetilde{q}}}$$

$$\cdot \left\{ \int_{B(z,2A_0 r)} [\varphi(x,t)]^{(\frac{\widetilde{q}}{q})'}\,d\mu(x) \right\}^{1/(\frac{\widetilde{q}}{q})'}$$

$$\lesssim \frac{1}{\varphi(B(z,2A_0 r),t)} \|S_k a - a\|_{L^{\widetilde{q}}(B(z,2A_0 r))}^q\, [V(z,2A_0 r)]^{1/(\frac{\widetilde{q}}{q})'-1}\,\varphi(B(z,2A_0 r),t)$$

$$\sim [V(z,2A_0 r)]^{-\frac{q}{\widetilde{q}}} \|S_k a - a\|_{L^{\widetilde{q}}(B(z,2A_0 r))}^q,$$

which, combined with [26, Proposition 2.7(iv)], further implies that, as $k \to \infty$,

$$\|S_k a - a\|_{L_\varphi^q(B(z,2A_0 r))} \lesssim [V(z,2A_0 r)]^{-\frac{q}{\widetilde{q}}} \|S_k a - a\|_{L^{\widetilde{q}}(B(z,2A_0 r))}^q \to 0.$$

This finishes the proof of (iii) and hence of Theorem 7.2. $\qquad\square$

As an application, we formulate some criteria for the boundedness of quasi-Banach valued sublinear operators on $H^\varphi(\mathcal{X})$, whose proofs are similar to that of [35, Theorem 3.5]; the details are omitted.

Theorem 7.3. *Let φ be a growth function satisfying the uniformly locally dominated convergence condition as in Definition 7.1. Let (φ, q) be admissible as in Definition 5.2 and φ of uniformly upper type $\gamma \in (0, 1]$. Suppose that $\mathcal{B}_\gamma$ is a γ-quasi-Banach space and one of the following statements hold true:*

(i) *$q \in (q(\varphi), \infty)$ and $T \colon H^{\varphi, q}_{\mathrm{fin}}(\mathcal{X}) \to \mathcal{B}_\gamma$ is a $\mathcal{B}_\gamma$-sublinear operator such that*

$$\sup\{\|Ta\|_{\mathcal{B}_\gamma} \colon a \text{ is a } (\varphi, q) - \mathrm{atom}\} < \infty;$$

(ii) *T is a $\mathcal{B}_\gamma$-sublinear operator defined on continuous (φ, ∞)-atoms such that*

$$\sup\{\|Ta\|_{\mathcal{B}_\gamma} \colon a \text{ is a continuous } (\varphi, \infty) - \mathrm{atom}\} < \infty.$$

Then T can be uniquely extended to a bounded $\mathcal{B}_\gamma$-sublinear operator from $H^\varphi(\mathcal{X})$ into $\mathcal{B}_\gamma$.

7.2. Finite atomic characterizations without having recourse to uniformly locally dominated convergence conditions. This subsection is devoted to a partial improvement of finite atomic characterizations obtained in Section 7.1 by removing the uniformly locally dominated convergence condition as in Definition 7.1 in case $q < \infty$.

By some arguments similar to those used in the proof of the Euclidean case in [7, Lemma 3.6] and (7.1), we immediately obtain the following result; the details are omitted.

Lemma 7.4. *Let φ be a growth function, $m(\varphi) \le 0$ and $r \in (q(\varphi)[r(\varphi)]', \infty]$, where $m(\varphi)$, $q(\varphi)$ and $r(\varphi)$ are, respectively, as in (3.15), (3.16) and (6.9). Then the space*

$$L^r_{b,0}(\mathcal{X}) := \left\{ f \in L^r(\mathcal{X}) \colon f \text{ has bounded support and } \int_{\mathcal{X}} f(x)\, d\mu(x) = 0 \right\}$$

is dense in $H^\varphi(\mathcal{X})$.

Now, we extend the Euclidean case in [7, Theorem 3.7] (see also [45, Theorem 4.2] for the anisotropic Euclidean case) to any space of homogeneous type (see Remark 1.1).

Theorem 7.5. *Let φ be a growth function, $m(\varphi) \le 0$ and $r \in (q(\varphi)[r(\varphi)]', \infty]$, where $m(\varphi)$, $q(\varphi)$ and $r(\varphi)$ are, respectively, as in (3.15), (3.16) and (6.9). Then, for any $q \in (q(\varphi), \frac{r}{[r(\varphi)]'})$ and $f \in L^r_{b,0}(\mathcal{X})$, there exist a finite sequence $\{a_j\}_{j=1}^N$ $(N \in \mathbf{N})$ of (φ, q)-atoms as in Definition 5.2 and numbers $\{\lambda_j\}_{j=1}^N \subset \mathbf{C}$ such that*

$$f = \sum_{j=1}^N \lambda_j a_j.$$

Moreover, there exists a positive constant C such that, for any $f \in L^r_{b,0}(\mathcal{X})$ with the decomposition as above,

$$\Lambda_q\left(\{\lambda_j a_j\}_{j=1}^N\right) \le C\|f\|_{H^\varphi(\mathcal{X})}.$$

Proof. Let $q \in (q(\varphi), \frac{r}{[r(\varphi)]'})$. Then $(\frac{r}{q})' \in (1, r(\varphi))$ and hence $\varphi \in \mathbf{RH}_{(\frac{r}{q})'}(\mathcal{X})$. For any $f \in L^r_{b,0}(\mathcal{X})$, there exists a ball $B := B(x_1, R)$, with $x_1 \in \mathcal{X}$ and $R \in (0, \infty)$,

such that supp $f \subset B$. By the Hölder inequality and $\varphi \in \mathbf{RH}_{(\frac{r}{q})'}(\mathcal{X})$, we know that, for any $t \in (0, \infty)$,

$$(7.7) \qquad \frac{1}{\varphi(B,t)} \int_B |f(x)|^q \varphi(x,t) \, d\mu(x)$$

$$\leq \frac{1}{\varphi(B,t)} \left[\int_B [\varphi(x,t)]^{(\frac{r}{q})'} \, d\mu(x) \right]^{1/(\frac{r}{q})'} \left[\int_B |f(x)|^r \, d\mu(x) \right]^{q/r}$$

$$\lesssim [\mu(B)]^{-q/r} \left[\int_B |f(x)|^r \, d\mu(x) \right]^{q/r} < \infty,$$

which, together with Lemma 7.4, further implies that $f \in H^{*,\varphi}(\mathcal{X}) \cap L^q_\varphi(B)$.

