COMPLEX OSCILLATION OF SOLUTIONS OF A THIRD ORDER ODE

Aimo Hinkkanen, Katsuya Ishizaki, Ilpo Laine and Kin Y. Li

University of Illinois at Urbana-Champaign, Department of Mathematics
1409 W. Green Street, Urbana, Illinois 61801-2975, U.S.A.; aimo@math.uiuc.edu

The Open University of Japan, Faculty of Liberal Arts
Mihama-ku, Chiba, Japan; ishizaki@ouj.ac.jp

University of Eastern Finland, Department of Physics and Mathematics
P.O. Box 111, FI-80101 Joensuu, Finland; ilpo.laine@uef.fi

Hong Kong University of Science and Technology, Department of Mathematics
Clear Water Bay, Kowloon, Hong Kong; makyli@ust.hk

Abstract. We solve a determinant problem related to a third order complex linear differential equation studied by Chiang, Laine and Wang. As a consequence, a simple procedure to explicit determination of the corresponding solutions is presented.

1. Introduction

For complex differential equations, there is an interesting line of investigations on the distribution of the zeros of the meromorphic solutions. If a non-constant meromorphic function $f(z)$ has the zeros $z_1, z_2, \ldots$ (counted according to multiplicity and ordered by non-decreasing moduli), the *exponent of convergence* $\lambda(f)$ of f is defined as $\inf\{\lambda \geq 0 \colon \sum_{z_j \neq 0} 1/|z_j|^\lambda < \infty\}$ if there is a real number $\lambda > 0$ such that the series $\sum_{z_j \neq 0} 1/|z_j|^\lambda$ converges, and $\lambda(f) = \infty$ otherwise. The *oscillation theory* of complex differential equations (see [3] and [11]) investigates the exponents of convergence of the solutions.

In [4], Bank and Laine studied the oscillation theory of certain second order differential equations with entire periodic coefficients and found explicit representations for the solutions with finite exponents of convergence.

As a concrete special case, Bank, Laine and Langley [5] proved the following result, see also [6]:

Theorem 1.1. *Let K be a non-zero complex number. The equation*

$$f'' + (e^z - K)f = 0$$

admits a non-trivial solution f that satisfies $\lambda(f) < \rho(f) = +\infty$ if, and only if,

$$K = \frac{(2n+1)^2}{16}$$

for some non-negative integer n, and

$$f(z) = \psi(e^{z/2}) \exp\left(de^{z/2} - \frac{2n+1}{4}z\right),$$

where $d^2 + 4 = 0$ and $\psi(\zeta) = \sum_{j=0}^{n} c_j \zeta^j$.

Actually, it is well known [11], [6] that the case $K = \frac{(2n+1)^2}{16}$ above determines two linearly independent solutions $f_\pm$ with $d_\pm = \pm 2i$ and the polynomial $\psi_\pm(e^{z/2})$ as described in Theorem 1.1. The true nature of the polynomials $\psi_\pm$ has been described in terms of special functions by Chiang and Ismail, see [7]:

Theorem 1.2. *Given $K = (2n+1)^2/16$ with $n \geq 0$, $n \in \mathbf{Z}$, the equation*

$$f'' + (e^z - K)f = 0$$

admits two linearly independent solutions

$$f_\pm(z) = \theta_n(\mp 2ie^{z/2}) \exp\left(\pm 2ie^{z/2} - \frac{2n+1}{4}z\right),$$

where $\theta_n(\zeta)$ is the reverse Bessel polynomial of degree n.

As for the notion of reverse Bessel polynomials, see [9] and [7]. Theorem 1.2 here is a special case of a more general result in [7], Theorem 1.2.

We assume that the reader is familiar with the notation and basic results of the Nevanlinna value distribution theory as given in [10]. In [8], Theorem 1, Chiang, Laine, and Wang established a somewhat similar result concerning the oscillation theory of a third order linear differential equation with entire periodic coefficients using the Nevanlinna theory:

Theorem 1.3. *Let $K \in \mathbf{C}$ and suppose that*

$$(1.1) \qquad\qquad f''' - Kf' + e^z f = 0$$

admits a non-trivial solution f such that

$$(1.2) \qquad\qquad \log^+ N(r, 1/f) = o(r) \quad \text{as } r \to \infty.$$

Then there exist integers r and s such that $r + s \geq 0$ and

$$(1.3) \qquad\qquad K = \frac{(r+s+1)^2}{9}.$$

Moreover, if $n = r + s > 0$, then n satisfies the following tridiagonal $(n+1) \times (n+1)$-determinant condition:

$$(1.4) \qquad\qquad \det \mathbf{A} = 0,$$

where the non-zero diagonals of $\mathbf{A}$ are determined by

$$(1.5) \qquad \begin{cases} a_{j,j-1} := (j-1)j(j+1) - 2jn - jn^2, & j = 1, \ldots, n, \\ a_{j,j} := -3j(j+1) + 2n + n^2, & j = 0, \ldots, n, \\ a_{j,j+1} := 3(j+1), & j = 0, \ldots, n-1. \end{cases}$$

Furthermore, f admits one of the following representations:

$$(1.6) \qquad\qquad f_i(z) = e^{-(s+1)z/3}\psi(e^{z/3})\exp(c_i e^{z/3}),$$

where $c_i^3 + 27 = 0$, $i = 1, 2, 3$, and

$$(1.7) \qquad\qquad \psi(\zeta) = \sum_{j=-r}^{s} d_j \zeta^j, \quad d_{-r}d_s \neq 0.$$

Conversely, suppose K takes the form (1.3) and $n = r + s > 0$ and n satisfies (1.4) and (1.5). Then there exists a rational function of the form (1.7) such that the three functions defined by (1.6) are linearly independent solutions of (1.1), each with exponent of convergence $\lambda(f_i) \leq 1$ for $i = 1, 2, 3$.

Remark. Theorem 1.3 above contains a slight inaccuracy. Namely, in the representations $f_i(z)$ in (1.6), the rational component ψ actually depends on i, $i = 1, 2, 3$. Therefore, (1.6) should in fact be written as

$$(1.8) \qquad f_i(z) = e^{-(s+1)z/3} \psi_i(e^{z/3}) \exp(c_i e^{z/3}),$$

and similarly (1.7) as

$$(1.9) \qquad \psi_i(\zeta) = \sum_{j=-r}^{s} d_{i,j} \zeta^j, \quad d_{i,-r} d_{i,s} \neq 0.$$

The converse conclusion in Theorem 1.3 then claims the existence of three rational functions of the form (1.9) that determine by (1.8) three linearly independent solutions of (1.1). For a new, more compact formulation of Theorem 1.3, see Theorem 5.1 below.

Remark. As seen in Theorem 5.1 below, the core of $\psi_i(\zeta)$ may be expressed as polynomials of ζ. It is possible that these polynomials may be expressed in terms of some special functions, as was the case in Theorem 1.1 above.

