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Dispersive estimates for the wave equation inside cylindrical convex domains

Len Meas

Abstract. The dispersive and Strichartz estimates are essential for establishing well posedness results for nonlinear equations as well as long time behaviour of solutions to the equation. While in the boundary-less case these estimates are well understood, the case of boundary the situation can become much more difficult. In this work, we establish local in time dispersive estimates for solutions of the model case Dirichlet wave equation inside cylindrical convex domains $\Omega \subset \mathbb{R}^3$ with smooth boundary $\partial\Omega \neq \emptyset$. In this paper, we provide detailed proofs of the results established in [16, 17]. Let us recall that dispersive estimates are key ingredients to prove Strichartz estimates. Strichartz estimates for waves inside an arbitrary domain Ω have been proved by Blair, Smith, Sogge [4, 5]. Optimal estimates in strictly convex domains have been obtained in [12]. Our case of cylindrical domains is an extension of the result of [12] in the case when the nonnegative curvature radius depends on the incident angle and vanishes in some directions.

Konveksin lieriön aaltoyhtälön hajonta-arvioita

Tiivistelmä. Hajonta-arviot ja Strichartzin arviot ovat oleellisia välineitä osoitettaessa, että epälineaarinen yhtälö on hyvinasetettu, tai määritettäessä sen ratkaisujen pitkän aikavälin käyttäytymistä. Nämä arviot ymmärretään hyvin reunattomassa tilanteessa, mutta reunallinen tapaus voi olla paljon vaikeampi. Tässä työssä johdamme ajallisesti rajattuja hajonta-arvioita mallitapauksessa, joka koskee Dirichlet's aaltoyhtälön ratkaisuita sileäreunaisissa ($\partial\Omega \neq \emptyset$) konvekseissa lieriöissä $\Omega \subset \mathbb{R}^3$. Esitämme yksityiskohtaiset todistukset aiemmissa töissä [16, 17] saaduille tuloksille. Hajonta-arviot ovat avaintyökalu Strichartzin arvioiden todistamisessa. Blair, Smith ja Sogge [4, 5] ovat todistaneet Strichartzin arvioita mielivaltaisessa alueessa Ω kulkeville aalloille. Optimaaliset arviot aidosti konvekseissa alueissa on saatu lähteessä [12]. Tarkastelemamme lieriöalueen tapaus yleistää lähteen [12] tulosta tapauksessa, jossa ei-negatiivinen kaarevuussäde riippuu tulokulmasta ja häviää joissakin suunnissa.

1. Introduction

1.1. The cylindrical model problem. Let $\Omega = \{x \geq 0, (y,z) \in \mathbb{R}^2\} \subset \mathbb{R}^3$ with smooth boundary $\partial\Omega = \{x = 0\}$, and let P be the wave operator

$$P = \partial_t^2 - (\partial_x^2 + (1+x)\partial_y^2 + \partial_z^2).$$

We consider solutions of the linear Dirichlet-wave equation inside Ω

$$Pu = 0, \quad u_{|t=0} = \delta_a, \quad \partial_t u_{|t=0} = 0, \quad u_{|x=0} = 0, \tag{1.1}$$

with $u = u(t,x,y,z)$, and for $a > 0$, $\delta_a = \delta_{x=a,y=0,z=0}$. We use the notation $\tau = \frac{h}{i}\partial_t, \eta = \frac{h}{i}\partial_y, \xi = \frac{h}{i}\partial_x, \zeta = \frac{h}{i}\partial_z$ for the Fourier variables and $h \in (0,1]$. The

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Riemannian manifold (Ω, Δ) with $\Delta = \partial_x^2 + (1+x)\partial_y^2 + \partial_z^2$ can be locally seen as a cylindrical domain in $\mathbb{R}^3$ by taking cylindrical coordinates (r, θ, z), where we set $r = 1 - x/2$, $\theta = y$, and $z = z$. The problem is local near the boundary $\partial\Omega = \{x = 0\}$. Let $(a, 0, 0) \in \Omega$, $a > 0$. In local coordinates, a is the distance from the source point to the boundary. We assume a is small enough as we are interested only in highly reflected waves, which we do not observe if the waves do not have time to hit the boundary. This gives us interesting phenomena such as caustics near the boundary.

We remark that when there is no z variable (or when $y \in \mathbb{R}^n$ and ∂_y^2 is replaced by Δ_y), it is the Friedlander model. In this case, the optimal dispersive estimates were recently obtained by Ivanovici, Lebeau, and Planchon in [12].

Recall that at time $t > 0$, the waves propagating from the source of light highly concentrate around a sphere of radius t. For a variable coefficients metric, if two different light rays emanating from the source do not cross (that is, if t is smaller than the injectivity radius), one may then construct parametrices using oscillatory integrals where the phase encodes the geometry of wave front. In our scenario, the geometry of the wave front becomes singular in arbitrarily small times which depend on the frequency of the source and its distance to the boundary. In fact, a caustic appears between the first and the second reflection of the wave front. Let us give a brief overview of what caustics are (see [12, Section 1.1]). Geometrically, caustics are defined as envelopes of light rays coming from the source of light. At the caustic point we expect the light to be singularly intense. Analytically, caustics can be characterized as points where usual bounds on oscillatory integrals are no longer valid. The classification of asymptotic behavior of the oscillatory integrals with caustics depends on the number and the order of their critical points that are real. Let us consider an oscillatory integral

$$u_h(z) = \frac{1}{(2\pi h)^{1/2}} \int e^{\frac{i}{h}\Phi(z,\zeta)} g(z, \zeta, h)\, d\zeta, \quad z \in \mathbb{R}^d, \quad \zeta \in \mathbb{R}, \quad h \in (0, 1].$$

We assume that Φ is smooth and that g is compactly support in z and ζ. If $\partial_\zeta \Phi \neq 0$ in an open neighborhood of the support of g, the repeated integration by parts yields $|u_h(z)| = O(h^N)$ for any $N > 0$. If $\partial_\zeta \Phi = 0$ and $\partial_\zeta^2 \Phi \neq 0$ (nondegenerate critical points), then the stationary phase method yields $\|u_h\|_{L^\infty} = O(1)$. If there are degenerate critical points, we define them to be caustics, as $\|u_h\|_{L^\infty}$ is no longer uniform bounded. The order of a caustic κ is defined as the infimum of κ' such that $\|u_h\|_{L^\infty} = O(h^{-\kappa'})$. Let us give some useful examples of degenerate phase functions. The phase function of the form $\Phi_F(z, \zeta) = \frac{\zeta^3}{3} + z_1\zeta + z_2$ corresponds to a fold with order $\kappa = \frac{1}{6}$. A typical example is the Airy function. The next canonical form is given by the phase function of the form $\Phi_C(z, \zeta) = \frac{\zeta^4}{4} + z_1\frac{\zeta^2}{2} + z_2\zeta + z_3$, which corresponds to a cusp singularity with order $\kappa = \frac{1}{4}$. A swallowtail canonical form is given by the phase $\Phi_S(z, \zeta) = \frac{\zeta^5}{5} + z_1\frac{\zeta^3}{3} + z_2\frac{\zeta^2}{2} + z_3\zeta + z_4$ with order $\kappa = \frac{3}{10}$.

The crucial result of this work is the extension of the result of [12] to the case of our model cylindrical convex domains which have the following property: the nonnegative curvature radius depends on the incident angle and vanishes in some directions.

The main goal of this work is to construct a local parametrix and establish a local in time dispersive estimates for solution u to (1.1).

1.2. Some known results. The dispersive estimates for the wave equation in $\mathbb{R}^d$ follows from the representation of the solution as a sum of Fourier integral operators [see [1, 6, 7]]. They read as follows:

$$\|\chi(hD_t)e^{\pm it\sqrt{-\Delta_{\mathbb{R}^d}}}\|_{L^1(\mathbb{R}^d)\to L^\infty(\mathbb{R}^d)} \le Ch^{-d}\min\left\{1, \left(\frac{h}{|t|}\right)^{\frac{d-1}{2}}\right\}, \tag{1.2}$$

where $\Delta_{\mathbb{R}^d}$ is the Laplace operator in $\mathbb{R}^d$. Here and in the sequel, the function χ belongs to $C_0^\infty(]0,\infty[)$ and is equal to 1 on $[1,2]$ and $D_t = \frac{1}{i}\partial_t$.

Inside strictly convex domains Ω_D of dimensions $d \ge 2$, the optimal (local in time) dispersive estimates for the wave equations have been established by Ivanovici, Lebeau, and Planchon in [12]. More precisely, they have proved that

$$\|\chi(hD_t)e^{\pm it\sqrt{-\Delta_D}}\|_{L^1(\Omega_D)\to L^\infty(\Omega_D)} \le Ch^{-d}\min\left\{1, \left(\frac{h}{|t|}\right)^{\frac{d-1}{2}-\frac{1}{4}}\right\}, \tag{1.3}$$

where Δ_D is the Laplace operator on Ω_D. Due to the caustics formation in arbitrarily small times, (1.3) induces a loss of $1/4$ powers of $(h/|t|)$ factor compared to (1.2).

Let us also recall a few results about Strichartz estimates [see [12], section 1]: let (Ω, g) be a Riemannian manifold without boundary of dimensions $d \ge 2$. Local in time Strichartz estimates state that

$$\|u\|_{L^q((-T,T);L^r(\Omega))} \le C_T\left(\|u_0\|_{\dot H^\beta(\Omega)} + \|u_1\|_{\dot H^{\beta-1}(\Omega)}\right), \tag{1.4}$$

where $\dot H^\beta$ denotes the homogeneous Sobolev space over Ω of order β and $2 \le q, r \le \infty$ satisfy

$$\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - \beta, \quad \frac{1}{q} \le \frac{d-1}{2}\left(\frac{1}{2} - \frac{1}{r}\right).$$

Here $u = u(t,x)$ is a solution to the wave equation

$$(\partial_t^2 - \Delta_g)u = 0 \text{ in } (-T,T)\times\Omega, \quad u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x),$$

where Δ_g denotes the Laplace–Beltrami operator on (Ω, g). The estimates (1.4) hold on $\Omega = \mathbb{R}^d$ and $g_{ij} = \delta_{ij}$.

In [5], Blair, Smith, Sogge proved the Strichartz estimates for the wave equation on (compact or noncompact) Riemannian manifold with boundary. They proved that the Strichartz estimates (1.4) hold if Ω is a compact manifold with boundary and (q, r, β) is a triple satisfying

$$\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - \beta, \quad \text{for} \quad \begin{cases} \frac{3}{q} + \frac{d-1}{r} \le \frac{d-1}{2}, & d \le 4, \\ \frac{1}{q} + \frac{1}{r} \le \frac{1}{2}, & d \ge 4. \end{cases}$$

Recently in [12], Ivanovici, Lebeau, and Planchon have deduced a local in time Strichartz estimates (1.4) from the optimal dispersive estimates inside strictly convex domains of dimensions $d \ge 2$ for a triple (d, q, β) satisfying

$$\frac{1}{q} \le \left(\frac{d-1}{2} - \frac{1}{4}\right)\left(\frac{1}{2} - \frac{1}{r}\right) \quad \text{and} \quad \beta = d\left(\frac{1}{2} - \frac{1}{r}\right) - \frac{1}{q}.$$

For $d \ge 3$ this improves the range of indices for which sharp Strichartz do hold compared to the result by Blair, Smith, Sogge in [5]. However, the results in [5] apply to any domains or manifolds with boundary.

The latest results in [14] on Strichartz estimates inside the Friedlander model domain have been obtained for pairs (q, r) such that

$$\frac{1}{q} \le \left(\frac{1}{2} - \frac{1}{9}\right)\left(\frac{1}{2} - \frac{1}{r}\right).$$

This result improves on the known results for strictly convex domains for $d = 2$, while in [12] only gives a loss of $\frac{1}{4}$.

In this paper, $A \lesssim B$ means that there exists a constant C such that $A \leq CB$ and this constant may change from line to line but is independent of all parameters. Similarly, $A \sim B$ means there exist constants C_1, C_2 such that $C_1 B \leq A \leq C_2 B$. We denote $f(\vartheta, h) \in O_{C^\infty}(h^\infty)$ for $\vartheta \in \Gamma$ if, uniformly in $a \in [h^{\frac{2}{3}-\varepsilon}, 1]$,

$$\forall \alpha, N, \ \exists C_{\alpha,N} \quad \text{such that} \quad \sup_{\vartheta \in \Gamma} |\partial_\vartheta^\alpha f(\vartheta, h)| \leq C_{\alpha,N} h^N,$$

and $O((x,y)^j)$ means any function of the form

$$x^l y^m f\left(\frac{x}{N}, \frac{y}{N}, a, N\right)$$

with f smooth uniformly in a, N and $l + m = j$.

By definition, a function $f(w)$ admits an asymptotic expansion for $w \to 0$ when there exists a (unique) sequence $(c_n)_n$ such that, for any n,

$$\lim_{w \to 0} w^{-(n+1)} \left(f(w) - \sum_0^n c_n w^n \right) = c_{n+1}.$$

We will denote $f(w) \sim_w \sum_n c_n w^n$.

1.3. Main results. Our main results concerning the local in time dispersive estimates and Strichartz estimates inside the cylindrical convex domain Ω are stated below. Let $\mathcal{G}_a$ be the Green function for (1.1).

Theorem 1.1. *There exists C such that for every $h \in]0, 1]$, every $t \in [-1, 1]$ and every $a \in]0, 1]$ the following holds:*

$$\|\chi(hD_t)\mathcal{G}_a(t, x, y, z)\|_{L^\infty} \leq C h^{-3} \min\left\{1, \left(\frac{h}{|t|}\right)^{3/4}\right\}. \tag{1.5}$$

As in [12], Theorem 1.1 states that a loss of 1/4 powers of $(h/|t|)$ appears compared to (1.2) . We will obtain in Theorems 1.3, 1.4, 1.5 better results, in particular near directions which are close to the axis of the cylinder.

As a consequence of Theorem 1.1, conservation of energy, interpolation and TT^* arguments, we obtain the following set of (local in time) Strichartz estimates.

Theorem 1.2. *Let (Ω, Δ) be defined as before. Let u be a solution of the wave equation on Ω:*

$$(\partial_t^2 - \Delta)u = 0 \quad \text{in } \Omega,$$
$$u_{|t=0} = u_0, \quad \partial_t u_{|t=0} = u_1, \quad u_{|x=0} = 0.$$

Then for all T there exists C_T such that

$$\|u\|_{L^q((0,T);L^r(\Omega))} \leq C_T \left(\|u_0\|_{\dot{H}^\beta(\Omega)} + \|u_1\|_{\dot{H}^{\beta-1}(\Omega)} \right),$$

with

$$\frac{1}{q} \leq \frac{3}{4}\left(\frac{1}{2} - \frac{1}{r}\right), \quad \text{and the scaling } \beta = 3\left(\frac{1}{2} - \frac{1}{r}\right) - \frac{1}{q}.$$

Theorem 1.2 improves the range of indices for which sharp Strichartz estimates do hold compared to [5]. Notice however that the results in [5] apply to arbitrary domains or manifolds with non-empty boundary. The proof of Theorem 1.2 follows the classical arguments, we first prove the frequency-localized Strichartz estimates by utilizing the frequency-localized dispersive estimates, interpolation and TT^* arguments. We then apply the Littlewood-Paley squarefunction estimates [see [2, 3, 15]] to get the Strichartz estimates [Theorem 1.2] in the context of cylindrical domains [see [18]].

1.4. Green function and precise dispersive estimates. The proofs of frequency-localized dispersive estimates are based on the construction of parametrices for the fundamental solution of the wave equation (1.1) and (possibly degenerate) stationary phase method.

We begin with the construction of the local parametrix for (1.1) by utilizing the spectral analysis of $-\Delta$ with Dirichlet condition on the boundary to obtain first the Green function associated to (1.1). The Laplace operator we work with on the half space Ω is equal to

$$\Delta = \partial_x^2 + (1+x)\partial_y^2 + \partial_z^2,$$

with the Dirichlet condition on the boundary $\partial\Omega$. We notice that a useful feature of this particular Laplace operator is that the coefficients of the metric do not depend on the vairables y, z and therefore this allows us to take the Fourier transform in y and z. Now taking the Fourier transform in y, z-variables yields

$$-\Delta_{\eta,\zeta} = -\partial_x^2 + (1+x)\eta^2 + \zeta^2.$$

For $\eta \neq 0, -\Delta_{\eta,\zeta}$ is a self-adjoint, positive operator on $L^2(\mathbb{R}_+)$ with a compact resolvent. Let $(e_k)_{k\geq 1}$ be an orthonormal basis in $L^2(\mathbb{R}_+)$ of Dirichlet eigenfunctions of $-\Delta_{\eta,\zeta}$ and let $(\lambda_k)_k$ be the associated eigenvalues. These eigenfunctions are explicit in term of Airy functions

$$e_k = e_k(x,\eta) = f_k \frac{|\eta|^{1/3}}{k^{1/6}} Ai(|\eta|^{2/3}x - \omega_k)$$

with associated eigenvalues

$$\lambda_k = \lambda_k(\eta,\zeta) = \eta^2 + \zeta^2 + \omega_k|\eta|^{4/3},$$

where $(-\omega_k)_k$ denote the zeros of Airy function in decreasing order and for all $k \geq 1$, f_k are constants so that $\|e_k(.,\eta)\|_{L^2(\mathbb{R}_+)} = 1$. Observe that $(f_k)_k$ is uniformly bounded in a fixed compact subset of $]0,\infty[$ as a consequence of

$$\int_{-\omega_k}^{-2} Ai^2(\omega)d\omega \sim \frac{1}{4\pi}\int_{-\omega_k}^{-2} |\omega|^{-1/2}(1+O(\omega^{-1}))d\omega \sim |\omega_k|^{1/2}$$

and

$$\omega_k \sim \left(\frac{3}{2}\pi k\right)^{2/3}(1+O(k^{-1})).$$

For $a \in \Omega$, let $g_a(t,x,\eta,\zeta)$ be the solution of

$$(\partial_t^2 - (\partial_x^2 - (1+x)\eta^2 - \zeta^2))g_a = 0,$$
$$g_{a|x=0} = 0,\ g_{a|t=0} = \delta_{x=a},\ \partial_t g_{a|t=0} = 0.$$

We have

$$g_a(t,x,\eta,\zeta) = \sum_{k\geq 1}\cos(t\lambda_k^{1/2})e_k(x,\eta)e_k(a,\eta). \tag{1.6}$$

Here $\delta_{x=a}$ denotes the Dirac distribution on $\mathbb{R}_+, a > 0$ and it may be decomposed as follows:

$$\delta_{x=a} = \sum_{k\geq 1} e_k(x,\eta)e_k(a,\eta).$$

Now taking the inverse Fourier transform, the Green function for (1.1) is given by

$$\begin{aligned}\mathcal{G}_a(t,x,y,z) &= \frac{1}{4\pi^2}\int e^{i(y\eta+z\zeta)} g_a(t,x,\eta,\zeta)\,d\eta\,d\zeta,\\ &= \frac{1}{4\pi^2h^2}\sum_{k\geq 1}\int e^{i(y\eta+z\zeta)/h}\cos(t\lambda_k^{1/2})e_k(x,\eta/h)e_k(a,\eta/h)\,d\eta\,d\zeta. \end{aligned} \tag{1.7}$$

We thus get the following formula for $2\chi(hD_t)\mathcal{G}_a$

$$\begin{aligned}2\chi(hD_t)\mathcal{G}_a(t,x,y,z) = \frac{1}{4\pi^2h^2}\sum_{k\geq 1}\int & e^{\frac{i}{h}(y\eta+z\zeta)}e^{i\frac{t}{h}(\eta^2+\zeta^2+\omega_k h^{2/3}|\eta|^{4/3})^{1/2}} e_k(x,\eta/h)\\ & \times e_k(a,\eta/h)\chi((\eta^2+\zeta^2+\omega_k h^{2/3}|\eta|^{4/3})^{1/2})d\eta d\zeta.\end{aligned} \tag{1.8}$$

On the wave front set of the above expression, one has $\tau = (\eta^2+\zeta^2+\omega_k h^{2/3}|\eta|^{4/3})^{1/2}$. In order to prove Theorem 1.1, we only need to work near tangential directions; therefore we will introduce an extra cutoff to insure $|\tau - (\eta^2+\zeta^2)^{1/2}|$ small, which is equivalent to $\omega_k h^{2/3}|\eta|^{4/3}$ small. Then, we are reduced to prove the dispersive estimate for $\mathcal{G}_{a,loc}$:

$$\begin{aligned}\mathcal{G}_{a,loc}(t,x,y,z) = \frac{1}{4\pi^2h^2}\sum_{k\geq 1}\int & e^{\frac{i}{h}(y\eta+z\zeta)}e^{i\frac{t}{h}(\eta^2+\zeta^2+\omega_k h^{2/3}|\eta|^{4/3})^{1/2}} e_k(x,\eta/h)e_k(a,\eta/h)\\ & \times \chi_0(\eta^2+\zeta^2)\chi_1(\omega_k h^{2/3}|\eta|^{4/3})d\eta d\zeta,\end{aligned} \tag{1.9}$$

where the cut-off functions χ_0 and χ_1 are defined in Section 2.

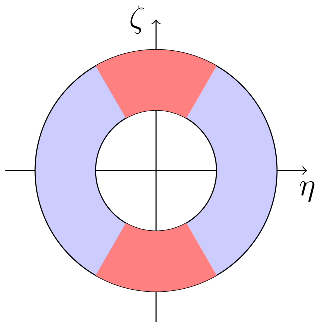


Figure 1. Phase space.

The phase space in Figure 1 illustrates the different regimes of η; that is η is bounded below by a constant c_0 and η is close to zero, where we will establish the dispersive estimates.

To obtain the local in time dispersive estimates, we will cut the η integration in (1.9) in different pieces (Figure 1). More precisely, we write

$$\mathcal{G}_{a,loc} = \mathcal{G}_{a,c_0} + \sum_{\epsilon_0\sqrt{a}\leq 2^m\sqrt{a}\leq c_0} \mathcal{G}_{a,m} + \mathcal{G}_{a,\epsilon_0}, \tag{1.10}$$

where $\mathcal{G}_{a,c_0}$ is associated with the integration for $|\eta|\geq c_0$, $\mathcal{G}_{a,m}$ is associated with the integration for $|\eta|\sim 2^m\sqrt{a}$, and $\mathcal{G}_{a,\epsilon_0}$ is associated with the integration for $0<|\eta|\leq \epsilon_0\sqrt{a}$.

We will prove the following results. Let $\epsilon\in\,]0,1/7[$.

Theorem 1.3. *There exists C such that for every $h \in]0,1]$, every $t \in [h,1]$, the following holds:*

$$\|\mathcal{G}_{a,c_0}(t,x,y,z)\|_{L^\infty(x\leq a)} \leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}\gamma(t,h,a), \tag{1.11}$$

with

$$\gamma(t,h,a) = \begin{cases} \left(\frac{h}{t}\right)^{1/3} & \text{if } a \leq h^{\frac{2}{3}(1-\epsilon)}, \\ \left(\frac{h}{t}\right)^{1/2} + a^{1/8}h^{1/4} & \text{if } a \geq h^{\frac{2}{3}(1-\epsilon')},\ \epsilon' \in]0,\epsilon[. \end{cases}$$

Observe that in Theorem 1.3 we get the same estimate as in Ivanovici–Lebeau–Planchon [12].

Theorem 1.4. *There exists C such that for every $h \in]0,1]$, every $t \in [h,1]$, the following holds :*

$$\|\mathcal{G}_{a,m}(t,x,y,z)\|_{L^\infty(x\leq a)} \leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}\gamma_m\big(t,h,a\big),$$

with

$$\gamma_m(t,h,a) = \begin{cases} \left(\frac{h}{t}\right)^{1/3}(2^m\sqrt{a})^{1/3} & \text{if } a \leq \left(\frac{h}{2^m\sqrt{a}}\right)^{\frac{2}{3}(1-\epsilon)}, \\ \min\left\{\left(\frac{h}{t}\right)^{1/2}, 2^m\sqrt{a}|\log(2^m\sqrt{a})|\right\} + a^{1/8}h^{1/4}(2^m\sqrt{a})^{3/4} & \\ & \text{if } a \geq \left(\frac{h}{2^m\sqrt{a}}\right)^{\frac{2}{3}(1-\epsilon')},\ \epsilon' \in]0,\epsilon[. \end{cases}$$

For $2^m\sqrt{a} \sim 1$, Theorem 1.4 yields the same result as in Theorem 1.3. We notice that the estimates get better when $|\eta|(\sim 2^m\sqrt{a})$ decreases. This is compatible with the intuition that less curvature implies better dispersion.

Theorem 1.5. *There exists C such that for every $h \in]0,1]$, every $t \in [h,1]$, the following holds:*

$$\|\mathcal{G}_{a,\epsilon_0}(t,x,y,z)\|_{L^\infty(x\leq a)} \leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}\min\left\{\left(\frac{h}{t}\right)^{1/2}, \sqrt{a}|\log(a)|\right\}. \tag{1.12}$$

Let us verify that Theorem 1.1 is a consequence of Theorems 1.3, 1.4 and 1.5. We may assume $|t| \geq h$, since for $|t| \leq h$, the best bound for the dispersive estimate is equal to Ch^{-3} by Sobolev inequality. Then, by symmetry of the Green function, we may assume $t \in [h,1]$ and $x \leq a$. Then Theorem 1.1 is a consequence of $\sum_{m\leq M}(2^m\sqrt{a})^\nu \sim (2^M\sqrt{a})^\nu$ for $\nu > 0$.

In Section 2, we prove Theorem 1.3. To do so, we use the representation of $\mathcal{G}_{a,c_0}$ as a sum of the eigenmodes (over k) which is used to prove the estimates for $a \leq h^{\frac{2}{3}(1-\epsilon)}$, $\epsilon \in]0,1/7[$. Using the Airy–Poisson summation formula (see Lemma 2.4), $\mathcal{G}_{a,c_0}$ can be also represented as a sum over $N \in \mathbb{Z}$ (its summands will be seen to be waves corresponding to the number of reflections on the boundary, indexed by N) for $a \geq h^{\frac{2}{3}(1-\epsilon')}$, for $\epsilon' \in]0,\epsilon[$. These local parametrices can be written in terms of a sum of oscillatory integrals with phase functions containing an Airy type terms with degenerate critical points. We give a precise analysis of the Lagrangian in the phase space associated to these oscillatory integrals. This geometric analysis allows us to track the degeneracy of the phases when we apply the stationary phase method.

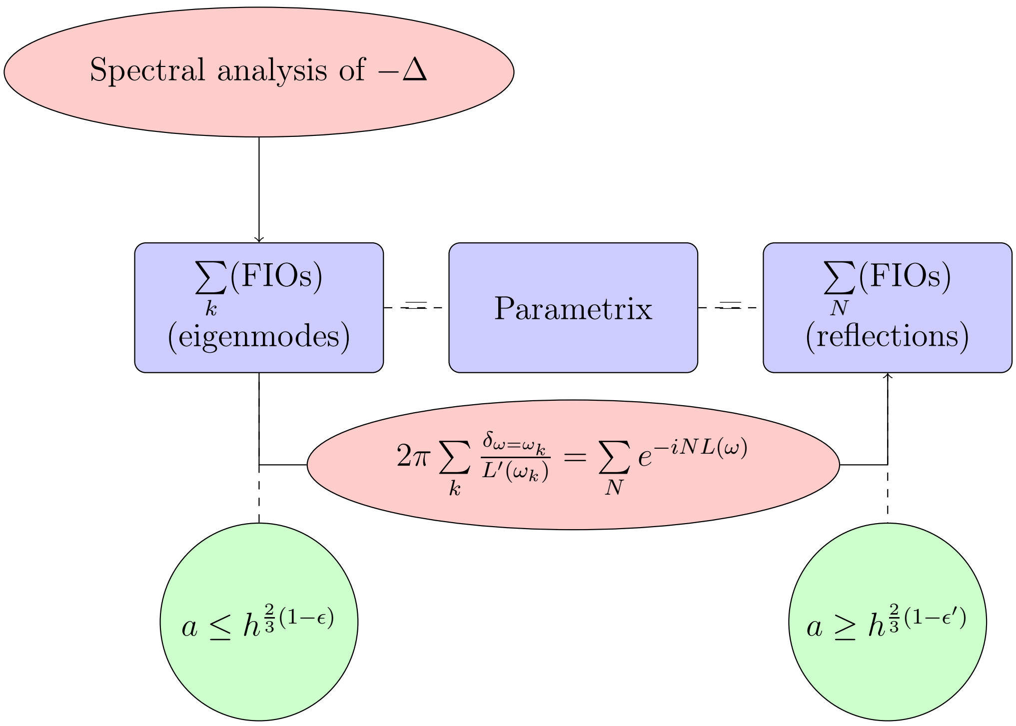
In Section 3, we prove Theorem 1.4. To get the estimates for $\mathcal{G}_{a,m}$, we distinguish between two different cases. The first case is $a \leq (\frac{h}{2^m\sqrt{a}})^{\frac{2}{3}(1-\epsilon)}$, $\epsilon \in]0,1/7[$: here, we follow ideas in section 2 and construct a local parametrix as a sum over eigenmodes. The second case is $a \geq (\frac{h}{2^m\sqrt{a}})^{\frac{2}{3}(1-\epsilon')}$, for $\epsilon' \in]0,\epsilon[$: there, the Airy–Poisson summation formula yields the representation of $\mathcal{G}_{a,m}$ as a sum over $N \in \mathbb{Z}$.

