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SMALL DISCS CONTAINING CONJUGATE ALGEBRAIC INTEGERS

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Abstract. In this note we show that, for any $\xi \in \mathbf{R}$, there is an infinite set of positive integers S such that, for each $d \in S$, the open disc with center at ξ and radius $1 + (\log\log d)^2/(2\log d)$ contains a full set of conjugates of an algebraic integer of degree d. A slightly better bound on the radius is established when $\xi \in \mathbf{Q} \setminus \mathbf{Z}$.

1. Introduction

For $E \subseteq \mathbf{C}$, the quantity

$$\tau(E) := \lim_{n\to\infty} \sup_{z_1,\dots,z_n \in E} \left(\prod_{1\le i<j\le n} |z_i - z_j| \right)^{2/n(n-1)}$$

is called the *transfinite diameter* (or *logarithmic capacity*) of E. It is known that a (closed or open) disc with radius R has transfinite diameter R, whereas an interval of lenght I has transfinite diameter $I/4$. In [7], Fekete has shown that every compact set E satisfying $\tau(E) < 1$ contains only finitely many full sets of conjugate algebraic integers over $\mathbf{Q}$. In particular, this result can be applied to every closed disc whose radius is smaller than 1 and to every real interval whose length is smaller than 4.

In the opposite direction, Fekete and Szegö [8] proved that if E is a compact set which is stable under complex conjugation and satisfies $\tau(E) \ge 1$, then its every complex neighborhood F (so that $E \subset F$ and F is an open set) contains infinitely many sets of conjugate algebraic integers. Furthermore, by the results of Robinson [15] and Ennola [4], every real interval of length strictly greater than 4 also contains infinitely many sets of conjugate algebraic integers.

In [18], Zaïmi gave a lower bound for the length of a real interval containing an algebraic integer of degree d and its conjugates. His result asserts that the length I of such an interval should be at least $4 - \phi(d)$, where $\phi(d)$ is some explicit positive function which tends to zero as $d \to \infty$. (For instance, one can take $\phi(d) = (c\log d)/d$ with some $c > 0$. Similar bound also follows from an earlier result of Schur [17].) On the other hand, the author has shown that, for infinitely many $d \in \mathbf{N}$, every real interval of length $4 + 4(\log\log d)^2/\log d$ contains an algebraic integer of degree d and its conjugates (see [2] and [3]). It is not known whether there is an interval $[t, t+4]$ with some $t \in \mathbf{R}\setminus\mathbf{Z}$ containing infinitely many full sets of algebraic integers. For $t \in \mathbf{Z}$, one can simply take infinitely many algebraic integers of the form $t+2\cos(\pi r)+2$, where $r \in \mathbf{Q}$. By Kronecker's theorem [13], these are the only such numbers in $[t, t+4]$ if $t \in \mathbf{Z}$.

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In this note, we shall consider discs with real centers and radii close 1. Suppose first that an algebraic integer α of degree d lies with its conjugates in the disc $|z-\xi| \le R_d$, where $\xi \in \mathbf{R}$ and $R_d > 0$. Then, we can write the discriminant

$$D = \prod_{1 \le i < j \le d} (\alpha_i - \alpha_j)^2 = \prod_{1 \le i < j \le d} \big((\alpha_i - \xi) - (\alpha_j - \xi)\big)^2$$

of α with conjugates $\alpha_1 = \alpha, \alpha_2, \dots, \alpha_d$ as the square of Vandermonde determinant with rows $(\alpha_1 - \xi)^{j-1}, \dots, (\alpha_d - \xi)^{j-1}$, where $j = 1, \dots, d$. Using the upper bound $\sum_{i=1}^{d} |\alpha_i - \xi|^{2(j-1)} \le dR_d^{2j-2}$, by Hadamard's inequality, we obtain

$$|D| \le \prod_{j=1}^{d} dR_d^{2j-2} = d^d R_d^{d(d-1)}.$$

This yields $R_d \ge |D_d|^{1/d(d-1)} d^{-1/(d-1)}$, where $|D_d|$ stands for the smallest discriminant of an algebraic number field of degree d. From $|D_d| > 1$ it follows that

$$R_d > d^{-1/(d-1)}$$

for $d \ge 2$. Moreover, $|D_d| > 22^d$ for d large enough (see [14]). Hence,

$$R_d > 1 - \frac{\log d}{d} \tag{1}$$

for d large enough. In the opposite direction we prove the following:

Theorem 1. *For any real number ξ, there is an infinite set of positive integers S such that, for each $d \in S$, the open disc with center at ξ and radius $1 + (\log\log d)^2/(2\log d)$ contains a full set of conjugates of an algebraic integer of degree d.*

Recall that the *diameter* of an algebraic integer α of degree d with conjugates α_i, $i = 1, \dots, d$, is defined by $\max_{1 \le i < j \le d} |\alpha_i - \alpha_j|$. In this context, Theorem 1 implies that the diameter of the algebraic integer of degree d whose existence is claimed in the theorem is less than $2 + (\log\log d)^2/\log d$.