Following the proof of Theorem 7.2(i), with the same notation as therein, we know that

$$f = \sum_{j \in \mathbf{Z}} \sum_{k \in I_j} h^j_k = \sum_{j \leq \widetilde{j}} \sum_{k \in I_j} h^j_k + \sum_{j > \widetilde{j}} \sum_{k \in I_j} h^j_k =: h + \ell$$

converges both in $(\mathcal{G}^\eta_0(\varrho, \vartheta))'$ and almost everywhere on $\mathcal{X}$, where $\widetilde{j}$ is as in (7.3), h^j_k is a multiple of a (φ, ∞)-atom as in Definition 5.2 for any $j \in \mathbf{Z}$ and $k \in I_j$, h is a multiple of a (φ, ∞)-atom supported in the ball $\widetilde{B} := B(x_1, 16A_0 R)$, $f^* \in L^1(\mathcal{X})$ and, for any $x \in \mathcal{X}$,

$$(7.8) \qquad \sum_{j > \widetilde{j}} \sum_{k \in I_j} \left| h^j_k(x) \right| \lesssim f^*(x) \mathbf{1}_{\Omega^j}(x) \in L^1(\mathcal{X}).$$

For any $N \in \mathbf{N}$, let

$$F_N := \left\{ (k,j) : j \in \mathbf{Z}, \ k \in I_j, \ j > \widetilde{j}, \ |k| + |j| \leq N \right\} \quad \text{and} \quad \ell_N := \sum_{(k,j) \in F_N} h^j_k.$$

Since $f \in L^r(\mathcal{X})$, then, from [27, Theorem 3.4(ii)], we deduce that $f^* \in L^r(\mathcal{X})$. By this, (7.8) and the Lebesgue dominated convergence theorem, we conclude that

$$\lim_{N \to \infty} \|\ell_N - \ell\|_{L^r(\mathcal{X})} = 0,$$

which, combined with (7.7) and supp $(\ell) \cup$ supp $(\ell - \ell_N) \subset \widetilde{B}$, further implies that

$$\|\ell_N - \ell\|_{L^q_\varphi(\widetilde{B})} \leq \left[\mu \left(\widetilde{B} \right) \right]^{-\frac{1}{r}} \|\ell_N - \ell\|_{L^r(\widetilde{B})} \to 0, \quad \text{as } N \to \infty.$$

This shows that, for any given $\varepsilon \in (0,1)$, there exists $N \in \mathbf{N}$ such that $\varepsilon^{-1}(\ell - \ell_N)$ is a (φ, q)-atom supported in the ball $\widetilde{B}$, which implies that $f = h + \ell_N + (\ell - \ell_N)$ is a finite linear combination of (φ, ∞)-atoms and (φ, q)-atoms and

$$\Lambda_q \left(\{ h, \ell - \ell_N, h^j_k \}_{(k,j) \in F_N} \right) \lesssim \|f\|_{H^{*,\varphi}(\mathcal{X})}.$$

This, together with (7.1), then finishes the proof of Theorem 7.5. $\qquad \square$

As an easy consequence of Theorem 7.5, we obtain the following result, which improves the result in Theorem 7.2(i) by removing the uniformly dominated convergence condition as in Definition 7.1; the details are omitted.

Corollary 7.6. *Let φ, r and q be as in Theorem 7.5. Then $\| \cdot \|_{H^{\varphi,q}_{\mathrm{fin}}(\mathcal{X})}$ and $\| \cdot \|_{H^\varphi(\mathcal{X})}$ are two equivalent quasi-norms on $L^r_{b,0}(\mathcal{X})$.*

As an application, we formulate an improved criterion for the boundedness of quasi-Banach valued sublinear operators on $H^\varphi(\mathcal{X})$ in case $q < \infty$, whose proof is similar to that of the Euclidean case in [35, Theorem 3.5]; the details are omitted.

Theorem 7.7. *Let φ be a growth function of the uniformly upper type $\gamma \in (0, 1]$, $m(\varphi) \leq 0$ and $q \in (q(\varphi), \infty)$, where $m(\varphi)$ and $q(\varphi)$ are, respectively, as in (3.15) and (3.16). Suppose that $\mathcal{B}_\gamma$ is a γ-quasi-Banach space. If $T: H_{\mathrm{fin}}^{\varphi,q}(\mathcal{X}) \to \mathcal{B}_\gamma$ is a $\mathcal{B}_\gamma$-sublinear operator such that*

$$\sup\{\|Ta\|_{\mathcal{B}_\gamma}: a \text{ is a } (\varphi, q)\text{-atom}\} < \infty,$$

then there exists a unique bounded $\mathcal{B}_\gamma$-sublinear operator $\widetilde{T}$ from $H^\varphi(\mathcal{X})$ into $\mathcal{B}_\gamma$ which extends T on $L_{b,0}^\infty(\mathcal{X})$.

Remark 7.8. (i) Theorem 7.7 improves the corresponding result in Theorem 7.3(i) by removing the uniformly locally dominated convergence condition as in Definition 7.1, which has wider applications. Observe that Theorem 7.3(i) is just a simple corollary of Theorem 7.2(i) and, moreover, the proofs of (ii) and (iii) of Theorem 7.2 need to use the proof of Theorem 7.2(i) which is also of independent interest.

(ii) But, in the case $q = \infty$, it is still unclear whether or not the uniformly locally dominated convergence condition in Theorem 7.3(ii) can be removed.

8. Dual of $H^\varphi(\mathcal{X})$

In this section, we show that the dual space of $H^\varphi(\mathcal{X})$ is just the space $\mathrm{BMO}^\varphi(\mathcal{X})$.

Now, we state the following main result of this section, which is a generalization of the corresponding results on Hardy spaces $H^p(\mathcal{X})$ in [10, Theorem B] [in this case, φ is the same as in (4.2)].

Theorem 8.1. *Let φ be a growth function as in Definition 2.5 satisfying $m(\varphi) \leq 0$ and $q \in (q(\varphi), \infty]$, where $m(\varphi)$ and $q(\varphi)$ are respectively as in (3.15) and (3.16). Then the dual space, $(H^\varphi(\mathcal{X}))^*$, of $H^\varphi(\mathcal{X})$ is $\mathrm{BMO}^\varphi(\mathcal{X})$ in the following sense:*

(i) *Suppose that $b \in \mathrm{BMO}^\varphi(\mathcal{X})$. Then the linear functional*

$$L_b: f \to L_b(f) := \int_{\mathcal{X}} f(x)b(x)\,d\mu(x),$$

initially defined for any $f \in H_{\mathrm{fin}}^{\varphi,q}(\mathcal{X})$, has a bounded linear extension to $H^\varphi(\mathcal{X})$.