In Section 4, we prove Theorem 1.5. Notice that as ϵ_0 is small, the estimates for $\mathcal{G}_{a,\epsilon_0}$ are in fact those in free case. To get that, we first compute the trajectories of the Hamiltonian flow for the operator P. At this frequency localization there is at most one reflection on the boundary of the cylinder. Moreover, we follow the techniques from section 2 and obtain an expression for $\mathcal{G}_{a,\epsilon_0}$ to which we apply the stationary phase method. It is particularly interesting that this localization gives us an oscillatory integral (the local parametrix) with nondegenerate phase function; this is due to the geometric study of the associated Lagrangian which rules out the cusps and swallowtails regimes for a given fixed time t, $|t| \leq 1$ if ϵ_0 is small.

In all these sections, we will assume that the integration with respect to η is restricted to $\eta > 0$, since the case $\eta < 0$ is exactly the same.

2. Dispersive estimates for $|\eta| \geq c_0$

In this section, we prove Theorem 1.3. The key ingredient is to construct local parametrices for the regimes $a \leq h^{\frac{2}{3}(1-\epsilon)}$ for $\epsilon \in]0,1/7[$, and for $a \geq h^{\frac{2}{3}(1-\epsilon')}$, for $\epsilon' \in]0,\epsilon[$ respectively. These are oscillatory integrals to which we apply the (degenerate) stationary phase type arguments to get the desired estimates. The Airy–Poisson summation formula [see Lemma 2.4] gives us the parametrix as a sum over multiple reflections on the boundary as illustrated in the following diagram.



2.1. Dispersive estimates for $0 < a \leq h^{\frac{2}{3}(1-\epsilon)}$, with $\epsilon \in]0,1/7[$. In this subsection, we prove local in time dispersive estimates for the function $\mathcal{G}_{a,c_0}$. In the regime $0 < a \leq h^{\frac{2}{3}(1-\epsilon)}$, with $\epsilon \in]0,1/7[$, the parametrix reads as a sum over eigenmodes k. Taking into account the asymptotic behaviour of the Airy functions, we deal with different values of k as follows: for small values of k, we use Lemma 3.5[12] to get the estimates; for large values of k, we use the asymptotic expansion of the Airy functions. The last case, the parametrix is a sum of oscillatory integrals to which we apply [12, Lemma 2.20]. Recall that the parametrix in this frequency localization and near tangential directions is equal to

$$\begin{aligned}\mathcal{G}_{a,c_0}(t,x,y,z) = \frac{1}{4\pi^2 h^2}\sum_{k\geq 1}\int e^{\frac{i}{h}(y\eta+z\zeta)} e^{i\frac{t}{h}(\eta^2+\zeta^2+\omega_k h^{2/3}|\eta|^{4/3})^{1/2}} e_k(x,\eta/h)e_k(a,\eta/h)\\ \times \chi_0(\zeta^2+\eta^2)\psi_0(\eta)\chi_1(\omega_k h^{2/3}|\eta|^{4/3})(1-\chi_1)(\varepsilon\omega_k)\,d\eta\,d\zeta. \qquad (2.1)\end{aligned}$$

Here

- $\chi_0 \in C_0^\infty, 0 \leq \chi_0 \leq 1, \chi_0$ is supported in the neighborhood of 1.
- $\psi_0 \in C_0^\infty(c_0/2,\infty), 0 \leq \psi_0 \leq 1$, $\psi_0(\eta) = 1$ for $\eta \geq c_0$.
- $\chi_1 \in C_0^\infty, 0 \leq \chi_1 \leq 1, \chi_1$ is supported in $(-\infty, 2\varepsilon], \chi_1 = 1$ on $(-\infty,\varepsilon]$, for $\varepsilon > 0$ small. χ_1 is used to localize in tangential directions. Notice that on the support of χ_1, we have $\omega_k h^{2/3}|\eta|^{4/3} \leq 2\varepsilon$ and since $\omega_k \sim k^{2/3}$; we obtain $k \leq \frac{\varepsilon}{h|\eta|^2}$. Thus since η is bounded from below, we may assume that $k \leq \varepsilon/h$. Moreover, we have $(1-\chi_1)(\varepsilon\omega_k) = 1$ for every $k \geq 1$ since $\omega_1 \approx 2.33$.

The main result of this section is the following proposition.

Proposition 2.1. *Let $\epsilon \in]0,1/7[$. There exists C such that for every $h \in]0,1]$, every $t \in [h,1]$ and every $0 < a \leq h^{\frac{2}{3}(1-\epsilon)}$, $y \in \mathbb{R}, z \in \mathbb{R}$, the following holds:*

$$\|\mathcal{G}_{a,c_0}(t,x,y,z)\|_{L^\infty(x\leq a)} \leq Ch^{-3}\left(\frac{h}{t}\right)^{5/6}. \qquad (2.2)$$

Proof. First, we study the integration in ζ. Let

$$J = \int e^{i\frac{t}{h}\phi_k}\chi_0(\zeta^2+\eta^2)\,d\zeta.$$

Recall that $\chi_0 \in C_0^\infty$ is supported near 1. The phase function ϕ_k is given by

$$\phi_k(\zeta) = \frac{z}{t}\zeta + \left(\eta^2+\zeta^2+\gamma\eta^2\right)^{1/2},$$

with $\gamma = h^{2/3}\omega_k|\eta|^{-2/3} > 0$. We introduce a change of variables $\zeta = |\eta|\tilde{\zeta}, z = t\tilde{z}$. Then we obtain

$$\phi_k(\zeta) = |\eta|\big(\tilde{z}\tilde{\zeta} + (1+\tilde{\zeta}^2+\gamma)^{1/2}\big).$$

Differentiating with respect to $\tilde{\zeta}$, we get

$$\partial_{\tilde{\zeta}}\phi_k = |\eta|\left(\tilde{z} + \frac{\tilde{\zeta}}{(1+\tilde{\zeta}^2+\gamma)^{1/2}}\right).$$

Because η is bounded from below, $\tilde{\zeta} = \zeta/|\eta|$ is also bounded, therefore we have $\left|\frac{\tilde{\zeta}}{(1+\tilde{\zeta}^2+\gamma)^{1/2}}\right| \leq 1-2\delta_1$, for some $\delta_1 > 0$ small. Then if $|\tilde{z}| \geq 1-\delta_1$, the contribution of $\tilde{\zeta}$-integration is $O_{C^\infty}((h/t)^\infty)$ by integration by parts. Thus we may assume that $|\tilde{z}| \leq 1-\delta_1$. In this case, the phase ϕ_k has a unique critical point on the support of χ_0. It is given by $\tilde{\zeta}_c = -\frac{\tilde{z}(1+\gamma)^{1/2}}{\sqrt{1-\tilde{z}^2}}$ and this critical point is nondegenerate since

$$\partial^2_{\tilde{\zeta}}\phi_k = |\eta|\left(\frac{1+\gamma}{(1+\tilde{\zeta}^2+\gamma)^{3/2}}\right) > 0.$$

Then we obtain by the stationary phase method (as $|\tilde{z}| < 1-\delta_1$)

$$J = \left(\frac{h}{t}\right)^{1/2} e^{i\frac{t}{h}|\eta|\sqrt{1-\tilde{z}^2}(1+\gamma)^{1/2}}\tilde{\chi}_0,$$

where $\tilde{\chi}_0$ is a classical symbol of order 0 with small parameter h/t. Hence we get

$$\mathcal{G}_{a,c_0}(t,x,y,z) = \frac{1}{4\pi^2 h^2}\left(\frac{h}{t}\right)^{1/2}\sum_{k\geq 1}\int e^{\frac{i}{h}\left(y\eta+|\eta|t\sqrt{1-\tilde{z}^2}(1+\gamma)^{1/2}\right)} e_k(x,\eta/h)e_k(a,\eta/h)$$
$$\times\, \tilde{\chi}_0\psi_0(\eta)\chi_1(\gamma|\eta|^2)(1-\chi_1)(\varepsilon\gamma h^{-2/3}|\eta|^{2/3})\, d\eta. \tag{2.3}$$

Next, we observe that $\mathcal{G}_{a,c_0}$ contains Airy functions which behave differently depending on the various values of k. To deal with it, we split the sum over k into $\mathcal{G}_{a,c_0} = \mathcal{G}_{a,<L} + \mathcal{G}_{a,>L}$, where in $\mathcal{G}_{a,<L}$ only the sum over $1\leq k\leq L$ is considered.

Estimates for $\mathcal{G}_{a,<L}$. To get the estimates for $\mathcal{G}_{a,<L}$, we need the following lemma, which follows from the bound $|Ai(s)| \leq C(1+|s|)^{-1/4}$.

Lemma 2.2. [12, Lemma 3.5] *There exists C_0 such that for $L\geq 1$, the following holds:*

$$\sup_{\mathbf{b}\in\mathbb{R}}\left(\sum_{1\leq k\leq L} k^{-1/3}Ai^2(\mathbf{b}-\omega_k)\right) \leq C_0 L^{1/3}.$$

We use the Cauchy–Schwarz inequality for (2.3) and Lemma 2.2 to get

$$\begin{aligned}
\|\mathcal{G}_{a,<L}\|_{L^\infty} &\lesssim h^{-2}\left(\frac{h}{t}\right)^{1/2}\sum_{1\leq k\leq L} h^{-2/3}k^{-1/3}Ai(|\eta/h|^{2/3}x-\omega_k)Ai(|\eta/h|^{2/3}a-\omega_k),\\
&\lesssim h^{-3}\left(\frac{h}{t}\right)^{1/2} h^{1/3}\left(\sum_{1\leq k\leq L} k^{-1/3}Ai^2(h^{-2/3}|\eta|^{2/3}x-\omega_k)\right)^{1/2}\\
&\quad\times\left(\sum_{1\leq k\leq L} k^{-1/3}Ai^2(h^{-2/3}|\eta|^{2/3}a-\omega_k)\right)^{1/2},\\
&\lesssim h^{-3}\left(\frac{h}{t}\right)^{1/2} h^{1/3}L^{1/3}.
\end{aligned}$$

We only have to prove (2.2) for $t > h$. Let $\epsilon\in\,]0,1/3[$ and $L = h^{-\epsilon}$. If $t\leq h^{\epsilon}$, then $L\leq \frac{1}{t}$, hence

$$\|\mathcal{G}_{a,<L}(t,x,y,z)\|_{L^\infty} \leq Ch^{-3}\left(\frac{h}{t}\right)^{5/6}.$$

We are reduced to the case $t > h^{\epsilon}\geq h^{1/3}$. Then we apply the stationary phase for η-integration of the form

$$\int e^{\frac{i}{h}\Phi_k} Ai(h^{-2/3}|\eta|^{2/3}x-\omega_k)Ai(h^{-2/3}|\eta|^{2/3}a-\omega_k)\, d\eta,$$

with the phase function

$$\Phi_k(\eta) = \eta\big(y + t\sqrt{1-\tilde{z}^2}(1+\gamma)^{1/2}\big).$$

To deal with this integral, we rewrite $\Phi_k = h\lambda\Psi_k$ where $\lambda = t\omega_k h^{-1/3}$ is a large parameter. We have $|\partial_\eta^2\Psi_k| \geq c > 0$. To apply the stationary phase, we need to check that one has for some $\nu > 0$ one has

$$|\partial_\eta^j Ai(h^{-2/3}|\eta|^{2/3}x - \omega_k)| \le C_j \lambda^{j(1/2-\nu)}.$$

Since one has $\sup_{b\ge 0} |b^l Ai^{(l)}(b-\omega_k)| \le C_l \omega_k^{3l/2}$, it is sufficient to check that there exists $\epsilon > 0$ such that for $t > h^\epsilon$ and $k \le h^{-\epsilon}$,

$$\omega_k^{3/2} \le (t\omega_k h^{-1/3})^{(1/2-\nu)}$$

This holds if $\epsilon < 1/7$. Therefore the estimate for $\epsilon < 1/7$ and $t > h^\epsilon$ is

$$\begin{aligned}
\|\mathbb{1}_{x\le a}\mathcal{G}_{a,<L}(t,x,y,z)\|_{L^\infty} &\le Ch^{-3}\left(\frac{h}{t}\right)^{1/2}\left[h^{1/3}\sum_{1\le k\le h^{-\epsilon}} k^{-1/3}\lambda^{-1/2}\right],\\
&\le Ch^{-3}\left(\frac{h}{t}\right)^{1/2}\left[h^{1/3}\sum_{1\le k\le h^{-\epsilon}} k^{-1/3}(t\omega_k h^{-1/3})^{-1/2}\right],\\
&\le Ch^{-3}\left(\frac{h}{t}\right)^{1/2}\left[\left(\frac{h}{t}\right)^{1/2} h^{-\epsilon/3}\right],\\
&\le Ch^{-3}\left(\frac{h}{t}\right)^{1/2}\left(\frac{h}{t}\right)^{1/3}.
\end{aligned}$$

Estimates for $\mathcal{G}_{a,>L}$. We now deal with large values of $k, L \le k \le \varepsilon/h$ with $L \ge D\max\{h^{-\epsilon}, 1/t\}$, $D > 0$ large constant. We are left to prove (2.2) holds true for $\mathcal{G}_{a,>L}$, defined by the sum over $L \le k \le \frac{\varepsilon}{h}$. For $k > Dh^{-\epsilon}$ and $0 \le x \le a \le h^{\frac{2}{3}(1-\epsilon)}$, we have

$$\omega_k - |\eta|^{2/3}h^{-2/3}x > \omega_k/2.$$

Therefore we can use the asymptotic expansion of the Airy function (see Appendix)

$$Ai(\vartheta) = \sum_{\pm} \omega^{\pm} e^{\mp\frac{2}{3}i(-\vartheta)^{3/2}}(-\vartheta)^{-1/4}\Psi_\pm(-\vartheta) \quad \text{for } -\vartheta > 1, \quad \text{where } \omega^\pm = e^{\pm i\pi/4}$$

and where $\Psi_\pm$ are given in the Appendix. By the definition of e_k, we have

$$\begin{aligned}
e_k(x,\eta/h) &= f_k \frac{|\eta|^{1/3}h^{-1/3}}{k^{1/6}} Ai(h^{-2/3}|\eta|^{2/3}x - \omega_k),\\
&= f_k \frac{|\eta|^{1/3}h^{-1/3}}{k^{1/6}} \sum_{\pm} \omega^\pm e^{\mp\frac{2}{3}i(\omega_k - |\eta|^{2/3}h^{-2/3}x)^{3/2}} \frac{\Psi_\pm(\omega_k - |\eta|^{2/3}h^{-2/3}x)}{(\omega_k - |\eta|^{2/3}h^{-2/3}x)^{1/4}}.
\end{aligned}$$

We can rewrite $\mathcal{G}_{a,>L}$ as follows:

$$\mathcal{G}_{a,>L}(t,x,y,z) = \sum_{L\le k\le \frac{\varepsilon}{h}} \frac{1}{4\pi^2 h^2}\left(\frac{h}{t}\right)^{1/2} \sum_{\pm,\pm} \int e^{\frac{i}{h}\Phi_k^{\pm,\pm}} \sigma_k^{\pm,\pm}\, d\eta, \tag{2.4}$$

with the phase functions are defined by

$$\Phi_k^{\pm,\pm}(t,x,y,z,a;\eta) = y\eta + |\eta| t\sqrt{1-\tilde{z}^2}(1+\gamma)^{1/2} \pm \frac{2}{3}|\eta|(\gamma - x)^{3/2} \pm \frac{2}{3}|\eta|(\gamma - a)^{3/2},$$

and the symbols are given by

$$\begin{aligned}
\sigma_k^{\pm,\pm}(x,a,h;\eta) &= h^{-1/3}|\eta|^{1/3}\tilde{\chi}_0\chi_1(\gamma\eta^2)(1-\chi_1)(\varepsilon\gamma h^{-2/3}|\eta|^{2/3})\frac{f_k^2}{k^{1/3}}\omega^\pm\omega^\pm\\
&\quad \times (\gamma - x)^{-1/4}(\gamma - a)^{-1/4}\Psi_\pm(|\eta|^{2/3}h^{-2/3}(\gamma - x))\Psi_\pm(|\eta|^{2/3}h^{-2/3}(\gamma - a)).
\end{aligned}$$

We have $3\eta\partial_\eta = -2\gamma\partial_\gamma$ and for $0 \le x \le a \le 2\gamma$,

$$|(\gamma\partial_\gamma)^j((\gamma-x)^{-1/4})| \le C_j\gamma^{-1/4} \le C'_j(hk)^{-1/6};$$

moreover, $\Psi_\pm$ are classical symbols of order 0 at infinity which is true in this case since we have

$$|\eta^{2/3}h^{-2/3}(\gamma-x)| \ge \omega_k/2 \ge Ch^{-2\epsilon/3},$$

since $k \ge L \ge h^{-\epsilon}$. Hence we obtain that for all j, there exists C_j such that

$$|\partial_\eta^j\sigma_k^{\pm,\pm}(x,a,h;\eta)| \le C_j(hk)^{-2/3},$$

since in the symbols $\sigma_k^{\pm,\pm}$ there is a factor $(hk)^{-1/3}$ and we apply η derivatives to the product $(\gamma-x)^{-1/4}(\gamma-a)^{-1/4}$ to get another factor $(hk)^{-1/3}$.

Therefore, to establish the dispersive estimates for $\mathcal{G}_{a,>L}$, it suffices to estimate the oscillatory integral of the form

$$\int e^{\frac{i}{h}\Phi_k^{\pm,\pm}}\sigma_k^{\pm,\pm}\,d\eta.$$

To get the estimates for this integral, we set $\Phi_k^{\pm,\pm} = h\lambda\psi_k^{\pm,\pm}$, where $\lambda = t\omega_k h^{-1/3}$. It defines a new large parameter since $\lambda \ge c > 0$ as $\omega_k \sim k^{2/3}, k \ge 1/t$, and $t \ge h$. The following result gives an estimate of these oscillatory integrals.

Proposition 2.3. *Let $\epsilon \in]0,1/7[$. For small ε, there exists a constant C independent of $a \in (0, h^{\frac{2}{3}(1-\epsilon)}]$, $t \in [h,1]$, $x \in [0,a]$, $y \in \mathbb{R}$, $z \in \mathbb{R}$ and $k \in [L, \frac{\varepsilon}{h}]$ such that the following holds:*

$$\left|\int e^{i\lambda\psi_k^{\pm,\pm}}\sigma_k^{\pm,\pm}\,d\eta\right| \le C(hk)^{-2/3}\lambda^{-1/3}.$$

Proof of Proposition 2.3. Since $(hk)^{2/3}\sigma_k^{\pm,\pm}$ are classical symbols of degree 0 compactly supported in η, we apply the stationary phase method to an integral of the form

$$J_1 = \int e^{i\lambda\psi_k^{\pm,\pm}}(hk)^{2/3}\sigma_k^{\pm,\pm}\,d\eta.$$

We have to prove that the following inequality holds uniformly with respect to the parameters:

$$|J_1| \le C\lambda^{-1/3}.$$

Let us recall that

$$h\lambda\psi_k^{\pm,\pm}(t,x,y,z;\eta) = y\eta + |\eta|t\sqrt{1-\tilde{z}^2}(1+\gamma)^{1/2} \pm \frac{2}{3}|\eta|(\gamma-x)^{3/2} \pm \frac{2}{3}|\eta|(\gamma-a)^{3/2}.$$

We compute

$$h\lambda\partial_\eta\psi_k^{\pm,\pm} = y + t\sqrt{1-\tilde{z}^2}\frac{1+\frac{2}{3}\gamma}{\sqrt{1+\gamma}} \pm \frac{2}{3}x(\gamma-x)^{1/2} \pm \frac{2}{3}a(\gamma-a)^{1/2},$$

and we need to consider four cases. Let $\delta = \frac{x}{a} \in [0,1], \alpha = \frac{a}{\omega_k h^{2/3}} \in [0,\alpha_0]$. Indeed, since $\omega_k \sim k^{2/3}$, $k \ge Dh^{-\epsilon}$ and $a \le h^{\frac{2}{3}(1-\epsilon)}$, it follows that $\alpha = ak^{-2/3}h^{-2/3} \le D^{-2/3}ah^{-\frac{2}{3}(1-\epsilon)} \le D^{-2/3} := \alpha_0$. Let $\rho = |\eta|^{-2/3}$, $V = \frac{y+t\sqrt{1-\tilde{z}^2}}{t\omega_k h^{2/3}}$ and define the function $F(\gamma)$ by

$$\frac{1+\frac{2}{3}\gamma}{\sqrt{1+\gamma}} = 1 + \gamma F(\gamma), \quad F(\gamma) = \frac{1}{6} + \frac{\gamma}{24} + O(\gamma^2).$$

With these notations we get:

$$\partial_\eta \psi_k^{\pm,\pm} = V + \sqrt{1-\tilde{z}^2}\rho F(h^{2/3}\omega_k\rho) + \frac{2}{3}\mu\left(\pm\delta(\rho-\delta\alpha)^{1/2} \pm (\rho-\alpha)^{1/2}\right),$$

where $\mu = \frac{ah^{-1/3}}{t\omega_k^{1/2}}$; it satisfies $0 \le \mu \le \frac{h^{\frac{1}{3}(1-\epsilon)}}{t}\min\{1, h^{-\epsilon/3}t^{1/3}\}$ and thus μ may be small or arbitrary large. In fact, if $t \ge h^\epsilon$, $\mu \le h^{\frac{1}{3}(1-\epsilon)}t^{-1} \le h^{1/3-4\epsilon/3}$, which is small if $\epsilon \le 1/4$. If $t \le h^\epsilon$, we have $\mu \le h^{1/3-2\epsilon/3}t^{-2/3}$ which could be large when $t \le h^{1/2-\epsilon}$. First, we consider the case where μ is bounded. We now study the critical points. We take $\rho = |\eta|^{-2/3}$ as variable, we get

$$\partial_\rho\partial_\eta \psi_k^{\pm,\pm} = \sqrt{1-\tilde{z}^2}(F(\gamma)+\gamma F'(\gamma)) + \frac{\mu}{3}\big(\pm\delta(\rho-\delta\alpha)^{-1/2} \pm (\rho-\alpha)^{-1/2}\big),$$
$$\partial_\rho^2\partial_\eta \psi_k^{\pm,\pm} = \sqrt{1-\tilde{z}^2}h^{2/3}\omega_k(2F'(\gamma)+\gamma F''(\gamma)) - \frac{\mu}{6}\big(\pm\delta(\rho-\delta\alpha)^{-3/2} \pm (\rho-\alpha)^{-3/2}\big).$$

For ε small enough, there exists $c > 0$ independent of $k \le \frac{\varepsilon}{h}$ such that

$$|\partial_\rho\partial_\eta\psi_k^{\pm,\pm}| + |\partial_\rho^2\partial_\eta\psi_k^{\pm,\pm}| \ge c. \tag{2.5}$$

Indeed, we observe that $(\rho-\alpha)^{-1/2} \ge \delta(\rho-\delta\alpha)^{-1/2}$ and $F(\gamma)+\gamma F'(\gamma) \sim \frac{1}{6}$. Thus we get $|\partial_\rho\partial_\eta\psi_k^{\pm,+}| \ge c_1 > 0$. Other cases, $\partial_\rho\partial_\eta\psi_k^{\pm,-}$ could vanish and when this happens we have

$$|\partial_\rho\partial_\eta\psi_k^{\pm,-}| \le 1/100 \implies \frac{\mu}{3}(\rho-\alpha)^{-1/2} \ge 0.05.$$

Then we have $|\partial_\rho^2\partial_\eta\psi_k^{-,-}| \ge c_2 > 0$. Moreover, for any function f, we have

$$f(\rho-\alpha) - \delta f(\rho-\delta\alpha) = (1-\delta)f(\rho-\delta\alpha) - \int_0^{\alpha(1-\delta)} f'(\rho-\delta\alpha-t)\,dt. \tag{2.6}$$

Taking $f(t) = t^{-1/2}$, we get that

$$|\partial_\rho\partial_\eta\psi_k^{+,-}| \le 1/100 \implies \mu(1-\delta) \ge c > 0.$$

Applying (2.6) with $f(t) = t^{-3/2}$, we obtain $|\partial_\rho^2\partial_\eta\psi_k^{+,-}| \ge c/2 > 0$. As a consequence of (2.5) together with [12, Lemma 2.20] (see Appendix), we get that the proposition holds true for μ bounded.

It remains to study the case where μ is large. For $(+,+)$ or $(-,+)$ case, we study again the critical points and we take $\Lambda = \lambda\mu$ as a large parameter. Since $\delta(\rho-\delta\alpha)^{-1/2} + (\rho-\alpha)^{-1/2} \ge c > 0$, we have $|\partial_\rho\partial_\eta\psi_k^{\pm,+}| \ge c > 0$. Hence $|J_1| \le C(\lambda\mu)^{-1/2}$. For $(+,-)$ and $(-,-)$ cases, we can use (2.6). We distinguish between two cases: if $\mu(1-\delta)$ is bounded, the computation of the derivatives of the phase functions $\psi_k^{\pm,-}$ yields the inequality (2.5) and the conclusion follows the [12, Lemma 2.20]. If $\mu(1-\delta)$ is large, we take $\Lambda' = \lambda\mu(1-\delta)$ as a large parameter in J_1 . Since by (2.6), we have

$$|(\rho-\alpha)^{-1/2} - \delta(\rho-\delta\alpha)^{-1/2}| \ge c(1-\delta)$$

with $c > 0$. We get that $|\partial_\rho\partial_\eta\psi_k^{\pm,-}| \ge c > 0$ and hence $|J_1| \le C(\lambda\mu(1-\delta))^{-1/2}$. □

To summarize, the Proposition 2.3 yields the dispersive estimates for $\mathcal{G}_{a,>L}$ for the large values of k, $L \leq k \leq \varepsilon/h$ as follows:

$$\begin{aligned}
\|\mathbb{1}_{x\leq a}\mathcal{G}_{a,>L}(t,x,y,z)\|_{L^\infty} &\leq Ch^{-2}\left(\frac{h}{t}\right)^{1/2}\sum_{k\leq\frac{\varepsilon}{h}}(hk)^{-2/3}\lambda^{-1/3},\\
&\leq Ch^{-2}\left(\frac{h}{t}\right)^{1/2}\sum_{k\leq\frac{\varepsilon}{h}}(hk)^{-2/3}(t\omega_k h^{-1/3})^{-1/3},\\
&\leq Ch^{-2}\left(\frac{h}{t}\right)^{1/2}\sum_{k\leq\frac{\varepsilon}{h}}(hk)^{-2/3}t^{-1/3}k^{-2/9}h^{1/9},\\
&\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}\left(\frac{h}{t}\right)^{1/3}h^{1/9}\left(\sum_{k\leq\frac{\varepsilon}{h}}k^{-8/9}\right),\\
&\leq Ch^{-3}\left(\frac{h}{t}\right)^{5/6},
\end{aligned}$$

where we used $\lambda = t\omega_k h^{-1/3}$ in the second line, and $\omega_k \sim k^{2/3}$ in the third line. This concludes the proof of Proposition 2.1. □

2.2. Airy–Poisson summation formula. Let $A_\pm(z) = e^{\mp i\pi/3}Ai(e^{\mp i\pi/3}z)$, we have $Ai(-z) = A_+(z) + A_-(z)$. For $\omega \in \mathbb{R}$, set

$$L(\omega) = \pi + i\log\left(\frac{A_-(\omega)}{A_+(\omega)}\right).$$

As in Lemma 2.7 in [10], the function L is analytic, strictly increasing and satisfies

$$L(0) = \pi/3, \quad \lim_{\omega\to-\infty} L(\omega) = 0, \quad L(\omega) = \frac{4}{3}\omega^{3/2} - B(\omega^{3/2}), \quad \text{for } \omega \geq 1,$$

with

$$B(\omega) \sim_{1/\omega} \sum_{j\geq 1} b_j\omega^{-j}, \quad b_j \in \mathbb{R}, \quad b_1 > 0, \tag{2.7}$$

and for all $k \geq 1$, the following holds

$$L(\omega_k) = 2\pi k \iff Ai(-\omega_k) = 0, \quad L'(\omega_k) = 2\pi\int_0^\infty Ai^2(x-\omega_k)\,dx.$$

Recall that f_k are constants such that $\|e_k(.,\eta)\|_{L^2(\mathbb{R}_+)} = 1$. This gives us

$$\int_0^\infty Ai^2(x-\omega_k)\,dx = \frac{k^{1/3}}{f_k^2} = \frac{L'(\omega_k)}{2\pi}.$$

The next lemma, whose proof can be found in [13], is the key tool to transform the sum over the eigenmodes k to the sum over N.