In [8], Chiang, Laine and Wang expressed the conjecture that the determinant condition $\det \mathbf{A} = 0$ in (1.4) and (1.5), might be equivalent to $n \not\equiv 2 \pmod 3$, meaning that K would be a ninth of a perfect square, but not an integer. This conjecture has remained open, although it has been verified numerically up to $n = 1000$, thanks to S. H. Lui at the Hong Kong University of Science and Technology. We also thank Y.-M. Chiang (Hong Kong University of Science and Technology) for his active participation in several discussions about this problem.

The key result in this paper is now to prove this conjecture:

Theorem 1.4. *The determinant condition* $\det \mathbf{A} = 0$ *in (1.4) and (1.5), holds if, and only if,* $n \not\equiv 2 \pmod 3$.

In what follows, if we wish to emphasize the dimension $n + 1$, we denote $\mathbf{A}$ by $\mathbf{A}_n$. To give a feeling for the determinants $D_n := \det \mathbf{A}_n$, note that if we adopt $N = n(n + 2)$ as a variable in (1.5), then D_n is a non-constant polynomial in N of degree $n + 1$. The first determinants D_n, for $n = 0, \ldots, 6$, may now be calculated to obtain

$$D_0 = N, \ D_1 = N(N - 3), \ D_2 = N(N - 3)(N - 6),$$
$$D_3 = N(N - 3)(N - 15)N, \ D_4 = N(N - 3)(N - 15)(N - 24)(N + 12),$$
$$D_5 = N(N - 3)(N - 15)(N - 24)(N^2 - 3N - 1080),$$
$$D_6 = N(N - 3)(N - 15)(N - 24)(N - 48)(N^2 + 27N - 1890).$$

2. Transformation of A

In what follows, we first apply elementary linear algebra, see, e.g., [12], to transform $\mathbf{A}$ by multiplication of finitely many elementary matrices to an upper triangular matrix $\mathbf{B}$. Then it is clear, by the elementary matrix transformations defined next, that $\det \mathbf{A} \neq 0$ if, and only if, all diagonal elements of $\mathbf{B}$ are non-vanishing.

As for the transformations, first denote by $P_i(c)$ a matrix obtained from the unit matrix by multiplying the i-th diagonal component with a number $c \neq 0$. Multiplication of a matrix A by $P_i(c)$ on the left then means to multiply by c the elements of the i-th row of A. Another elementary matrix $P_{ij}(c)$, to be needed below, is defined by setting i, j-component as c for $i \neq j$, and all other components 0 except the diagonal

components which are equal to 1. Multiplication of a matrix A by $P_{i,j}(c)$ on the left then means to add c times the j-th row of A to the i-th row of A.

We now consider an $(n+1) \times (n+1)$ tridiagonal matrix $A = (a_{i,j})$ such that $a_{ij} = 0$, whenever $|i - j| \geq 2$:

$$
(2.1) \qquad A = \begin{pmatrix}
a_{0,0} & a_{0,1} & 0 & & & & \\
a_{1,0} & a_{1,1} & a_{1,2} & 0 & & \text{\Large 0} & \\
0 & a_{2,1} & a_{2,2} & a_{2,3} & 0 & & \\
& 0 & a_{3,2} & a_{3,3} & \ddots & \ddots & \\
& & 0 & \ddots & \ddots & \ddots & 0 \\
& \text{\Large 0} & & \ddots & \ddots & \ddots & a_{n-1,n} \\
& & & & 0 & a_{n,n-1} & a_{n,n}
\end{pmatrix},
$$

with $a_{0,0} \neq 0$. By means of elementary matrix multiplications defined above, we transform A to an upper triangular matrix A_n as follows:

As the first step,

$$A \to A_1 = P_{1,0}(-a_{1,0}) P_1(a_{0,0}) A$$

$$
= \begin{pmatrix}
a_{0,0} & a_{0,1} & 0 & & & & \\
0 & a_{1,1}a_{0,0} - a_{0,1}a_{1,0} & a_{1,2}a_{0,0} & 0 & & \text{\Large 0} & \\
0 & a_{2,1} & a_{2,2} & a_{2,3} & 0 & & \\
& 0 & a_{3,2} & a_{3,3} & \ddots & \ddots & \\
& & 0 & \ddots & \ddots & \ddots & 0 \\
& \text{\Large 0} & & \ddots & \ddots & \ddots & a_{n-1,n} \\
& & & & 0 & a_{n,n-1} & a_{n,n}
\end{pmatrix}.
$$

For what follows, we write $a_{0,0} = \beta_{0,0}$ and $a_{1,1}a_{0,0} - a_{0,1}a_{1,0} = \beta_{1,1}$. Then

$$
A_1 = \begin{pmatrix}
\beta_{0,0} & a_{0,1} & 0 & & & & \\
0 & \beta_{1,1} & a_{1,2}\beta_{0,0} & 0 & & \text{\Large 0} & \\
0 & a_{2,1} & a_{2,2} & a_{2,3} & 0 & & \\
& 0 & a_{3,2} & a_{3,3} & \ddots & \ddots & \\
& & 0 & \ddots & \ddots & \ddots & 0 \\
& \text{\Large 0} & & \ddots & \ddots & \ddots & a_{n-1,n} \\
& & & & 0 & a_{n,n-1} & a_{n,n}
\end{pmatrix}.
$$

As for the next step, writing $\beta_{11}a_{22} - a_{21}a_{12}\beta_{00} = \beta_{22}$, we obtain

$$A_1 \to A_2 = P_{2,1}(-a_{2,1}) P_2(\beta_{1,1}) A_1$$

$$
= \begin{pmatrix}
\beta_{0,0} & a_{0,1} & 0 & & & & \\
0 & \beta_{1,1} & a_{1,2}\beta_{0,0} & 0 & & \text{\Large 0} & \\
0 & 0 & \beta_{1,1}a_{2,2} - a_{2,1}a_{1,2}\beta_{0,0} & a_{2,3}\beta_{1,1} & 0 & & \\
& & & 1 & & & \\
0 & & a_{3,2} & a_{3,3} & \ddots & \ddots & \\
& & 0 & \ddots & \ddots & \ddots & 0 \\
& \text{\Large 0} & & \ddots & \ddots & \ddots & a_{n-1,n} \\
& & & & 0 & a_{n,n-1} & a_{n,n}
\end{pmatrix}
$$

$$= \begin{pmatrix} \beta_{0,0} & a_{0,1} & 0 & & & & \\ 0 & \beta_{1,1} & a_{1,2}\beta_{0,0} & 0 & & \mathbf{0} & \\ 0 & 0 & \beta_{2,2} & a_{2,3}\beta_{1,1} & 0 & & \\ & 0 & a_{3,2} & a_{3,3} & \ddots & \ddots & \\ & & 0 & \ddots & \ddots & \ddots & 0 \\ & \mathbf{0} & & \ddots & \ddots & \ddots & a_{n-1,n} \\ & & & & 0 & a_{n,n-1} & a_{n,n} \end{pmatrix}.$$