(ii) *Conversely, every continuous linear functional on $H^\varphi(\mathcal{X})$ arises as in (i) with a unique $b \in \mathrm{BMO}^\varphi(\mathcal{X})$. Moreover, $\|b\|_{\mathrm{BMO}^\varphi(\mathcal{X})} \sim \|L_b\|_{(H^\varphi(\mathcal{X}))'}$, where the positive equivalence constants are independent of b.*

Before proving Theorem 8.1, we first state the following lemma which is similar to the Euclidean case in [35, Lemma 4.4] and we omit its proof here.

Lemma 8.2. *Let (φ, q) be admissible as in Definition 5.2 and $N \in \mathbf{N} \cup \{\infty\}$. Then there exists a positive constant C such that, for any finite sequence $\{b_j\}_{j=1}^N$ of multiples of (φ, q)-atoms,*

$$\sum_{j=1}^N \|b\|_{L_\varphi^q(B_j)} \|\mathbf{1}_{B_j}\|_{L^\varphi(\mathcal{X})} \leq C\Lambda\left(\{b_j\}_{j=1}^N\right).$$

Now, we start to prove Theorem 8.1.

Proof of Theorem 8.1. To prove $\mathrm{BMO}^{\varphi}(\mathcal{X}) = \mathrm{Lip}_{\varphi,1}(\mathcal{X}) \subset (H^{\varphi}(\mathcal{X}))^*$, by Theorem 3.4 and (7.1), it suffices to show that $\mathrm{Lip}_{\varphi,q'}(\mathcal{X}) \subset (H^{\varphi,q}_{\mathrm{at}}(\mathcal{X}))^*$, where $q \in (q(\varphi), \infty)$ with $q(\varphi)$ as in (3.16).

Let $g \in \mathrm{Lip}_{\varphi,q'}(\mathcal{X})$ and a be a (φ, q)-atom supported in a ball $B \subset \mathcal{X}$ as in Definition 5.2. Then, by the vanishing moment and the size conditions of a, together with the Hölder inequality and $\varphi(B, \|\mathbf{1}_B\|_{L^{\varphi}(\mathcal{X})}^{-1}) = 1$, we conclude that

$$(8.1) \quad \left| \int_{\mathcal{X}} a(x) g(x) \, d\mu(x) \right|$$

$$= \left| \int_{\mathcal{X}} a(x) [g(x) - m_B(g)] \, d\mu(x) \right|$$

$$\leq \int_{\mathcal{X}} |a(x)| \left[\varphi\left(x, \|\mathbf{1}_B\|_{L^{\varphi}(\mathcal{X})}^{-1} \right) \right]^{1/q} \left[\frac{|g(x) - m_B(g)|}{\varphi(x, \|\mathbf{1}_B\|_{L^{\varphi}(\mathcal{X})}^{-1})} \right]$$

$$\cdot \left[\varphi\left(x, \|\mathbf{1}_B\|_{L^{\varphi}(\mathcal{X})}^{-1} \right) \right]^{1/q'} d\mu(x)$$

$$\leq \|a\|_{L^q_{\varphi}(B)} \int_{\mathcal{X}} \left[\frac{|g(x) - m_B(g)|}{\varphi(x, \|\mathbf{1}_B\|_{L^{\varphi}(\mathcal{X})}^{-1})} \right]^{q'} \varphi\left(x, \|\mathbf{1}_B\|_{L^{\varphi}(\mathcal{X})}^{-1} \right) d\mu(x)$$

$$\leq \frac{1}{\|\mathbf{1}_B\|_{L^{\varphi}(\mathcal{X})}} \int_{\mathcal{X}} \left[\frac{|g(x) - m_B(g)|}{\varphi(x, \|\mathbf{1}_B\|_{L^{\varphi}(\mathcal{X})}^{-1})} \right]^{q'} \varphi\left(x, \|\mathbf{1}_B\|_{L^{\varphi}(\mathcal{X})}^{-1} \right) d\mu(x) = \|g\|_{\mathrm{Lip}_{\varphi,q'}(\mathcal{X})}.$$

Now, for any $f \in H^{\varphi,q}_{\mathrm{fin}}(\mathcal{X}) = L^q_{b,0}(\mathcal{X})$, by Definition 5.2, we know that there exists a finite sequence $\{b_j\}_{j=1}^m$ of multiplies of (φ, q)-atoms supported, respectively, in balls $\{B_j\}_{j \in \mathbf{N}}$ such that $f := \sum_{j=1}^m b_j$ and $\Lambda(\{b_j\}_{j=1}^m) \lesssim \|f\|_{H^{\varphi,q}_{\mathrm{fin}}(\mathcal{X})}$. From this, (8.1), Lemma 8.2 and Theorem 7.2, we deduce that

$$\left| \int_{\mathcal{X}} f(x) g(x) \, d\mu(x) \right| \leq \sum_{j=1}^m \|b\|_{L^q_{\varphi}(B_j)} \|\mathbf{1}_{B_j}\|_{L^{\varphi}(\mathcal{X})} \|g\|_{\mathrm{Lip}_{\varphi,q'}(\mathcal{X})} \lesssim \Lambda_q(\{b_j\}_{j=1}^m) \|g\|_{\mathrm{Lip}_{\varphi,q'}(\mathcal{X})}$$

$$\lesssim \|f\|_{H^{\varphi,q}_{\mathrm{fin}}(\mathcal{X})} \|g\|_{\mathrm{Lip}_{\varphi,q'}(\mathcal{X})} \sim \|f\|_{H^{\varphi,q}_{\mathrm{at}}(\mathcal{X})} \|g\|_{\mathrm{Lip}_{\varphi,q'}(\mathcal{X})},$$

which, combined with the fact that $L^q_{b,0}(\mathcal{X})$ is dense in $H^{\varphi}(\mathcal{X})$ (see Lemma 7.4), Theorem 7.6 and a standard density argument, further completes the proof of (i).