Lemma 2.4. (Airy–Poisson summation formula) *The following equality holds true in* $\mathcal{D}'(\mathbb{R}_\omega)$,

$$\sum_{N\in\mathbb{Z}} e^{-iNL(\omega)} = 2\pi\sum_{k\in\mathbb{N}^*}\frac{1}{L'(\omega_k)}\delta_{\omega=\omega_k}.$$

That is, for $\phi(\omega) \in C_0^\infty$,

$$\sum_{N\in\mathbb{Z}} \int e^{-iNL(\omega)}\phi(\omega)\, d\omega = 2\pi \sum_{k\in\mathbb{N}^*} \frac{1}{L'(\omega_k)}\phi(\omega_k).$$

Now we rewrite (1.9) using the definition of the eigenfunctions e_k and we replace the factor $\frac{f_k^2}{k^{1/3}}$ by $\frac{2\pi}{L'(\omega_k)}$. We get

$$\begin{aligned}
\mathcal{G}_{a,c_0}(t,x,y,z) &= \frac{1}{(2\pi)^2 h^{8/3}} \int e^{\frac{i}{h}(y\eta+z\zeta)} \sum_{k\geq 1} \frac{|f_k|^2}{k^{1/3}} e^{i\frac{t}{h}(\eta^2+\zeta^2+\omega_k h^{2/3}|\eta|^{4/3})^{1/2}} |\eta|^{2/3} \\
&\quad \times \chi_0(\eta^2+\zeta^2)\psi_0(\eta)\chi_1(\omega_k h^{2/3}|\eta|^{4/3})(1-\chi_1)(\varepsilon\omega_k) \\
&\quad \times Ai\Big(h^{-2/3}|\eta|^{2/3}x-\omega_k\Big) Ai\Big(h^{-2/3}|\eta|^{2/3}a-\omega_k\Big)\, d\eta\, d\zeta, \\
&= \frac{1}{(2\pi)^2 h^{8/3}} \int e^{\frac{i}{h}(y\eta+z\zeta)} \sum_{k\geq 1} \frac{2\pi}{L'(\omega_k)} e^{i\frac{t}{h}(\eta^2+\zeta^2+\omega_k h^{2/3}|\eta|^{4/3})^{1/2}} |\eta|^{2/3} \\
&\quad \times \chi_0(\eta^2+\zeta^2)\psi_0(\eta)\chi_1(\omega_k h^{2/3}|\eta|^{4/3})(1-\chi_1)(\varepsilon\omega_k) \\
&\quad \times Ai\Big(h^{-2/3}|\eta|^{2/3}x-\omega_k\Big) Ai\Big(h^{-2/3}|\eta|^{2/3}a-\omega_k\Big)\, d\eta\, d\zeta, \\
&= \frac{1}{(2\pi)^2 h^{8/3}} \int e^{\frac{i}{h}(y\eta+z\zeta)} 2\pi \sum_{k\geq 1} \frac{\delta_{\omega=\omega_k}}{L'(\omega_k)} e^{i\frac{t}{h}(\eta^2+\zeta^2+\omega h^{2/3}|\eta|^{4/3})^{1/2}} |\eta|^{2/3} \\
&\quad \times \chi_0(\eta^2+\zeta^2)\psi_0(\eta)\chi_1(\omega h^{2/3}|\eta|^{4/3})(1-\chi_1)(\varepsilon\omega) \\
&\quad \times Ai\Big(h^{-2/3}|\eta|^{2/3}x-\omega\Big) Ai\Big(h^{-2/3}|\eta|^{2/3}a-\omega\Big)\, d\omega\, d\eta\, d\zeta.
\end{aligned}$$

Using Lemma 2.4, $\mathcal{G}_{a,c_0}$ becomes

$$\begin{aligned}
\mathcal{G}_{a,c_0}(t,x,y,z) &= \frac{1}{(2\pi)^2 h^{8/3}} \int e^{\frac{i}{h}(y\eta+z\zeta)} \sum_{N\in\mathbb{Z}} e^{-iNL(\omega)} e^{i\frac{t}{h}(\eta^2+\zeta^2+\omega h^{2/3}|\eta|^{4/3})^{1/2}} |\eta|^{2/3} \\
&\quad \times \chi_0(\zeta^2+\eta^2)\psi_0(\eta)\chi_1(\omega h^{2/3}|\eta|^{4/3})(1-\chi_1)(\varepsilon\omega) \\
&\quad \times Ai\Big(h^{-2/3}|\eta|^{2/3}x-\omega\Big) Ai\Big(h^{-2/3}|\eta|^{2/3}a-\omega\Big)\, d\omega\, d\eta\, d\zeta.
\end{aligned}$$

From definition of the Airy function (see Appendix)

$$\left(\frac{A_-(\omega)}{A_+(\omega)}\right)^N = i^N e^{-\frac{4}{3}iN\omega^{3/2}} e^{iNB(\omega^{3/2})},$$

where for $\omega \in \mathbb{R}_+$, we recall that $B(\omega) \in \mathbb{R}$ is defined as in (2.7). It follows that

$$\begin{aligned}\mathcal{G}_{a,c_0}(t,x,y,z) &= \sum_{N\in\mathbb{Z}} \frac{(-1)^N}{(2\pi)^2 h^{8/3}} \int e^{\frac{i}{h}(y\eta+z\zeta)} e^{i\frac{t}{h}(\eta^2+\zeta^2+\omega h^{2/3}|\eta|^{4/3})^{1/2}} |\eta|^{2/3} \\ &\quad \times \chi_0(\zeta^2+\eta^2)\psi_0(\eta)\chi_1(\omega h^{2/3}|\eta|^{4/3})(1-\chi_1)(\varepsilon\omega)\left(\frac{A_-(\omega)}{A_+(\omega)}\right)^N \\ &\quad \times Ai\Big(h^{-2/3}|\eta|^{2/3}x-\omega\Big) Ai\Big(h^{-2/3}|\eta|^{2/3}a-\omega\Big)\, d\omega\, d\eta\, d\zeta, \\ &= \sum_{N\in\mathbb{Z}} \frac{(-i)^N}{(2\pi)^4 h^{10/3}} \\ &\quad \times \int e^{\frac{i}{h}\left(y\eta+z\zeta+t(\eta^2+\zeta^2+\omega h^{2/3}|\eta|^{4/3})^{1/2}+\frac{s^3}{3}+s(|\eta|^{2/3}x-\omega h^{2/3})+\frac{\sigma^3}{3}+\sigma(|\eta|^{2/3}a-\omega h^{2/3})\right)} \\ &\quad \times |\eta|^{2/3}\chi_0(\zeta^2+\eta^2)\psi_0(\eta)\chi_1(\omega h^{2/3}|\eta|^{4/3})(1-\chi_1)(\varepsilon\omega) \\ &\quad \times e^{-\frac{4}{3}iN\omega^{3/2}+iNB(\omega^{3/2})}\, ds\, d\sigma\, d\omega\, d\eta\, d\zeta, \end{aligned} \tag{2.8}$$

From the first to the second line, we made a change of variables $s = Sh^{-1/3}$ and $\sigma = \Sigma h^{-1/3}$ in the Airy functions; but for simplicity we keep the notations s, σ.

Therefore, (2.8) is a local parametrix that reads as a sum over N. Notice that our parametrix coincides with the constructed sum over reflected waves in [12] since each term has essentially the same phase. In the sequel, we refer the sum over $N \in \mathbb{Z}$ as the summands of waves corresponding to the number of reflections on the boundary, indexed by N.

2.3. Dispersive estimates for $a \geq h^{\frac{2}{3}(1-\epsilon')}$, $\epsilon' \in]0,\epsilon[$. In this subsection, we establish the local in time dispersive estimates for the parametrix in the form (2.8) as a sum over $N \in \mathbb{Z}$ in the regime $a \geq h^{\frac{2}{3}(1-\epsilon')}$, for $\epsilon' \in]0,\epsilon[$. Recall that our local parametrix under the form (2.8) is constructed from (1.9) together with the Lemma 2.4. It is a sum of oscillatory integrals with phase functions containing an Airy type terms with degenerate critical points. We give a precise analysis of the Lagrangian in the phase space associated to these oscillatory integrals. This geometric analysis allows us to track the degeneracy of the phases when we apply the stationary phase method.

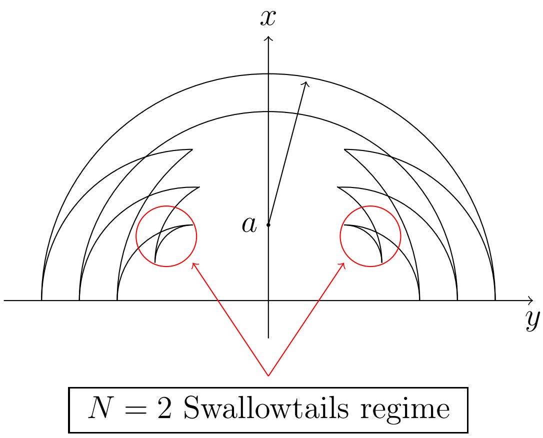


Figure 2. Swallowtails.

To deal with (2.8), we introduce a change of variables

$$a\tilde{\omega} = h^{2/3}\omega|\eta|^{-2/3}, \quad x = aX, \zeta = |\eta|\tilde{\zeta}, \quad s = a^{1/2}|\eta|^{1/3}\tilde{s}, \quad \sigma = a^{1/2}|\eta|^{1/3}\tilde{\sigma}.$$

Then we can rewrite $\mathcal{G}_{a,c_0}$ as follows:

$$\mathcal{G}_{a,c_0}(t,x,y,z) = \sum_{N\in\mathbb{Z}} G_{a,N}, \tag{2.9}$$

with for each $N \in \mathbb{Z}$,

$$\begin{aligned} G_{a,N}(t,x,y,z) = \frac{(-i)^N a^2}{(2\pi)^4 h^4} \int & e^{\frac{i}{h}\Phi_{N,a,h}} |\eta|^3 \chi_0(\eta^2(1+|\tilde{\zeta}|^2))\psi_0(\eta)\chi_1(a\tilde{\omega}\eta^2) \\ & \times (1-\chi_1)(\varepsilon a h^{-2/3}|\eta|^{2/3}\tilde{\omega})\, d\tilde{s}\, d\tilde{\sigma}\, d\tilde{\omega}\, d\tilde{\zeta}\, d\eta, \end{aligned} \tag{2.10}$$

with the phase function $\Phi_{N,a,h} = \Phi_{N,a,h}(t,x,y,z;\tilde{s},\tilde{\sigma},\tilde{\omega},\tilde{\zeta},\eta)$,

$$\begin{aligned} \Phi_{N,a,h} = y\eta + |\eta| z\tilde{\zeta} + |\eta| t(1+\tilde{\zeta}^2 + a\tilde{\omega})^{1/2} + a^{3/2}|\eta| \bigg(\frac{\tilde{s}^3}{3} + \tilde{s}(X-\tilde{\omega}) + \frac{\tilde{\sigma}^3}{3} \\ + \tilde{\sigma}(1-\tilde{\omega}) - \frac{4}{3} N \tilde{\omega}^{3/2} + \frac{h}{a^{3/2}|\eta|} N B\big(\tilde{\omega}^{3/2} a^{3/2}|\eta|/h\big) \bigg). \end{aligned}$$

The main result of this subsection is Theorem 2.5. It gives the estimate of the sum over N of the oscillatory integrals of the form (2.10) by using the stationary phase type estimates with degenerate critical points.

Theorem 2.5. *Let $\alpha < 2/3$. There exists C such that for all $h \in]0,h_0]$, all $a \in [h^\alpha, a_0]$, all $X \in [0,1]$, all $T \in]0, a^{-1/2}]$, all $Y \in \mathbb{R}$, all $z \in \mathbb{R}$, the following holds:*

$$\bigg| \sum_{0 \le N \le C_0 a^{-1/2}} G_{a,N}(T,X,Y,z;h) \bigg| \le C h^{-3} \Big(\frac{h}{t}\Big)^{1/2} \bigg(\Big(\frac{h}{t}\Big)^{1/2} + a^{1/8} h^{1/4} \bigg). \tag{2.11}$$

Notice that the first part on the right hand side of (2.11) corresponds to the free space estimates in $\mathbb{R}^3$, while the contribution in the second part appears as a consequence of the presence of caustics (cusps and swallowtails type).

First of all, we observe that when $N = 0, G_{a,0}$ satisfies $PG_{a,0} = 0$ and the associated data at time $t = 0$ is a localized Dirac at $x = a, y = 0, z = 0$. Therefore, $G_{a,0}$ satisfies the classical dispersive estimate for the wave equation in three-dimensional space; that is,

$$\big| G_{a,0}(T,X,Y,z,h) \big| \le C h^{-3} \Big(\frac{h}{t} \Big).$$

Thus it remains to prove the Theorem 2.5 for the sum over $1 \le N \le C_0 a^{-1/2}$.

First, we can apply the stationary phase method to deal with the $\tilde{\zeta}$-integration appearing in $G_{a,N}$ as the following lemma.

Lemma 2.6. *One has*

$$\begin{aligned} J_{N,a,h} &= \int e^{\frac{i}{h}|\eta|(z\tilde{\zeta} + t(1+\tilde{\zeta}^2 + a\tilde{\omega})^{1/2})} \chi_0(\eta^2(1+|\tilde{\zeta}|^2))\, d\tilde{\zeta} \\ &= \Big(\frac{h}{t}\Big)^{1/2} e^{\frac{i}{h}|\eta|\sqrt{t^2-z^2}(1+a\tilde{\omega})^{1/2}} \tilde{\chi}_0, \end{aligned}$$

where $\tilde{\chi}_0$ is a classical symbol of order 0 *with small parameter h/t.*

Proof. We apply the classical stationary phase method for $J_{N,a,h}$. First we make a change of variable $z = t\tilde{z}$. Let the phase function ϕ be

$$\phi(\tilde{\zeta}; \tilde{z}, \tilde{\omega}, a) = \tilde{z}\tilde{\zeta} + (1+\tilde{\zeta}^2 + a\tilde{\omega})^{1/2}.$$

Differentiating with respect to $\tilde{\zeta}$, we get

$$\partial_{\tilde{\zeta}}\phi = \tilde{z} + \frac{\tilde{\zeta}}{(1+\tilde{\zeta}^2+a\tilde{\omega})^{1/2}}.$$

On the support of χ_0, we have $\left|\frac{\tilde{\zeta}}{(1+\tilde{\zeta}^2+a\tilde{\omega})^{1/2}}\right| \leq 1-2\delta_1$ for some $\delta_1 > 0$ small. If $|\tilde{z}| \geq 1-\delta_1$, then the contribution of $\tilde{\zeta}$-integration is $O_{C^\infty}((h/t)^\infty)$ by integration by parts. Thus we may assume that $|\tilde{z}| < 1-\delta_1$. In this case, the phase ϕ admits a unique critical point on the support of χ_0. It is given by $\tilde{\zeta}_c = -\frac{\tilde{z}(1+a\tilde{\omega})^{1/2}}{\sqrt{1-\tilde{z}^2}}$ and this critical point is nondegenerate since

$$\partial^2_{\tilde{\zeta}}\phi = \frac{1+a\tilde{\omega}}{(1+\tilde{\zeta}^2+a\tilde{\omega})^{3/2}} > 0.$$

Then by the stationary phase method (as $|\tilde{z}| < 1-\delta_1$),

$$J_{N,a,h} = \left(\frac{h}{t}\right)^{1/2} e^{i|\eta|\frac{t}{h}\sqrt{1-\tilde{z}^2}(1+a\tilde{\omega})^{1/2}}\tilde{\chi}_0. \qquad \square$$

By Lemma 2.6, (2.9) becomes

$$\begin{aligned}
&\mathcal{G}_{a,c_0}(t,x,y,z) \\
&= \sum_{N\in\mathbb{Z}} \frac{(-i)^N a^2}{(2\pi)^4 h^4}\left(\frac{h}{t}\right)^{1/2} \int e^{\frac{i}{h}\tilde{\Phi}_{N,a,h}}|\eta|^3\tilde{\chi}_0\psi_0\chi_1(1-\chi_1)\,d\tilde{s}\,d\tilde{\sigma}\,d\tilde{\omega}\,d\eta,
\end{aligned} \tag{2.12}$$

where $\tilde{\Phi}_{N,a,h} = \Phi_{N,a,h}(.,\tilde{\zeta}_c,.)$; that is,

$$\begin{aligned}
\tilde{\Phi}_{N,a,h} &= y\eta + |\eta|t\sqrt{1-\tilde{z}^2}(1+a\tilde{\omega})^{1/2} + a^{3/2}|\eta|\Bigg(\frac{\tilde{s}^3}{3} + \tilde{s}(X-\tilde{\omega}) \\
&\quad + \frac{\tilde{\sigma}^3}{3} + \tilde{\sigma}(1-\tilde{\omega}) - \frac{4}{3}N\tilde{\omega}^{3/2} + \frac{h}{a^{3/2}|\eta|}NB\big(\tilde{\omega}^{3/2}a^{3/2}|\eta|/h\big)\Bigg).
\end{aligned} \tag{2.13}$$

Now we introduce the change of variables

$$t = a^{1/2}T, \quad y + t\sqrt{1-\tilde{z}^2} = a^{3/2}Y,$$

$$(1+a\tilde{\omega})^{1/2} - 1 = a\gamma_a(\tilde{\omega}) = \frac{a\tilde{\omega}}{1+(1+a\tilde{\omega})^{1/2}} \quad \text{and} \quad \lambda = \frac{a^{3/2}}{h}|\eta|.$$

We get (2.13) as follows:

$$\begin{aligned}
\tilde{\Phi}_{N,a,h} &= a^{3/2}|\eta|\Bigg\{Y + T\sqrt{1-\tilde{z}^2}\gamma_a(\tilde{\omega}) + \frac{\tilde{s}^3}{3} + \tilde{s}(X-\tilde{\omega}) + \frac{\tilde{\sigma}^3}{3} + \tilde{\sigma}(1-\tilde{\omega}) \\
&\quad - \frac{4}{3}N\tilde{\omega}^{3/2} + \frac{h}{a^{3/2}|\eta|}NB\big(\tilde{\omega}^{3/2}a^{3/2}|\eta|/h\big)\Bigg\}.
\end{aligned} \tag{2.14}$$

First, we study geometrically the set of critical points $\mathcal{C}_{N,a,h}$ of the associated Lagrangian manifold $\mathbf{\Lambda}_{N,a,h}$ for the phase function $\tilde{\Phi}_{N,a,h}$. The set of critical points is defined by

$$\mathcal{C}_{a,N,h} = \{(t,x,y,\tilde{s},\tilde{\sigma},\tilde{\omega},\eta) \mid \partial_{\tilde{s}}\tilde{\Phi}_{N,a,h} = \partial_{\tilde{\sigma}}\tilde{\Phi}_{N,a,h} = \partial_{\tilde{\omega}}\tilde{\Phi}_{N,a,h} = \partial_\eta\tilde{\Phi}_{N,a,h} = 0\}.$$

Then $\mathcal{C}_{a,N,h}$ is defined by a system of equations

$$
\begin{aligned}
&X = \tilde\omega - \tilde s^2, \quad \tilde\omega = 1 + \tilde\sigma^2,\\
&T = \frac{2(1+a\tilde\omega)^{1/2}}{\sqrt{1-\tilde z^2}}\Big(\tilde s + \tilde\sigma + 2N\tilde\omega^{1/2}\Big(1 - \frac{3}{4}B'\Big(\tilde\omega^{3/2}\lambda\Big)\Big)\Big),\\
&Y = -T\sqrt{1-\tilde z^2}\gamma_a(\tilde\omega) - \frac{\tilde s^3}{3} - \tilde s(X-\tilde\omega) - \frac{\tilde\sigma^3}{3} - \tilde\sigma(1-\tilde\omega) + N\tilde\omega^{3/2}\Big(\frac{4}{3} - B'\Big(\tilde\omega^{3/2}\lambda\Big)\Big).
\end{aligned}
$$

We may parametrize $\mathcal{C}_{a,N,h}$ by $(\tilde s, \tilde\sigma)$ near origin:

$$
\begin{aligned}
&X = 1 + \tilde\sigma^2 - \tilde s^2, \quad \tilde\omega = 1 + \tilde\sigma^2,\\
&T = \frac{2}{\sqrt{1-\tilde z^2}}(1+a+a\tilde\sigma^2)^{1/2}\Big(\tilde s + \tilde\sigma + 2N(1+\tilde\sigma^2)^{1/2}\Big(1 - \frac{3}{4}B'\Big((1+\tilde\sigma^2)^{3/2}\lambda\Big)\Big)\Big),\\
&Y = H_1(a,\tilde\sigma)(\tilde s + \tilde\sigma) + \frac{2}{3}(\tilde s^3 + \tilde\sigma^3) + \frac{4}{3}NH_2(a,\tilde\sigma)\left(1 - \frac{3}{4}B'\Big((1+\tilde\sigma^2)^{3/2}\lambda\Big)\right),
\end{aligned}
$$

with

$$
\begin{aligned}
H_1(a,\tilde\sigma) &= -(1+\tilde\sigma^2)\frac{(1+a+a\tilde\sigma^2)^{1/2}}{1+(1+a+a\tilde\sigma^2)^{1/2}},\\
H_2(a,\tilde\sigma) &= (1+\tilde\sigma^2)^{3/2}\frac{-3-4a-4a\tilde\sigma^2}{2+a+a\tilde\sigma^2+3(1+a+a\tilde\sigma^2)^{1/2}}.
\end{aligned}
$$

Let $\mathbf{\Lambda}_{a,N,h} \subset T^*\mathbb{R}^3$ be the image of $\mathcal{C}_{a,N,h}$ by the map

$$(t,x,y,\tilde s,\tilde\sigma,\tilde\omega,\eta) \longmapsto (x,t,y,\xi = \partial_x\tilde\Phi_{N,a,h}, \tau = \partial_t\tilde\Phi_{N,a,h}, \eta = \partial_y\tilde\Phi_{N,a,h}).$$

Then $\mathbf{\Lambda}_{a,N,h} \subset T^*\mathbb{R}^3$ is a Lagrangian submanifold parametrized by $(\tilde s, \tilde\sigma, \eta)$

$$
\begin{aligned}
&X = 1 + \tilde\sigma^2 - \tilde s^2,\\
&T = \frac{2}{\sqrt{1-\tilde z^2}}(1+a+a\tilde\sigma^2)^{1/2}\Big(\tilde s + \tilde\sigma + 2N(1+\tilde\sigma^2)^{1/2}\Big(1 - \frac{3}{4}B'\Big((1+\tilde\sigma^2)^{3/2}\lambda\Big)\Big)\Big),\\
&Y = H_1(a,\tilde\sigma)(\tilde s + \tilde\sigma) + \frac{2}{3}(\tilde s^3 + \tilde\sigma^3) + \frac{4}{3}NH_2(a,\tilde\sigma)\Big(1 - \frac{3}{4}B'\Big((1+\tilde\sigma^2)^{3/2}\lambda\Big)\Big),\\
&\xi = \eta\tilde s a^{1/2}, \quad \tau = \eta\sqrt{1-\tilde z^2}(1+a+a\tilde\sigma^2)^{1/2}, \quad \eta = \eta.
\end{aligned}
$$

On $\mathcal{C}_{a,N,h}$, we have $\tilde\omega = 1 + \tilde\sigma^2$, thus the projection of $\mathbf{\Lambda}_{a,N,h}$ onto $\mathbb{R}^3$ is

$$
\begin{aligned}
&X = 1 + \tilde\sigma^2 - \tilde s^2,\\
&T = \frac{2}{\sqrt{1-\tilde z^2}}(1+a+a\tilde\sigma^2)^{1/2}\Big(\tilde s + \tilde\sigma + 2N(1+\tilde\sigma^2)^{1/2}\\
&\qquad \times\left(1 - \frac{3}{4}B'\Big((1+\tilde\sigma^2)^{3/2}\lambda\Big)\right)\Big),\\
&Y = H_1(a,\tilde\sigma)(\tilde s + \tilde\sigma) + \frac{2}{3}(\tilde s^3 + \tilde\sigma^3) + \frac{4}{3}NH_2(a,\tilde\sigma)\Big(1 - \frac{3}{4}B'\Big((1+\tilde\sigma^2)^{3/2}\lambda\Big)\Big).
\end{aligned}
\tag{2.15}
$$

As in [12], we rewrite the system (2.15) in the following form

$$
\begin{aligned}
X &= 1+\tilde{\sigma}^2-\tilde{s}^2, \\
Y &= H_1(a,\tilde{\sigma})(\tilde{s}+\tilde{\sigma})+\frac{2}{3}(\tilde{s}^3+\tilde{\sigma}^3) \\
&\quad+\frac{2}{3}H_2(a,\tilde{\sigma})(1+\tilde{\sigma}^2)^{-1/2}\left(\frac{T\sqrt{1-\tilde{z}^2}}{2(1+a+a\tilde{\sigma}^2)^{1/2}}-\tilde{s}-\tilde{\sigma}\right),
\end{aligned}
\tag{2.16}
$$

and

$$
2N\left(1-\frac{3}{4}B'\left(\tilde{\omega}^{3/2}\lambda\right)\right)=(1+\tilde{\sigma}^2)^{-1/2}\left(\frac{T\sqrt{1-\tilde{z}^2}}{2(1+a+a\tilde{\sigma}^2)^{1/2}}-\tilde{s}-\tilde{\sigma}\right). \tag{2.17}
$$

Remark 2.7. Notice that from (2.17) in the range of $T\in\,]0,a^{-1/2}]$, we can reduce the sum over $N\in\mathbb{Z}$ of $G_{a,N}$ in (2.9) to the sum over $1\le N\le C_0a^{-1/2}$.

For a given a and $(X,Y,T)\in\mathbb{R}^3$, (2.16) is a system of two equations for unknown $(\tilde{s},\tilde{\sigma})$ and (2.17) gives an equation for N. We are looking for a solutions of (2.16) in the range

$$
a\in[h^\alpha,a_0],\ \ \alpha<2/3,\ \ a|\tilde{\sigma}|^2\le\epsilon_0,\ \ 0<T\le a^{-1/2},\ \ X\in[0,1]\quad\text{with } a_0,\ \epsilon_0 \text{ small.}
$$

Then for a given point $(X,Y,T)\in[-2,2]\times\mathbb{R}\times[0,a^{-1/2}]$, let us denote by $\mathcal{N}(X,Y,T)$ the set of integers $N\ge 1$ such that (2.15) admits at least one real solution $(\tilde{\sigma},\tilde{s},\lambda)$ with $a|\tilde{\sigma}|^2\le\epsilon_0$ and $\lambda\ge\lambda_0$. We denote by $\mathcal{N}^{\mathbb{C}}(X,Y,T)$ the set of complex N such that (2.15) admits at least one complex solution $(\tilde{\sigma},\tilde{s},\lambda)$ with $\tilde{\sigma}\in U$, where $U=\{\tilde{\sigma}\in\mathbb{C},|\tilde{\sigma}|\le 0.5 \text{ or } |\mathrm{Im}(\tilde{\sigma})|\le|\mathrm{Re}(\tilde{\sigma})|/\sqrt{3}\}$ and $a|\tilde{\sigma}|^2\le\epsilon_0$ and $\lambda\ge\lambda_0$.

We have the following lemma on the geometric estimates whose proof follows the same line as in the proof of Lemma 2.18 and Lemma 2.19 in [12].

Lemma 2.8. *There exists a constant C_0 such that the followings hold:*

1. *For all $(X,Y,T)\in[0,1]\times\mathbb{R}\times[0,a^{-1/2}]$, one has the cardinal of $\mathcal{N}(X,Y,T)$, $|\mathcal{N}(X,Y,T)|\le C_0$, and $\mathcal{N}^{\mathbb{C}}(X,Y,T)$ is a subset of the union of four disks of radius C_0.*
2. *For all $(X,Y,T)\in[0,1]\times\mathbb{R}\times[0,a^{-1/2}]$, the subset of $\mathbb{N}$,*

$$
\mathcal{N}_1(X,Y,T)=\bigcup_{|Y'-Y|+|T'-T|\le 1,|X'-X|\le 1}\mathcal{N}(X',Y',T')
$$

has cardinality satisfying

$$
|\mathcal{N}_1(X,Y,T)|\le C_0(1+T\lambda^{-2}\tilde{\omega}^{-3}).
$$

We notice that for $\tilde{\omega}\le 3/4$, we get rapid decay in λ by integration by part in $\tilde{\sigma}$. In particular, we may replace $1-\chi_1$ by 1 in (2.12). Moreover, the swallowtails will appear when $\tilde{s}=\tilde{\sigma}=0$ i.e. for $\tilde{\omega}=1$. For this reason, we introduce a cutoff function $\chi_2(\tilde{\omega})\in C_0^\infty(]1/2,3/2[)$, $0\le\chi_2\le 1$, $\chi_2=1$ on $]\frac{3}{4},\frac{5}{4}[$ in the integral (2.12) and we denote by $G_{a,N,2}$ the corresponding integral. This $G_{a,N,2}$ corresponds to the regime of swallowtails. We write $G_{a,N}=G_{a,N,1}+G_{a,N,2}$. $G_{a,N,1}$ is defined by introducing χ_3 in (2.12). We will have $\tilde{\omega}\ge 5/4$ on the support of χ_3.

To summarize, we have $\mathcal{G}_{a,c_0}$ as follows:

$$
\mathcal{G}_{a,c_0}=\sum_{1\le N\le C_0a^{-1/2}}G_{a,N}=\sum_{1\le N\le C_0a^{-1/2}}\left(G_{a,N,1}+G_{a,N,2}\right),
$$

where

$$G_{a,N,1} = \frac{(-i)^N a^2}{(2\pi)^4 h^4} \left(\frac{h}{t}\right)^{1/2} \int e^{\frac{i}{h}\tilde{\Phi}_{N,a,h}} |\eta|^3 \tilde{\chi}_0 \psi_0 \chi_1 \chi_3(\tilde{\omega}) d\tilde{s} d\tilde{\sigma} d\tilde{\omega} d\eta,$$
$$G_{a,N,2} = \frac{(-i)^N a^2}{(2\pi)^4 h^4} \left(\frac{h}{t}\right)^{1/2} \int e^{\frac{i}{h}\tilde{\Phi}_{N,a,h}} |\eta|^3 \tilde{\chi}_0 \psi_0 \chi_1 \chi_2(\tilde{\omega}) d\tilde{s} d\tilde{\sigma} d\tilde{\omega} d\eta.$$

In what follows, we get the estimates for these oscillatory integrals based on the (degenerate) stationary phase type result which consists in the precise study of where the phase $\tilde{\Phi}_{N,a,h}$ may be stationary.