We now continue the process inductively by

$$(2.2) \qquad \beta_{j+1,j+1} = \beta_{j,j}\, a_{j+1,j+1} - a_{j+1,j}\, a_{j,j+1}\, \beta_{j-1,j-1}, \quad j = 1, 2, \ldots, n-1$$

with $\beta_{0,0} = a_{0,0} \neq 0$ and $\beta_{1,1} = \beta_{0,0}a_{1,1} - a_{0,1}a_{1,0}$, to obtain

$$(2.3) \qquad B = \begin{pmatrix} \beta_{0,0} & a_{0,1} & 0 & & & \mathbf{0} & \\ 0 & \beta_{1,1} & a_{1,2}\beta_{0,0} & 0 & & & \\ 0 & 0 & \beta_{2,2} & a_{2,3}\beta_{1,1} & 0 & & \\ & 0 & 0 & \beta_{3,3} & \ddots & \ddots & \\ & & 0 & \ddots & \ddots & \ddots & 0 \\ & \mathbf{0} & & \ddots & \ddots & \ddots & a_{n-1,n}\,\beta_{n-1,n-1} \\ & & & & 0 & 0 & \beta_{n,n} \end{pmatrix}.$$

We now specify A to be our original tridiagonal matrix $\mathbf{A}_n$ defined in (1.5), where n is a fixed integer. Define as B the transformed matrix, obtained from A for this special case $A = \mathbf{A}_n$. By (1.5), we have $\beta_{0,0} = n(n+2)$ and

$$\beta_{1,1} = \beta_{0,0}a_{1,1} - a_{0,1}a_{1,0}$$
$$= n(n+2)(-6 + 2n + n^2) + (2n + n^2) \cdot 3$$
$$(2.4) \qquad\qquad = (n-1)n(n+2)(n+3).$$

Defining $N := \beta_{0,0} = n(n+2)$, we have $\beta_{1,1} = N(N-3)$ by (2.4). Making use of N, we may write (1.5) as

$$(2.5) \qquad \begin{cases} a_{j,j-1} := (j-1)j(j+1) - jN, & j = 1, \ldots, n, \\ a_{j,j} := -3j(j+1) + N, & j = 0, \ldots, n, \\ a_{j,j+1} := 3(j+1), & j = 0, \ldots, n-1. \end{cases}$$

It follows from $\beta_{0,0} = N$, $\beta_{1,1} = N(N-3)$, (2.2) and (2.5) that $\beta_{j,j}$, $j = 0, 1, \ldots, n$ are polynomials in N. Furthermore, we have

Lemma 2.1. *Let* $\beta_{j,j}$, $j = 0, 1, 2\ldots$ *be defined by (2.2) with* $\beta_{0,0} = N$ *and* $\beta_{1,1} = N(N-3)$. *Then* $\beta_{j,j}$, $j = 0, 1, 2\ldots$ *are polynomials in* N *of degree* $j+1$, *and coefficient of* N^{j+1} *in* $\beta_{j,j}$ *equals* 1.

Proof. Assume that the assertion holds for the cases j and $j - 1$. By (2.5), $\beta_{j,j}\, a_{j+1,j+1}$ is a polynomial in N of degree $j + 2$ and the coefficient of N^{j+2} is 1. Moreover, $-a_{j+1,j}a_{j,j+1}\beta_{j-1,j-1}$ is a polynomial of degree $j + 1$ in N. This implies that $\beta_{j+1,j+1}$ is a polynomial of degree $j + 2$ in N, and the coefficient of N^{j+2} equals 1 by (2.2), proving the lemma by induction. $\qquad\square$

3. Proof of Theorem 1.4, the first part

We first prove the following proposition, which is the easy part in proving Theorem 1.4:

Proposition 3.1. *If* $\det \mathbf{A} = \det \mathbf{B} = 0$, *then* $n \not\equiv 2 \pmod 3$.

Proof. We will show that $n \equiv 2, (\text{or} -1) \pmod 3$ implies $\det \mathbf{A} \neq 0$. Suppose that $n \equiv 2 \pmod 3$. Then there exists an integer k such that $n = 3k - 1$, which gives

$$\beta_{0,0} \equiv m = n(n+2) = (3k-1)(3k+1) = 9k^2 - 1 \equiv -1 \pmod 3$$

and

$$\beta_{1,1} = \beta_{00}a_{11} - a_{01}a_{10} = (3k-2)(3k-1)(3k+1)(3k+2)$$
$$= (9k^2 - 4)(9k^2 - 1) = 81k^4 - 45k^2 + 4 \equiv 1 \pmod 3.$$

We complete the proof by induction. Assume that for $j \leq \ell$, $\beta_{j,j} \equiv -1 \pmod 3$ if j is even, and $\beta_{j,j} \equiv 1 \pmod 3$ if j is odd. We note that $a_{j,j} = -3j(j+1) + m = -1 \pmod 3$. In case ℓ is odd, we have some integers $k_1, \ldots, k_4$ such that

$$\beta_{\ell+1,\ell+1} = \beta_{\ell,\ell}\, a_{\ell+1,\ell+1} - a_{\ell+1,\ell}\, a_{\ell,\ell+1}\, \beta_{\ell-1,\ell-1}$$
$$= (3k_1 + 1)(3k_2 - 1) - 3k_3 = 3k_4 - 1 \equiv -1 \pmod 3.$$

Similarly, we obtain $\beta_{\ell+1,\ell+1} \equiv 1 \pmod 3$ when ℓ is even. This gives that $\beta_{j,j}$, $j = 0, 1, \ldots, n$ do not vanish in (2.3), and hence $\det \mathbf{A} \neq 0$. $\square$

4. Proof of Theorem 1.4, the converse part

4.1. Notations and definitions. We define the set E of certain integers by

(4.1) $$E = \{n \in \mathbf{Z} \colon n \geq 0,\ n \not\equiv 2 \pmod 3\}.$$

Thus $E = \{0, 1, 3, 4, 6, 7, \ldots\}$.