To prove (ii), let $L \in (H^{\varphi}(\mathcal{X}))^* = (H^{\varphi,q}_{\mathrm{at}}(\mathcal{X}))^*$ [see (7.1)]. For any ball $B \subset \mathcal{X}$ and $q \in (q(\varphi), \infty]$, let

$$L^q_{\varphi,0}(B) := \left\{ f \in L^q_{\varphi}(B) : \int_{\mathcal{X}} f(x) \, d\mu(x) = 0 \right\}.$$

Obviously, $L^q_{\varphi,0}(B) \subset H^{\varphi}(\mathcal{X})$ and, for any $f \in L^q_{\varphi,0}(B)$, $a := \|\mathbf{1}_B\|_{L^{\varphi}(\mathcal{X})}^{-1} \|f\|_{L^q_{\varphi}(B)}^{-1} f$ is a (φ, q)-atom supported in B and

$$\|f\|_{H^{\varphi,q}_{\mathrm{at}}(\mathcal{X})} \leq \|\mathbf{1}_B\|_{L^{\varphi}(\mathcal{X})} \|f\|_{L^q_{\varphi}(B)},$$

which further implies that, for any $L \in (H^{\varphi,q}_{\mathrm{at}}(\mathcal{X}))^*$ and $f \in L^q_{\varphi,0}(B)$,

$$|Lf| \leq \|L\|_{(H^{\varphi,q}_{\mathrm{at}}(\mathcal{X}))^*} \|f\|_{H^{\varphi,q}_{\mathrm{at}}(\mathcal{X})}.$$

Thus, L is a bounded linear functional on $L^q_{\varphi,0}(B)$. From the Hahn–Banach theorem (see, for instance, [54, p. 77, Corollary 1]), it follows that L can be linearly boundedly extended to the whole space $L^q_{\varphi}(B)$ without increasing its norm, which, together

with the Lebesgue–Radon–Nikodym theorem (see, for instance, [14, Theorem 3.8]), we conclude that there exists $h \in L^1(B) := \{f \in L^1(\mathcal{X}) : \operatorname{supp} f \subset B\}$ such that, for any $f \in L^q_{\varphi,0}(B)$,

$$L(f) = \int_{\mathcal{X}} f(x)h(x)\, d\mu(x).$$

We now choose an increasing sequence $\{B_j\}_{j \in \mathbf{N}}$ of balls such that $B_1 \subset B_2 \subset \cdots \subset B_j \subset \cdots$ and $\bigcup_{j=1}^{\infty} B_j = \mathcal{X}$. From the above arguments, we deduce that there exists a sequence $\{h_j\}_{j \in \mathbf{N}}$ of measurable functions such that, for any $j \in \mathbf{N}$, $h_j \in L^1(B_j)$ and, for any $f \in L^q_{\varphi,0}(B_j)$,

$$(8.2) \qquad\qquad L(f) = \int_{\mathcal{X}} f(x)h_j(x)\, d\mu(x).$$

Therefore, for any $f \in L^q_{\varphi,0}(B_1)$,

$$\int_{B_1} f(x)[h_1(x) - h_2(x)]\, d\mu(x) = 0,$$

which, combined with the fact that $g - m_{B_1}(g) \in L^q_{\varphi,0}(B_1)$ for any $g \in L^q_{\varphi}(B_1)$, further implies that, for any $g \in L^q_{\varphi}(B_1)$,

$$\int_{B_1} [g(x) - m_{B_1}(g)][h_1(x) - h_2(x)]\, d\mu(x) = 0,$$

here and hereafter, $m_{B_1}(g)$ denotes the mean of g on B_1 as in (2.3) with f and B replaced, respectively, by g and B_1 here.

Observe that, for any ball B and $f,\, g \in L^q_{\varphi}(B)$,

$$\int_B m_B(g)[h_1(x) - h_2(x)]\, d\mu(x) = \int_B g(x)m_B(h_1 - h_2)\, d\mu(x).$$

Thus, for any $g \in L^q_{\varphi}(B_1)$,

$$\int_{B_1} g(x)[h_1(x) - h_2(x) - m_{B_1}(h_1 - h_2)]\, d\mu(x) = 0,$$

which further shows that, for μ-almost every $x \in B_1$,

$$h_1(x) - h_2(x) = m_{B_1}(h_1 - h_2).$$

Let $\widetilde{h}_1 := h_1$ and, for any $j \in \mathbf{N}$,

$$\widetilde{h}_{j+1} := h_{j+1} + m_{B_j}\left(\widetilde{h}_j - h_{j+1}\right).$$

Then the sequence $\{\widetilde{h}_j\}_{j \in \mathbf{N}}$ satisfies that, for μ-almost every $x \in B_j$, $\widetilde{h}_{j+1}(x) = \widetilde{h}_j(x)$ and $\widetilde{h}_j \in L^1(B_j)$. Let b be a measurable function satisfying that, if $x \in B_j$, then $b(x) = \widetilde{h}_j(x)$. To finish the proof of Theorem 8.1(ii), it still needs to show that $b \in \operatorname{Lip}_{\varphi,1}(\mathcal{X})$ and, for any $f \in H^{\varphi,q}_{\mathrm{fin}}(\mathcal{X})$,

$$L(f) = \int_{\mathcal{X}} f(x)b(x)\, d\mu(x).$$

Indeed, for any $f \in H^{\varphi,q}_{\mathrm{fin}}(\mathcal{X})$, we easily know that there exists $j \in \mathbf{N}$ such that $\operatorname{supp} f \subset B_j$. Thus, $f \in L^q_{\varphi,0}(B_j)$, which, together with (8.2), further implies that

$$L(f) = \int_{\mathcal{X}} f(x)b(x)\, d\mu(x).$$

Furthermore, for any ball $B \subset \mathcal{X}$, let $f := \text{sign}(b - m_B(b))$, where $\text{sign}(\cdot)$ denotes the *sign function*, and

$$a := \frac{1}{2} \|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1} [f - m_B(f)] \mathbf{1}_B.$$

Then a is a (φ, q)-atom supported in B and

$$
\begin{aligned}
\frac{1}{\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}} \int_B |b(x) - m_B(b)| \, d\mu(x) &= \frac{1}{\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}} \left| \int_B [b(x) - m_B(b)] f(x) \, d\mu(x) \right| \\
&= \frac{1}{\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}} \left| \int_B [f(x) - m_B(f)] b(x) \, d\mu(x) \right| \\
&\lesssim L(a) \lesssim \|L\|_{(H_{\text{at}}^{\varphi,q}(\mathcal{X}))^*} \|a\|_{H_{\text{at}}^{\varphi,q}(\mathcal{X})} \lesssim \|L\|_{(H_{\text{at}}^{\varphi,q}(\mathcal{X}))^*},
\end{aligned}
$$

which further implies that $b \in \text{Lip}_{\varphi,1}(\mathcal{X})$ and

$$\|b\|_{\text{Lip}_{\varphi,1}(\mathcal{X})} \lesssim \|L\|_{(H_{\text{at}}^{\varphi,q}(\mathcal{X}))^*}.$$

This finishes the proof of (ii) and hence of Theorem 8.1. $\qquad\square$

9. Boundedness of Calderón–Zygmund operators

In this section, we establish the boundedness of Calderón–Zygmund operators from $H^\varphi(\mathcal{X})$ into $L^\varphi(\mathcal{X})$ or from $H^\varphi(\mathcal{X})$ to itself, respectively. To this end, we first recall the following notion of Calderón–Zygmund operators from [9] (see also [2, 11]).