2.3.1. The analysis of $G_{a,N,1}$. Let us recall that the $G_{a,N,1}$ is the oscillatory integral which corresponds to the regime where there are no swallowtails.

The estimates of $G_{a,N,1}$ can be obtained by combining the estimates of the following oscillatory integrals.

- First, for $(\tilde{s}, \tilde{\sigma})$-integrations, we use the stationary phase method.
- Then, for $\tilde{\omega}$-integration, we apply the degenerate phase method.
- Finally, for η-integration, we distinguish by cases that contribute to the estimates when we apply the stationary phase method. Meanwhile, the contribution in the estimates also comes from the cardinality of $\mathcal{N}_1$ defined in Lemma 2.8.

Our main results of this subsection are Proposition 2.9 and Proposition 2.10.

Proposition 2.9. *Let $\alpha < 2/3$. There exists C such that for all $h \in]0, h_0]$, all $a \in [h^\alpha, a_0]$, all $X \in [0,1]$, all $T \in]0, a^{-1/2}]$, all $Y \in \mathbb{R}$, all $z \in \mathbb{R}$, the following holds:*

$$\left| \sum_{2 \le N \le C_0 a^{-1/2}} G_{a,N,1}(T, X, Y, z; h) \right| \le C h^{-3} \left(\frac{h}{t}\right)^{1/2} h^{1/3}.$$

Proof. First of all, we apply the stationary phase method to $(\tilde{s}, \tilde{\sigma})$-integrations since on the support of χ_3 we have $\tilde{\omega} > 1$. Let I be defined by

$$\begin{aligned} I &= \int e^{i\lambda\left(\frac{\tilde{s}^3}{3} - \tilde{s}(\tilde{\omega} - X) + \frac{\tilde{\sigma}^3}{3} - \tilde{\sigma}(\tilde{\omega} - 1)\right)} \, d\tilde{s} \, d\tilde{\sigma} \\ &= (\tilde{\omega} - X)^{1/2} (\tilde{\omega} - 1)^{1/2} \int e^{i\lambda(\tilde{\omega} - X)^{3/2}\left(\frac{\tilde{s}^3}{3} - \tilde{s}\right)} e^{i\lambda(\tilde{\omega} - 1)^{3/2}\left(\frac{\tilde{\sigma}^3}{3} - \tilde{\sigma}\right)} \, d\tilde{s} \, d\tilde{\sigma}, \end{aligned}$$

where we made a change of variables $\tilde{s} = (\tilde{\omega} - X)^{1/2}\bar{s}, \tilde{\sigma} = (\tilde{\omega} - 1)^{1/2}\bar{\sigma}$ in the second line but for simplicity, we keep the notations $\tilde{s}, \tilde{\sigma}$. Thus by the stationary phase near the critical points $\tilde{s} = \pm 1, \tilde{\sigma} = \pm 1$ and integration by parts in $\tilde{s}, \tilde{\sigma}$ elsewhere we get

$$I = \lambda^{-1} (\tilde{\omega} - X)^{-1/4} (\tilde{\omega} - 1)^{-1/4} e^{i\lambda\left(\pm\frac{2}{3}(\tilde{\omega} - X)^{3/2} \pm \frac{2}{3}(\tilde{\omega} - 1)^{3/2}\right)} b_\pm c_\pm + O_{C^\infty}(\lambda^{-\infty}),$$

with $b_\pm, c_\pm$ are classical symbols of degree 0 in large parameter $\lambda(\tilde{\omega} - X)^{3/2}$ and $\lambda(\tilde{\omega} - 1)^{3/2}$ respectively. Notice that I is a part of the $G_{a,N,1}$ corresponding to the integrations in $\tilde{s}, \tilde{\sigma}$. Therefore, we obtain

$$G_{a,N,1}(T, X, Y, z; h) = \frac{(-i)^N a^2 \lambda^{-1}}{(2\pi)^4 h^4} \left(\frac{h}{t}\right)^{1/2} \int e^{i\frac{a^{3/2}}{h} Y\eta} |\eta|^3 \tilde{G}_{a,N,1} \, d\eta,$$
$$\tilde{G}_{a,N,1}(T, X, Y, z; h) = \sum_{\epsilon_1, \epsilon_2} \int e^{i\lambda \tilde{\Phi}_{N,\epsilon_1,\epsilon_2}} \Theta_{\epsilon_1,\epsilon_2} \, d\tilde{\omega} + O_{C^\infty}(\lambda^{-\infty}),$$

where $\epsilon_j = \pm, \Theta_{\epsilon_1,\epsilon_2}(\tilde\omega, a, \lambda) = \tilde\chi_0\psi_0\chi_1\chi_3(\tilde\omega)(\tilde\omega - X)^{-1/4}(\tilde\omega-1)^{-1/4}b_{\epsilon_1}c_{\epsilon_2}$ which satisfy $\left|\tilde\omega^l\partial^l_{\tilde\omega}\Theta_{\epsilon_1,\epsilon_2}\right| \leq C_l\tilde\omega^{-1/2}$, and the phase functions are given by

$$\begin{aligned}\tilde\Phi_{N,\epsilon_1,\epsilon_2}(T,X,z;\tilde\omega) &= T\sqrt{1-\tilde z^2}\gamma_a(\tilde\omega) + \frac{2}{3}\epsilon_1(\tilde\omega - X)^{3/2}\\ &\quad + \frac{2}{3}\epsilon_2(\tilde\omega-1)^{3/2} - \frac{4}{3}N\tilde\omega^{3/2} + \frac{N}{\lambda}B(\tilde\omega^{3/2}\lambda).\end{aligned} \tag{2.18}$$

Let us denote

$$\begin{aligned}G_{a,N,1,\epsilon_1,\epsilon_2}(T,X,Y,z;h) &= \frac{(-i)^N a^2\lambda^{-1}}{(2\pi)^4h^4}\left(\frac{h}{t}\right)^{1/2}\int e^{i\frac{a^{3/2}}{h}Y\eta}|\eta|^3\tilde G_{a,N,1,\epsilon_1,\epsilon_2}\,d\eta,\\ \tilde G_{a,N,1,\epsilon_1,\epsilon_2}(T,X,z;\lambda) &= \int e^{i\lambda\tilde\Phi_{N,\epsilon_1,\epsilon_2}}\Theta_{\epsilon_1,\epsilon_2}(\tilde\omega,a,\lambda)\,d\tilde\omega.\end{aligned} \tag{2.19}$$

We are reduced to proving the following inequality:

$$\left|\sum_{2\leq N\leq C_0a^{-1/2}} G_{a,N,1,\epsilon_1,\epsilon_2}(T,X,Y,z;h)\right| \leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}h^{1/3}, \tag{2.20}$$

with a constant C independent of $h\in\,]0,h_0]$, $a\in[h^{2/3},a_0]$, $X\in[0,1]$, $T\in[0,a^{-1/2}]$. For convenience, let $\Omega = \tilde\omega^{3/2}$ be a new variable of integration and we get

$$\tilde G_{a,N,1,\epsilon_1,\epsilon_2}(T,X,z;\lambda) = \int e^{i\lambda\tilde\Phi_{N,\epsilon_1,\epsilon_2}}\tilde\Theta_{\epsilon_1,\epsilon_2}(\Omega,a,\lambda)\,d\Omega; \tag{2.21}$$

$\tilde\Theta_{\epsilon_1,\epsilon_2}(\Omega,a,\lambda)$ are smooth functions with compact support in Ω. As $d\tilde\omega = \frac{2}{3}\Omega^{-1/3}d\Omega$, we get $\left|\Omega^l\partial^l_\Omega\tilde\Theta_{\epsilon_1,\epsilon_2}\right| \leq C_l\Omega^{-2/3}$ with C_l independent of a, λ and the phases (2.18) become

$$\begin{aligned}\tilde\Phi_{N,\epsilon_1,\epsilon_2}(T,X,z,\tilde\omega;a,\lambda) &= T\sqrt{1-\tilde z^2}\gamma_a(\Omega) + \frac{2}{3}\epsilon_1(\Omega^{2/3}-X)^{3/2}\\ &\quad + \frac{2}{3}\epsilon_2(\Omega^{2/3}-1)^{3/2} - \frac{4}{3}N\Omega + \frac{N}{\lambda}B(\Omega\lambda).\end{aligned}$$

We now study the critical points. We have

$$\begin{aligned}\partial_\Omega\tilde\Phi_{N,\epsilon_1,\epsilon_2} &= \frac{2}{3}\Big(H_{a,\epsilon_1,\epsilon_2}(T,X,z;\Omega) - 2N\Big(1-\frac{3}{4}B'(\Omega\lambda)\Big)\Big),\\ H_{a,\epsilon_1,\epsilon_2} &= \Omega^{-1/3}\Big(\frac{T}{2}\sqrt{1-\tilde z^2}(1+a\Omega^{2/3})^{-1/2} + \epsilon_1(\Omega^{2/3}-X)^{1/2}\\ &\qquad + \epsilon_2(\Omega^{2/3}-1)^{1/2}\Big),\\ \partial_\Omega H_{a,\epsilon_1,\epsilon_2} &= \frac{1}{3}\Omega^{-4/3}\Big(-\frac{T}{2}\sqrt{1-\tilde z^2}(1+a\Omega^{2/3})^{-3/2}(1+2a\Omega^{2/3})\\ &\qquad + \epsilon_1X(\Omega^{2/3}-X)^{-1/2} + \epsilon_2(\Omega^{2/3}-1)^{-1/2}\Big).\end{aligned} \tag{2.22}$$

We will first prove that (2.20) holds true in the case $(\epsilon_1,\epsilon_2) = (+,+)$. We have that the equation $\partial_\Omega H_{a,+,+}(\Omega) = 0$ admits a unique solution $\Omega_q = \Omega_q^+(T,X,z,a) > 1$ such that

$$\lim_{T\to\infty}\Omega_q^+(T,X,z,a)=1 \quad \text{uniformly in } X,z,a, \tag{2.23}$$

$$0>\frac{9}{2}\Omega_q^{5/3}\partial_\Omega^2 H_{a,+,+}(\Omega_q)=-\frac{aT}{2}\sqrt{1-\tilde z^2}\Big(1+a\Omega_q^{2/3}\Big)^{-5/2}\left(\frac12-a\Omega_q^{2/3}\right)$$
$$-\frac12\big(\Omega_q^{2/3}-1\big)^{-3/2}-\frac12 X\big(\Omega_q^{2/3}-X\big)^{-3/2}.$$

Thus, the function $H_{a,+,+}(\Omega)$ is strictly increasing on $[1,\Omega_q[$ and strictly decreasing on $]\Omega_q,\infty[$. Observe that

$$H_{a,+,+}(1)=\frac{T}{2}\sqrt{1-\tilde z^2}(1+a)^{-1/2}+(1-X)^{1/2},\quad \lim_{\Omega\to\infty}H_{a,+,+}=2. \tag{2.24}$$

For all k, there exist constant C_k such that

$$\forall\Omega\geq 1,\quad |\partial_\Omega^k(NB'(\Omega\lambda))|\leq C_k N\lambda^{-2}\Omega^{-(k+2)}. \tag{2.25}$$

Let $T_0\gg 1$. First, suppose that $0\leq T\leq T_0$. Since $H_{a,+,+}(\Omega)\leq C(1+T)$ and for $N\geq N(T_0)=C(1+T_0)$ for some constant C, we get $|\partial_\Omega\tilde\Phi_{N,+,+}|\geq c_0N$ with the constant $c_0>0$. Then by integration by parts, we get $|\tilde G_{a,N,1,+,+}|\in O(N^{-\infty}\lambda^{-\infty})$ and this implies

$$\sup_{T\leq T_0,X\in[0,1],Y\in\mathbb{R},z\in\mathbb{R}}\left|\sum_{N(T_0)\leq N\leq C_0a^{-1/2}}G_{a,N,1,+,+}(T,X,Y,z)\right|\in O_{C^\infty}(h^\infty).$$

Next, for $0\leq T\leq T_0$ and $2\leq N\leq N(T_0)$, we may estimate the sum by the sup of each term. In this case, we see that $\tilde\Phi_{N,+,+}$ has at most a critical point of order 2 near $\Omega=\Omega_q$ and

$$|\partial_\Omega\tilde\Phi_{N,+,+}|+|\partial_\Omega^2\tilde\Phi_{N,+,+}|+|\partial_\Omega^3\tilde\Phi_{N,+,+}|\geq c>0.$$

Moreover, if $N\geq 2$, we have a positive lower bound for $|\partial_\Omega\tilde\Phi_{N,+,+}(\Omega)|$ for large values of Ω; thus the contribution of $\tilde G_{a,N,1,+,+}$ is $O_{C^\infty}(\lambda^{-\infty})$ for large values of Ω. The critical point of order 2 near $\Omega=\Omega_q$, the estimate of $\tilde G_{a,N,1,+,+}$ is given by the Lemma 2.20 [12] which yields $|\tilde G_{a,N,1,+,+}(T,X,z;\lambda)|\leq C\lambda^{-1/3}$ with C independent of $T\in[0,T_0]$, $X\in[0,1]$. Hence from (2.19), we get

$$\sup_{X\in[0,1],Y\in\mathbb{R},z\in\mathbb{R}}\left|\sum_{2\leq N\leq N(T_0)}G_{a,N,1,+,+}(T,X,Y,z,h)\right|\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}\big(h^{-1}a^2\lambda^{-1}\lambda^{-1/3}\big),$$
$$\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}h^{1/3}.$$

Then we prove that (2.20) holds true for $T_0\leq T\leq a^{-1/2}$. Like before, we may assume $N\leq C_1T$ with C_1 large, the contribution of the sum on N such that $C_1T\leq N\leq C_0a^{-1/2}$ being negligible. From (2.23), we may choose T_0 large enough so that $\Omega_q^+(T,X,z,a)<\Omega_0$ with $\Omega_0>1$ for $T\geq T_0$ and we may assume with a constant $c>0$ that

$$|\partial_\Omega^2\tilde\Phi_{N,+,+}(\Omega)|\geq cT\Omega^{-4/3},\quad \forall\Omega\geq\Omega_0,\ \forall T\geq T_0,\ \forall N\leq C_0a^{-1/2}.$$

Therefore, on the support of $\tilde\Theta_{+,+}$, the phase $\tilde\Phi_{N,+,+}$ admits at most one critical point $\Omega_c=\Omega_c(T,X,z,N,\lambda,a)$ and this critical point is nondegenerate. Because $N\geq 2$, from the first item of (2.22) we get $\Omega_c^{1/3} \le T$ and this implies $\Omega_c^{1/3} \sim T/N$. As a consequence, if $T/N \sim 1$ then $\Omega_c \sim 1$. By stationary phase method, we get

$$|\tilde{G}_{a,N,1,+,+}(T,X,z;\lambda)| \le C\lambda^{-1/2}T^{-1/2} \quad \text{with } C \text{ independent of } N.$$

If $T/N \gg 1$, then we perform the change of variable $\Omega = \tilde{\Omega}(T/N)^3$ in (2.21); the unique critical point $\tilde{\Omega}_c$ remains in a fixed compact interval of $]0,\infty[$. We have

$$\partial_{\tilde{\Omega}}^k \tilde{\Theta}_{+,+}(\tilde{\Omega}(T/N)^3, a, \lambda) \le c_k (N/T)^2 \tilde{\Omega}^{-2/3-k}.$$

Thus by the stationary phase method, we get

$$\sup_{2\le N\le C_1 T, X\in[0,1], z\in\mathbb{R}} |\tilde{G}_{a,N,1,+,+}(T,X,z;\lambda)| \le C\lambda^{-1/2}T^{-1/2}.$$

It remains to estimate the sum

$$\left| \sum_{2\le N\le C_0 a^{-1/2}} G_{a,N,1,+,+}(T,X,Y,z;h) \right|.$$

Let $G_N(T,X,z,\lambda,a) = \tilde{\Phi}_{N,+,+}(T,X,z,\Omega_c(T,X,z,N,\lambda,a),\lambda,a)$. Therefore, by the stationary phase method at the critical point $\Omega_c = \Omega_c(T,X,z,N,\lambda,a)$ in (2.21), we obtain

$$\tilde{G}_{a,N,1,+,+}(T,X,z,h) = \lambda^{-1/2}T^{-1/2} e^{i\lambda G_N(T,X,z,\lambda,a)} \psi_N(T,X,\lambda,a),$$

with $\psi_N(T,X,\lambda,a)$ is a classical symbol of order 0 in λ. Therefore, if we denote $\tilde{\lambda} = a^{3/2}/h = \lambda/\eta$, we have

$$\begin{aligned} G_{a,N,1,+,+}(T,X,Y,z;h) &= \frac{(-i)^N a^2 \lambda^{-1}}{(2\pi)^4 h^4} \left(\frac{h}{t}\right)^{1/2} \lambda^{-1/2} T^{-1/2} \\ &\quad \times \int e^{i\tilde{\lambda}|\eta|(Y+G_N(T,X,z,\tilde{\lambda}\eta,a))} \psi_N |\eta|^3 \, d\eta. \end{aligned} \tag{2.26}$$

It is an oscillatory integral with large parameter $\tilde{\lambda}$ and phase

$$L_N(T,X,Y,z,\eta\tilde{\lambda}) = |\eta| \left(Y + G_N(T,X,z,\tilde{\lambda}\eta,a) \right).$$

By construction, the equation

$$\partial_\eta L_N = Y + G_N(T,X,z,\lambda,a) + \lambda \partial_\lambda G_N(T,X,z,\lambda,a) = 0$$

implies that (X,Y,T) belongs to the projection of $\mathbf{\Lambda}_{a,N,h}$ on $\mathbb{R}^3$. As in the proof of Proposition 2.14 [12], we see that the contribution of $G_{a,N,1,+,+}$ for the sum over N such that $N \notin \mathcal{N}_1(X,Y,T)$ is $O(\lambda^{-\infty})$. Thus it remains to estimate the sum

$$\left| \sum_{N\in\mathcal{N}_1(X,Y,T)} G_{a,N,1,+,+}(T,X,Y,z,h) \right|. \tag{2.27}$$

We apply the stationary phase method for η-integral with the phase function L_N. We have

$$\partial_\eta L_N = Y + G_N + \lambda\partial_\lambda G_N,$$

with

$$\lambda\partial_\lambda G_N = \lambda\partial_\lambda \tilde{\Phi}_{N,+,+}(T,X,\Omega_c,a,\lambda) = \frac{N}{\lambda}\big(- B(\lambda\Omega_c) + \lambda\Omega_c B'(\lambda\Omega_c)\big).$$

Then we obtain

$$\partial_\eta^2 L_N = \frac{N}{\eta}(\lambda\Omega_c)\partial_\lambda(\lambda\Omega_c) B''(\lambda\Omega_c).$$

On the other hand, $\partial_\lambda \Omega_c$ satisfies

$$\partial_\lambda \Omega_c \partial_\Omega^2 \tilde{\Phi}_{N,+,+}(\Omega_c) = -\partial_\lambda \partial_\Omega \tilde{\Phi}_{N,+,+}(\Omega_c) = -N\Omega_c B''(\lambda\Omega_c).$$

As we have $\partial_\Omega^2 \tilde{\Phi}_{N,+,+}(\Omega_c) \geq cT\Omega_c^{-4/3}$, $\Omega_c^{1/3} \sim T/N$, and for ω large, we have $B''(\omega) \sim \omega^{-3}$. We get

$$|\partial_\lambda \Omega_c| \leq cT^{-1}\Omega_c^{4/3} N\Omega_c(\lambda^{-3}\Omega_c^{-3}) \leq c\lambda^{-3}\Omega_c^{-1}.$$

This yields

$$|\partial_\lambda(\lambda\Omega_c)| = |\lambda\partial_\lambda\Omega_c + \Omega_c| \geq c\Omega_c(1 - c\lambda^{-2}\Omega_c^{-2}) \geq c'\Omega_c.$$

Hence we deduce that

$$|\partial_\eta^2 L_N| \geq CN\lambda^{-2}\Omega_c^{-1}.$$

Therefore η-integration produces a factor $q^{-1/2}$ with $q = N\lambda^{-1}\Omega_c^{-1}$. Let us recall that

$$|\mathcal{N}_1(X,Y,T)| \leq C_0(1 + T\lambda^{-2}\Omega_c^{-2}).$$

We get the estimates of the sum in (2.27) by distinguishing between many cases which depend on whether there are contributions from η-integration and the cardinality of $\mathcal{N}_1$, $|\mathcal{N}_1(X,Y,T)|$ as follows:

First case, if $\Omega_c^{1/3} \sim T/N \sim 1$, then $T \sim N$ and

- if $N \leq \lambda$, then there is no contribution from η-integration and we have $|\mathcal{N}_1| \leq C_0$. Hence the estimate is

$$\left| \sum_{N\in\mathcal{N}_1} G_{a,N,1,+,+} \right| \leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2} \left(h^{-1}\lambda^{-1}a^2\lambda^{-1/2}T^{-1/2}\right)$$
$$\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2} a^{-1/4}h^{1/2} \leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2} h^{1/3},$$

  since $a^{-1/4}h^{1/2} \leq h^{1/3}$ when $a \geq h^{2/3}$.
- if $\lambda < N \leq \lambda^2$, then there is a contribution $q^{-1/2}$ factor from η-integration and we also have $|\mathcal{N}_1| \leq C_0$. We get

$$\left| \sum_{N\in\mathcal{N}_1} G_{a,N,1,+,+} \right| \leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2} \left(h^{-1}\lambda^{-1}a^2\lambda^{-1/2}T^{-1/2}N^{-1/2}\lambda^{1/2}\right)$$
$$\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2} \left(h^{-1}a^2\lambda^{-2}\right) \leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2} h^{1/3}.$$

- if $N > \lambda^2$, then there are contributions from both $q^{-1/2}$ factor from η-integration and $|\mathcal{N}_1| \leq C_0T\lambda^{-2}$. Thus the estimate is

$$\left| \sum_{N\in\mathcal{N}_1} G_{a,N,1,+,+} \right| \leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2} \sum_{N\in\mathcal{N}_1} \left(h^{-1}\lambda^{-1}a^2\lambda^{-1/2}T^{-1/2}N^{-1/2}\lambda^{1/2}\right)$$
$$\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2} \left(h^{-1}\lambda^{-1}a^2T^{-1}|\mathcal{N}_1(X,Y,T)|\right)$$
$$\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2} \left(a^{-5/2}h^2\right) \leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2} h^{1/3}.$$

Second case, if $T/N \gg 1$, then $\Omega_c \gg 1$. We have

- if $N \leq \lambda\Omega_c$, then there is no contribution from η-integration. Moreover, we have $|\mathcal{N}_1| \leq C_0$. To see this point, assume by contradiction $T \geq \lambda^2\Omega_c^2$; this implies $\Omega_c^{1/3} \sim T/N \geq \lambda\Omega_c$ which is impossible since $\Omega_c \gg 1$. Thus the estimate is

$$\Big|\sum_{N\in\mathcal{N}_1} G_{a,N,1,+,+}\Big| \leq Ch^{-3}\Big(\frac{h}{t}\Big)^{1/2}\big(h^{-1}\lambda^{-1}a^2\lambda^{-1/2}T^{-1/2}\big)$$
$$\leq Ch^{-3}\Big(\frac{h}{t}\Big)^{1/2} a^{-1/4}h^{1/2} \leq Ch^{-3}\Big(\frac{h}{t}\Big)^{1/2} h^{1/3}.$$

- if $N > \lambda\Omega_c$ and $\lambda\Omega_c^{2/3} < T \leq \lambda^2\Omega_c^2$, then there is a contribution $q^{-1/2}$ factor from η-integration and we also have $|\mathcal{N}_1| \leq C_0$. We get

$$\Big|\sum_{N\in\mathcal{N}_1} G_{a,N,1,+,+}\Big| \leq Ch^{-3}\Big(\frac{h}{t}\Big)^{1/2}\big(h^{-1}\lambda^{-1}a^2\lambda^{-1/2}T^{-1/2}N^{-1/2}\lambda^{1/2}\Omega_c^{1/2}\big)$$
$$\leq Ch^{-3}\Big(\frac{h}{t}\Big)^{1/2}\big(h^{-1}a^2\lambda^{-2}\big) \leq Ch^{-3}\Big(\frac{h}{t}\Big)^{1/2} h^{1/3}.$$

- if $N > \lambda\Omega_c$ and $T > \lambda^2\Omega_c^2$, then there are contributions from both $q^{-1/2}$ factor from η-integration and $|\mathcal{N}_1| \leq C_0 T\lambda^{-2}\Omega_c^{-2}$. We get

$$\Big|\sum_{N\in\mathcal{N}_1} G_{a,N,1,+,+}\Big| \leq Ch^{-3}\Big(\frac{h}{t}\Big)^{1/2}\sum_{N\in\mathcal{N}_1}\big(h^{-1}\lambda^{-1}a^2\lambda^{-1/2}T^{-1/2}N^{-1/2}\lambda^{1/2}\Omega_c^{1/2}\big)$$
$$\leq Ch^{-3}\Big(\frac{h}{t}\Big)^{1/2}\big(h^{-1}\lambda^{-1}a^2T^{-1}\Omega_c^{2/3}|\mathcal{N}_1(X,Y,T)|\big)$$
$$\leq Ch^{-3}\Big(\frac{h}{t}\Big)^{1/2}\big(h^{-1}a^2\lambda^{-3}\big)(T/N)^{-4} \leq Ch^{-3}\Big(\frac{h}{t}\Big)^{1/2} h^{1/3}.$$

Next, we prove that (2.20) holds true in the case $(\epsilon_1,\epsilon_2) = (+,-)$. In this case, from the last item of (2.22), $X \in [0,1]$, and $B''(\lambda\Omega) \sim \lambda^{-3}\Omega^{-3}$ we get that for $T > 0$, $\partial_\Omega H_{a,+,-}(\Omega) + \frac{3N}{2}\lambda B''(\lambda\Omega) < 0$; that is, the function $H_{a,+,-}(\Omega) + \frac{3N}{2}B'(\lambda\Omega)$ decreases on $[1,\infty[$ from $H_{a,+,-}(1) + \frac{3N}{2}B'(\lambda) = \frac{T}{2}\sqrt{1-\tilde{z}^2}(1+a)^{-1/2} + (1-X)^{1/2} + \frac{3N}{2}B'(\lambda)$ to $\big(H_{a,+,-} + \frac{3N}{2}B'(\lambda.)\big)(\infty) = 0$. The equation $\partial_\Omega \Phi_{N,+,-} = 0$ admits a unique solution Ω_c and it is nondegenerate; thus we can argue as $(+,+)$ case. Finally, the case $(\epsilon_1,\epsilon_2) = (-,+)$ is similar to $(+,+)$ case and $(\epsilon_1,\epsilon_2) = (-,-)$ is similar to $(+,-)$ case. The proof of proposition is complete. □

Now we prove the estimates for $N = 1$.