It is clear that the diameter of a root of unity shifted by an integer, namely, $e^{2\pi i/n} + t$, where $t \in \mathbf{Z}$ and $n \in \mathbf{N}$, of degree $d = \varphi(n)$ is less than or equal to 1. In [10], Grandcolas computed the smallest possible diameters of algebraic integers of degree d up to 10. These computations show that, for each d from 2 to 10 except for $d = 9$, there is an algebraic integer, other than $e^{2\pi i/n} + t$, whose diameter is less than 2. Apparently, the are no such numbers of degree $d \ge 11$, but this is very far from being proved. If proved, this would imply that if an algebraic integer α of degree d is not a shifted root of unity and lies with its conjugates in a closed disc with radius R_d, then instead of (1) the stronger inequality $R_d > 1$ holds for $d \ge 11$. Some related results can be found in [11], [12]. See also [1], [9] for the calculations of small diameters of totally real algebraic integers and [5], [6] for the constructions of conjugate algebraic numbers lying on a circle $|z - \xi| = R$.

The main ingredient in the proof of Theorem 1 is its version for rational ξ with a slightly better estimate in d.

Theorem 2. *Let $p \ne 0$ and $q \ge 2$ be two coprime integers. Then, there is an infinite set of positive integers S such that, for each $d \in S$, the open disc with center at p/q and radius*

$$1 + \frac{\log(3q)\log\log d}{\log d}$$

contains a full set of conjugates of an algebraic integer of degree d.

Note that in case $p = 0$ or $q = 1$, we have $t = p/q \in \mathbf{Z}$. Then, as we already remarked above, for any $n \in \mathbf{N}$, there is is an algebraic integer $\alpha = e^{2\pi i/n} + t$ of degree $d = \varphi(n)$ lying with its conjugates on the circle with center at t and radius 1.

In the next section we prove Theorem 2. Then, in Section 3 we will prove Theorem 1.

2. Proof of Theorem 2

Fix two positive integers $K < d$ and write

$$\left(x - \frac{p}{q}\right)^d = x^d + \sum_{k=1}^{K}(-1)^k \binom{d}{k} \left(\frac{p}{q}\right)^k x^{d-k} + \sum_{k=0}^{d-K-1} (-1)^{d-k} \binom{d}{k} \left(\frac{p}{q}\right)^{d-k} x^k.$$

Let D_K be the least common multiple of $1, 2, \dots, K$. For each $k \in \{1, \dots, K\}$, the coefficient

$$a_{d-k} := (-1)^k \binom{d}{k} \left(\frac{p}{q}\right)^k = (-1)^k \frac{d}{kq^K} \binom{d-1}{k-1} p^k q^{K-k} \tag{2}$$

is an even integer if

$$2D_K q^K \quad \text{divides} \quad d. \tag{3}$$

The proof of the theorem consists in the construction of the polynomial of the form

$$\begin{aligned} f(x) &= \left(x - \frac{p}{q}\right)^d + \sum_{k=0}^{d-K-1} b_k \left(x - \frac{p}{q}\right)^k \\ &= x^d + a_{d-1}x^{d-1} + \dots + a_{d-K}x^{d-K} + \sum_{k=0}^{d-K-1} a_k x^k \end{aligned}$$

with some specially chosen $b_0, \dots, b_{d-K-1} \in \mathbf{Q}$. Observe that a_{d-k} are as in (2) for $k = 1, \dots, K$. Also, for each k in the range $0 \le k \le d - K - 1$, one has

$$a_k = b_k + (-1)^{d-k} \binom{d}{k} \left(\frac{p}{q}\right)^{d-k} + \sum_{j=k+1}^{d-K-1} (-1)^{j-k} b_j \binom{j}{k} \left(\frac{p}{q}\right)^{j-k}.$$

Thus, step by step, we can first choose $b_{d-K-1} \in \mathbf{Q}$, then $b_{d-K-2} \in \mathbf{Q}$, etc. up to $b_0 \in \mathbf{Q}$ so that the coefficients $a_{d-K-1}, \dots, a_0$ are all integers. Furthermore, iteratively we can select $b_{d-K-1}, \dots, b_1 \in (-1, 1] \cap \mathbf{Q}$ so that the integers $a_{d-K-1}, \dots, a_1$ all even, and after that select $b_0 \in (-2, 2] \cap \mathbf{Q}$ so that that the integer a_0 is 2 modulo 4. With this choice, by Eisenstein's criterion with respect to the prime 2, the above monic polynomial $f(x) \in \mathbf{Z}[x]$ of degree d will be irreducible over $\mathbf{Q}$ provided (3) holds.