We set $\alpha_{-1} = \alpha_{-1}(N) = 1$ and define for $j \geq 0$,

(4.2) $$\alpha_j = \alpha_j(N) = \prod_{0 \leq k \leq j,\ k \in E} (N - k(k+2)).$$

Thus α_j is a polynomial of degree $[2(j+2)/3]$ in N (where for a real number x, the notation $[x]$ means the largest integer $\leq x$) and

$$\alpha_0 = N,\ \alpha_1 = \alpha_2 = N(N-3),\ \alpha_3 = N(N-3)(N-15),$$
$$\alpha_4 = \alpha_5 = N(N-3)(N-15)(N-24),$$
$$\alpha_6 = N(N-3)(N-15)(N-24)(N-48).$$

We write $\det \mathbf{A}_n$ in the form

(4.3) $$\det \mathbf{A}_n = \alpha_n Q_n(N).$$

Then $Q_n(N)$ is a rational function of N. We will prove later that Q_n is a polynomial in N, but as long as we do not know that, we should think of Q_n as a rational function. Since α_n is a polynomial in N (and so is $\det \mathbf{A}_n$), there are at most finitely many real values of N for which $Q_n(N)$ is not a well-defined real number.

From the formulas for D_n above we see that

$$Q_0 = Q_1 = 1,\ Q_2 = N - 6,\ Q_3 = N,\ Q_4 = N + 12,$$
$$Q_5 = N^2 - 3N - 1080,\ Q_6 = N^2 + 27N - 1890.$$

We next derive recursion formulas for the functions D_j and after that for the functions Q_j.

The last row of $\mathbf{A}_n$ has only two non-zero elements, and they are the last two elements. Developing $\det \mathbf{A}_n$ according to the last row, we get

$$D_n = \det \mathbf{A}_n = a_{nn} D_{n-1} - a_{n,n-1} \det B,$$

where the matrix B is obtained by deleting from $\mathbf{A}$ the last row and the second last column. The last row of B has only two non-zero elements, and they are the last two elements, namely, $a_{n-1,n-2}$ and $a_{n-1,n}$. Developing $\det B$ according to the last row of B, we get

$$\det B = a_{n-1,n} D_{n-2} - a_{n-1,n-2} \det B',$$

where the matrix B' is obtained by deleting from B the last row and the second last column. Thus the last column of B' is a zero column, so that $\det B' = 0$. This yields

$$(4.4) \qquad D_n = a_{nn} D_{n-1} - a_{n,n-1} a_{n-1,n} D_{n-2}.$$

Substituting $D_j = \alpha_j Q_j(N)$ for $j \in \{n-2, n-1, n\}$ into this equation, and noting that $a_{nn} = N - 3n(n+1)$, $a_{n,n-1} = (n-1)n(n+1) - nN$, and $a_{n-1,n} = 3n$, we obtain

$$(4.5) \qquad \begin{aligned} \alpha_n Q_n(N) &= (N - 3n(n+1))\alpha_{n-1} Q_{n-1}(N) \\ &\quad - n((n-1)(n+1) - N)3n\alpha_{n-2} Q_{n-2}(N). \end{aligned}$$

We divide the further development of this formula into three cases depending on the residue class of n modulo 3.

Suppose that $n = 3k$ for some integer k. Then

$$\alpha_{n-1}/\alpha_n = \alpha_{n-2}/\alpha_n = 1/(N - n(n+2)).$$

Thus by (4.5),

$$(4.6) \qquad (N - n(n+2))Q_n = (N - 3n(n+1))Q_{n-1} + 3n^2(N - n^2 + 1)Q_{n-2}.$$

This is compatible with all the functions Q_j being rational functions of N. We note that if Q_{n-2} and Q_{n-1} are polynomials, then so is Q_n if, and only if, the right hand side of (4.6) is divisible by $N - n(n+2)$, which is true if, and only if, the right hand side of (4.6) is equal to zero when $N = n(n+2)$. This is indeed the case, as we will show later, but right now we do not know that so that we only think of Q_n as a rational function of N.

Suppose that $n = 3k + 1$ for some integer k. Then

$$\frac{\alpha_{n-1}}{\alpha_n} = \frac{1}{N - n(n+2)}, \quad \frac{\alpha_{n-2}}{\alpha_n} = \frac{1}{(N - n(n+2))(N - (n-1)(n+1))}.$$

Thus by (4.5),

$$(4.7) \qquad (N - n(n+2))Q_n = (N - 3n(n+1))Q_{n-1} + 3n^2 Q_{n-2}.$$

The comments regarding the possibility of the Q_j being polynomials that were made in the case $n = 3k$ apply also when $n = 3k + 1$.

Suppose that $n = 3k + 2$ for some integer k. Then

$$\alpha_{n-1}/\alpha_n = 1, \quad \alpha_{n-2}/\alpha_n = \frac{1}{N - (n-1)(n+1)}.$$

Thus by (4.5),

$$(4.8) \qquad Q_n = (N - 3n(n+1))Q_{n-1} + 3n^2 Q_{n-2}.$$

So when $n = 3k + 2$, if Q_{n-2} and Q_{n-1} are polynomials, then so is Q_n, and there are no extra conditions.

We have now established the recursion formulas for the functions D_j and the functions Q_j. We collect the results of this subsection together in the following lemma.

Lemma 4.1. *Suppose that $n \geq 1$ and write $D_n = \det \mathbf{A}_n$, so that D_n is a polynomial in the variable N. With α_n given by (4.2) (where E is as in (4.1)), write $D_n = \alpha_n Q_n$ where Q_n is a rational function of N. Then $Q_0(N) = Q_1(N) = 1$, and the functions Q_n satisfy the following recursion formulas. If $n = 3k$ for some integer k, then*

$$(4.9) \qquad (N - n(n+2))Q_n = (N - 3n(n+1))Q_{n-1} + 3n^2(N - n^2 + 1)Q_{n-2}.$$

If $n = 3k + 1$ for some integer k, then

$$(4.10) \qquad (N - n(n+2))Q_n = (N - 3n(n+1))Q_{n-1} + 3n^2 Q_{n-2}.$$

If $n = 3k + 2$ for some integer k, then

$$(4.11) \qquad Q_n = (N - 3n(n+1))Q_{n-1} + 3n^2 Q_{n-2}.$$

4.2. Formulas for the solution v of Av = 0. Suppose that $\mathbf{v} = (x_0, x_1, \ldots, x_n)$ and $\mathbf{Av} = 0$. We may express the equations to be satisfied by $x_0, x_1, \ldots, x_n$ as follows. Considering the jth component of the equation $\mathbf{Av} = 0$, we obtain

$$(4.12) \qquad a_{j,j-1}x_{j-1} + a_{jj}x_j + a_{j,j+1}x_{j+1} = 0.$$

When $j = 0$, this reads

$$(4.13) \qquad Nx_0 + 3x_1 = 0.$$

Thus

$$(4.14) \qquad x_1 = -\frac{N}{3}x_0.$$

When $j = n$, this reads

$$(4.15) \qquad ((n-1)n(n+1) - nN)x_{n-1} + (N - 3n(n+1))x_n = 0.$$

In the special case when $N = n(n+2)$, this reads

$$(4.16) \qquad x_n = -x_{n-1}.$$

When $1 \leq j \leq n - 1$, we get the equation

$$(4.17) \qquad ((j-1)j(j+1) - jN)x_{j-1} + (N - 3j(j+1))x_j + 3(j+1)x_{j+1} = 0.$$

Using the equation (4.17) inductively, starting with $j = 1$, we can proceed up to $j = n - 1$ to get each x_j as x_0 multiplied by a constant depending on j. This is also true for x_1 by (4.14). It is then clear that the equation $\mathbf{Av} = \mathbf{0}$ (when $N = n(n+2)$) has a non-zero solution $\mathbf{v}$ if, and only if, the values of x_{n-1} and x_n so obtained satisfy (4.16) identically, that is, for all values of x_0.