Proposition 2.10. *Let $\alpha < 2/3$. There exists C such that for all $h \in]0,h_0]$, all $a \in [h^\alpha, a_0]$, all $X \in [0,1]$, all $T \in]0, a^{-1/2}]$, all $Y \in \mathbb{R}$, all $z \in \mathbb{R}$, the following holds:*

$$\Big|G_{a,1,1}(T,X,Y,z;h)\Big| \leq Ch^{-3}\Big(\frac{h}{t}\Big)^{1/2}\bigg(\Big(\frac{h}{t}\Big)^{1/2} + h^{1/3}\bigg).$$

Proof. Let us recall that

$$G_{a,1,1} = \frac{(-i)a^2\lambda^{-1}}{(2\pi)^4 h^4}\left(\frac{h}{t}\right)^{1/2} \int e^{i\frac{a^{3/2}}{h}Y\eta}|\eta|^3 \tilde{G}_{a,1,1}\, d\eta,$$
$$\tilde{G}_{a,1,1} = \sum_{\epsilon_1,\epsilon_2} \int e^{i\lambda\tilde{\Phi}_{1,\epsilon_1,\epsilon_2}}\Theta_{\epsilon_1,\epsilon_2}\, d\tilde{\omega} + O_{C^\infty}(h^\infty).$$

We recall that $\epsilon_j = \pm, \Theta_{\epsilon_1,\epsilon_2}(\tilde{\omega},a,\lambda) = \tilde{\chi}_0\psi_0\chi_1\chi_3(\tilde{\omega})(\tilde{\omega}-X)^{-1/4}(\tilde{\omega}-1)^{-1/4}b_{\epsilon_1}c_{\epsilon_2}$ which satisfy $\left|\tilde{\omega}^l\partial^l_{\tilde{\omega}}\Theta_{\epsilon_1,\epsilon_2}\right| \le C_l\tilde{\omega}^{-1/2}$. The only difference with the case $N \ge 2$ is in the study of the phase $\tilde{\Phi}_{1,+,+}$ since in the case $N=1$ we may have a critical point $\tilde{\omega}_c$ large. Let

$$\tilde{G}_{a,1,1,+,+} = \int e^{i\lambda\tilde{\Phi}_{1,+,+}}\Theta_{+,+}(\tilde{\omega},a,\lambda)\, d\tilde{\omega}, \tag{2.28}$$

with the phase function

$$\tilde{\Phi}_{1,+,+}(T,X,z;\tilde{\omega}) = T\sqrt{1-\tilde{z}^2}\gamma_a(\tilde{\omega}) + \frac{2}{3}(\tilde{\omega}-X)^{3/2} + \frac{2}{3}(\tilde{\omega}-1)^{3/2} - \frac{4}{3}\tilde{\omega}^{3/2} + \frac{1}{\lambda}B(\lambda\tilde{\omega}^{3/2}),$$

and $\Theta_{+,+}(\tilde{\omega},a,\lambda)$ is a classical symbol of order $-1/2$ with respect to $\tilde{\omega}$. Let denote $\chi_3(\tilde{\omega}) \in C_0^\infty(]\tilde{\omega}_1,\infty[)$ with $\tilde{\omega}_1$ large and set

$$\tilde{J}_{1,+,+} = \int e^{i\lambda\tilde{\Phi}_{1,+,+}}\Theta_{+,+}(\tilde{\omega},a,\lambda)\chi_3(\tilde{\omega})\, d\tilde{\omega}. \tag{2.29}$$

To prove the proposition, it suffices to verify $|\tilde{J}_{1,+,+}| \le C\lambda^{-1/2}T^{-1/2}$. We have

$$\partial_{\tilde{\omega}}\tilde{\Phi}_{1,+,+} = \frac{T}{2}\sqrt{1-\tilde{z}^2}(1+a\tilde{\omega})^{-1/2} - \frac{\tilde{\omega}^{-1/2}}{2}(1+X) + O_{C^\infty}(\tilde{\omega}^{-3/2}),$$
$$\partial^2_{\tilde{\omega}\tilde{\omega}}\tilde{\Phi}_{1,+,+} = \frac{-Ta}{4}\sqrt{1-\tilde{z}^2}(1+a\tilde{\omega})^{-3/2} + \frac{\tilde{\omega}^{-3/2}}{4}(1+X) + O_{C^\infty}(\tilde{\omega}^{-5/2}).$$

Thus, we see that to get a large critical point $\tilde{\omega}_c, T$ must be small. It follows that $\tilde{\omega}_c^{-1/2} \sim T$ and thus $\partial^2_{\tilde{\omega}\tilde{\omega}}\tilde{\Phi}_{1,+,+}(\tilde{\omega}_c) \sim T^3$. Now we can make a change of variable $\tilde{\omega} = T^{-2}\tilde{\upsilon}$ in (2.29). Because $\Theta_{+,+}(\tilde{\omega},a,\lambda)$ is a classical symbol in $\tilde{\omega}$ of order $-1/2$; thus $\tilde{\Theta}_{+,+}(T^{-2}\tilde{\upsilon},a,\lambda) = T^{-1}\tilde{\upsilon}^{1/2}\Theta_{+,+}(T^{-2}\tilde{\upsilon},a,\lambda)$ is a classical symbol of order 0 in $\tilde{\upsilon} \ge \tilde{\upsilon}_0 > 0$ uniformly in $T \in]0,T_0]$ and we also have $\partial^2_{\tilde{\upsilon}\tilde{\upsilon}}\tilde{\Phi}_{1,+,+} \sim T^{-1}$ or $T\partial^2_{\tilde{\upsilon}\tilde{\upsilon}}\tilde{\Phi}_{1,+,+} \sim 1$. Therefore, the stationary phase method yields

$$|\tilde{J}_{1,+,+}| = \left|\frac{1}{T^2}\int e^{i(\frac{\lambda}{T})T\tilde{\Phi}_{1,+,+}}T\upsilon^{-1/2}\tilde{\Theta}_{+,+}(\tilde{\omega},a,\lambda)\chi_3(T^{-2}\tilde{\upsilon})\, d\tilde{\upsilon}\right| \le C\frac{1}{T}\left(\frac{\lambda}{T}\right)^{-1/2}$$
$$|\tilde{J}_{1,+,+}| \le C\lambda^{-1/2}T^{-1/2},$$

which is the desired result. □

2.3.2. The Analysis of $G_{a,N,2}$. Recall that the $G_{a,N,2}$ is a sum of oscillatory integrals which corresponds to the regime where there are swallowtails; that is, corresponding to the case when $\tilde{s} = \tilde{\sigma} = 0$ i.e. for $\tilde{\omega} = 1$.

The estimates of $G_{a,N,2}$ can be obtained by combining the estimates of the following oscillatory integrals.

- First, we consider the $\tilde{\omega}$-integration by using the stationary phase method.

- Then, for η-integration, we distinguish by cases that we can apply the stationary phase method, namely there is an η-integration contribution when $N \gg \lambda$, while there is no η-integration contribution as $N \lesssim \lambda$. Meanwhile, the contribution in the estimates also comes from the cardinality of $\mathcal{N}_1$ defined in Lemma 2.8.
- Finally, for $(\tilde{s}, \tilde{\sigma})$-integrations, we consider 2 cases that contribute to the estimates, namely $N \geq \lambda^{1/3}$ (Lemma 2.12) and $N < \lambda^{1/3}$ (Lemma 2.13).

Our result of this subsection is Proposition 2.11.

Proposition 2.11. *Let $\alpha < 2/3$. There exists C such that for all $h \in]0, h_0]$, all $a \in [h^\alpha, a_0]$, all $X \in [0,1]$, all $T \in]0, a^{-1/2}]$, all $Y \in \mathbb{R}$, all $z \in \mathbb{R}$, the following holds:*

$$\left| \sum_{1 \leq N \leq C_0 a^{-1/2}} G_{a,N,2}(T,X,Y,z;h) \right| \leq C h^{-3} \left(\frac{h}{t}\right)^{1/2} a^{1/8} h^{1/4}.$$

Proof. First, we rewrite $G_{a,N,2}$ in the form

$$\begin{aligned} G_{a,N,2} &= \frac{(-i)^N a^2}{(2\pi)^4 h^4} \left(\frac{h}{t}\right)^{1/2} \int e^{i \frac{a^{3/2}}{h} Y \eta} |\eta|^3 \tilde{G}_{a,N,2}\, d\eta, \\ \tilde{G}_{a,N,2} &= \int e^{i\lambda \tilde{\phi}_{N,a,h}} \tilde{\chi}_0 \psi_0 \chi_1 \chi_2(\tilde{\omega})\, d\tilde{s}\, d\tilde{\sigma}\, d\tilde{\omega}, \end{aligned} \tag{2.30}$$

with the phase

$$\begin{aligned} \tilde{\phi}_{N,a,h}(T,X,z;\tilde{s},\tilde{\sigma},\tilde{\omega}) = T\sqrt{1-\tilde{z}^2}\gamma_a(\tilde{\omega}) + \frac{\tilde{s}^3}{3} + \tilde{s}(X - \tilde{\omega}) \\ + \frac{\tilde{\sigma}^3}{3} + \tilde{\sigma}(1-\tilde{\omega}) - \frac{4}{3} N \tilde{\omega}^{3/2} + \frac{N}{\lambda} B(\tilde{\omega}^{3/2}\lambda). \end{aligned}$$

Since $\tilde{\omega}$ is close to 1 on the support of χ_2, we may localize $\tilde{s}, \tilde{\sigma}$ in a compact set. Let $K = \{\tilde{s}, \tilde{\sigma} \in [-1,1], \tilde{\omega} = 1\}$ and K_1 be a suitable neighborhood of K depending on the support of χ_2. Introduce a cutoff function $\chi_4(\tilde{s}, \tilde{\sigma}, \tilde{\omega}) \in C_0^\infty$ equal to 1 near K_1. Then the contribution of $\tilde{G}_{a,N,2}$ outside K_1 is $O_{C^\infty}(\lambda^{-\infty})$ as a result of integration by parts. Therefore we obtain

$$\begin{aligned} \tilde{G}_{a,N,2}(T,X,z,h) &= \int e^{i\lambda \tilde{\phi}_{N,a,h}} \chi(\tilde{s}, \tilde{\sigma}, \tilde{\omega}, a)\, d\tilde{s}\, d\tilde{\sigma}\, d\tilde{\omega} + O_{C^\infty}(\lambda^{-\infty}), \\ \chi(\tilde{s}, \tilde{\sigma}, \tilde{\omega}, a, h) &= \tilde{\chi}_0 \psi_0 \chi_1 \chi_2(\tilde{\omega}) \chi_4(\tilde{s}, \tilde{\sigma}, \tilde{\omega}), \end{aligned} \tag{2.31}$$

with $O_{C^\infty}(\lambda^{-\infty})$ uniform in T, X, z, N, a and χ is a classical symbol of order 0 in h with support near K_1. We first perform the integration with respect to $\tilde{\omega}$. We have

$$\begin{aligned} \partial_{\tilde{\omega}} \tilde{\phi}_{N,a,h} &= \frac{T}{2}\sqrt{1-\tilde{z}^2}(1+a\tilde{\omega})^{-1/2} - \tilde{s} - \tilde{\sigma} - 2N\tilde{\omega}^{1/2}\Big(1 - \frac{3}{4}B'(\tilde{\omega}^{3/2}\lambda)\Big), \\ \partial^2_{\tilde{\omega}\tilde{\omega}} \tilde{\phi}_{N,a,h} &= -N\tilde{\omega}^{-1/2}(1 + O_{C^\infty}(\lambda^{-2}\tilde{\omega}^{-3})) + O_{C^\infty}(a^{1/2}). \end{aligned}$$

Because $\partial^2_{\tilde{\omega}\tilde{\omega}} \tilde{\phi}_{N,a,h} < 0$, it follows that $\partial_{\tilde{\omega}} \tilde{\phi}_{N,a,h}$ decreases from $\partial_{\tilde{\omega}} \tilde{\phi}_{N,a,h}(1) > 0$ to $\partial_{\tilde{\omega}} \tilde{\phi}_{N,a,h}(\infty) < 0$. Therefore $\tilde{\phi}_{N,a,h}$ admits a unique nondegenerate critical point $\tilde{\omega}_c$ and we are interested in the values of parameters such that $\tilde{\omega}_c$ close to 1; then we must have $\tilde{T} = T/4N \in$ compact set of $\mathbb{R}_+$, namely $[1/2, 3/2]$. In addition, from the equation $\partial_{\tilde{\omega}} \tilde{\phi}_{N,a,h} = 0$, we get

$$\frac{T}{2}\sqrt{1-\tilde{z}^2}(1+a\tilde{\omega})^{-1/2} = \tilde{s} + \tilde{\sigma} + 2N\tilde{\omega}^{1/2}\Big(1 - \frac{3}{4}B'(\tilde{\omega}^{3/2}\lambda)\Big). \tag{2.32}$$

Now we study the solution of (2.32) with $\lambda = \infty$; in this case, we have

$$\tilde{\omega}^{1/2}(1+a\tilde{\omega})^{1/2} = \tilde{T}\sqrt{1-\tilde{z}^2} - \frac{1}{2N}(\tilde{s}+\tilde{\sigma})(1+a\tilde{\omega})^{1/2}.$$

The solution of this equation is of the form $\tilde{\omega}_c = \sum F_k(a, \tilde{T}, \tilde{s}/N, \tilde{\sigma}/N)$ where F_k are homogeneous functions of degree k in $(\tilde{s}/N, \tilde{\sigma}/N)$ (see [12, Lemma 2.23]). By comparing the terms with the same homogeneous degree in $(\tilde{s}/N, \tilde{\sigma}/N)$, we get

$$F_0(1+aF_0) = \tilde{T}^2(1-\tilde{z}^2) \ \text{ which gives } \ F_0 = \frac{2\tilde{T}^2(1-\tilde{z}^2)}{1+\sqrt{1+4a\tilde{T}^2(1-\tilde{z}^2)}},$$

and

$$(1+2aF_0)F_1 = -\frac{\tilde{T}}{N}\sqrt{1-\tilde{z}^2}(\tilde{s}+\tilde{\sigma})(1+aF_0)^{1/2}.$$

We define

$$F_1 = -\frac{E_0}{N}(\tilde{s}+\tilde{\sigma})(1+aF_0)^{1/2},$$
$$E_0^{-1} = \sqrt{F_0}\sqrt{1+aF_0}\left(\frac{1}{F_0} + \frac{a}{1+aF_0}\right).$$

Therefore $\tilde{\omega}_c = F_0 + F_1 + \mathcal{O}_2$ with the notation $\mathcal{O}_j$ means any function of the form $F = \sum\limits_{k \geq j} F_k$. Then by the implicit function theorem, we get that the equation

$$\tilde{\omega}^{1/2}(1+a\tilde{\omega})^{1/2}\Big(1 - \frac{3}{4}B'(\tilde{\omega}^{3/2}\lambda)\Big) = \tilde{T}\sqrt{1-\tilde{z}^2} - \frac{1}{2N}(\tilde{s}+\tilde{\sigma})(1+a\tilde{\omega})^{1/2}$$

has solution of the form $\tilde{\omega}_c = F_0 + F_1 + \mathcal{O}_2 + \frac{g_0}{\lambda^2}$ with g_0 is a function of degree 0 in λ. By substituting $\tilde{\omega}_c$ into $\tilde{\phi}_{N,a,h}$, we get a phase function which is denoted by $\tilde{\Psi}_{N,a,h} = \tilde{\phi}_{N,a,h}(.,\tilde{\omega}_c,.)$. It is given by

$$\begin{aligned}\tilde{\Psi}_{N,a,h} &= T\sqrt{1-\tilde{z}^2}\gamma_a(F_0) + \frac{\tilde{s}^3}{3} + \tilde{s}(X-F_0) + \frac{\tilde{\sigma}^3}{3} + \tilde{\sigma}(1-F_0)\\ &\quad + \frac{E_0}{N}(1+aF_0)^{1/2}(\tilde{s}+\tilde{\sigma})^2 - \frac{1}{4N^2}(\tilde{s}+\tilde{\sigma})^3 + aN\mathcal{O}_3\\ &\quad + \frac{g_0}{\lambda^2} + N\Big(-\frac{4}{3}F_0^{3/2} + \frac{g_1}{\lambda^2}\Big).\end{aligned}$$

Therefore, by applying the stationary phase method for (2.31), we get

$$\tilde{G}_{a,N,2} = \frac{1}{\sqrt{\lambda N}}\int e^{i\lambda\tilde{\Psi}_{N,a,h}}\tilde{\chi}(\tilde{T},\tilde{s},\tilde{\sigma},1/N,a,h)\,d\tilde{s}\,d\tilde{\sigma} + O_{C^\infty}(\lambda^{-\infty}),$$

with $\tilde{\chi}$ is a classical symbol of order zero in h. Now, with $\tilde{\lambda} = \lambda/\eta$, (2.30) becomes

$$G_{a,N,2} = \frac{(-i)^N a^2}{(2\pi)^4 h^4}\left(\frac{h}{t}\right)^{1/2}\frac{1}{\sqrt{\lambda N}}\int e^{i\tilde{\lambda}|\eta|(Y+\tilde{\Psi}_{N,a,h})}|\eta|^3\tilde{\chi}\,d\tilde{s}\,d\tilde{\sigma}\,d\eta + O_{C^\infty}(\lambda^{-\infty}).$$

We study the η-integration with the phase function $L_N = \eta(Y + \tilde{\Psi}_{N,a,h})$ and a large parameter $\tilde{\lambda}$. Follow the arguments in the proof of Proposition 2.9, we have

$$\partial_\eta L_N = Y + \tilde{\Psi}_{N,a,h} + \lambda\partial_\lambda\tilde{\Psi}_{N,a,h} = 0$$

implies that (X,Y,T) belongs to the projection of $\mathbf{\Lambda}_{N,a,h}$ on $\mathbb{R}^3$ and the sum for N such that $N \notin \mathcal{N}_1(X,Y,T)$ gives $O_{C^\infty}(\lambda^{-\infty})$ (see [12, Lemma 2.24]). Hence it remains to estimate the sum

$$\left|\sum_{N\in\mathcal{N}_1} G_{a,N,2}(T,X,Y,z;h)\right|.$$

We also have $|\partial_\eta^2 L_N| \geq CN\lambda^{-2}\tilde{\omega}_c^{-3/2}$. It follows that there are 2 cases to consider.

- If $N \lesssim \lambda$, then the contribution of the η-integration is $O_{C^\infty}(\lambda^{-\infty})$ as a consequence of integration by parts.
- If $N \gg \lambda$, then by the stationary phase method, the η-integration gives a contribution of a factor $(N\lambda^{-1})^{-1/2}$ since $\tilde{\omega}_c \sim 1$.

Therefore, for $N \gg \lambda$, it yields

$$G_{a,N,2} = \frac{(-i)^N a^2}{(2\pi)^4 h^4}\left(\frac{h}{t}\right)^{1/2}\frac{1}{\sqrt{\lambda N}}\lambda^{1/2}N^{-1/2}\int e^{i\tilde{\lambda}L_N(\eta_c)}|\eta|^3\tilde{\chi}_1\,d\tilde{s}\,d\tilde{\sigma} + O_{C^\infty}(\lambda^{-\infty}). \tag{2.33}$$

Moreover, we note that the phase function $L_N(\eta_c)$ satisfies $\partial_{\tilde{s}}L_N(\eta_c) = \eta_c\partial_{\tilde{s}}\tilde{\Psi}_{N,a,h}$, $\partial_{\tilde{\sigma}}L_N(\eta_c) = \eta_c\partial_{\tilde{\sigma}}\tilde{\Psi}_{N,a,h}$. In addition, when $\partial_{\tilde{s}}L_N(\eta_c) = \partial_{\tilde{s}}^2 L_N(\eta_c) = 0$; that is, when $\partial_{\tilde{s}}\tilde{\Psi}_{N,a,h} = \partial_{\tilde{s}}^2\tilde{\Psi}_{N,a,h} = 0$, we have $\partial_{\tilde{s}}^3 L_N(\eta_c) = \eta_c\partial_{\tilde{s}}^3\tilde{\Psi}_{N,a,h}$ and similar for $\tilde{\sigma}$. Thus the study the critical points of the phase $L_N(\eta_c)$ in $(\tilde{s},\tilde{\sigma})$-integrations is the same as ones with the phase $\tilde{\Psi}_{N,a,h}$. As in [12], to avoid multiplication of symbol by a classical symbol of order 0 in λ, we can replace $\tilde{\Psi}_{N,a,h}$ by $\tilde{\psi}_{a,N,h}$, where

$$\begin{aligned}\tilde{\psi}_{N,a,h}(T,X;\tilde{s},\tilde{\sigma}) = T\sqrt{1-\tilde{z}^2}\gamma_a(F_0) + \frac{\tilde{s}^3}{3} + \tilde{s}(X-F_0) + \frac{\tilde{\sigma}^3}{3} + \tilde{\sigma}(1-F_0)\\ + \frac{E_0}{N}(1+aF_0)^{1/2}(\tilde{s}+\tilde{\sigma})^2 - \frac{1}{4N^2}(\tilde{s}+\tilde{\sigma})^3 + aN\mathcal{O}_3.\end{aligned}$$

Let us recall that $\mathcal{O}_3$ represents any function of the form $F = \sum_{k\geq 3} F_k$, where F_k are homogeneous functions of degree k in $(\tilde{s}/N, \tilde{\sigma}/N)$.

In what follows, we get the estimates of the oscillatory integral associated with the phase function $\tilde{\psi}_{a,N,h}$ for different values of N, namely for $N \geq \lambda^{1/3}$ and $N < \lambda^{1/3}$. Our results are Lemma 2.12 and Lemma 2.13.

Lemma 2.12. *There exists a constant C such that for all $N \geq \lambda^{1/3}$,*

$$\frac{1}{\sqrt{N}}\left|\int e^{i\lambda\tilde{\psi}_{N,a,h}}\tilde{\chi}_1\,d\tilde{s}\,d\tilde{\sigma}\right| \leq C\lambda^{-5/6}. \tag{2.34}$$

Here C is a constant independent of $N \geq 1, X \in [0,1], T \in\,]0, a^{-1/2}]$, $a \in [h^\alpha, a_0]$ and $\lambda \in [\lambda_0, \infty[$ with a_0 small and λ_0 large.

Proof. Adapting the arguments in the proof of Lemma 2.25 [12]. It is sufficient to prove that for all $N \geq \lambda^{1/3}$,

$$\left|\int e^{i\lambda\tilde{\psi}_{N,a,h}}\tilde{\chi}_1\,d\tilde{s}\,d\tilde{\sigma}\right| \leq C\lambda^{-2/3}. \tag{2.35}$$

Set $X - F_0 = -A\lambda^{-2/3}, 1 - F_0 = -B\lambda^{-2/3}, \tilde{s} = \lambda^{-1/3}x', \tilde{\sigma} = \lambda^{-1/3}y'$. It remains to prove that

$$\left|\int e^{i\hat{\psi}_{N,a,h}}\tilde{\chi}_1(\lambda^{-1/3}x', \lambda^{-1/3}y', \dots)\,dx'\,dy'\right| \leq C, \tag{2.36}$$

with the phase function $\hat{\psi}_{N,a,h}$ given by

$$\begin{aligned}\hat{\psi}_{N,a,h} &= T\lambda\sqrt{1-\tilde{z}^2}\gamma_a(F_0) - Ax' + \frac{x'^3}{3} - By' + \frac{y'^3}{3} \\ &\quad + \frac{E_0\lambda^{1/3}}{N}(1+aF_0)^{1/2}(x'+y')^2 - \frac{1}{4N^2}(x'+y')^3 + aN\mathcal{O}_3.\end{aligned}$$

Then (2.36) is an oscillatory integral over a domain of integration of size $\lambda^{2/3}$ whose parameters $F_0, E_0, \lambda^{1/3}/N$ are bounded.

We will prove that the constant C is uniform with respect to (A,B). We introduce new variables (r,θ) and write $(A,B) = (r\cos\theta, r\sin\theta)$ with $r \le c_0\lambda^{2/3}$. We have

$$\begin{aligned}\partial_{x'}\hat{\psi}_{N,a,h} &= -A + x'^2 + \frac{2E_0}{N}(1+aF_0)^{1/2}\lambda^{1/3}(x'+y') \\ &\quad - \frac{3}{4N^2}(x'+y')^2 + aN^{-2}\lambda^{-1}O\big((x',y')^2\big), \\ \partial_{y'}\hat{\psi}_{N,a,h} &= -B + y'^2 + \frac{2E_0}{N}(1+aF_0)^{1/2}\lambda^{1/3}(x'+y') \\ &\quad - \frac{3}{4N^2}(x'+y')^2 + aN^{-2}\lambda^{-1}O\big((x',y')^2\big).\end{aligned}$$

Moreover, the compactly support of $\tilde{\chi}_1$ in $(\tilde{s},\tilde{\sigma})$ yields

$$\sup_{(x',y')}\left|\partial^\gamma_{(x',y')}\tilde{\chi}_1(\lambda^{-1/3}x', \lambda^{-1/3}y', \dots)\right| \le C_\gamma(1+|x'|+|y'|)^{-|\gamma|},$$

with C_γ independent of T, a, N, λ. Therefore, the oscillatory integral is bounded for $0 \le r \le r_0$, where r_0 is a fixed constant, and for large (x',y') as a consequence of integration by parts.

For $r \in [r_0, c_0\lambda^{2/3}]$, we rescale variables $(x',y') = r^{1/2}(x'',y'')$ and we set $\hat{\psi}_{N,a,h} = r^{3/2}\psi^*_{N,a,h}$ and $\chi'(x'',y'',\dots) = \tilde{\chi}_1(r^{1/2}\lambda^{-1/3}x'', r^{1/2}\lambda^{-1/3}y'',\dots)$. Since $r^{1/2}\lambda^{-1/3}$ is bounded, we still have

$$\sup_{(x'',y'')}\left|\partial^\gamma_{(x'',y'')}\chi'\right| \le C_\gamma(1+|x''|+|y''|)^{-|\gamma|}.$$

It remains to prove

$$r\left|\int e^{ir^{3/2}\psi^*_{N,a,h}}\chi'\,dx''\,dy''\right| \le C. \tag{2.37}$$

Now we study the critical points of $\psi^*_{N,a,h}$. We have

$$\begin{aligned}\partial_{x''}\psi^*_{N,a,h} &= -\cos\theta + x''^2 - \frac{3}{4N^2}(x''+y'')^2 \\ &\quad + r^{-1/2}O((x'',y'')) + aN^{-1}\lambda^{-1}r^{3/2}O((x'',y'')^2), \\ \partial_{y''}\psi^*_{N,a,h} &= -\sin\theta + y''^2 - \frac{3}{4N^2}(x''+y'')^2 \\ &\quad + r^{-1/2}O((x'',y'')) + aN^{-1}\lambda^{-1}r^{3/2}O((x'',y'')^2).\end{aligned}$$

and

$$\begin{aligned}
\partial^2_{x''x''}\psi^*_{N,a,h} &= 2x'' - \frac{3}{2}(x''+y'') + r^{-1/2}O(1) + aN^{-1}\lambda^{-1}r^{3/2}O((x'',y'')),\\
\partial^2_{x''y''}\psi^*_{N,a,h} &= \partial^2_{y''x''}\psi^*_{N,a,h} = -\frac{3}{2}(x''+y'') + r^{-1/2}O(1) + aN^{-1}\lambda^{-1}r^{3/2}O((x'',y'')),\\
\partial^2_{y''y''}\psi^*_{N,a,h} &= 2y'' - \frac{3}{2}(x''+y'') + r^{-1/2}O(1) + aN^{-1}\lambda^{-1}r^{3/2}O((x'',y'')).
\end{aligned}$$

For small a and large r_0, we may localize the integral to a compact set in (x'',y'') as a result of integration by parts for large (x'',y''). The Hessian of $\psi^*_{N,a,h}$, that we denote by $\mathcal{H}_N(x'',y'')$, takes the form

$$\begin{aligned}
\mathcal{H}_N(x'',y'') &= \det\begin{pmatrix}\partial^2_{x''x''}\psi^*_{N,a,h} & \partial^2_{x''y''}\psi^*_{N,a,h}\\ \partial^2_{y''x''}\psi^*_{N,a,h} & \partial^2_{y''y''}\psi^*_{N,a,h}\end{pmatrix}\\
&= 4x''y'' - \frac{3}{N^2}(x''+y'')^2 + r^{-1/2}O(1) + aN^{-1}\lambda^{-1}r^{3/2}O((x'',y'')).
\end{aligned}$$

Thus for $N\geq 2, a$ small and r_0 large, outside $(x'',y'')=(0,0)$, define a smooth curve $\Gamma=\{(x'',y'')$ such that $\mathcal{H}_N(x'',y'')=0\}$; that is, Γ is close to the union of two lines $c(x''+y'')\pm(x''-y'')=0$, $c^2=\frac{N^2-3}{N^2}\in[1/4,1]$. Then we have 2 cases to consider

- The contribution of points (x'',y'') outside Γ to the integral is $O_{C^\infty}(r^{-3/2})$ by the usual stationary phase method and we get
$$r\left|\int e^{ir^{3/2}\psi^*_{N,a,h}}\chi'\,dx''\,dy''\right| \leq Cr^{-1/2}.$$
- The contribution of points (x'',y'') close to Γ is given by Lemma 2.21 [12]. For any values of θ, the hypothesis of part (a) Lemma 2.21 [12] holds true, then we get
$$r\left|\int e^{ir^{3/2}\psi^*_{N,a,h}}\chi'\,dx''\,dy''\right| \leq Cr(r^{3/2})^{-5/6} = Cr^{-1/4}.$$

Hence in any cases, (2.37) is satisfied. □

To summarize, recall that $T\sim N$ in this case and hence the cardinality of $\mathcal{N}_1$, $|\mathcal{N}_1(X,Y,T)|\leq C_0(1+T\lambda^{-2})$. We deduce the estimates for the sum of $G_{a,N,2}$ with Lemma 2.12 for $N\geq\lambda^{1/3}$ as follows:

- If $\lambda^{1/3}\leq N\leq\lambda$, there is no contribution from η-integration and we have the cardinality of $\mathcal{N}_1$, $|\mathcal{N}_1|\leq C_0$. We obtain

$$\begin{aligned}
\left|\sum_{N\in\mathcal{N}_1} G_{a,N,2}(T,X,Y,z;h)\right| &\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}\left(h^{-1}a^2\lambda^{-1/2}\lambda^{-5/6}\right)\\
&\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}h^{1/3}.
\end{aligned}$$

- If $\lambda \le N \le \lambda^2$, then there is a $(N\lambda^{-1})^{-1/2}$ factor contribution from η-integration and we also have the cardinality of $\mathcal{N}_1$, $|\mathcal{N}_1| \le C_0$. We get

$$\begin{aligned}\Big|\sum_{N\in\mathcal{N}_1} G_{a,N,2}(T,X,Y,z;h)\Big| &\le Ch^{-3}\Big(\frac{h}{t}\Big)^{1/2}\big(h^{-1}a^2T^{-1/2}\lambda^{-5/6}\big)\\ &\le Ch^{-3}\Big(\frac{h}{t}\Big)^{1/2}\big(h^{-1}a^2\lambda^{-1/2}\lambda^{-5/6}\big)\\ &\le Ch^{-3}\Big(\frac{h}{t}\Big)^{1/2}h^{1/3}.\end{aligned}$$

- If $N > \lambda^2$, then there are contributions from both $(N\lambda^{-1})^{-1/2}$ from η-integration and the cardinality of $\mathcal{N}_1$, $|\mathcal{N}_1| \le C_0T\lambda^{-2}$. We get

$$\begin{aligned}\Big|\sum_{N\in\mathcal{N}_1} G_{a,N,2}(T,X,Y,z;h)\Big| &\le Ch^{-3}\Big(\frac{h}{t}\Big)^{1/2}\sum_{N\in\mathcal{N}_1}\Big(h^{-1}a^2\frac{1}{N}\lambda^{-2/3}\Big)\\ &\le Ch^{-3}\Big(\frac{h}{t}\Big)^{1/2}\big(h^{-1}a^2\lambda^{-2/3}T^{-1}|\mathcal{N}_1(X,Y,T)|\big)\\ &\le Ch^{-3}\Big(\frac{h}{t}\Big)^{1/2}\big(a^{-2}h^{5/3}\big) \le Ch^{-3}\Big(\frac{h}{t}\Big)^{1/2}h^{1/3}.\end{aligned}$$

Lemma 2.13. *There exists a constant C such that for all $N < \lambda^{1/3}$,*

$$\frac{1}{\sqrt{N}}\Big|\int e^{i\lambda\tilde\psi_{N,a,h}}\tilde\chi_1\,d\tilde s\,d\tilde\sigma\Big| \le CN^{-1/4}\lambda^{-3/4}. \tag{2.38}$$

Notice that Lemma 2.13 says that for N large it gives a better estimate and it is compatible with the estimate (2.34) for $N \sim \lambda^{1/3}$.