Let us consider d of the form $d = 2D_K q^K$, so that (3) surely holds. Fix a small positive number δ. Then, by the Prime Number Theorem, for each $K \ge K(\delta)$, we have $\log D_K < (1 + \delta)K$ (see, e.g., [16]), and hence

$$\log d = \log(2D_K) + K \log q < \log 2 + (1 + \delta + \log q)K < K \log(2.8q).$$

Accordingly,

$$K > \frac{\log d}{\log(2.8q)}. \tag{4}$$

By Rouché's theorem, the polynomials $f(x)$ and $(x-p/q)^d$ have the same number of roots in the open disc

$$\left|x - \frac{p}{q}\right| < R := 1 + \frac{\log(3q)\log\log d}{\log d}$$

(i.e., they both have d roots) if, for their difference

$$\varphi(x) = f(x) - \left(x - \frac{p}{q}\right)^d = \sum_{k=0}^{d-K-1} b_k \left(x - \frac{p}{q}\right)^k,$$

the inequality $|\varphi(x)| < |(x-p/q)^d| = R^d$ is true for every $x \in \mathbf{C}$ on the circle $|x-p/q| = R$. Then, $f(x) \in \mathbf{Z}[x]$ is an irreducible monic polynomial of degree d with all d roots in $|x-p/q| < R$, and so it defines an algebraic integer of degree $d = 2D_K q^K$ with required properties.

Since $|b_0| \le 2$ and $|b_1|, \dots, |b_{d-K-1}| \le 1$, it remains to verify that

$$2 + \sum_{k=1}^{d-K-1} R^k < R^d. \tag{5}$$

Notice that (5) is equivalent to $R^{d-K} - 1 < (R-1)(R^d - 1)$. Multiplying by R^{K-d} we obtain $1 - R^{K-d} < R^K(R-1)(1-R^{-d})$. This is clearly true if $R^K(R-1) \ge 1$. By (4), for K large enough (and so d large enough), we deduce that

$$R^K > \left(1 + \frac{\log(3q)\log\log d}{\log d}\right)^{\log d/\log(2.8q)} > e^{\log\log d} = \log d.$$

Hence, $R^K(R-1) > \log(3q)\log\log d > 1$, as claimed. This implies (5).

3. Proof of Theorem 1

There is nothing to prove if $\xi \in \mathbf{Z}$. We can simply take the disc with radius 1. If $\xi = p/q$ with coprime integers $p \ne 0$ and $q \ge 2$, then the result follows by Theorem 2.

From now on, we assume that ξ is irrational. Then, by Dirichlet's theorem, there is an infinite sequence of positive integers $q_1 < q_2 < q_3 < \dots$ such that, for every $n \in \mathbf{N}$,

$$\left|\xi - \frac{p_n}{q_n}\right| < \frac{1}{q_n^2} \tag{6}$$

with some $p_n \in \mathbf{Z}$. For each $n \in \mathbf{N}$, we select

$$K_n := q_n^2 \quad \text{and} \quad d_n := 2D_{K_n} q_n^{K_n} = 2D_{q_n^2} q_n^{q_n^2}. \tag{7}$$

Then, $d_n > q_n^{K_n} = e^{q_n^2 \log q_n}$, and hence

$$q_n^2 \log q_n < \log d_n. \tag{8}$$

Also, as in (4), from (7) it follows that

$$K_n > \frac{\log d_n}{\log(2.8 q_n)} \tag{9}$$

for each sufficiently large n.