When we solve for x_{j+1} from (4.17), we are dividing by the number $3(j+1)$ that is independent of N. Hence it is clear that regarding N as a variable, we obtain each x_j for $1 \leq j \leq n$ as a polynomial in N (where we have not used (4.15) yet).

We further note that when we use N as a variable, instead of taking for N a specific value that depends on n, the formulas that we obtain for x_j are independent of n. That is, if we specify any value for j, and if we then choose any value for n with $n \geq j$, we get the same formula for x_j regardless of the choice of n.

We set $\beta_{-1} = 1$, and for $j \geq 0$, we define

$$(4.18) \qquad \beta_j = \prod_{k=0}^{j}(3(k+1)) = 3^{j+1}((j+1)!).$$

Recall that α_j is defined by (4.2) and that $\alpha_{-1} = 1$.

We now write x_j for $1 \leq j \leq n$ in the form

$$(4.19) \qquad x_j = (-1)^j \frac{\alpha_{j-1}}{\beta_{j-1}} x_0 P_{j-1}(N).$$

As is compatible with the rest of this notation, we set $P_{-1}(N) = 1$. Then (4.19) is valid for $j = 0$ also. It then follows that $P_j(N)$, for $-1 \leq j \leq n-1$, is a rational function of N. We will prove later that $P_j = Q_j$.

From (4.14) we obtain

$$(4.20) \qquad P_0(N) = 1.$$

We next derive recursion formulas for the functions P_j. Substituting (4.19) into (4.17), we obtain

$$-3(j+1)(-1)^{j+1}\frac{\alpha_j}{\beta_j} x_0 P_j(N)$$

$$= ((j-1)j(j+1) - jN)(-1)^{j-1}\frac{\alpha_{j-2}}{\beta_{j-2}} x_0 P_{j-2}(N)$$

$$(4.21) \qquad + (N - 3j(j+1))(-1)^j \frac{\alpha_{j-1}}{\beta_{j-1}} x_0 P_{j-1}(N).$$

For this to be valid when $x_0 \neq 0$, we must have

$$(4.22) \qquad \frac{\alpha_j}{\beta_j} P_j = \frac{N - 3j(j+1)}{3(j+1)}\frac{\alpha_{j-1}}{\beta_{j-1}} P_{j-1} + \frac{N - j^2 + 1}{3(j+1)} j \frac{\alpha_{j-2}}{\beta_{j-2}} P_{j-2}.$$

We again divide the further development of this formula into three cases depending on the residue class of j modulo 3.

Suppose that $j = 3k$ for some integer k. As we saw before, then

$$(4.23) \qquad \alpha_{j-1}/\alpha_j = \alpha_{j-2}/\alpha_j = 1/(N - j(j+2)).$$

Also

$$(4.24) \qquad \frac{\beta_{j-1}}{\beta_j} = \frac{1}{3(j+1)}, \quad \frac{\beta_{j-2}}{\beta_j} = \frac{1}{9j(j+1)}.$$

Thus by (4.22),

$$(4.25) \qquad (N - j(j+2))P_j = (N - 3j(j+1))P_{j-1} + 3j^2(N - j^2 + 1)P_{j-2}.$$

Suppose that $j = 3k + 1$ for some integer k. As we saw before, then

$$(4.26) \qquad \frac{\alpha_{j-1}}{\alpha_j} = \frac{1}{N - j(j+2)}, \quad \frac{\alpha_{j-2}}{\alpha_j} = \frac{1}{(N - j(j+2))(N - (j-1)(j+1))}.$$

Also

$$(4.27) \qquad \frac{\beta_{j-1}}{\beta_j} = \frac{1}{3(j+1)}, \quad \frac{\beta_{j-2}}{\beta_j} = \frac{1}{9j(j+1)}.$$

Thus by (4.22),

$$(4.28) \qquad (N - j(j+2))P_j = (N - 3j(j+1))P_{j-1} + 3j^2 P_{j-2}.$$

When $j = 3k$ or $j = 3k+1$, the comments regarding the possibility of the various P_ℓ being polynomials are similar to the comments that were made for the Q_ℓ.

Suppose that $j = 3k + 2$ for some integer k. As we saw before, then

$$(4.29) \qquad \alpha_{j-1}/\alpha_j = 1, \ \alpha_{j-2}/\alpha_j = \frac{1}{N - (j-1)(j+1)}.$$

Also

$$(4.30) \qquad \frac{\beta_{j-1}}{\beta_j} = \frac{1}{3(j+1)}, \ \frac{\beta_{j-2}}{\beta_j} = \frac{1}{9j(j+1)}.$$

Thus by (4.22),

$$(4.31) \qquad P_j = (N - 3j(j+1))P_{j-1} + 3j^2 P_{j-2}.$$

When $j = 3k + 2$, the function P_j is a polynomial if both P_{j-1} and P_{j-2} are polynomials, without any extra conditions.

We have now established the recursion formulas for the functions P_j. We have found P_0 before (we have $P_0 = 1$). With $P_{-1} = 1$, (4.19) is valid with $j = 0$. We calculate a few more functions P_j. Taking $j = 1$ in (4.28), we obtain

$$(N - 3)P_1 = (N - 6)P_0 + 3P_{-1} = N - 6 + 3 = N - 3,$$

so that

$$P_1 = 1.$$

Taking $j = 2$ in (4.31), we obtain

$$P_2 = N - 6.$$

Continuing in the same way, we get

$$P_3 = N, \ P_4 = N + 12.$$

We see that

$$(4.32) \qquad P_j(N) = Q_j(N)$$

at least for $0 \le j \le 4$. We will see later that (4.32) is valid for all $j \ge 1$.

By now we have seen that the equation $\mathbf{A}\mathbf{v} = 0$ implies that for $1 \le j \le n$, each x_j is a certain multiple of x_0. In addition the equation $\mathbf{A}\mathbf{v} = 0$ for $\mathbf{v} \ne 0$ implies that (4.15) must be valid also when $x_0 \ne 0$, and for $N = n(n+2)$, (4.15) is equivalent to (4.16). Since x_{n-1} and x_n have already been determined, this imposes a further condition on P_{n-1} and P_{n-2}. We now determine that condition when $N = n(n+2)$.