Definition 9.1. A function $K \in L^1_{\text{loc}}(\{\mathcal{X} \times \mathcal{X}\} \setminus \{(x,x) \colon x \in \mathcal{X}\})$ is called a *Calderón–Zygmund kernel* if there exists a positive constant $C_{(K)}$, depending on K, such that

(i) for any $x, y \in \mathcal{X}$ with $x \neq y$,

(9.1)
$$|K(x,y)| \leq C_{(K)} \frac{1}{V(x,y)};$$

(ii) there exists a positive constant $s \in (0,1]$, depending on K, such that
   $(\text{ii})_1$ for any $x, \widetilde{x}, y \in \mathcal{X}$ with $d(x,y) \geq 2A_0 d(x, \widetilde{x}) > 0$,

(9.2)
$$|K(x,y) - K(\widetilde{x},y)| \leq C_{(K)} \left[\frac{d(x,\widetilde{x})}{d(x,y)} \right]^s \frac{1}{V(x,y)};$$

   $(\text{ii})_2$ for any $x, \widetilde{x}, y \in \mathcal{X}$ with $d(x,y) \geq 2A_0 d(y, \widetilde{y}) > 0$,

(9.3)
$$|K(x,y) - K(x,\widetilde{y})| \leq C_{(K)} \left[\frac{d(y,\widetilde{y})}{d(x,y)} \right]^s \frac{1}{V(x,y)},$$

where A_0 is the same as in (1.2).

Let $C(\mathcal{X})$ denote the *space of all continuous functions on $\mathcal{X}$* and $s \in (0,1]$. Recall that the *space $C_b^s(\mathcal{X})$* is defined by setting

$$C_b^s(\mathcal{X}) := \{f \in C^s(\mathcal{X}) \colon f \text{ has bounded support}\},$$

where the *space $C^s(\mathcal{X})$* is defined by setting

$$C^s(\mathcal{X}) := \{f \in C(\mathcal{X}) \colon \|f\|_{C^s(\mathcal{X})} < \infty\}$$

with

$$\|f\|_{C^s(\mathcal{X})} := \|f\|_{L^\infty(\mathcal{X})} + \sup_{\{x,y \in \mathcal{X} \colon x \neq y\}} \frac{|f(x) - f(y)|}{[d(x,y)]^s}.$$

We equip $C_b^s(\mathcal{X})$ with the strict inductive limit topology (see, for instance, [50, p. 273] and [26, p. 23]). Moreover, the *dual space* $(C_b^s(\mathcal{X}))'$ is defined to be the set of all linear functionals on $C_b^s(\mathcal{X})$ equipped with the weak-$*$ topology.

Let $T\colon C_b^s(\mathcal{X}) \to (C_b^s(\mathcal{X}))'$ be a linear continuous operator. Then T is called a *Calderón–Zygmund operator* with the kernel K satisfying (9.1), (9.2) and (9.3) if, for any $f \in C_b^s(\mathcal{X})$,

$$(9.4) \qquad Tf(x) := \int_{\mathcal{X}} K(x,y) f(y) \, d\mu(y), \quad \forall\, x \in (\operatorname{supp} f)^{\complement}.$$

Then we state the main result of this section as follows.

Theorem 9.2. *Let ω be as in (1.4) and $s \in (0,1]$ as in Definition 9.1. Suppose that φ is a growth function as in Definition 2.5 with $q(\varphi) < \frac{p(s+\omega)}{\omega}$, where $p \in (0,1]$ is as in Definition 2.5(iii) and $q(\varphi)$ as in (3.16), and that T is a Calderón–Zygmund operator as in (9.4) which is bounded on $L^2(\mathcal{X})$.*

   (i) *Then there exists a positive constant C such that, for any $f \in H^\varphi(\mathcal{X})$, $Tf \in L^\varphi(\mathcal{X})$ and $\|Tf\|_{L^\varphi(\mathcal{X})} \leq C\|f\|_{H^\varphi(\mathcal{X})}$.*

   (ii) *If we further assume that $T^*1 = 0$, then there exists a positive constant $\widetilde{C}$ such that, for any $f \in H^\varphi(\mathcal{X})$, $Tf \in H^\varphi(\mathcal{X})$ and $\|Tf\|_{H^\varphi_{\mathrm{at}}(\mathcal{X})} \leq \widetilde{C}\|f\|_{H^\varphi(\mathcal{X})}$.*

Proof. Choose $r \in [2, \infty)$ such that $r' < r(\varphi)$. Let (φ, q) be admissible as in Definition 5.2 and a a (φ, q)-atom with $q \in (rq(\varphi), \infty)$, supported in the ball $B_0 := B(x_0, r_0)$ for some $x_0 \in \mathcal{X}$ and $r_0 \in (0, \infty)$ as in Definition 5.2.

We first prove (i). By Theorem 7.7, it suffices to show that $\|Ta\|_{L^\varphi(\mathcal{X})} \lesssim 1$. Indeed, we write

$$\|Ta\|_{L^\varphi(\mathcal{X})} \lesssim \|\mathbf{1}_{2A_0 B_0} Ta\|_{L^\varphi(\mathcal{X})} + \left\|\mathbf{1}_{(2A_0 B_0)^{\complement}} Ta\right\|_{L^\varphi(\mathcal{X})} =: \mathrm{I} + \mathrm{II}.$$

For I, by the boundedness of T on $L^r(\mathcal{X})$, we know that

$$(9.5) \qquad \|Ta\|_{L^r(\mathcal{X})} \lesssim \|a\|_{L^r(\mathcal{X})}.$$

Meanwhile, from $\frac{q}{r} \in (q(\varphi), \infty)$, it follows that $\varphi \in \mathbf{A}_{\frac{q}{r}}(\mathcal{X})$, which, combined with Lemma 2.6(iii), implies that

$$\left\{\frac{1}{\mu(B_0)} \int_{B_0} |a(x)|^r \, d\mu(x)\right\}^{1/r} \lesssim \left\{\frac{1}{\varphi(B_0, t)} \int_{B_0} |a(x)|^q \varphi(x, t) \, d\mu(x)\right\}^{1/q}$$

$$\lesssim \|a\|_{L^q_\varphi(\mathcal{X})} \lesssim \frac{1}{\|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}}.$$

By this and (9.5), we conclude that

$$(9.6) \qquad \|Ta\|_{L^r(\mathcal{X})} \lesssim \|a\|_{L^r(\mathcal{X})} \lesssim \frac{[\mu(B_0)]^{1/r}}{\|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}},$$

which, together with the arguments used in the estimation of A in the proof of Theorem 6.15, we conclude that

$$\int_{2A_0 B} \varphi(x, T(\lambda a)(x)) \, d\mu(x) \lesssim \varphi\left(B, \frac{|\lambda|}{\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}}\right).$$

This further shows that $\mathrm{I} \lesssim 1$.