Proof. Let $\frac{\lambda}{N^3} = \Lambda \ge 1$ and we take Λ as a new large parameter. To get the estimates of our oscillatory integral, we set

$$X - F_0 = -pN^{-2},\quad 1 - F_0 = -qN^{-2},\quad \tilde s = -\bar x/N,\quad \tilde\sigma = -\bar y/N.$$

It yields $\tilde\psi_{N,a,h} = N^{-3}\bar\psi_{N,a,h}$. Then it remains to prove that

$$\Big|\int e^{i\Lambda\bar\psi_{N,a,h}}\tilde\chi_1(\bar x/N,\bar y/N,\dots)\,d\bar x\,d\bar y\Big| \le C\Lambda^{-3/4}, \tag{2.39}$$

with the phase $\bar\psi_{N,a,h}$ takes the form

$$\begin{aligned}\bar\psi_{N,a,h} = p\bar x - \frac{\bar x^3}{3} + q\bar y - \frac{\bar y^3}{3} + E_0(1+aF_0)^{1/2}(\bar x+\bar y)^2\\ + \frac{1}{4N^2}(\bar x+\bar y)^3 + TN^3\sqrt{1-\tilde z^2}\gamma_a(F_0) + aN^{-2}O\big((\bar x,\bar y)^3\big).\end{aligned}$$

We have

$$\begin{aligned}\partial_{\bar x}\bar\psi_{N,a,h} &= p - \bar x^2 + 2E_0(1+aF_0)^{1/2}(\bar x+\bar y)\\ &\quad + \frac{3}{4N^2}(\bar x+\bar y)^2 + aN^{-2}O\big((\bar x,\bar y)^2\big),\\ \partial_{\bar y}\bar\psi_{N,a,h} &= q - \bar y^2 + 2E_0(1+aF_0)^{1/2}(\bar x+\bar y)\\ &\quad + \frac{3}{4N^2}(\bar x+\bar y)^2 + aN^{-2}O\big((\bar x,\bar y)^2\big),\end{aligned} \tag{2.40}$$

and

$$\begin{aligned}
\partial^2_{\bar{x}\bar{x}}\bar{\psi}_{N,a,h} &= -2\bar{x} + 2E_0(1+aF_0)^{1/2} + \frac{3}{2N^2}(\bar{x}+\bar{y}) + aN^{-2}O((\bar{x},\bar{y})),\\
\partial^2_{\bar{x}\bar{y}}\bar{\psi}_{N,a,h} &= \partial^2_{\bar{y}\bar{x}}\bar{\psi}_{N,a,h} = 2E_0(1+aF_0)^{1/2} + \frac{3}{2N^2}(\bar{x}+\bar{y}) + aN^{-2}O((\bar{x},\bar{y})),\\
\partial^2_{\bar{y}\bar{y}}\bar{\psi}_{N,a,h} &= -2\bar{y} + 2E_0(1+aF_0)^{1/2} + \frac{3}{2N^2}(\bar{x}+\bar{y}) + aN^{-2}O((\bar{x},\bar{y})).
\end{aligned}$$

The Hessian of $\bar{\psi}_{N,a,h}$, that we denote by $\mathcal{H}_N(\bar{x},\bar{y},a)$, takes the form

$$\begin{aligned}
\mathcal{H}_N(\bar{x},\bar{y},a) &= \det\begin{pmatrix} \partial^2_{\bar{x}\bar{x}}\bar{\psi}_{N,a,h} & \partial^2_{\bar{x}\bar{y}}\bar{\psi}_{N,a,h} \\ \partial^2_{\bar{y}\bar{x}}\bar{\psi}_{N,a,h} & \partial^2_{\bar{y}\bar{y}}\bar{\psi}_{N,a,h} \end{pmatrix}\\
&= 4\bar{x}\bar{y} - 4E_0(1+aF_0)^{1/2}(\bar{x}+\bar{y}) - \frac{3}{N^2}(\bar{x}+\bar{y})^2 + aN^{-2}O\big((\bar{x},\bar{y})\big). \qquad (2.41)
\end{aligned}$$

Lemma 2.14. *There exist constants r_0 and C such that for all (p,q) such that $|(p,q)| \geq r_0$,*

$$\left| \int e^{i\Lambda\bar{\psi}_{N,a,h}} \tilde{\chi}_1(\bar{x}/N, \bar{y}/N, \dots)\, d\bar{x}\, d\bar{y} \right| \leq C\Lambda^{-5/6}. \qquad (2.42)$$

Proof of Lemma 2.14. Apply the arguments in the proof of Lemma 2.26 [12]. Set $(p,q) = (r\cos\theta, r\sin\theta)$ with $r \geq r_0$. Let $\chi \in C_0^\infty(|(\bar{x},\bar{y})| < c)$ with small c and $\chi = 1$ near 0. Then from (2.40), we get by integration by parts in $(\bar{x},\bar{y})$, for all k,

$$\left| \int e^{i\Lambda\bar{\psi}_{N,a,h}} \chi(r^{-1/2}(\bar{x},\bar{y}))\tilde{\chi}_1(\bar{x}/N, \bar{y}/N, \dots)\, d\bar{x}\, d\bar{y} \right| \leq Cr^{-k}\Lambda^{-k}.$$

For $|(\bar{x},\bar{y})|$ large, we make a change of variable $(\bar{x},\bar{y}) = r^{1/2}(x',y')$ and we set $\bar{\psi}'_{N,a,h} = r^{-3/2}\bar{\psi}_{N,a,h}$. Then it remains to prove

$$\left| r\int e^{ir^{3/2}\Lambda\bar{\psi}'_{N,a,h}} (1-\chi)(x',y')\tilde{\chi}_1(r^{1/2}x'/N, r^{1/2}y'/N, \dots)\, dx'\, dy' \right| \leq C\Lambda^{-5/6}.$$

We observe that since $(1-\chi)(x',y') = 0$ near $0, (1-\chi)(x',y') = 1$ for $|(x',y')| \geq c$ and $\tilde{\chi}_1$ is compactly support, we still have

$$\sup_{(x',y')} \left| \partial^\gamma_{(x',y')}(1-\chi)(x',y')\tilde{\chi}_1(r^{1/2}x'/N, r^{1/2}y'/N, \dots) \right| \leq C_\gamma(1+|x'|+|y'|)^{-|\gamma|}.$$

The phase $\bar{\psi}'_{N,a,h}$ is of the form

$$\begin{aligned}
\bar{\psi}'_{N,a,h} &= (\cos\theta)x' - \frac{x'^3}{3} + (\sin\theta)y' - \frac{y'^3}{3} + \frac{1}{4N^2}(x'+y')^3\\
&\quad + \frac{TN^3}{r^{3/2}}\sqrt{1-\tilde{z}^2}\gamma_a(F_0) + aN^{-2}O((x',y')^3).
\end{aligned}$$

We get that

$$\begin{aligned}
\partial_{x'}\bar{\psi}'_{N,a,h} &= \cos\theta - x'^2 + \frac{3}{4N^2}(x'+y')^2 + aN^{-2}O((x',y')^2),\\
\partial_{y'}\bar{\psi}'_{N,a,h} &= \sin\theta - y'^2 + \frac{3}{4N^2}(x'+y')^2 + aN^{-2}O((x',y')^2).
\end{aligned}$$

and

$$\partial^2_{x'x'}\bar{\psi}'_{N,a,h} = -2x' + \frac{3}{2N^2}(x'+y') + aN^{-2}O((x',y')),$$
$$\partial^2_{x'y'}\bar{\psi}'_{N,a,h} = \partial^2_{y'x'}\bar{\psi}'_{N,a,h} = \frac{3}{2N^2}(x'+y') + aN^{-2}O((x',y')),$$
$$\partial^2_{y'y'}\bar{\psi}'_{N,a,h} = -2y' + \frac{3}{2N^2}(x'+y') + aN^{-2}O((x',y')).$$

Thus for small a and large r_0, by integration by parts, we may localize the integral to a compact set in (x',y'). The Hessian of $\bar{\psi}'_{N,a,h}$, that we denote by $\mathcal{H}'_N(x',y',a)$, takes the form

$$\begin{aligned}\mathcal{H}'_N(x',y',a) &= \det\begin{pmatrix}\partial^2_{x'x'}\bar{\psi}'_{N,a,h} & \partial^2_{x'y'}\bar{\psi}'_{N,a,h}\\ \partial^2_{y'x'}\bar{\psi}'_{N,a,h} & \partial^2_{y'y'}\bar{\psi}'_{N,a,h}\end{pmatrix}\\ &= 4x'y' - \frac{3}{N^2}(x'+y')^2 + aN^{-2}O((x',y')).\end{aligned}$$

We apply the same argument as before, for $N \geq 2, a$ small and r_0 large, outside $(x',y') = (0,0)$, we set $\Gamma = \{(x',y') \text{ such that } \mathcal{H}'_N(x',y') = 0\}$ and there are 2 cases to consider:

- The contribution of points (x',y') outside Γ to the integral is $O(r^{-3/2}\Lambda^{-1})$ by the usual stationary phase method; that is,

$$\left| r\int e^{ir^{3/2}\Lambda\bar{\psi}'_{N,a,h}}(1-\chi)(x',y')\tilde{\chi}_1(r^{1/2}x'/N, r^{1/2}y'/N,\dots)\,dx'\,dy'\right| \leq Cr^{-1/2}\Lambda^{-1}.$$

- The contribution of points (x',y') close to Γ given by Lemma 2.21[12]. For any values of θ, the hypothesis of part (a) Lemma 2.21[12] holds true, then we get

$$\begin{aligned}\left| r\int e^{ir^{3/2}\Lambda\bar{\psi}'_{N,a,h}}(1-\chi)(x',y')\tilde{\chi}_1(r^{1/2}x'/N, r^{1/2}y'/N,\dots)\,dx'\,dy'\right| &\leq Cr(r^{3/2}\Lambda)^{-5/6}\\ &\leq Cr^{-1/4}\Lambda^{-5/6}. \quad\square\end{aligned}$$

Lemma 2.15. *There exist constants r_0 and C such that for all (p,q) such that $|(p,q)| \leq r_0$,*

$$\left|\int e^{i\Lambda\bar{\psi}_{N,a,h}}\tilde{\chi}_1(\bar{x}/N, \bar{y}/N, \dots)d\bar{x}d\bar{y}\right| \leq C\Lambda^{-3/4}. \tag{2.43}$$

Proof of Lemma 2.15. Now we consider the case $|(p,q)| \leq r_0$. There exists $c > 0$ independent of $N \geq 2$ such that

$$\forall(\bar{x},\bar{y}) \in \mathbb{R}^2, \quad \left|\bar{x}^2 - \frac{3}{4N^2}(\bar{x}+\bar{y})^2\right| + \left|\bar{y}^2 - \frac{3}{4N^2}(\bar{x}+\bar{y})^2\right| \geq c(\bar{x}^2+\bar{y}^2). \tag{2.44}$$

Then by integration by parts, (2.40) gives a contribution $O_{C^\infty}(\Lambda^{-\infty})$ to the integral (2.39) for large values $(\bar{x},\bar{y})$. Then we may assume that $(\bar{x},\bar{y})$ is in compact set. It remains to prove

$$\left|\int e^{i\Lambda\bar{\psi}_{N,a,h}}\tilde{\chi}_1 d\bar{x}d\bar{y}\right| \leq C\Lambda^{-3/4},$$

with the phase

$$\bar\psi_{N,a,h} = p\bar x - \frac{\bar x^3}{3} + q\bar y - \frac{\bar y^3}{3} + E_0(1+aF_0)^{1/2}(\bar x+\bar y)^2 + \frac{1}{4N^2}(\bar x+\bar y)^3 \\ + TN^3\sqrt{1-\tilde z^2}\gamma_a(F_0) + aN^{-2}O\big((\bar x,\bar y)^3\big)$$

and the Hessian $\mathcal{H}_N(\bar x,\bar y,a)$ of $\bar\psi_{N,a,h}$ is given by (2.41).

For a small, the set $\Gamma = \{(\bar x,\bar y,a)$ such that $\mathcal{H}_N(\bar x,\bar y)=0\}$ is a smooth curve that is close to the elliptic $4\bar x\bar y - 4E_0(1+aF_0)^{1/2}(\bar x+\bar y) - 3(\bar x+\bar y)^2 = 0$ for $N=1$ and close to hyperbola $4\bar x\bar y - 4E_0(1+aF_0)^{1/2}(\bar x+\bar y) - \frac{3}{N^2}(\bar x+\bar y)^2 = 0$ for $N\ge 2$. It remains to use [12, Lemma 2.21] (see Appendix) for $(\bar x,\bar y)$ near (p,q) with $|(p,q)|\le r_0$. Hence, there are 3 cases to consider:

- If (p,q) is outside Γ, then the contribution to the integral is Λ^{-1} by usual stationary phase method.
- If $(0,0)\ne(p,q)$ is close to Γ, the contribution to the integral is given by Lemma 2.21 [12]. Since the hypothesis of part (a) in Lemma 2.21 [12] holds true, then near (p,q) the contribution to the integral is $\Lambda^{-5/6}$.
- If $(p,q)=(0,0)$, we have $(\bar x,\bar y)$ near $(0,0)$ and hypothesis of part (b) in Lemma 2.21 [12] holds true. Then the contribution to the integral is $\Lambda^{-3/4}$. □

Lemma 2.14 and Lemma 2.15 yield the proof of Lemma 2.13. □

Notice that when $N<\lambda^{1/3}$, there is no contribution from η-integration and the cardinality of $\mathcal{N}_\infty$, $|\mathcal{N}_1|\le C_0$. As a consequence, we obtain the estimates for the sum of $G_{a,N,2}$ for $N<\lambda^{1/3}$ as follows:

$$\Big|\sum_{N\in\mathcal{N}_1} G_{a,N,2}(T,X,Y,z;h)\Big| \le Ch^{-3}\Big(\frac{h}{t}\Big)^{1/2}\big(h^{-1}a^2\lambda^{-1/2}N^{-1/4}\lambda^{-3/4}\big) \\ \le Ch^{-3}\Big(\frac{h}{t}\Big)^{1/2}\big(a^{1/8}h^{1/4}N^{-1/4}\big).$$

We notice that we get the same estimates for $N=1$,

$$\Big|G_{a,1,2}(T,X,Y,z;h)\Big| \le Ch^{-3}\Big(\frac{h}{t}\Big)^{1/2}\big(h^{-1}a^2\lambda^{-1/2}\lambda^{-3/4}\big) \\ \le Ch^{-3}\Big(\frac{h}{t}\Big)^{1/2}\big(a^{1/8}h^{1/4}\big).$$

To summarize, putting these estimates together we proved that

$$\Big|\sum_{1\le N\le C_0a^{-1/2}} G_{a,N,2}(T,X,Y,z;h)\Big| \le Ch^{-3}\Big(\frac{h}{t}\Big)^{1/2}\big(h^{1/3}+a^{1/8}h^{1/4}\big).$$

Notice that $h^{1/3}\le a^{1/8}h^{1/4}$ when $a\ge h^{2/3}$; hence the proof of the Proposition 2.11 is complete. □

Proof of Theorem 2.5. Putting the estimates in Proposition 2.9, 2.10 and 2.11 together yields the desired result. □

3. Dispersive estimates for $\epsilon_0\sqrt{a}\le\eta\le c_0$

In this section, we prove Theorem 1.4. To get the estimates for $\mathcal{G}_{a,m}$, we distinguish between two different cases. The first case deals with $a\le\big(\frac{h}{2^m\sqrt{a}}\big)^{\frac{2}{3}(1-\epsilon)}$,

$\epsilon \in]0,1/7[$, where we follow ideas in section 2 and construct a local parametrix as a sum over eigenmodes. The second case is $a \geq \left(\frac{h}{2^m\sqrt{a}}\right)^{\frac{2}{3}(1-\epsilon')}$, for $\epsilon' \in]0,\epsilon[$, where the Airy–Poisson summation formula yields the representation of $\mathcal{G}_{a,m}$ as a sum over $N \in \mathbb{Z}$, representing waves corresponding to the number of reflections on the boundary.

Recall that we have

$$\mathcal{G}_a(t,x,y,z) = \frac{1}{4\pi^2 h^2}\sum_{k\geq 1}\int e^{\frac{i}{h}\Phi_k}\sigma_k\,d\eta\,d\zeta, \tag{3.1}$$

where the phase Φ_k and the function σ_k are defined by

$$\Phi_k = y\eta + z\zeta + t(\eta^2+\zeta^2+\omega_k h^{2/3}\eta^{4/3})^{1/2},$$
$$\sigma_k = e_k(x,\eta/h)e_k(a,\eta/h)\chi_0(\zeta^2+\eta^2)\chi_1(\omega_k h^{2/3}\eta^{4/3})(1-\chi_1)(\varepsilon\omega_k).$$

We have to get L^∞ estimates for $\mathcal{G}_a$ in the range $t\in[h,1]$, when the integral in (3.1) is restricted to values of $\eta \in [\epsilon_0\sqrt{a}, c_0]$ with c_0 small. Let μ^2 be defined by

$$\mu^2 = \eta^2 + \omega_k h^{2/3}\eta^{4/3}.$$

Observe that μ^2 is small since $\omega_k h^{2/3}\eta^{4/3}$ is small by the truncation χ_1 and η is small. Let $\chi_4 \in C_0^\infty]-1,1[$ with $\chi_4 = 1$ on $[-1/2,1/2]$ and $D\geq 1$. Let $\mathcal{J}_a(t,x,y,z)$ be defined by

$$\mathcal{J}_a(t,x,y,z) = \frac{1}{4\pi^2h^2}\sum_{k\geq 1}\int e^{\frac{i}{h}\Phi_k}\chi_4\left(\frac{t\mu^2}{Dh}\right)\sigma_k\,d\eta\,d\zeta.$$

The following lemma tells us that $\mathcal{J}_a$ satisfies the free dispersive estimate.

Lemma 3.1. *There exists a constant C independent of a constant D such that*

$$|\mathcal{J}_a(t,x,y,z)| \leq Ch^{-3}\left(\frac{h}{t}\right)D.$$

Proof. On the support of χ_4, one has $\eta^2 \leq Dh/t$ and $h\omega_k^{3/2}\eta^2 \leq (Dh/t-\eta^2)^{3/2}$. This implies that the sum over k is restricted to $k \leq c_0\frac{(Dh/t-\eta^2)^{3/2}}{h\eta^2}$. Since one has $e_k(x,\eta/h) = f_k k^{-1/6}(\eta/h)^{1/3}Ai((\eta/h)^{2/3}x-\omega_k)$, Lemma 2.2 gives

$$\begin{aligned}|\mathcal{J}_a(t,x,y,z)| &\leq Ch^{-2}\int_{\eta^2\leq Dh/t}(\eta/h)^{2/3}\frac{(Dh/t-\eta^2)^{1/2}}{h^{1/3}\eta^{2/3}}\,d\eta \\ &= Ch^{-3}\int_{\eta^2\leq Dh/t}(Dh/t-\eta^2)^{1/2}\,d\eta\end{aligned} \tag{3.2}$$

and the result follows from $\int_{\eta^2\leq Dh/t}(Dh/t-\eta^2)^{1/2}\,d\eta = (Dh/t)\int_{x^2\leq 1}(1-x^2)^{1/2}\,dx$. □

Observe that in the range $\eta \geq c_0$, one has $\mu^2 \geq c_0^2$, so the condition $t\mu^2/h \leq D$ is equivalent to $t \leq Ch$ and the above lemma is irrelevant. But in the range $\eta \in [\epsilon_0\sqrt{a}, c_0]$, the above lemma becomes useful since it tells us that we may now assume that $\lambda = t\mu^2/h$ is a large parameter. Since we allow some loss in the dispersive estimate with respect to the free case, we may even assume that we have $\lambda = t\mu^2/h \geq (\frac{h}{t})^{-\epsilon}$ for some $\epsilon > 0$ (take $D = (\frac{h}{t})^{-\epsilon}$), and therefore in the sequel a term like $O_{C^\infty}(\lambda^{-\infty})$ will be negligible. We are now in position to eliminate the ζ-integration in (3.1). This is the purpose of the following lemma. Recall that the truncation $\chi_0(\zeta^2+\eta^2)$ localizes $\zeta^2+\eta^2$ near 1. Therefore, for η small, ζ will be close to 1 or -1. In the sequel, we assume ζ near 1.

Lemma 3.2. *Let $\lambda = t\mu^2/h \geq 1$, $\tilde{z} = z/t$ and $\phi(\tilde{z},\mu^2,\zeta) = \frac{1}{\mu^2}(\tilde{z}\zeta + (\zeta^2+\mu^2)^{1/2})$. Let*

$$I(\tilde{z},\mu^2,\eta;\lambda) = \int_{\zeta\sim 1} e^{i\lambda\phi(\tilde{z},\mu^2,\zeta)}\chi_0(\zeta^2+\eta^2)\,d\zeta.$$

There exists $0 < c_1 < C_1$ such that the following holds true:

$$\text{For } \tilde{z} \notin [-1+c_1\mu^2, -1+C_1\mu^2] \text{ one has } \sup_{\tilde{z},\mu^2,\eta} |I(\tilde{z},\mu^2,\eta;\lambda)| \in O_{C^\infty}(\lambda^{-\infty}). \tag{3.3}$$

For $\tilde{z} \in [-1+c_1\mu^2, -1+C_1\mu^2]$, set $\tilde{z} = -1+z^*\mu^2$. There exists a classical symbol of degree 0 in λ, $\sigma_0(z^*,\eta,\mu^2;\lambda)$, such that one has

$$I(\tilde{z},\mu^2,\eta;\lambda) = \left(\frac{h}{t\mu^2}\right)^{1/2} e^{i\frac{t\mu}{h}(1-\tilde{z}^2)^{1/2}}\sigma_0(z^*,\eta,\mu^2;\lambda). \tag{3.4}$$

Proof. One has

$$\partial_\zeta\phi = \frac{1}{\mu^2}(\tilde{z} + \zeta(\zeta^2+\mu^2)^{-1/2}), \quad \partial_\zeta^2\phi = (\zeta^2+\mu^2)^{-3/2} \geq c > 0$$

and $\partial_\zeta^j\phi$ is bounded for all $j \geq 2$. Since

$$\zeta(\zeta^2+\mu^2)^{-1/2} = 1 - \frac{\mu^2}{2\zeta^2} + O(\mu^4),$$

(3.3) follows by integration by parts. For $\tilde{z} \in [-1+c_1\mu^2, -1+C_1\mu^2]$, and with $\tilde{z} = -1+z^*\mu^2$, one has

$$\phi = z^*\zeta + (\zeta + (\zeta^2+\mu^2)^{1/2})^{-1}$$

and a unique critical point $\zeta_c = -\mu\tilde{z}(1-\tilde{z}^2)^{1/2}$ with critical value

$$\phi(\zeta_c) = \frac{\zeta_c}{\mu^2}(\tilde{z} - 1/\tilde{z}) = (1-\tilde{z}^2)^{1/2}/\mu \in O(1).$$

Therefore, by stationary phase we get that (3.4) holds true. □

Using Lemmas 3.1 and 3.2, we are now reduced to the study of

$$\frac{1}{4\pi^2h^2}\left(\frac{h}{t}\right)^{1/2}\sum_{k\geq 1}\int e^{\frac{i}{h}(y\eta + t\mu(1-\tilde{z}^2)^{1/2})}\frac{\tilde{\sigma}_k}{\mu}\,d\eta \tag{3.5}$$

where $\tilde{\sigma}_k$ is defined by

$$\tilde{\sigma}_k = \sigma_0(z^*,\eta,\mu^2;\lambda)\Big(1-\chi_4\big(\frac{t\mu^2}{Dh}\big)\Big)e_k(x,\eta/h)e_k(a,\eta/h)\chi_1(\omega_k h^{2/3}\eta^{4/3})(1-\chi_1)(\varepsilon\omega_k).$$

To get L^∞ estimate for the parametrix in the range $\eta \in [\epsilon_0\sqrt{a}, c_0]$, we will use a Litttlewood–Paley decomposition in η. We choose

$$\psi_1 \in C_0^\infty(]0.5, 2.5[), 0 \leq \psi_1 \leq 1 \text{ such that } \sum_{m\in\mathbb{Z}}\psi_1(2^m x) = 1 \text{ for all } x > 0,$$

and we introduce the cut-off function $\psi_1(\frac{\eta}{2^m\sqrt{a}})$ in (3.5). In the sequel, we will therefore have

$$\epsilon_0 \leq 2^m \leq c_0/\sqrt{a}.$$

We will use the notations

$$\eta = 2^m\sqrt{a}\ \tilde\eta, \quad h = 2^m\sqrt{a}\ \tilde h,$$
$$\mu^2 = \eta^2 + \omega_k h^{2/3}\eta^{4/3} = (2^m\sqrt{a})^2(\tilde\eta^2 + \omega_k\tilde h^{2/3}\tilde\eta^{4/3}) = (2^m\sqrt{a})^2\tilde\mu^2,$$
$$\gamma = \omega_k h^{2/3}\eta^{-2/3} = \omega_k\tilde h^{2/3}\tilde\eta^{-2/3}.$$

We define $\mathcal{G}_{a,m}$ by the formula

$$\mathcal{G}_{a,m}(t,x,y,z) = \frac{1}{4\pi^2h^2}\left(\frac{h}{t}\right)^{1/2}\sum_{k\geq 1}\int e^{\frac{i}{h}\left(y\eta + t\mu(1-\tilde z^2)^{1/2}\right)}\psi_1\left(\frac{\eta}{2^m\sqrt{a}}\right)\frac{\tilde\sigma_k}{\mu}\,d\eta. \tag{3.6}$$

Observe that due to the truncation χ_1, we have $k \leq \frac{\varepsilon}{h\eta^2}$ in the above sum. Using the change of variable $\eta = 2^m\sqrt{a}\tilde\eta$, we get with $\tilde y = y/t$, since $d\eta/\mu = d\tilde\eta/\tilde\mu$

$$\mathcal{G}_{a,m}(t,x,y,z) = \frac{1}{4\pi^2h^2}\left(\frac{h}{t}\right)^{1/2}\sum_{1\leq k\leq \frac{\varepsilon}{(2^m\sqrt{a})^3\tilde h}}\int e^{\frac{it}{\tilde h}\left(\tilde y\tilde\eta + \tilde\mu(1-\tilde z^2)^{1/2}\right)} g_k\psi_1(\tilde\eta)\,d\tilde\eta, \tag{3.7}$$

where g_k is defined by

$$g_k = \frac{1}{\tilde\mu}\sigma_0(z^*,\eta,\mu^2;\lambda)\left(1-\chi_4\left(\frac{t\mu^2}{Dh}\right)\right)e_k(x,\tilde\eta/\tilde h)e_k(a,\tilde\eta/\tilde h)\chi_1(\omega_k h^{2/3}\eta^{4/3})(1-\chi_1)(\varepsilon\omega_k).$$

Lemma 3.3. *Let $M \geq 1$ be given. There exists C_M such that for all m, a, h such that $2^m\sqrt{a} \leq hM$, the following holds true:*

$$|\mathcal{G}_{a,m}| \leq C_M h^{-3}\left(\frac{h}{t}\right)^{1/2}2^m\sqrt{a}|\log(2^m\sqrt{a})|. \tag{3.8}$$

Proof. One has $\tilde h \geq 1/M$ and hence,

$$|e_k(x,\tilde\eta/\tilde h)| \leq Ck^{-1/6}\left(\frac{\tilde\eta}{\tilde h}\right)^{1/3}\omega_k^{-1/4}.$$

Moreover, we have $\tilde\mu \geq \omega_k^{1/2}\tilde h^{1/3}\tilde\eta^{2/3}$. Therefore, we get

$$|\mathcal{G}_{a,m}| \leq C''h^{-2}\left(\frac{h}{t}\right)^{1/2}\sum_{1\leq k\leq \frac{\varepsilon}{(2^m\sqrt{a})^3\tilde h}}\omega_k^{-1/2}\tilde h^{-1/3}k^{-1/3}\left(\frac{1}{\tilde h}\right)^{2/3}\omega_k^{-1/2}$$
$$\leq C'h^{-3}\left(\frac{h}{t}\right)^{1/2}2^m\sqrt{a}|\log(2^{2m}ah)| \leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}2^m\sqrt{a}|\log(2^m\sqrt{a})|. \quad \square \tag{3.9}$$

From the above lemma, we get in the range $\tilde h \geq 1/M$ the estimate

$$|\mathcal{G}_{a,m}| \leq C_M h^{-3}(h/t)^{1/2}(2^m\sqrt{a})^{1/3}(hM)^{2/3}|\log(hM)|. \tag{3.10}$$

This estimate is even better than the free estimate $Ch^{-3}(h/t)$. Therefore, in the sequel we will assume $\tilde h \leq \tilde h_0$ with $\tilde h_0$ small and recall that $\tilde h = h/(2^m\sqrt{a})$. To establish the local in time estimates for the $\mathcal{G}_{a,m}$, we follow the strategy of Section 2. We distinguish between two different cases. First case, if $a \leq \tilde h^{\frac{2}{3}(1-\epsilon)}$, for a given $\epsilon \in]0,1/7[$, we use the sum over eigenmodes. Second case, if $a \geq \tilde h^{\frac{2}{3}(1-\epsilon')}$, with $\epsilon' \in]0,\epsilon[$, we use the Airy–Poisson summation formula [see Lemma 2.4] and we rewrite $\mathcal{G}_{a,m}$ as a sum over $N \in \mathbb{Z}$.