Set

$$R_n := 1 + \frac{\log(3q_n)\log\log d_n}{\log d_n}. \tag{10}$$

Evidently, in view of (6) the disc with center at ξ and radius $R_n + 1/q_n^2$ covers the disc with center at p_n/q_n and radius R_n. By Theorem 2, the latter disc contains a full set of conjugates of an algebraic integer of degree d_n. Thus, by (10), it suffices to show that

$$\frac{\log(3q_n)\log\log d_n}{\log d_n} + \frac{1}{q_n^2} < \frac{(\log\log d_n)^2}{2\log d_n}. \tag{11}$$

Now, we will verify (11) using (8) and (9). Evidently, $q_2 > 1$ so for $n \geq 2$ we can write (8) in the form $2\log q_n + \log\log q_n < \log\log d_n$. Consequently,

$$2\log(3q_n) = \log 9 + 2\log q_n < \log 9 + \log\log d_n - \log\log q_n < \log\log d_n - 2$$

for all sufficiently large n. This gives an upper bound for the first term in (11):

$$\frac{\log(3q_n)\log\log d_n}{\log d_n} < \frac{(\log\log d_n)^2 - 2\log\log d_n}{2\log d_n} = \frac{(\log\log d_n)^2}{2\log d_n} - \frac{\log\log d_n}{\log d_n}.$$

Next, by (9), we find that $\log d_n < K_n \log q_n = q_n^2 \log(2.8q_n)$. This yields $q_n > \sqrt{\log d_n/\log\log d_n}$ for n large enough. Therefore,

$$\frac{1}{q_n^2} < \frac{\log\log d_n}{\log d_n}.$$

By adding both displayed estimates we obtain (11). This, by (7), completes the proof of the theorem with the set

$$S := \{2D_{q_n^2} q_n^{q_n^2},\ n = n_0, n_0+1, \dots\},$$

where n_0 is large enough.

We remark that by using Hurwitz theorem instead of that of Dirichlet (with $1/q^2$ replaced by $1/\sqrt{5}q^2$ in (6)) one gets no advantage, since it implies the same result.

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References

[1] Capparelli, S., A. Del Fra, and C. Sciò: On the span of polynomials with integer coefficients. - Math. Comp. 79, 2010, 967–981.

[2] Dubickas, A.: On intervals containing full sets of conjugates of algebraic integers. - Acta Arith. 91, 1999, 379–386.

[3] Dubickas, A.: Totally real algebraic integers in small intervals. - Lith. Math. J. 40, 2000, 236–240.

[4] Ennola, V.: Conjugate algebraic integers in an interval. - Proc. Amer. Math. Soc. 53, 1975, 259–261.

[5] Ennola, V., and C. J. Smyth: Conjugate algebraic numbers on a circle. - Ann. Acad. Sci. Fenn. Ser. A I Math. 582, 1974, 1–31.

[6] Ennola, V., and C. J. Smyth: Conjugate algebraic numbers on circles. - Acta Arith. 29, 1976, 147–157.

[7] Fekete, M.: Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten. - Math. Z. 17, 1923, 228–249.

[8] Fekete, M., and G. Szegö: On algebraic equations with integral coefficients whose roots belong to a given point set. - Math. Z. 63, 1955, 158–172.

[9] Flammang, V., G. Rhin, and Q. Wu: The totally real algebraic integers with diameter less than 4. - Mosc. J. Comb. Number Theory 1, 2011, 17–25.

[10] Grandcolas, M.: Diameters of complete sets of conjugate algebraic integers of small degree. - Math. Comp. 67, 1998, 821–831.

[11] Grandcolas, M.: Weighted diameters of complete sets of conjugate algebraic integers. - Bull. Austral. Math. Soc. 57, 1998, 25–36.

[12] Grandcolas, M.: Problèmes de Favard généralisés. - Bull. Math. Soc. Sci. Math. Roumanie (N.S.) 57:105, 2014, 403–407.

[13] Kronecker, L.: Zwei Sätze über Gleichungen mit ganzzahligen Coefficienten. - J. Reine Angew. Math. 53, 1857, 173–175.

[14] Odlyzko, A. M.: Lower bounds for discriminants of number fields. II. - Tôhoku Math. J. 29, 1977, 209–216.

[15] Robinson, R. M.: Intervals containing infinitely many sets of conjugate algebraic integers. - In: Studies in Mathematical Analysis and Related Topics, Stanford, 1962, 305–315.

[16] Rosser, J. B., and L. Schoenfeld: Approximate formulas for some functions of prime numbers. - Illinois J. Math. 6, 1962, 64–94.

[17] Schur, I.: Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten. - Math. Z. 1, 1918, 377–402.

[18] Zaïmi, T.: Minoration du diamètre d'un entier algébrique totalement réel. - C. R. Acad. Sci. Paris Sér. I Math. 319, 1994, 417–419.

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