To estimate II, from $\int_{\mathcal{X}} a(y)\,d\mu(y) = 0$, (9.3), the Hölder inequality and (9.6), we deduce that, for any $x \in (2A_0 B_0)^{\complement}$,

$$
(9.7) \qquad
\begin{aligned}
|Ta(x)| &= \left| \int_{\mathcal{X}} [K(x,y) - K(x,x_0)]\, a(y)\, d\mu(y) \right| \\
&\leq \int_{\mathcal{X}} |K(x,y) - K(x,x_0)|\, |a(y)|\, d\mu(y) \\
&\lesssim \int_{B_0} \left[\frac{d(y,x_0)}{d(x,x_0)} \right]^s \frac{1}{V(x,x_0)} |a(y)|\, d\mu(y) \\
&\lesssim \left[\frac{r_0}{d(x,x_0)} \right]^s \frac{1}{V(x,x_0)} \|a\|_{L^1(\mathcal{X})} \\
&\lesssim \left[\frac{r_0}{d(x,x_0)} \right]^s \frac{1}{V(x,x_0)} \|a\|_{L^r(\mathcal{X})} [\mu(B_0)]^{\frac{1}{r'}} \\
&\lesssim \left[\frac{r_0}{d(x,x_0)} \right]^s \frac{1}{V(x,x_0)} \frac{\mu(B_0)}{\|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}} \\
&\lesssim \left[\frac{\mu(B_0)}{V(x,x_0)} \right]^{\frac{s}{\omega}+1} \frac{1}{\|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}} \lesssim \frac{[M(\mathbf{1}_{B_0})(x)]^{\frac{s}{\omega}+1}}{\|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}},
\end{aligned}
$$

which, combined with Theorem 4.11 and $q(\varphi) < \frac{p(s+\omega)}{\omega}$, further implies that

$$
\begin{aligned}
\mathrm{II} &\lesssim \left\| \left[M\left(\frac{\mathbf{1}_{B_0}}{\|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{\frac{\omega}{\omega+s}}} \right) \right]^{\frac{s}{\omega}+1} \right\|_{L^\varphi(\mathcal{X})} \sim \left\| M\left(\frac{\mathbf{1}_{B_0}}{\|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{\frac{\omega}{\omega+s}}} \right) \right\|_{L^{\widetilde{\varphi}}(\mathcal{X})}^{\frac{\omega+s}{\omega}} \\
&\lesssim \left\| \frac{\mathbf{1}_{B_0}}{\|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}^{\frac{\omega}{\omega+s}}} \right\|_{L^{\widetilde{\varphi}}(\mathcal{X})}^{\frac{\omega+s}{\omega}} \sim \left\| \frac{\mathbf{1}_{B_0}}{\|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}} \right\|_{L^\varphi(\mathcal{X})} \sim 1,
\end{aligned}
$$

where, for any $x \in \mathcal{X}$ and $t \in (0,\infty)$, $\widetilde{\varphi}(x,t) := \varphi(x, t^{\frac{s+\omega}{\omega}})$ is of uniformly lower type $\frac{p(s+\omega)}{\omega}$.

Combining the estimates of I and II, we have

$$
\|Ta\|_{L^\varphi(\mathcal{X})} \lesssim \mathrm{I} + \mathrm{II} \lesssim 1,
$$

which completes the proof of (i).

Now, we show (ii). Similarly to (i), by Theorem 7.7, we only need to prove that $\|Ta\|_{H_{\mathrm{at}}^\varphi(\mathcal{X})} \lesssim 1$. Indeed, we show that Ta is a (φ, r, s)-molecule related to the ball $2A_0 B_0$ as in Definition 6.7. To this end, we write

$$
Ta = \sum_{j=0}^{\infty} \mathbf{1}_{U_j(2A_0 B_0)} Ta,
$$

where $U_0(2A_0 B_0) := 2A_0 B_0$ and, for any $j \in \mathbf{N}$, $U_j(2A_0 B_0) := \delta^{-j} 2A_0 B_0 \backslash \delta^{-j+1} 2A_0 B_0$. From (9.6) and some arguments similar to those used in the proof of [43, Theorem 2.7], it follows that

$$
\|Ta\|_{L^r(\mathcal{X})} \lesssim \frac{[\mu(B_0)]^{1/r}}{\|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}} \lesssim \frac{[\mu(2A_0 B_0)]^{1/r}}{\|\mathbf{1}_{2A_0 B_0}\|_{L^\varphi(\mathcal{X})}}.
$$

Moreover, by the proof of (9.7) and some arguments similar to those used in the proof of [43, Theorem 2.7], we know that, for any $x \in (2A_0B_0)^{\complement}$,

$$|Ta(x)| \lesssim \left[\frac{r_0}{d(x,x_0)}\right]^s \frac{1}{V(x,x_0)} \frac{\mu(B_0)}{\|\mathbf{1}_{B_0}\|_{L^\varphi(\mathcal{X})}} \lesssim \left[\frac{r_0}{d(x,x_0)}\right]^s \frac{1}{V(x,x_0)} \frac{\mu(2A_0B_0)}{\|\mathbf{1}_{2A_0B_0}\|_{L^\varphi(\mathcal{X})}},$$

which further implies that, for any $j \in \mathbf{N}$,

$$\|Ta\|_{L^r(U_j(2A_0B_0))}$$

$$\lesssim \left\{\int_{\delta^{-j}2A_0B_0 \setminus \delta^{-j+1}2A_0B_0} \left[\frac{r_0}{d(x,x_0)}\right]^{rs} \frac{1}{[V(x,x_0)]^r} \, d\mu(x)\right\}^{\frac{1}{r}} \frac{\mu(2A_0B_0)}{\|\mathbf{1}_{2A_0B_0}\|_{L^\varphi(\mathcal{X})}}$$

$$\lesssim \delta^{js} \frac{[\mu(\delta^{-j}2A_0B_0)]^{1/r}}{V(x_0, \delta^{-j+1}2A_0r_0)} \frac{\mu(2A_0B_0)}{\|\mathbf{1}_{2A_0B_0}\|_{L^\varphi(\mathcal{X})}} \lesssim \delta^{js} \frac{[\mu(\delta^{-j}2A_0B_0)]^{1/r}}{\|\mathbf{1}_{2A_0B_0}\|_{L^\varphi(\mathcal{X})}}.$$