Using (4.19) with $j = n - 1$ and $j = n$ and substituting the results into (4.16), we obtain, when $x_0 \ne 0$,

$$(4.33) \qquad \frac{\alpha_{n-2}}{\beta_{n-2}} P_{n-2}(N) = \frac{\alpha_{n-1}}{\beta_{n-1}} P_{n-1}(N).$$

We require this to be true only when n is of the form $3k$ or $3k + 1$.

Suppose that $n = 3k$ for some integer k, so that $n-1$ is of the form $3k+2$. Using (4.29) and (4.30) with j replaced by $n - 1$ in those formulas, we can write (4.33) in the form

$$3nP_{n-2}(N) = P_{n-1}(N).$$

Since we want this to be true when $N = n(n+2)$, we require, more precisely, that

$$(4.34) \qquad 3nP_{n-2}(n(n+2)) = P_{n-1}(n(n+2)).$$

Suppose that $n = 3k+1$ for some integer k, so that $n-1$ is of the form $3k$. Using (4.23) and (4.24) with j replaced by $n-1$ in those formulas, we can write (4.33) in the form

$$(4.35) \qquad 3nP_{n-2}(n(n+2)) = (N - (n-1)(n+1))P_{n-1}(n(n+2)).$$

When $N = n(n+2)$, we have

$$N - (n-1)(n+1) = n(n+2) - (n-1)(n+1) = 2n + 1.$$

Since we want (4.35) to be true when $N = n(n+2)$, we require, more precisely, that

$$(4.36) \qquad 3nP_{n-2}(n(n+2)) = (2n+1)P_{n-1}(n(n+2)).$$

We collect the results of this subsection together in the following lemma.

Lemma 4.2. *Suppose that $n \geq 1$. Then the equation $\mathbf{A}_n\mathbf{v} = 0$ has a non-zero solution $\mathbf{v} = (x_0, x_1, \ldots, x_n)$ if, and only if, with the notation (4.19), the rational functions $P_j(N)$ of N satisfy the recursion formulas given below for $1 \leq j \leq n-1$ and, in addition, (4.15) holds (hence (4.16) when $N = n(n+2)$). The recursion formulas are as follows. We have $P_{-1}(N) = P_0(N) = P_1(N) = 1$. If $j = 3k$ for some integer k, then*

$$(4.37) \qquad (N - j(j+2))P_j = (N - 3j(j+1))P_{j-1} + 3j^2(N - j^2 + 1)P_{j-2}.$$

If $j = 3k + 1$ for some integer k, then

$$(4.38) \qquad (N - j(j+2))P_j = (N - 3j(j+1))P_{j-1} + 3j^2 P_{j-2}.$$

If $j = 3k + 2$ for some integer k, then

$$(4.39) \qquad P_j = (N - 3j(j+1))P_{j-1} + 3j^2 P_{j-2}.$$

For the rest of this lemma, we assume that $N = n(n+2)$. If $n = 3k$ for some integer k, then (4.16) is equivalent to

$$(4.40) \qquad 3nP_{n-2}(n(n+2)) = P_{n-1}(n(n+2)).$$

If $n = 3k + 1$ for some integer k, then (4.16) is equivalent to

$$(4.41) \qquad 3nP_{n-2}(n(n+2)) = (2n+1)P_{n-1}(n(n+2)).$$

4.3. Connection between the formulas for $\det \mathbf{A}_n$ and $\mathbf{v}$. We note that the functions Q_j and P_j are defined in the same way, no matter which value of n with $n \geq j+1$ is considered to define Q_j and P_j. The following lemma expresses the relationship between Q_j and P_j.

Lemma 4.3. *For all $j \geq 0$, we have*

$$(4.42) \qquad P_j(N) = Q_j(N).$$

To prove Lemma 4.3, we merely have to note that (4.42) is valid for $j = 0$ and $j = 1$, and that the recursion formulas for Q_j are identical to those for P_j.

4.4. The final statements to be proved by induction. For $n \geq 1$ with $n \in E$, we define the statement $S(n)$ as follows.

If $n = 3k$ for some integer k, then $S(n)$ states that (4.40) holds.

If $n = 3k + 1$ for some integer k, then $S(n)$ states that (4.41) holds.

We will prove the following result by induction on n. Recall that the set E is defined by (4.1).

Lemma 4.4. *For every integer $n \geq 1$, all of the following statements are valid:*

(i) *For all j with $1 \le j \le n$ such that $j \in E$, we have*

(4.43) $$\det \mathbf{A}_j = 0$$

   when $N = j(j+2)$.
(ii) *For all j with $1 \le j \le n-1$, the function $P_j(N)$ (and hence also the function $Q_j(N)$) is a polynomial in N.*
(iii) *For all j with $1 \le j \le n$ such that $j \in E$, the statement $S(j)$ is valid.*

Clearly, after Lemma 4.4 has been proved, Theorem 1.4 follows immediately.

4.5. Proof of Lemma 4.4. To be able to use the recursion formulas for P_j, we need to prove two base cases first. If $n = 1$, then (ii) and (iii) of Lemma 4.4 are vacuous, while (i) states that $\det \mathbf{A}_1 = 0$, that is, $D_1 = 0$ when $N = n(n+2) = 3$. Since $D_1 = N(N-3)$, this is true.

As a side-remark, we note that if $n = 0$ and $N = n(n+2) = 0$, then $D_0 = N = 0$ also. We further know that $P_0(N) = Q_0(N) = 1$ is a polynomial in N.

Suppose that $n = 2$, so $n \notin E$. Then (i) imposes no further requirements beyond what was demanded for $n = 1$. For (ii), we need to know that $P_1(N)$ is a polynomial in N, which is true since $P_1(N) \equiv 1$. The condition (iii) requires that $S(1)$ is true, that is, that (4.41) holds when we take $n = 1$ in (4.41). This reads $3P_{-1}(3) = 3P_0(3)$, which is true since $P_{-1}(N) \equiv 1$ and $P_0(N) \equiv 1$.

For more convenient notation, suppose that $n \ge 3$ and that the claims of Lemma 4.4 are valid for $n - 1$ instead of n. For the induction step, we need to prove that these claims are valid for n. To do this, we need to prove the following. For (i), if $n \in E$, we need to prove that $\det \mathbf{A}_n = 0$ when $N = n(n+2)$ (while if $n \notin E$, we do not need to prove anything for (i)). For (ii), we need to prove that $P_{n-1}(N)$ is a polynomial in N. For (iii), we need to prove that if $n \in E$, then $S(n)$ is true (while if $n \notin E$, we do not need to prove anything for (iii)).

Let us begin with the proof of (ii), which we divide into three cases according to the residue class of n modulo 3. Thus we are assuming that $P_\ell(N)$ is a polynomial in N for $1 \le \ell \le n - 2$.