3.1. Dispersive estimates for $0 < a \leq \tilde h^{\frac{2}{3}(1-\epsilon)}$, with $\epsilon \in]0,1/7[$. The following Proposition 3.4 gives a local in time dispersive estimates for $\mathcal{G}_{a,m}$ and is the main result of this subsection.

Proposition 3.4. *Let $\epsilon \in]0, 1/7[$. There exists C such that for all $h \in]0, 1]$, all $0 < a \le \tilde{h}^{\frac{2}{3}(1-\epsilon)}$, and all $t \in [h, 1]$, the following holds true:*

$$\|\mathbb{1}_{x\le a}\mathcal{G}_{a,m}(t,x,y,z)\|_{L^\infty} \le Ch^{-3}(2^m\sqrt{a})^{1/3}\left(\frac{h}{t}\right)^{5/6}. \tag{3.11}$$

Proof. Recall that $\mathcal{G}_{a,m}$ is defined by

$$\mathcal{G}_{a,m}(t,x,y,z) = \frac{1}{4\pi^2h^2}\left(\frac{h}{t}\right)^{1/2}\sum_{1\le k\le \frac{\varepsilon}{(2^m\sqrt{a})^3\tilde{h}}}\int e^{\frac{it}{\tilde{h}}(\tilde{y}\tilde{\eta}+\tilde{\mu}(1-\tilde{z}^2)^{1/2})}g_k\psi_1(\tilde{\eta})\,d\tilde{\eta}, \tag{3.12}$$

with g_k equal to

$$g_k = \frac{1}{\tilde{\mu}}\sigma_0(z^*,\eta,\mu^2;\lambda)\left(1-\chi_4\left(\frac{t\mu^2}{Dh}\right)\right)e_k(x,\tilde{\eta}/\tilde{h})e_k(a,\tilde{\eta}/\tilde{h})\chi_1(\omega_k h^{2/3}\eta^{4/3})(1-\chi_1)(\varepsilon\omega_k).$$

Recall from (3.10) that we may assume $\tilde{h} \le \tilde{h}_0$ with $\tilde{h}_0$ small. Since $\mathcal{G}_{a,m}$ contains Airy functions which behave differently depending on the various values of k, we split the sum over k in (3.12) in two pieces. We fix a large constant D and we write $\mathcal{G}_{a,m} = \mathcal{G}_{a,m,<}+\mathcal{G}_{a,m,>}$, where in $\mathcal{G}_{a,m,<}$ only the sum over $1 \le k \le D\tilde{h}^{-\epsilon}$ is considered.

Proof of (3.11) *for* $\mathcal{G}_{a,m,<}$. Recall the definition of $\mathcal{G}_{a,m,<}$:

$$\mathcal{G}_{a,m,<}(t,x,y,z) = \frac{1}{4\pi^2h^2}\left(\frac{h}{t}\right)^{1/2}\sum_{1\le k\le D\tilde{h}^{-\epsilon}}\int e^{\frac{it}{\tilde{h}}\left(\tilde{y}\tilde{\eta}+\tilde{\mu}(1-\tilde{z}^2)^{1/2}\right)}g_k\psi_1(\tilde{\eta})\,d\tilde{\eta}, \tag{3.13}$$

$$g_k = f_k^2k^{-1/3}\left(\frac{\tilde{\eta}^{2/3}}{\tilde{\mu}\tilde{h}^{2/3}}\right)\sigma_0(z^*,\eta,\mu^2;\lambda)\left(1-\chi_4\left(\frac{t\mu^2}{Dh}\right)\right)\chi_1(\omega_k h^{2/3}\eta^{4/3})(1-\chi_1)(\varepsilon\omega_k)n_k,$$

$$n_k = Ai((\tilde{\eta}/\tilde{h})^{2/3}x-\omega_k)Ai((\tilde{\eta}/\tilde{h})^{2/3}a-\omega_k).$$

Let us first assume $t2^m\sqrt{a} \le \tilde{h}^\epsilon$. Since we have $\tilde{\mu} = (\tilde{\eta}^2+\omega_k\tilde{h}^{2/3}\tilde{\eta}^{4/3})^{1/2} \ge \tilde{\eta}$, we get the estimate

$$|g_k| \le C\tilde{h}^{-2/3}k^{-1/3}\left|Ai((\tilde{\eta}/\tilde{h})^{2/3}x-\omega_k)Ai((\tilde{\eta}/\tilde{h})^{2/3}a-\omega_k)\right|.$$

By Lemma 2.2, this implies

$$\sum_{1\le k\le D\tilde{h}^{-\epsilon}}|g_k| \le C\tilde{h}^{-2/3}(\tilde{h}^{-\epsilon})^{1/3} \le C(\tilde{h})^{-2/3}(t2^m\sqrt{a})^{-1/3} = Ch^{-1}(2^m\sqrt{a})^{1/3}\left(\frac{h}{t}\right)^{1/3}$$

and (3.11) follows from (3.13).

Let us now assume $t2^m\sqrt{a} \ge \tilde{h}^\epsilon$. Observe that in the range $k \le \tilde{D}h^{-\epsilon}$, we have

$$\omega_k\tilde{h}^{2/3} \le C\tilde{h}^{2/3(1-\epsilon)} \le C\tilde{h}_0^{2/3(1-\epsilon)}$$

small. Hence, $\gamma = \omega_k\tilde{h}^{2/3}\tilde{\eta}^{-2/3}$ is small and

$$\tilde{\mu} = \tilde{\eta}(1+\gamma)^{1/2} = \tilde{\eta}+\tilde{\eta}^{1/3}\omega_k\tilde{h}^{2/3}/2+O((\omega_k\tilde{h}^{2/3})^2).$$

Therefore, we get $\left|\frac{\partial^2\tilde{\mu}}{\partial\tilde{\eta}^2}\right| \ge c\omega_k\tilde{h}^{2/3}$ with $c > 0$, and for all $j \ge 2$,

$$\left|\frac{\partial^j\tilde{\mu}}{\partial\tilde{\eta}^j}\right| \le C_j\omega_k\tilde{h}^{2/3}.$$

We will apply the stationary phase in $\tilde{\eta}$ in each term of the sum in (3.13) with the phase function

$$\Phi_k(\tilde{\eta}) = \frac{t}{\tilde{h}}\left(\tilde{y}\tilde{\eta}+\tilde{\mu}(1-\tilde{z}^2)^{1/2}\right).$$

Let $\Lambda_k = t\tilde{h}^{-1/3}\omega_k 2^m\sqrt{a}$, and let $\Psi_k(\tilde{\eta})$ the phase function defined by

$$\frac{t\Phi_k}{\tilde{h}} = \Lambda_k\Psi_k.$$

Lemma 3.5. *Let $\tilde{g}_k = k^{1/3}\tilde{h}^{2/3}g_k$. There exists C such that for all $1 \le k \le D\tilde{h}^{-\epsilon}$, the following holds true:*

$$\left|\int e^{i\Lambda_k\Psi_k}\tilde{g}_k\psi_1(\tilde{\eta})\,d\tilde{\eta}\right| \le C\min\left\{1, \Lambda_k^{-1/2}\right\}. \tag{3.14}$$

Proof. We may assume $\Lambda_k \ge 1$ since we have $|\tilde{g}_k| \le C$. Recall from Lemma 3.2 that we may assume $\sqrt{1-\tilde{z}^2} \sim \mu = 2^m\sqrt{a}\tilde{\mu} \sim 2^m\sqrt{a}$. Therefore, there exists $c > 0$ such that for all $1 \le k \le D\tilde{h}^{-\epsilon}$ one has

$$\left|\frac{\partial^2\Psi_k}{\partial\tilde{\eta}^2}\right| = \left|\frac{t}{\tilde{h}\Lambda_k}\frac{\partial^2\tilde{\mu}}{\partial\tilde{\eta}^2}\sqrt{1-\tilde{z}^2}\right| \ge c > 0$$

and for all $j \ge 2$, $\left|\frac{\partial^j\Psi_k}{\partial\tilde{\eta}^j}\right| \le C_j$. Thus, to apply the stationary phase, we just need to check that there exist $\nu > 0$ and for all j, a constant C_j such that

$$\left|\frac{\partial^j\tilde{g}_k}{\partial\tilde{\eta}^j}\right| \le C_j\Lambda_k^{j(1/2-\nu)}, \quad \forall k \le D\tilde{h}^{-\epsilon}\,. \tag{3.15}$$

In Lemma 3.2, z^* is defined by $\tilde{z} = -1 + z^*\mu^2$, but since we have here $\mu \sim 2^m\sqrt{a}$ we may as well define z^* by $\tilde{z} = -1 + z^*2^{2m}a$. Then z^* becomes independent of $\tilde{\eta}$. Recall $\lambda = t\mu^2/h$. Since $\eta = 2^m\sqrt{a}\tilde{\eta}$ and all the derivatives of γ and $\tilde{\mu}$ with respect to $\tilde{\eta}$ are bounded, we get $\left|\frac{\partial^j\lambda}{\partial\tilde{\eta}^j}\right| \le C_j\lambda$ for all j. Since λ is bounded on the support of derivatives of χ_4, the term

$$f_k^2\tilde{\eta}^{2/3}\sigma_0(z^*,\eta,\mu^2;\lambda)\left(1-\chi_4\left(\frac{t\mu^2}{Dh}\right)\right)\chi_1(\omega_k h^{2/3}\eta^{4/3})(1-\chi_1)(\varepsilon\omega_k)$$

satisfies the estimate (3.15), and therefore, it remains to show that the function $Ai((\frac{\tilde{\eta}}{\tilde{h}})^{2/3}x - \omega_k)$ satisfies the estimate (3.15) uniformly in $x \in [0,a]$. Let set $\theta = x\tilde{h}^{-2/3} \ge 0$ and $r = \tilde{\eta}^{2/3}$ which belongs to a compact subset of $]0,\infty[$. One has $\partial_r^l(Ai(r\theta-\omega_k)) \sim (r\theta)^l Ai^{(l)}(r\theta-\omega_k)$. Since for all l one has

$$\sup_{b\ge 0}|b^l Ai^{(l)}(b-\omega_k)| \le C_l\omega_k^{3l/2},$$

we get that (3.15) holds true if

$$\exists\beta > 3, c > 0, \quad c\omega_k^\beta \le \Lambda_k = t\tilde{h}^{-1/3}\omega_k 2^m\sqrt{a}$$

We have $t2^m\sqrt{a} \ge \tilde{h}^\epsilon$, and $c\omega_k^2 \le \tilde{h}^{-4\epsilon/3}$, thus this holds for $\epsilon < 1/7$. □

Therefore, we get the following estimate for $\mathcal{G}_{a,m,<}$ and $t2^m\sqrt{a} \ge \tilde{h}^\epsilon$

$$\|\mathbb{1}_{x\le a}\mathcal{G}_{a,m,<}(t,x,y,z)\|_{L^\infty} \le Ch^{-2}\left(\frac{h}{t}\right)^{1/2}\left(\sum_{1\le k\le D\tilde{h}^{-\epsilon}} k^{-1/3}\tilde{h}^{-2/3}(t\tilde{h}^{-1/3}\omega_k 2^m\sqrt{a})^{-1/2}\right)$$

$$\le Ch^{-2}\left(\frac{h}{t}\right)^{1/2}(t2^m\sqrt{a})^{-1/2}\tilde{h}^{-(1/2+\epsilon/3)}$$

$$\le Ch^{-3}(2^m\sqrt{a})^{1/3}\left(\frac{h}{t}\right)^{5/6}\left(h\left(\frac{t}{h}\right)^{1/3}(2^m\sqrt{a})^{-1/3}(t2^m\sqrt{a})^{-1/2}\tilde{h}^{-(1/2+\epsilon/3)}\right).$$

This concludes the proof of Proposition 3.4 for $\mathcal{G}_{a,m,<}$ since $t2^m\sqrt{a} \geq \tilde{h}^{\epsilon}$ implies

$$h^{2/3}t^{-1/6}(2^m\sqrt{a})^{-5/6}\tilde{h}^{-(1/2+\epsilon/3)} \leq \tilde{h}^{1/6-\epsilon/2}.$$

Proof of (3.11) *for* $\mathcal{G}_{a,m,>}$. For $k \geq D\tilde{h}^{-\epsilon}$ with D large and $a \leq \tilde{h}^{2/3(1-\epsilon)}$ one has

$$\omega_k - \tilde{h}^{-2/3}\tilde{\eta}^{2/3}a \geq \omega_k/2.$$

Since $\gamma = \omega_k\tilde{h}^{2/3}\tilde{\eta}^{-2/3}$, we get $\gamma - a \geq a$ and $\gamma - a \geq \gamma/2$. Then by the definition of e_k and asymptotic of the Airy functions, we obtain

$$\mathcal{G}_{a,m,>}(t,x,y,z) = \sum_{\tilde{h}^{-\epsilon} \leq k \leq \frac{\varepsilon}{(2^m\sqrt{a})^3\tilde{h}}} \frac{1}{4\pi^2h^2}\left(\frac{h}{t}\right)^{1/2}\sum_{\pm,\pm}\int e^{\frac{i}{\tilde{h}}\Phi_k^{\pm,\pm}}\sigma_k^{\pm,\pm}\psi_1(\tilde{\eta})d\tilde{\eta}, \tag{3.16}$$

with phase functions defined by

$$\Phi_k^{\pm,\pm}(\tilde{\eta}) = \tilde{\eta}\left[y + t\sqrt{1-\tilde{z}^2}(1+\gamma)^{1/2} \pm \frac{2}{3}(\gamma - x)^{3/2} \pm \frac{2}{3}(\gamma - a)^{3/2}\right], \tag{3.17}$$

and the symbols are given by

$$\begin{aligned}\sigma_k^{\pm,\pm}(\tilde{\eta}) = {} & f_k^2k^{-1/3}\tilde{h}^{-1/3}\tilde{\eta}^{-2/3}\sigma_0(z^*,\eta,\mu^2,\lambda)\left(1-\chi_4\left(\frac{t\mu^2}{Dh}\right)\right)\chi_1(\omega_kh^{2/3}\eta^{4/3})\\ & \times(1-\chi_1(\epsilon\omega_k))(\gamma-x)^{-1/4}(\gamma-a)^{-1/4}(1+\gamma)^{-1/2}\omega^{\pm}\omega^{\pm}\\ & \times\Psi_{\pm}\Big(\tilde{\eta}^{2/3}\tilde{h}^{-2/3}(\gamma-x)\Big)\Psi_{\pm}\Big(\tilde{\eta}^{2/3}\tilde{h}^{-2/3}(\gamma-a)\Big),\end{aligned}$$

where $\Psi_{\pm}$ are classical symbols of order 0 at infinity. In Lemma 3.2, z^* is defined by $\tilde{z} = -1 + z^*\mu^2$, but since we have here $\mu \sim 2^m\sqrt{a}(1+\omega_k\tilde{h}^{2/3})^{1/2}$ we may as well define z^* by $\tilde{z} = -1 + z^*2^{2m}a(1+\omega_k\tilde{h}^{2/3})$. Then z^* becomes independent of $\tilde{\eta}$. Observe that for all j, there exists C_j, C'_j such that for all k one has

$$|\partial_{\tilde{\eta}}^j\gamma| \sim C_j\gamma, \quad |\partial_{\tilde{\eta}}^j\tilde{\mu}| \leq C'_j\tilde{\mu}, \quad |\partial_{\tilde{\eta}}^j\mu^2| \leq C'_j\mu^2 \leq C'_j.$$

Since $\lambda = t\mu^2/h = \frac{t2^m\sqrt{a}}{\tilde{h}}(1+\gamma)$, we get $\left|\frac{\partial^j\lambda}{\partial\tilde{\eta}^j}\right| \leq C_j\lambda$ for all j. Finally, λ is bounded on the support of derivatives of χ_4 and there exists $c_1 > 0$ such that $\tilde{\eta}^{2/3}\tilde{h}^{-2/3}(\gamma - a) \geq c_1$. Since $\gamma \sim (k\tilde{h})^{2/3}$, we get that for all j, there exists C_j such that for all k one has

$$|\partial_{\tilde{\eta}}^j\sigma_k^{\pm,\pm}(\tilde{\eta})| \leq C_j(k\tilde{h})^{-2/3}(1+\gamma)^{-1/2}. \tag{3.18}$$

We notice that for the values of k, $D\tilde{h}^{-\epsilon} \leq k \leq \frac{1}{h\eta^2}$, we get $\gamma \in \left[2a, \frac{1}{2^{2m}a}\right]$. In what follows, we distinguish between the two cases: $\gamma \in [2a,1]$ and $\gamma \in \left[1, \frac{1}{2^{2m}a}\right]$.

- The first case $\gamma \in [2a,1]$ corresponds to $\tilde{h}^{-\epsilon} \leq k \leq \tilde{h}^{-1}$. Let denote $\Lambda_k = t2^m\sqrt{a}\omega_k\tilde{h}^{-1/3}$ and $\Phi_k^{\pm,\pm} = \tilde{h}\Lambda_k\Psi_k^{\pm,\pm}$.

Proposition 3.6. *There exists a constant C independent of $a \in]0, \tilde{h}^{2/3(1-\epsilon)}]$, $t \in [h,1]$, $x \in [0,a]$, $y \in \mathbb{R}$, $z \in \mathbb{R}$, and $k \in [\tilde{h}^{-\epsilon}, \tilde{h}^{-1}]$ such that the following holds:*

$$\left|\int e^{i\Lambda_k\Psi_k^{\pm,\pm}}\sigma_k^{\pm,\pm}\psi_1(\tilde{\eta})\,d\tilde{\eta}\right| \leq C(\tilde{h}k)^{-2/3}\Lambda_k^{-1/3}.$$

Proof of Proposition 3.6. By (3.18), Proposition 3.6 is obvious for $\Lambda_k \leq 1$. In the case $\Lambda_k \geq 1$, we use $\tilde{\mu} \sim 2^m\sqrt{a}$ which implies $t\sqrt{1-\tilde{z}^2} \sim t2^m\sqrt{a}$. Then the proof is the same as the proof of Proposition 2.3, if one replaces (h,t) in Proposition 2.3 by $(\tilde{h}, t2^m\sqrt{a})$. □

Hence the corresponding estimate of $\mathcal{G}_{a,m,>}$ for $\tilde h^{-\epsilon} \le k \le \tilde h^{-1}$ is given by

$$\begin{aligned}\|\mathbb{1}_{x\le a}\mathcal{G}_{a,m,>}(t,x,y,z)\|_{L^\infty} &\le Ch^{-2}\left(\frac{h}{t}\right)^{1/2} \sum_{\tilde h^{-\epsilon}\le k\le \tilde h^{-1}} (\tilde h k)^{-2/3}(t2^m\sqrt{a}\omega_k \tilde h^{-1/3})^{-1/3} \\ &\le Ch^{-2}\left(\frac{h}{t}\right)^{1/2} \tilde h^{-2/3}(t2^m\sqrt{a})^{-1/3}\tilde h^{1/9}\sum_{k\le 1/\tilde h} k^{-8/9} \\ &\le Ch^{-3}\left(\frac{h}{t}\right)^{5/6}(2^m\sqrt{a})^{1/3}.\end{aligned}$$

- The second case $\gamma \in \left[1, \frac{1}{2^{2m}a}\right]$ corresponds to $\tilde h^{-1} \le k \le \frac{1}{2^{2m}ah}$. We still define Λ_k and $\Psi_k^{\pm,\pm}$ by $\Lambda_k = t2^m\sqrt{a}\omega_k\tilde h^{-1/3}$ and $\Phi_k^{\pm,\pm} = \tilde h\Lambda_k\Psi_k^{\pm,\pm}$.

Proposition 3.7. *There exists a constant C independent of $a \in]0, \tilde h^{2/3(1-\epsilon)}]$, $t \in [h,1]$, $x \in [0,a]$, $y \in \mathbb{R}$, $z \in \mathbb{R}$, and $k \in [\tilde h^{-1}, \frac{1}{2^{2m}ah}]$ such that the following holds:*

$$\left|\int e^{i\Lambda_k\Psi_k^{\pm,\pm}}\sigma_k^{\pm,\pm}\psi_1(\tilde\eta)\,d\tilde\eta\right| \le C(\tilde h k)^{-1}\Lambda_k^{-1/3}.$$

Proof of Proposition 3.7. One has $\gamma \sim (k\tilde h)^{2/3}$. Thus $\gamma \ge 1$ and (3.18) imply $|\partial_{\tilde\eta}^j\sigma_k^{\pm,\pm}(\tilde\eta)| \le C_j(k\tilde h)^{-1}$. Therefore, Proposition 3.7 is obvious for $\Lambda_k \le 1$. In the case $\Lambda_k \ge 1$ we proceed as in the Proposition 2.3 . Recall that $\tilde z$ is close to -1 and $\tilde z = -1 + z^*2^{2m}a(1+\omega_k\tilde h^{2/3})$ with z^* in a compact set of $]0,\infty[$. We write

$$\frac{t\sqrt{1-\tilde z^2}}{\tilde h\Lambda_k}\frac{1+2\gamma/3}{(1+\gamma)^{1/2}} = \sqrt{z^*}(1-\tilde z)^{1/2}\tilde\eta^{-2/3}\tilde F(\gamma),$$

with

$$\tilde F(\gamma) = \frac{2(1+\omega_k\tilde h^{2/3})^{1/2}}{3(1+\gamma)^{1/2}}(1+1/\gamma).$$

For γ large one has $\tilde F(\gamma) \sim 1, \tilde F(\gamma) + \gamma\tilde F'(\gamma) \sim 1$. Moreover, one has

$$\omega_k\tilde h^{2/3}(2\tilde F'(\gamma) + \gamma\tilde F''(\gamma)) \sim \omega_k\tilde h^{2/3}\gamma^{-1} \sim 1.$$

Hence the proof is the same as the proof of Proposition 2.3, if one replaces (h,F) in Proposition 2.3 by $(\tilde h, \tilde F)$. □

Using Proposition 3.7, we get the estimate of $\mathcal{G}_{a,m,>}$ for $\tilde h^{-1} \le k \le \frac{1}{2^{2m}ah}$:

$$\begin{aligned}\|\mathbb{1}_{x\le a}\mathcal{G}_{a,m,>}(t,x,y,z)\|_{L^\infty} &\le Ch^{-2}\left(\frac{h}{t}\right)^{1/2} \sum_{\tilde h^{-1}\le k} (\tilde h k)^{-1}(t2^m\sqrt{a}\omega_k \tilde h^{-1/3})^{-1/3} \\ &\le Ch^{-2}\left(\frac{h}{t}\right)^{1/2} t^{-1/3}(2^m\sqrt{a})^{-1/3}\tilde h^{-8/9}\sum_{\tilde h^{-1}\le k} k^{-11/9} \\ &\le Ch^{-3}\left(\frac{h}{t}\right)^{5/6}(2^m\sqrt{a})^{1/3}.\end{aligned}$$

This concludes the proof of Proposition 3.4. □

3.2. Dispersive estimates for $a \ge \tilde h^{\frac{2}{3}(1-\epsilon')}$, for $\epsilon' \in]0,\epsilon[$. In this subsection, we assume $a \ge \tilde h^{\frac{2}{3}(1-\epsilon')}$, for some $\epsilon' \in]0,\epsilon[$ and we establish a local in time dispersive estimates for $\mathcal{G}_{a,m}$. Observe that $\Lambda = a^{3/2}/\tilde{h} \geq \tilde{h}^{-\epsilon'}$ is a large parameter. Recall from (3.12) that $\mathcal{G}_{a,m}$ is defined by

$$\begin{aligned}(3.19)\qquad &\mathcal{G}_{a,m}(t,x,y,z)\\ &= \frac{1}{4\pi^2h^2}\left(\frac{h}{t}\right)^{1/2}\sum_{1\leq k\leq \frac{\varepsilon}{(2^m\sqrt{a})^3\tilde{h}}}\int e^{\frac{i}{\tilde{h}}\left(y\tilde{\eta}+t\tilde{\mu}(1-\tilde{z}^2)^{1/2}\right)}g(\omega_k,\tilde{\eta},\tilde{h})\psi_1(\tilde{\eta})\,d\tilde{\eta},\end{aligned}$$

with $g(\omega_k,\tilde{\eta},\tilde{h})$ equal to

$$g = \frac{1}{\tilde{\mu}}\sigma_0(z^*,\eta,\mu^2;\lambda)\left(1-\chi_4\left(\frac{t\mu^2}{Dh}\right)\right)e_k(x,\tilde{\eta}/\tilde{h})e_k(a,\tilde{\eta}/\tilde{h})\chi_1(\omega_k h^{2/3}\eta^{4/3})(1-\chi_1)(\varepsilon\omega_k),$$

and we recall $h = 2^m\sqrt{a}\tilde{h}, \eta = 2^m\sqrt{a}\tilde{\eta}, \mu = 2^m\sqrt{a}\tilde{\mu}$, and

$$\gamma = \omega\tilde{h}^{2/3}\tilde{\eta}^{-2/3}, \quad \tilde{\mu} = \tilde{\eta}(1+\gamma)^{1/2}\ .$$

We will use the same notations as in section 2,

$$t = a^{1/2}T, \quad x = aX, \quad y + t\sqrt{1-\tilde{z}^2} = a^{3/2}Y.$$

Let $\omega = \tilde{\eta}^{2/3}\tilde{h}^{-2/3}a\tilde{\omega}$. We get $\gamma = a\tilde{\omega}$ and $(1+a\tilde{\omega})^{1/2}-1 = a\gamma_a(\tilde{\omega}) = \frac{a\tilde{\omega}}{1+(1+a\tilde{\omega})^{1/2}}$. Then we use the Airy–Poisson summation formula, and we get

$$\mathcal{G}_{a,m} = \sum_N G_{a,m,N}$$

with

$$(3.20)\quad G_{a,m,N}(t,x,y,z) = \frac{(-1)^N}{(2\pi)^4h^4}\left(\frac{h}{t}\right)^{1/2}a^2(2^m\sqrt{a})^2\int e^{i\Lambda\Phi_N}\chi_m\tilde{\eta}^2\psi_1(\tilde{\eta})\,d\tilde{s}\,d\tilde{\sigma}\,d\tilde{\omega}\,d\tilde{\eta}$$

with the phase function

$$\begin{aligned}\Phi_N(\tilde{s},\tilde{\sigma},\tilde{\omega},\tilde{\eta}) = \tilde{\eta}\Big[&Y + T(1-\tilde{z}^2)^{1/2}\gamma_a(\tilde{\omega}) + \frac{\tilde{s}^3}{3} + \tilde{s}(X-\tilde{\omega})\\ &+\frac{\tilde{\sigma}^3}{3} + \tilde{\sigma}(1-\tilde{\omega}) - \frac{4}{3}N\tilde{\omega}^{3/2} + \frac{N}{\Lambda\tilde{\eta}}B\left(\tilde{\omega}^{3/2}\Lambda\tilde{\eta}\right)\Big],\end{aligned}$$

and symbol $\chi_m(a,t,z;\tilde{\eta},\tilde{\omega},\tilde{h})$ equal to, with $\lambda = t2^m\sqrt{a}\tilde{\mu}^2/\tilde{h}$,

$$(3.21)\quad \chi_m = \frac{1}{\tilde{\mu}}\sigma_0(z^*,\eta,\mu^2;\lambda)(1-\chi_4(\lambda/D))\chi_1((2^m\sqrt{a})^2\tilde{\eta}^2a\tilde{\omega})(1-\chi_1)(\varepsilon\tilde{\eta}^{2/3}\tilde{h}^{-2/3}a\tilde{\omega}).$$

Observe that we get the same phase function Φ_N as in section 2, but we have to take care of the fact that now $(1-\tilde{z}^2)^{1/2}$ may be small. Therefore, in order to use the results of section 2, we introduce the notation $\tilde{T} = T(1-\tilde{z}^2)^{1/2}$. Set

$$\mathcal{C}_{a,m,N,h} = \{(t,x,y,\tilde{s},\tilde{\sigma},\tilde{\omega},\tilde{\eta}) \text{ such that } \partial_{\tilde{s}}\Phi_N = \partial_{\tilde{\sigma}}\Phi_N = \partial_{\tilde{\omega}}\Phi_N = \partial_{\tilde{\eta}}\Phi_N = 0\}.$$

Hence $\mathcal{C}_{a,m,N,h}$ is defined by the system of equations

$$\begin{aligned}&X = \tilde{\omega} - \tilde{s}^2, \quad \tilde{\omega} = 1 + \tilde{\sigma}^2,\\ &\tilde{T} = 2(1+a\tilde{\omega})^{1/2}\left(\tilde{s} + \tilde{\sigma} + 2N\tilde{\omega}^{1/2}\left(1 - \frac{3}{4}B'\left(\tilde{\omega}^{3/2}\Lambda\tilde{\eta}\right)\right)\right),\\ &Y = -\tilde{T}\gamma_a(\tilde{\omega}) - \frac{\tilde{s}^3}{3} - \tilde{s}(X-\tilde{\omega}) - \frac{\tilde{\sigma}^3}{3} - \tilde{\sigma}(1-\tilde{\omega}) + N\tilde{\omega}^{3/2}\left(\frac{4}{3} - B'\left(\tilde{\omega}^{3/2}\Lambda\tilde{\eta}\right)\right).\end{aligned}$$

We define the Lagrangian submanifold $\mathbf{\Lambda}_{a,m,N,h} \subset T^*\mathbb{R}^3$ as the image of $\mathcal{C}_{a,m,N,h}$ by the map

$$(t,x,y,\tilde{s},\tilde{\sigma},\tilde{\omega},\tilde{\eta}) \longmapsto (x,t,y,\xi = \partial_x\Phi_N, \tau = \partial_t\Phi_N, \eta = \partial_y\Phi_N).$$

Then the projection of $\mathbf{\Lambda}_{a,m,N,h}$ onto $\mathbb{R}^3$ is defined by the system of equations

$$\begin{aligned} X &= 1+\tilde{\sigma}^2-\tilde{s}^2, \\ Y &= H_1(a,\tilde{\sigma})(\tilde{s}+\tilde{\sigma}) + \frac{2}{3}(\tilde{s}^3+\tilde{\sigma}^3) \\ &\quad + \frac{2}{3}H_2(a,\tilde{\sigma})(1+\tilde{\sigma}^2)^{-1/2}\left(\frac{\tilde{T}}{2(1+a+a\tilde{\sigma}^2)^{1/2}} - \tilde{s} - \tilde{\sigma}\right), \end{aligned} \tag{3.22}$$

where H_1, H_2 are defined in Section 2 and

$$2N\left(1-\frac{3}{4}B'\left(\tilde{\omega}^{3/2}\Lambda\tilde{\eta}\right)\right) = (1+\tilde{\sigma}^2)^{-1/2}\left(\frac{\tilde{T}}{2(1+a+a\tilde{\sigma}^2)^{1/2}} - \tilde{s} - \tilde{\sigma}\right). \tag{3.23}$$

Remark 3.8. We notice from (3.23) in the range of $T \in]0, a^{-1/2}]$, we can still reduce the sum over $N \in \mathbb{Z}$ to the sum over $1 \le N \le C_0 a^{-1/2}$ since $\tilde{T} \le T$.