On another hand, by $T^*1 = 0$, we immediately obtain $\int_{\mathcal{X}} Ta(x) \, d\mu(x) = 0$. Thus, Ta is a harmless positive constant multiple of a (φ, r, s)-molecule related to the ball $2A_0B_0$, which, together with Theorem 6.8, further implies that

$$\|Ta\|_{H_{\mathrm{at}}^\varphi(\mathcal{X})} \sim \|Ta\|_{H_{\mathrm{mol}}^{\varphi,r,s}(\mathcal{X})} \lesssim 1.$$

This finishes the proof of (ii) and hence of Theorem 9.2. $\qquad\qquad\square$

Remark 9.3. Let all the notation be the same as in Theorem 9.2. We point out that the condition $q(\varphi) < \frac{p(s+\omega)}{\omega}$ in Theorem 9.2 is sharp in the sense that, if $\varphi(x,t) := t^p$ for any $(x,t) \in \mathcal{X} \times [0,\infty)$, then $q(\varphi) = 1$ and hence $q(\varphi) < \frac{p(s+\omega)}{\omega}$ if and only if $p > \frac{\omega}{s+\omega}$, which returns to the classical case (see, for instance, [47, Theorem 3.4]).

10. Class of pointwise multipliers for BMO($\mathcal{X}$)

In this section, we establish a new characterization of PWM(BMO($\mathcal{X}$)), namely, the set of all pointwise multipliers of the space BMO($\mathcal{X}$).

It was known by [31, p. 1925] that φ in (2.4) satisfies $\omega q(\varphi) < (\omega + 1)i(\varphi)$. More precisely, $\varphi \in \mathbf{A}_1(\mathcal{X})$ and, for almost every fixed $x \in \mathcal{X}$, $\varphi(x, \cdot)$ is concave with $i(\varphi) = 1$.

The following definition of log-atoms on $\mathcal{X}$ is a variant of log-atoms in the Euclidean case from [35, Section 7].

Definition 10.1. A measurable function a is called a *log-atom* if it satisfies the following three conditions:

(i) supp $a \subset B$ for some ball B in $\mathcal{X}$;
(ii) $\|a\|_{L^\infty(\mathcal{X})} \leq \frac{1}{\mu(B)}[\log(e + \frac{1}{\mu(B)}) + \sup_{x \in B} \log(e + d(x_0, x))]$ with x_0 as in Remark 2.10;
(iii) $\int_{\mathcal{X}} a(x) \, d\mu(x) = 0$.

We first show the following technical proposition, which is a generalization of the Euclidean case in [35, Proposition 7.1] to any space of homogeneous type. In what follows, we always let φ be as in (2.4).

Proposition 10.2. *There exists a positive constant $\widetilde{C}$ such that, if f is a (φ, ∞)-atom as in Definition 5.2 [resp., log-atom as in Definition 10.1], then $\widetilde{C}^{-1}f$ is a log-atom [resp., (φ, ∞)-atom].*

Proof. Let f be a (φ, ∞)-atom supported in a ball $B \subset \mathcal{X}$. We first show that $\frac{1}{4}f$ is a log-atom. Indeed, to this end, it suffices to show that $\frac{1}{4}f$ satisfies Definition 10.1(ii). Observe that

$$4\left[\log\left(e + \frac{1}{\mu(B)}\right) + \sup_{x \in B}\log\left(e + d(x_0, x)\right)\right]\left[\log\left(e + \frac{4}{\mu(B)}\left\{\log\left(e + \frac{1}{\mu(B)}\right)\right.\right.\right.$$
$$\left.\left.\left. + \sup_{x \in B}\log\left(e + d(x_0, x)\right)\right\}\right) + \sup_{x \in B}\log\left(e + d(x_0, x)\right)\right]^{-1} \geq 1.$$

By this, we conclude that

$$\int_B \varphi\left(x, \frac{4}{\mu(B)}\left[\log\left(e + \frac{1}{\mu(B)}\right) + \sup_{x \in B}\log\left(e + d(x_0, x)\right)\right]\right) d\mu(x) \geq 1,$$

which implies that

$$\left\|\frac{1}{4}f\right\|_{L^\infty(\mathcal{X})} \leq \frac{1}{4}\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1} \leq \frac{1}{4}\frac{4}{\mu(B)}\left[\log\left(e + \frac{1}{\mu(B)}\right) + \sup_{x \in B}\log\left(e + d(x_0, x)\right)\right]$$
$$= \frac{1}{\mu(B)}\left[\log\left(e + \frac{1}{\mu(B)}\right) + \sup_{x \in B}\log\left(e + d(x_0, x)\right)\right].$$

This is the desired conclusion.

On another hand, let f be a log-atom. Then we prove that f is a multiple of a (φ, ∞)-atom. We also observe that

$$\left[\log\left(e + \frac{1}{\mu(B)}\right) + \sup_{x \in B}\log\left(e + d(x_0, x)\right)\right]\left[\log\left(e + \frac{1}{\mu(B)}\left\{\log\left(e + \frac{1}{\mu(B)}\right)\right.\right.\right.$$
$$\left.\left.\left. + \sup_{x \in B}\log\left(e + d(x_0, x)\right)\right\}\right) + \sup_{x \in B}\log\left(e + d(x_0, x)\right)\right]^{-1} \leq 1.$$

By this and $\varphi \in \mathbf{A}_1(\mathcal{X})$, we find that

$$\int_B \varphi\left(x, \frac{1}{\mu(B)}\left[\log\left(e + \frac{1}{\mu(B)}\right) + \sup_{x \in B}\log\left(e + d(x_0, x)\right)\right]\right) d\mu(x) \lesssim 1,$$

which implies that

$$\|f\|_{L^\infty(\mathcal{X})} \leq \frac{1}{\mu(B)}\left[\log\left(e + \frac{1}{\mu(B)}\right) + \sup_{x \in B}\log\left(e + d(x_0, x)\right)\right] \lesssim \|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1}.$$

This finishes the proof of Proposition 10.2. $\qquad\square$

The following definition of weighted BMO spaces is taken from [52].

Definition 10.3. Let $\mathrm{BMO}_\phi(\mathcal{X})$ be the space of all $f \in L^1_{\mathrm{loc}}(\mathcal{X})$ such that, for any $a \in \mathcal{X}$ and $r \in (0, \infty)$,

$$\mathrm{MO}_\phi(f, B(a, r)) := \frac{1}{\phi(a, r)}\frac{1}{V(a, r)}\int_{B(a,r)}\left|f(x) - m_{B(a,r)}(f)\right| d\mu(x) < \infty,$$

where $\phi(a, r) := \frac{1}{\log(d(x_0, a) + r + 1/r)}$ and

$$m_{B(a,r)}(f) := \frac{1}{\mu(B(a, r))}\int_{B(a,r)} f(y) \, d\mu(y).$$

For any $f \in L^1_{\mathrm{loc}}(\mathcal{X})$, let $\|f\|_{\mathrm{BMO}_\phi(\mathcal{X})} := \sup_{\{a \in \mathcal{X}, \, r \in (0,\infty)\}} \mathrm{MO}_\phi(f, B(a, r))$ and

$$\|f\|_{\mathrm{bmo}_\phi(\mathcal{X})} := \|f\|_{\mathrm{BMO}_\phi(\mathcal{X})} + |m_{B(x_0, 1)}(f)|,$$

where x_0 is as in (2.4).