Suppose first that n is of the form $n = 3k$, so $n-1 = 3(k-1)+2$. Write $j = n-1$. Then the formula (4.31) is applicable, and it gives $P_j(N)$, that is, $P_{n-1}(N)$, as

(4.44) $$P_j(N) = (N - 3j(j+1))P_{j-1}(N) + 3j^2 P_{j-2}(N).$$

By the induction assumption, P_{j-1} and P_{j-2} are polynomials in N, so that $P_j(N)$ is also trivially a polynomial in N.

Suppose next that n is of the form $n = 3k + 1$, so $n - 1 = 3k$. Write $j = n - 1$. Then the formula (4.25) is applicable, and it gives $P_j(N)$, that is, $P_{n-1}(N)$, as

(4.45) $$(N - j(j+2))P_j(N) = (N - 3j(j+1))P_{j-1}(N) + 3j^2(N - j^2 + 1)P_{j-2}(N).$$

To prove that P_j is a polynomial is equivalent to proving that the right hand side of (4.45) is equal to zero when $N = j(j+2)$. We write the right hand side of (4.45) as

$$(N - j(j+2))(P_{j-1} + 3j^2 P_{j-2})$$
$$+ (j(j+2) - 3j(j+1))P_{j-1} + 3j^2(j(j+2) - j^2 + 1)P_{j-2}$$
$$= (N - j(j+2))(P_{j-1} + 3j^2 P_{j-2}) - j(2j+1)(P_{j-1} - 3j P_{j-2}).$$

Thus we need to prove that for $j = n - 1 = 3k \in E$, we have

(4.46) $$P_{j-1}(j(j+2)) = 3j P_{j-2}(j(j+2)).$$

Our induction assumption regarding (i) implies that $\det \mathbf{A}_{n-1} = 0$. Thus the equation $\mathbf{A}_{n-1}\mathbf{v} = 0$ has a non-zero solution $\mathbf{v}$, and now (4.40) of Lemma 4.2 implies (4.46) on replacing n by j in (4.40).

Suppose then that n is of the form $n = 3k+2$, so $n-1 = 3k+1$. Write $j = n-1$. Then the formula (4.28) is applicable, and it gives $P_j(N)$, that is, $P_{n-1}(N)$, as

$$(4.47) \qquad (N - j(j+2))P_j(N) = (N - 3j(j+1))P_{j-1}(N) + 3j^2 P_{j-2}(N).$$

To prove that P_j is a polynomial is equivalent to proving that the right hand side of (4.47) is equal to zero when $N = j(j+2)$. We write the right hand side of (4.47) as

$$(N - j(j+2))P_{j-1} + (j(j+2) - 3j(j+1))P_{j-1} + 3j^2 P_{j-2}$$
$$= (N - j(j+2))P_{j-1} - j((2j+1)P_{j-1} - 3j P_{j-2}).$$

Thus we need to prove that for $j = n-1 = 3k+1 \in E$, we have

$$(4.48) \qquad (2j+1)P_{j-1}(j(j+2)) = 3j P_{j-2}(j(j+2)).$$

Our induction assumption regarding (i) implies that $\det \mathbf{A}_{n-1} = 0$. Thus the equation $\mathbf{A}_{n-1}\mathbf{v} = 0$ has a non-zero solution $\mathbf{v}$, and now (4.41) of Lemma 4.2 implies (4.48) on replacing n by j in (4.41). This completes the proof that $P_{n-1}(N)$ is a polynomial in N, so that (ii) is proved for n.

Next, we prove (i), that is, we prove that $\det \mathbf{A}_n = 0$ if $n \in E$ and $N = n(n+2)$. For this, we only need to use (4.3) and note that since $P_n = Q_n$ by Lemma 4.3, also $Q_n(N)$ is a polynomial in N. Now, since $n \in E$, the product α_n contains the factor $N - n(n+2)$, so that $\alpha_n = 0$ and hence $\det \mathbf{A}_n = 0$ when $N = n(n+2)$. This proves (i) for n.

For (iii), suppose that $n \in E$. By what we have proved already, we know that $\det \mathbf{A}_n = 0$ when $N = n(n+2)$, so that for this N, the equation $\mathbf{A}_n\mathbf{v} = 0$ has a non-zero solution $\mathbf{v}$. By Lemma 4.2, the equation (4.40) holds if $n = 3k$, and the equation (4.41) holds if $n = 3k+1$. This gives $S(n)$. This completes the proof of Lemma 4.4 and hence also the converse part of Theorem 1.4.

5. Discussion

In [2], Bank constructed a method to test the second order linear differential equation

$$f'' + A(z)f = 0,$$

where $A(z)$ is a non-constant periodic entire function, for the existence of solutions f satisfying $\lambda(f) < \infty$, as well as to construct these solutions explicitly. The method in [2] involves about finding polynomial solutions of four second order linear differential equations. This method has been extended by Baesch [1] to test higher order linear differential equations of type

$$f^{(n)} + \sum_{j=1}^{n-2} A_j(z)f^{(j)} + A_0(z)f = 0,$$

where $A_1, \ldots, A_{n-2}$ are constants, $n \geq 3$, and A_0 is a non-constant periodic entire function. In this higher order case, n^2 linear differential equations of order n have to be tested for polynomial solutions. For a more general consideration where the coefficients are permitted to be rational functions of e^z, see [13].

Unfortunately, the procedure described in [2], [1] is somewhat difficult to be applied in practice, although it is certainly possible to implement a suitable software package for such a purpose. In the special case of (1.1) with the determinant condition

(1.4), it is much easier to get an explicit construction of solutions f with $\lambda(f) \leq 1$ from a reformulation of Theorem 1.3:

Theorem 5.1. *The linear differential equation (1.1) with $K \in \mathbf{C}$ admits a non-trivial solution f such that*

$$\log^+ N(r, 1/f) = o(r) \quad \text{as } r \to \infty,$$

if, and only if,

$$K = \frac{(n+1)^2}{9}$$

for some integer $n \geq 0$ such that $n \not\equiv 2 \pmod 3$. Moreover, in this case, three linearly independent such solutions f exist, corresponding to the three complex numbers c such that $c^3 + 27 = 0$. With $\zeta := e^{z/3}$, these solutions f have a representation of the form

$$(5.1) \qquad f(z) = e^{c\zeta} \zeta^{-1} Q(c^{-1}\zeta^{-1}),$$

where Q is a polynomial of degree n, independent of c. The function $Q(z)$ satisfies the differential equation

$$(5.2) \quad z^4 Q''' + z^2 (6z - 3) Q'' - (6z - 3 + (n(n+2) - 6)z^2) Q' - n(n+2)(z-1)Q = 0.$$

Furthermore, with $P_j(N)$ defined as in Section 4, with $\beta_j = 3^{j+1}((j+1)!)$ defined as in (4.18), and with $\alpha_j(N)$ defined as in (4.2), we may write

$$(5.3) \qquad Q(z) = 1 + \sum_{j=1}^{n} d_j z^j,$$

where

$$(5.4) \qquad d_j = (-1)^j \frac{\alpha_{j-1}(n(n+2))}{\beta_{j-1}} P_{j-1}(n(n+2)),$$

for $1 \leq j \leq n$. In particular, $\lambda(f) = 0$ for $n = 0$ and $\lambda(f) = 1$ for $n > 0$, in each case for all three choices of c.

Remark. Thus the numbers d_j are the same as the numbers x_j given by (4.19), when taking $x_0 = 1$ and $N = n(n+2)$ in (4.19). By Lemma 4.1 and Lemma 4.3, we may also write

$$(5.5) \qquad d_j = (-1)^j \frac{\det \mathbf{A}_{j-1}}{3^j(j!)}$$

where we take $N = n(n+2)$ when computing $\det \mathbf{A}_{j-1}$. Thus it has been useful for us to solve the equation $\mathbf{A}\mathbf{v} = 0$ not only to prove Theorem 1.4 but also to obtain the coefficients of Q for a more explicit solution of (1.1). In spite of their appearance, the numbers d_j are integers, but we will not prove that here.

Proof. Suppose that $n \geq 0$, $n \in E$, and $K = (n+1)^2/9$. We look for a solution f of (1.1) in the form $f(z) = F(ce^{z/3})$, where $c^3 = -27$. Substituting this f into (1.1), denoting in the resulting equation the quantity $ce^{z/3}$ by t and using the fact that $c^3 = -27$, we obtain for the function $F(t)$ the equation

$$(5.6) \qquad t^3 F'''(t) + 3t^2 F''(t) - n(n+2)t F'(t) - t^3 F(t) = 0.$$

This equation is independent of c, so that for each choice of c, a solution F of (5.6) yields a solution f of (1.1). We look for F in the form $F(t) = e^t g(1/t)$. Substituting

this F into (5.6) and denoting in the resulting equation $1/t$ by z, we obtain for $g(z)$ the equation

$$(5.7) \qquad z^3 g''' + 3z(z-1)g'' - \frac{3}{z}\left(\frac{g}{z} - g'\right) + n(n+2)\left(\frac{g}{z} - zg'\right) = 0.$$

Writing $g(z) = zQ(z)$ we get for the function $Q(z)$ the equation (5.2). Thus with $\zeta = e^{z/3}$, a solution Q of (5.2) gives the solution $f(z) = e^{c\zeta}(c\zeta)^{-1}Q(1/(c\zeta))$ of (1.1). Then also cf is a solution of (1.1), and this gives the right hand side of (5.1).

We next verify that (5.2) has a solution Q that is a polynomial. When $n = 0$, it is obvious that $Q(z) \equiv 1$ is a solution of (5.2). Suppose that $n \geq 1$ and $n \in E$. Write $N = n(n+2)$. Substituting

$$Q(z) = \sum_{j=0}^{n} d_j z^j$$

into (5.2) we get the recursion formulas for the d_j. We need to consider the coefficient of z^j on the left hand side of (5.2), for $0 \leq j \leq n+1$.

Considering the constant coefficient on the left hand side of (5.2), we get

$$(5.8) \qquad d_1 = -(N/3)d_0.$$

The coefficient of z^j for $1 \leq j \leq n-1$ yields

$$(5.9) \qquad 3(j+1)d_{j+1} + (N - 3j(j+1))d_j + (j^2 - 1 - N)jd_{j-1} = 0.$$

The coefficient of z^n gives

$$(5.10) \qquad (N - 3n(n+1))d_n + (n^2 - 1 - N)nd_{n-1} = 0,$$

and since $N = n(n+2)$, this is equivalent to

$$(5.11) \qquad d_n = -d_{n-1}.$$

The coefficient of z^{n+1} vanishes identically, regardless of the value of d_n.

The equations (5.8), (5.9), and (5.11) are the same as the equations (4.14), (4.17), and (4.16) for the x_j, simply with x_j replaced by d_j. As we have seen in Section 4, when $n \in E$ and $N = n(n+2)$, the equations for the x_j have a solution for all values of x_0, and hence the equations for the d_j have a solution for all values of d_0. Since a solution f can be multiplied by any non-zero complex number, we may take $d_0 = 1$, and then, considering (4.19), we obtain (5.4).

Thus we see that when $n \geq 0$, $n \in E$, and $K = (n+1)^2/9$, the equation (1.1) indeed has the solution f given by (5.1) where c, ζ, and Q are as stated, for each c with $c^3 = -27$. Clearly the three functions f obtained for the three values of c are linearly independent.

The first claim of Theorem 5.1 follows by combining Theorem 1.4 and Theorem 1.1 (taking into account the subsequent remark to this theorem). This completes the proof of Theorem 5.1. $\qquad \square$

Examples. We consider what the functions f look like for some small values of n when $n \geq 0$, $n \in E$, and $K = (n+1)^2/9$. Without further indexing, we allow c to be any one of the three values with $c^3 = -27$. Again, $\zeta = e^{z/3}$.

We first express solutions f in the style of (1.6). If $n = 0$, then $f(z) = \zeta^{-1}e^{c\zeta}$. If $n = 1$, then

$$f(z) = \zeta^{-2}(1 - c\zeta)e^{c\zeta}.$$

If $n = 4$, then

$$(5.12) \qquad f(z) = \zeta^{-5}\left(1 - c\zeta + \frac{1}{2}c^2\zeta^2 - \frac{1}{7}c^3\zeta^3 + \frac{1}{56}c^4\zeta^4\right)e^{c\zeta}.$$

Compare (5.12) to the fact that when $n = 4$, we have

$$Q(z) = 1 - 8z + 28z^2 - 56z^3 + 56z^4.$$

The difference is due to a different coefficient being normalized to be equal to 1. Apart from the normalizations, the polynomial Q is essentially the reverse (or the reciprocal polynomial) of the polynomial in $c\zeta$ in (5.12).

Regarding the expression (5.1) for the solution f, we note that when n gets larger, the coefficients of the polynomial Q become more complicated. For example, if $n = 7$, then

$$Q(z) = 1 - 21z + 210z^2 - 1330z^3 + 5880z^4 - 18200z^5 + 36400z^6 - 36400z^7.$$

If $n = 9$, then

$$Q(z) = 1 - 33z + 528z^2 - 5456z^3 + 40656z^4 - 227920z^5 + 960960z^6$$
$$ - 2932160z^7 + 5864320z^8 - 5864320z^9.$$

The formula (5.4) for the coefficients of Q might be used to consider the question mentioned before, see the Remark preceding Theorem 1.4, of whether ψ (or Q) is related to some known special functions.

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