In what follows, we need the following *Ahlfors n-regular condition* on the measure μ, namely, there exists $n \in (0, \infty)$ such that, for any ball $B(x, r)$ with $x \in \mathcal{X}$ and $r \in (0, \infty)$,

$$(10.1) \qquad\qquad V(x, r) \sim r^n,$$

where the positive equivalence constants are independent of x and r.

The following result is a generalization of the corresponding Euclidean case in [35, Proposition 7.2].

Proposition 10.4. *Assume that μ satisfies the Ahlfors n-regular condition* (10.1). *For any $f \in \mathrm{BMO}_\phi(\mathcal{X})$, it holds true that*

$$\|f\|_{\mathrm{BMO}_\phi(\mathcal{X})}$$
$$\sim \sup_{B:\,\mathrm{ball}} \frac{1}{\mu(B)} \left[\log\left(e + \frac{1}{\mu(B)}\right) + \sup_{x \in B} \log\left(e + d(x_0, x)\right) \right] \int_B |f(x) - m_B(f)| \, d\mu(x) < \infty$$

with the positive equivalence constants independent of f.

Proof. It suffices to show that, for any ball $B := B(a, r)$ with $a \in \mathcal{X}$ and $r \in (0, \infty)$,

$$\log\left(d(x_0, a) + r + \frac{1}{r}\right) \sim \log\left(e + \frac{1}{\mu(B)}\right) + \sup_{x \in B} \log(e + d(x_0, x)).$$

We first show that

$$\log\left(e + \frac{1}{\mu(B)}\right) + \sup_{x \in B} \log(e + d(x_0, x)) \lesssim \log\left(d(x_0, a) + r + \frac{1}{r}\right).$$

Indeed, by (10.1), we have

$$\log\left(e + \frac{1}{\mu(B)}\right) \sim \log\left(e + \frac{1}{r^n}\right) \lesssim \log\left(e + \frac{1}{r}\right) \lesssim \log\left(d(x_0, a) + r + \frac{1}{r}\right)$$

and we also observe that

$$\sup_{x \in B} \log(e + d(x_0, x)) \leq \log\left(e + A_0 d(x_0, a) + A_0 r\right) \lesssim \log\left(d(x_0, a) + r + \frac{1}{r}\right),$$

which implies the desired conclusion.

Conversely, we show that

$$\log\left(d(x_0, a) + r + \frac{1}{r}\right) \lesssim \log\left(e + \frac{1}{\mu(B)}\right) + \sup_{x \in B} \log(e + d(x_0, x)).$$

It is easy to see that $(\mathcal{X}, d, \mu)$ with μ satisfying (10.1) is an RD-space, which further implies that there exists a constant $a_0 \in (1, \infty)$ such that, for any $a \in \mathcal{X}$ and $r \in (0, \infty)$, $B(a, r) \setminus B(a, \frac{r}{a_0}) \neq \emptyset$. Thus, we are able to choose $y_0 \in B(a, r) \setminus B(a, \frac{r}{a_0})$. Therefore, we have

$$(10.2) \qquad\qquad \frac{r}{a_0} \leq d(y_0, a) < r.$$

We further consider the following two cases.

Case (i) $r \in (0, 1]$. In this case, by (10.1), we obtain

$$\log\left(d(x_0, a) + r + \frac{1}{r}\right) \leq \log\left(d(x_0, a) + \frac{2}{r}\right) \lesssim \log\left(e + \frac{1}{r}\right) + \log(e + d(x_0, a))$$

$$\lesssim \log\left(e + \frac{1}{\mu(B)}\right) + \sup_{x \in B} \log(e + d(x_0, x)).$$

Case (ii) $r \in (1, \infty)$. In this case, by (10.2), we have

$$\log\left(d(x_0, a) + r + \frac{1}{r}\right) \leq \log\left(d(x_0, a) + 2r\right) \lesssim \log\left(e + d(x_0, a) + d(y_0, a)\right)$$
$$\lesssim \sup_{x \in B} \log(e + d(x_0, x)),$$

which then completes the proof of Proposition 10.4. □

The following corollary gives an equivalent characterization of the $\mathrm{BMO}^{\log}(\mathcal{X})$ norm.

Corollary 10.5. *Let μ satisfy the Ahlfors n-regular condition (10.1). Then both $\mathrm{BMO}^{\log}(\mathcal{X})$ and $\mathrm{BMO}_\phi(\mathcal{X})$ coincide with equivalent norms.*

Proof. Let φ be as in (2.4). It was shown by [31, Remark 4.1(iii)] that, for any ball B,

$$\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})} \sim \frac{\mu(B)}{\log(e + \frac{1}{\mu(B)}) + \sup_{x \in B} \log(e + d(x_0, x))},$$

which, combined with Proposition 10.4, immediately completes the proof of Corollary 10.5. □

Now, we state the main result of this section as follows, which is a generalization of the corresponding Euclidean case in [35, Theorem 3.3].

Theorem 10.6. *Assume that μ satisfies the Ahlfors n-regular condition (10.1). The set of pointwise multipliers for $\mathrm{BMO}(\mathcal{X})$, denoted by $\mathrm{PWM}(\mathrm{BMO}(\mathcal{X}))$, is the dual space of $L^1(\mathcal{X}) + H^{*,\varphi}(\mathcal{X})$, where φ is as in (2.4).*

Proof. From [52, Example 2.8], it follows that $\mathrm{PWM}(\mathrm{BMO}(\mathcal{X})) = [L^\infty(\mathcal{X}) \cap \mathrm{BMO}^\varphi(\mathcal{X})]$. Thus, we only need to prove that $[L^\infty(\mathcal{X}) \cap \mathrm{BMO}^\varphi(\mathcal{X})] = (L^1(\mathcal{X}) + H^{*,\varphi}(\mathcal{X}))^*$.

From Propositions 10.2 and 10.4, Corollary 10.5 and some arguments similar to those used in the proof of [35, Theorem 3.3], we deduce that the above conclusion obviously holds true, which completes the proof of Theorem 10.6. □

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