This system yields the cardinality of $\mathcal{N}$ and $\mathcal{N}_1$ such that $|\mathcal{N}(X,Y,T)| \le C_0$ and $|\mathcal{N}_1(X,Y,T)| \le C_0\left(1+\tilde{T}\Lambda^{-2}\tilde{\omega}^{-3}\right)$, respectively. Recall that here the notations $\mathcal{N}, \mathcal{N}_1$ are those defined in Section 2.

Our main result of this subsection is Theorem 3.9, which gives dispersive estimates for the sum over N of $G_{a,m,N}$.

Theorem 3.9. *Let $\alpha < 2/3$ and $\tilde{h} = h/(2^m\sqrt{a})$. There exists C such that for all $h \in]0,h_0]$, all $a \in \left[\tilde{h}^\alpha, a_0\right]$, all $x \in [0,a]$, all $t \in]h,1]$, all $y \in \mathbb{R}$, all $z \in \mathbb{R}$, the following holds:*

$$\begin{aligned} &\left|\sum_{1\le N\le C_0 a^{-1/2}} G_{a,m,N}(t,x,y,z)\right| \\ &\le Ch^{-3}\left(\frac{h}{t}\right)^{1/2}\left(\min\left\{\left(\frac{h}{t}\right)^{1/2}, 2^m\sqrt{a}\right\} + a^{1/8}h^{1/4}(2^m\sqrt{a})^{3/4}\right). \end{aligned}$$

We notice as in section 2, that for $\tilde{\omega} \le 3/4$, we get rapid decay in Λ by integration by parts in $\tilde{\sigma}$. In particular, we may replace $1-\chi_1$ by 1 in (3.21). As in section 2, we introduce a cutoff function $\chi_2(\tilde{\omega}) \in C_0^\infty(]1/2,3/2[), 0 \le \chi_2 \le 1, \chi_2 = 1$ on $]\frac{3}{4},\frac{5}{4}[$ and we denote by $G_{a,m,N,2}$ the corresponding integral. Hence, we get

$$G_{a,m,N} = G_{a,m,N,1} + G_{a,m,N,2} + O_{C^\infty}(\Lambda^{-\infty}),$$

where $G_{a,m,N,1}$ is defined by a cutoff χ_3 with $\tilde{\omega} \ge 5/4$ on the support of χ_3.

3.2.1. The analysis of $\boldsymbol{G_{a,m,N,1}}$. The main results in this subsection are Proposition 3.10 and Proposition 3.11.

Proposition 3.10. *Let $\alpha < 2/3$ and $\tilde{h} = h/(2^m\sqrt{a})$. There exists C such that for all $h \in]0,h_0]$, all $a \in \left[\tilde{h}^\alpha, a_0\right]$, all $x \in [0,a]$, all $t \in]h,1]$, all $y \in \mathbb{R}$, all $z \in \mathbb{R}$, the following holds:*

$$\left|\sum_{2\le N\le C_0 a^{-1/2}} G_{a,m,N,1}(t,x,y,z;h)\right| \le Ch^{-3}\left(\frac{h}{t}\right)^{1/2} h^{1/3}(2^m\sqrt{a})^{2/3}.$$

Proof. On the support of χ_3, we can apply the stationary phase method for $(\tilde{s},\tilde{\sigma})$-integrations with large parameter $\Lambda\tilde{\eta}$; hence we get

$$G_{a,m,N,1} = \frac{(-1)^N a^2\Lambda^{-1}}{(2\pi)^4 h^4}(2^m\sqrt{a})^2\left(\frac{h}{t}\right)^{1/2}\int e^{i\Lambda Y\tilde{\eta}}\tilde{\eta}\psi_1(\tilde{\eta})\tilde{G}_{a,m,N,1}\,d\tilde{\eta},$$
$$\tilde{G}_{a,m,N,1} = \sum_{\epsilon_1,\epsilon_2}\int e^{i\Lambda\tilde{\eta}\Phi_{N,m,\epsilon_1,\epsilon_2}}\Theta_{\epsilon_1,\epsilon_2}(1+a\tilde{\omega})^{-1/2}\,d\tilde{\omega},$$

with symbols $\Theta_{\epsilon_1,\epsilon_2}$, where $\epsilon_j=\pm$, have a support in $\tilde{\omega}\le(2^m\sqrt{a})^{-2}/a$, and such that $|\tilde{\omega}^l\partial^l_{\tilde{\omega}}\Theta_{\epsilon_1,\epsilon_2}|\le C_l\tilde{\omega}^{-1/2}$ with C_l independent of a,m. The phase functions are

$$\Phi_{N,m,\epsilon_1,\epsilon_2}(\tilde{\omega}) = \tilde{T}\gamma_a(\tilde{\omega})+\frac{2}{3}\epsilon_1(\tilde{\omega}-X)^{3/2}+\frac{2}{3}\epsilon_2(\tilde{\omega}-1)^{3/2}-\frac{4}{3}N\tilde{\omega}^{3/2}+\frac{N}{\Lambda\tilde{\eta}}B\big(\tilde{\omega}^{3/2}\Lambda\tilde{\eta}\big)\,.$$

Let us define

$$G_{a,m,N,1,\epsilon_1,\epsilon_2} = \frac{(-1)^N a^2\Lambda^{-1}}{(2\pi)^4 h^4}(2^m\sqrt{a})^2\left(\frac{h}{t}\right)^{1/2}\int e^{i\Lambda Y\tilde{\eta}}\tilde{\eta}\psi_1(\tilde{\eta})\tilde{G}_{a,m,N,1,\epsilon_1,\epsilon_2}\,d\tilde{\eta},$$
$$\tilde{G}_{a,m,N,1,\epsilon_1,\epsilon_2} = \sum_{\epsilon_1,\epsilon_2}\int e^{i\Lambda\tilde{\eta}\Phi_{N,m,\epsilon_1,\epsilon_2}}\Theta_{\epsilon_1,\epsilon_2}(1+a\tilde{\omega})^{-1/2}\,d\tilde{\omega}.$$

We are reduce to prove the following inequality

$$\left|\sum_{2\le N\le C_0a^{-1/2}} G_{a,m,N,1,\epsilon_1,\epsilon_2}(t,x,y,z,h)\right| \le Ch^{-3}\left(\frac{h}{t}\right)^{1/2}h^{1/3}(2^m\sqrt{a})^{2/3}, \tag{3.24}$$

with a constant C independent of m, $h\in\,]0,h_0]$, $a\in\big[\tilde{h}^{2/3},a_0\big]$, $x\in[0,a]$, $t\in[h,1]$. We proceed as in the proof of Proposition 2.9. Let us recall that on the support of χ_1 we have $a\tilde{\omega}\le\varepsilon/2^{2m}a$; hence $a\tilde{\omega}$ could be small or large. We distinguish between two cases.

The first case is $a\tilde{\omega}\le 1$. Let $\tilde{T}_0\gg 1$. We get the following results:

- For $0\le\tilde{T}\le\tilde{T}_0$, $N\ge N(\tilde{T}_0)$, then we apply the integration by parts to get $|\tilde{G}_{a,m,N,1,+,+}|\in O_{C^\infty}(N^{-\infty}\Lambda^{-\infty})$ and

$$\sup_{\tilde{T}\le\tilde{T}_0,X\in[0,1],(y,z)\in\mathbb{R}^2}\left|\sum_{N(\tilde{T}_0)\le N\le Ca^{-1/2}} G_{a,m,N,1,+,+}\right|\in O_{C^\infty}(h^\infty).$$

- For $0\le\tilde{T}\le\tilde{T}_0$, $2\le N\le N(\tilde{T}_0)$, [12, Lemma 2.20] yields the following estimate $|\tilde{G}_{a,m,N,1,+,+}|\le C\Lambda^{-1/3}$ and

$$\sup_{\tilde{T}\le\tilde{T}_0,X\in[0,1],(y,z)\in\mathbb{R}^2}\left|\sum_{2\le N\le N(\tilde{T}_0)} G_{a,m,N,1,+,+}\right| \le Ch^{-3}\left(\frac{h}{t}\right)^{1/2}\big(h^{-1}a^2(2^m\sqrt{a})^2\Lambda^{-4/3}\big)$$
$$\le Ch^{-3}\left(\frac{h}{t}\right)^{1/2}h^{1/3}(2^m\sqrt{a})^{2/3}.$$

- For $\tilde{T}_0\le\tilde{T}\le a^{-1/2}(1-\tilde{z}^2)^{1/2}$, we still use the same notation as before $\Omega=\tilde{\omega}^{3/2}$; we have $|\partial^2_\Omega\Phi_{N,m,+,+}|\ge c\tilde{T}\Omega^{-4/3}$ and a nondegenerate critical point Ω_c which satisfies for $N\ge 2$, $\Omega_c^{1/3}\sim\frac{\tilde{T}}{N}$. Hence, we have also either $\tilde{T}/N$ bounded or large, the stationary phase yields

$$|\tilde{G}_{a,m,N,1,+,+}|\le C\Lambda^{-1/2}\tilde{T}^{-1/2}.$$

Moreover, the $\tilde\eta$-integration produces a $q^{-1/2}$ factor contribution where $q = N\Lambda^{-1}\Omega_c^{-1}$ when $q \geq 1$. Thus, we get the estimates as follows:

If $\tilde T/N$ is bounded, Ω_c stays in a compact subset of $[1,\infty[$, and we get $\tilde T \sim N$.

- If $N \leq \Lambda^2$, we have $|\mathcal{N}_1| \leq C_0$. Hence, the estimate is

$$\begin{aligned}\left|\sum_{N\in\mathcal{N}_1} G_{a,m,N,1,+,+}\right| &\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}\left(h^{-1}\Lambda^{-1}a^2(2^m\sqrt{a})^2\Lambda^{-1/2}\tilde T^{-1/2}\right)\\ &\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2} a^{-1/4}h^{1/2}(2^m\sqrt{a})^{1/2}\tilde T^{-1/2}\\ &\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2} h^{1/3}(2^m\sqrt{a})^{2/3},\end{aligned}$$

since $\tilde T \geq \tilde T_0$ and $a^{-1/4}h^{1/2} \leq h^{1/3}(2^m\sqrt{a})^{1/6}$ when $a \geq \tilde h^{2/3}$.

- If $N > \Lambda^2$, then there is the contribution $q^{-1/2}$ from $\tilde\eta$-integration and $|\mathcal{N}_1| \leq C_0\tilde T\Lambda^{-2}$. Thus, the estimate is

$$\begin{aligned}\left|\sum_{N\in\mathcal{N}_1} G_{a,m,N,1,+,+}\right| &\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}\sum_{N\in\mathcal{N}_1}\left(h^{-1}\Lambda^{-1}a^2(2^m\sqrt{a})^2\Lambda^{-1/2}\tilde T^{-1/2}N^{-1/2}\Lambda^{1/2}\right)\\ &\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}\left(h^{-1}\Lambda^{-1}a^2(2^m\sqrt{a})^2\tilde T^{-1}|\mathcal{N}_1(X,Y,T)|\right)\\ &\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}\left(a^{-5/2}\tilde h^2 2^m\sqrt{a}\right)\\ &\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}\tilde h^{1/3}2^m\sqrt{a}\\ &\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2} h^{1/3}(2^m\sqrt{a})^{2/3}.\end{aligned}$$

Next, if $\tilde T/N$ is large then Ω_c is large.

- If $N \leq \Lambda\Omega_c$, then there is no contribution from $\tilde\eta$-integration. Moreover, we have $|\mathcal{N}_1| \leq C_0$ since $\tilde T \geq \Lambda^2\Omega_c^2$ implies $\Omega_c^{1/3} \sim \tilde T/N \geq \lambda\Omega_c$ which is impossible since Ω_c is large. Thus, the estimate is

$$\left|\sum_{N\in\mathcal{N}_1} G_{a,m,N,1,+,+}\right| \leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2} h^{1/3}(2^m\sqrt{a})^{2/3}.$$

- If $N > \Lambda\Omega_c$ and $\tilde T \leq \Lambda^2\Omega_c^2$, we also have $|\mathcal{N}_1| \leq C_0$. Thus, we get the estimate

$$\left|\sum_{N\in\mathcal{N}_1} G_{a,m,N,1,+,+}\right| \leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2} h^{1/3}(2^m\sqrt{a})^{2/3}.$$

- If $N > \lambda\Omega_c$ and $\tilde{T} > \Lambda^2\Omega_c^2$, then there is the contribution $q^{-1/2}$ from $\tilde{\eta}$-integration and $|\mathcal{N}_1| \leq C_0\tilde{T}\Lambda^{-2}\Omega_c^{-2}$. We get

$$\begin{aligned}\left|\sum_{N\in\mathcal{N}_1} G_{a,m,N,1,+,+}\right| &\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}\sum_{N\in\mathcal{N}_1}\left(h^{-1}\Lambda^{-1}a^2(2^m\sqrt{a})^2\tilde{T}^{-1/2}N^{-1/2}\Omega_c^{1/2}\right)\\ &\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}\left(h^{-1}\Lambda^{-1}a^2(2^m\sqrt{a})^2\tilde{T}^{-1}\Omega_c^{2/3}|\mathcal{N}_1(X,Y,T)|\right)\\ &\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}\left(h^{-1}a^2(2^m\sqrt{a})^2\Lambda^{-3}\right)\\ &\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}h^{1/3}(2^m\sqrt{a})^{2/3}.\end{aligned}$$

The result of the other cases of (ϵ_1,ϵ_2) can be achieved by proceeding along the same lines as in the proof for $G_{a,N,1}$ in Section 2.

The second case if $a\tilde{\omega} \geq 1$, then a critical point Ω_c satisfies $\Omega_c^{1/3}(1+a\Omega_c^{2/3})^{1/2} \sim \frac{\tilde{T}}{N}$ for $N \geq 2$. This yields, since $T \geq C\tilde{T}$ with C large,

$$T \geq C\tilde{T} \geq CN\Omega_c^{1/3} = CN\omega_c^{1/2} \geq CNa^{-1/2}$$

which contradicts $t \leq 1$. □

Now we prove the following estimate for $N = 1$.

Proposition 3.11. *Let $\alpha < 2/3$ and $\tilde{h} = h/(2^m\sqrt{a})$. There exists C such that for all $h \in]0,h_0]$, all $a \in \left[\tilde{h}^\alpha, a_0\right]$, all $x \in [0,a]$, all $t \in [h,1]$, all $y \in \mathbb{R}$, all $z \in \mathbb{R}$, the following holds:*

$$\begin{aligned}&\left|G_{a,m,1,1}(t,x,y,z;h)\right|\\ &\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}\left(\min\left\{\left(\frac{h}{t}\right)^{1/2}, 2^m\sqrt{a}|\log(2^m\sqrt{a})|\right\} + h^{1/3}(2^m\sqrt{a})^{2/3}\right).\end{aligned}$$

Proof. Let us recall

$$\begin{aligned}G_{a,m,1,1} &= \frac{(-1)a^2\Lambda^{-1}}{(2\pi)^4h^4}(2^m\sqrt{a})^2\left(\frac{h}{t}\right)^{1/2}\int e^{i\Lambda Y\tilde{\eta}}\tilde{\eta}\psi_1(\tilde{\eta})\tilde{G}_{a,m,1,1}\,d\tilde{\eta},\\ \tilde{G}_{a,m,1,1} &= \sum_{\epsilon_1,\epsilon_2}\int e^{i\Lambda\tilde{\eta}\Phi_{1,m,\epsilon_1,\epsilon_2}}\Theta_{\epsilon_1,\epsilon_2}(1+a\tilde{\omega})^{-1/2}\,d\tilde{\omega}.\end{aligned}$$

The only difference with the case $N \geq 2$ is in the study of the phase $\Phi_{1,m,+,+}$ since in the case $N = 1$ we may have a critical point $\tilde{\omega}_c$ large. Let

$$\tilde{G}_{a,m,1,1,+,+} = \int e^{i\Lambda\Phi_{1,m,+,+}}\Theta_{+,+}(1+a\tilde{\omega})^{-1/2}\,d\tilde{\omega}, \tag{3.25}$$

with the phase function

$$\Phi_{1,m,+,+} = \tilde{T}\gamma_a(\tilde{\omega}) + \frac{2}{3}(\tilde{\omega}-X)^{3/2} + \frac{2}{3}(\tilde{\omega}-1)^{3/2} - \frac{4}{3}\tilde{\omega}^{3/2} + \frac{1}{\Lambda\tilde{\eta}}B(\Lambda\tilde{\omega}^{3/2}\tilde{\eta}),$$

and $\Theta_{+,+}$ is a classical symbol of order $-1/2$ with respect to $\tilde{\omega}$ which satisfies $|\tilde{\omega}^l\partial_{\tilde{\omega}}^l\Theta_{+,+}| \leq C_l\tilde{\omega}^{-1/2}$. Let $\chi_3(\tilde{\omega}) \in C_0^\infty(]\tilde{\omega}_1,\infty[)$ with $\tilde{\omega}_1$ large and set

$$J = \int e^{i\Lambda\Phi_{1,m,+,+}}\Theta_{+,+}\chi_3(\tilde{\omega})(1+a\tilde{\omega})^{-1/2}\,d\tilde{\omega}. \tag{3.26}$$

To prove the proposition, we just have to verify

$$a^{1/2}2^m\sqrt{a}|J| \le C\min\left\{\left(\frac{h}{t}\right)^{1/2}, 2^m\sqrt{a}|\log(2^m\sqrt{a})|\right\}. \tag{3.27}$$

We first observe that on the support of the integral defined in (3.26), one has $a\tilde{\omega} \le (2^m\sqrt{a})^{-2} = L$. Hence, we get

$$|J| \le C\left(1 + \int_1^{L/a} \frac{1}{\sqrt{x(1+ax)}}\,dx\right) = C\left(1 + a^{-1/2}\int_a^L \frac{1}{\sqrt{y(1+y)}}\,dy\right) \le Ca^{-1/2}\log L.$$

This implies

$$a^{1/2}2^m\sqrt{a}|J| \le C2^m\sqrt{a}|\log(2^m\sqrt{a})|.$$

We have

$$\partial_{\tilde{\omega}}\Phi_{1,m,+,+} = \frac{\tilde{T}}{2}(1+a\tilde{\omega})^{-1/2} - \frac{\tilde{\omega}^{-1/2}}{2}(1+X) + O_{C^\infty}(\tilde{\omega}^{-3/2}),$$
$$\partial^2_{\tilde{\omega}\tilde{\omega}}\Phi_{1,m,+,+} = \frac{-\tilde{T}a}{4}(1+a\tilde{\omega})^{-3/2} + \frac{\tilde{\omega}^{-3/2}}{4}(1+X) + O_{C^\infty}(\tilde{\omega}^{-5/2}).$$

At a large critical point we have $\tilde{T}^2 \sim (a+\tilde{\omega}_c^{-1})(1+X)^2$. Hence, $\tilde{T}$ is small and

$$\partial^2_{\tilde{\omega}\tilde{\omega}}\Phi_{1,+,+}(\tilde{\omega}_c) \sim \tilde{T}^3(1+a\tilde{\omega}_c)^{-5/2}.$$

Let $S = (\tilde{T}/(1+X))^2 - a$. Then, we have $S \sim \tilde{\omega}_c^{-1}$, and by stationary phase method, we get

$$|J| \le C(1+a\tilde{\omega}_c)^{3/4}\Lambda^{-1/2}\tilde{T}^{-3/2}S^{1/2}.$$

We have to take care in this section that $a\tilde{\omega}_c$ may be large.

- In the case $a\tilde{\omega}_c \le 1$, we have $S \sim \tilde{T}^2$, and therefore we obtain as before $|J| \le C\Lambda^{-1/2}\tilde{T}^{-1/2}$, which gives

$$a^{1/2}2^m\sqrt{a}|J| \le C\left(\frac{h}{t}\right)^{1/2}.$$

- In the case $a\tilde{\omega}_c \ge 1$, we must have $\tilde{T} \sim \sqrt{a}$, and $S = a\rho$ with $\rho > 0$ small. Hence, we get $|J| \le C\rho^{-1/4}a^{-1/4}\Lambda^{-1/2}$. This yields

$$a^{1/2}2^m\sqrt{a}|J| \le Ch^{1/2}\left((2^m\sqrt{a})^{1/2}a^{-1/2}\rho^{-1/4}\right).$$

Finally, we observe that we have

$$\sqrt{a} \sim \tilde{T} \sim ta^{-1/2}2^m\sqrt{a}(1+a\tilde{\omega}_c)^{1/2} \text{ implies } t \sim a(2^m\sqrt{a})^{-1}\rho^{1/2},$$

which gives $a^{1/2}2^m\sqrt{a}|J| \le C(h/t)^{1/2}$. The proof of Proposition 3.11 is complete. □

3.2.2. The analysis of $G_{a,m,N,2}$. The main result in this subsection is Proposition 3.12.

Proposition 3.12. *Let $\alpha < 2/3$ and $\tilde{h} = h/(2^m\sqrt{a})$. There exists C such that for all $h \in]0,h_0]$, all $a \in [\tilde{h}^\alpha, a_0]$, all $x \in [0,a]$, all $t \in]h,1]$, all $y \in \mathbb{R}$, all $z \in \mathbb{R}$, the following holds:*

$$\left|\sum_{1\le N\le C_0a^{-1/2}} G_{a,m,N,2}(t,x,y,z;h)\right| \le Ch^{-3}\left(\frac{h}{t}\right)^{1/2}a^{1/8}h^{1/4}(2^m\sqrt{a})^{3/4}.$$

Proof. Recall

$$G_{a,m,N,2}(t,x,y,z) = \frac{(-1)^N}{(2\pi)^4 h^4}\left(\frac{h}{t}\right)^{1/2} a^2(2^m\sqrt{a})^2 \int e^{i\Lambda\Phi_N} f_m \tilde{\eta}^2 \psi_1(\tilde{\eta})\chi_2(\tilde{\omega})\, d\tilde{s}\, d\tilde{\sigma}\, d\tilde{\omega}\, d\tilde{\eta} \tag{3.28}$$

with the phase function

$$\Phi_N(\tilde{s},\tilde{\sigma},\tilde{\omega},\tilde{\eta}) = \tilde{\eta}\Big[Y + \tilde{T}\gamma_a(\tilde{\omega}) + \frac{\tilde{s}^3}{3} + \tilde{s}(X-\tilde{\omega}) + \frac{\tilde{\sigma}^3}{3} + \tilde{\sigma}(1-\tilde{\omega}) - \frac{4}{3}N\tilde{\omega}^{3/2} + \frac{N}{\Lambda\tilde{\eta}}B\left(\tilde{\omega}^{3/2}\Lambda\tilde{\eta}\right)\Big].$$

To start with, we rewrite $G_{a,m,N,2}$ in the following form

$$G_{a,m,N,2} = \frac{(-1)^N}{(2\pi)^4 h^4}\left(\frac{h}{t}\right)^{1/2} a^2(2^m\sqrt{a})^2 \int e^{i\Lambda Y\tilde{\eta}}\tilde{\eta}^2\psi_1(\tilde{\eta})\tilde{G}_{a,m,N,2}\, d\tilde{\eta},$$

$$\tilde{G}_{a,m,N,2} = \int e^{i\Lambda\tilde{\eta}\tilde{\phi}_{N,m}}\chi_m\chi_2(\tilde{\omega})\, d\tilde{s}\, d\tilde{\sigma}\, d\tilde{\omega},$$

with the phase function

$$\tilde{\phi}_{N,m}(\tilde{s},\tilde{\sigma},\tilde{\omega}) = \tilde{T}\gamma_a(\tilde{\omega}) + \frac{\tilde{s}^3}{3} + \tilde{s}(X-\tilde{\omega}) + \frac{\tilde{\sigma}^3}{3} + \tilde{\sigma}(1-\tilde{\omega}) - \frac{4}{3}N\tilde{\omega}^{3/2} + \frac{N}{\Lambda\tilde{\eta}}B\left(\tilde{\omega}^{3/2}\Lambda\tilde{\eta}\right).$$

Now we can proceed as in the analysis of $G_{a,N,2}$ in section 2. More precisely, we apply the stationary phase method for $(\tilde{\omega},\tilde{\eta})$-integrations. It yields $\Lambda^{-1/2}$ and $(N\Lambda^{-1})^{-1/2}$, respectively. We have the following facts (see Section 2):

- Lemma 2.12: For $N \geq \Lambda^{1/3}$, there exists C such that

  $$\left|\int e^{i\Lambda\tilde{\psi}_{N,m}}\tilde{\chi}\, d\tilde{s}\, d\tilde{\sigma}\right| \leq C\Lambda^{-2/3} \quad\text{and}\quad \frac{1}{\sqrt{N}}\left|\int e^{i\Lambda\tilde{\psi}_{N,m}}\tilde{\chi}\, d\tilde{s}\, d\tilde{\sigma}\right| \leq C\Lambda^{-5/6},$$

  with $\tilde{\psi}_{N,m}$ is a perturbation of the phase function obtained from $\tilde{\phi}_{N,m}$ at the critical point $\tilde{\omega}_c$. Hence, we obtain the following estimates:
  - When $|\mathcal{N}_1(X,Y,T)| \leq C_0$, we get

    $$\left|\sum_{N\in\mathcal{N}_1} G_{a,m,N,2}\right| \leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}\left((2^m\sqrt{a})^2 h^{-1}a^2\Lambda^{-1/2}\Lambda^{-5/6}\right) \leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2} h^{1/3}(2^m\sqrt{a})^{2/3}.$$

  - When $|\mathcal{N}_1(X,Y,T)| \leq C_0\tilde{T}\Lambda^{-2}$, the $(N\Lambda^{-1})^{-1/2}$ factor contributes to the $\tilde{\eta}$-integration, and we get

    $$\left|\sum_{N\in\mathcal{N}_1} G_{a,m,N,2}\right| \leq \sum_{N\in\mathcal{N}_1} Ch^{-3}\left(\frac{h}{t}\right)^{1/2}\left((2^m\sqrt{a})^2 h^{-1}a^2 N^{-1}\Lambda^{-2/3}\right) \leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}\left((2^m\sqrt{a})^2 h^{-1}a^2\Lambda^{-8/3}\right) \leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2} h^{1/3}(2^m\sqrt{a})^{2/3}.$$

  Recall that we used $N \sim \tilde{T}$, $|\mathcal{N}_1| \leq C_0(1+\tilde{T}\Lambda^{-2})$ and $a \geq \tilde{h}^{2/3}$.

- Lemma 2.13: For $N \leq \Lambda^{1/3}$, we have
$$\frac{1}{\sqrt{N}}\left|\int e^{i\Lambda\tilde{\psi}_{N,m}}\tilde{\chi}\,d\tilde{s}\,d\tilde{\sigma}\right| \leq CN^{-1/4}\Lambda^{-3/4}.$$
Therefore, the estimate in this case is given by
$$\left|\sum_{N\in\mathcal{N}_1} G_{a,m,N,2}\right| \leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}\left((2^m\sqrt{a})^2h^{-1}a^2\Lambda^{-1/2}\Lambda^{-3/4}\right)$$
$$\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2} a^{1/8}h^{1/4}(2^m\sqrt{a})^{3/4}.$$

Hence putting these estimates together, we get
$$\left|\sum_{1\leq N\leq C_0a^{-1/2}} G_{a,m,N,2}\right| \leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}\left(h^{1/3}(2^m\sqrt{a})^{2/3} + a^{1/8}h^{1/4}(2^m\sqrt{a})^{3/4}\right).$$

We notice that $h^{1/3}(2^m\sqrt{a})^{2/3} \leq a^{1/8}h^{1/4}(2^m\sqrt{a})^{3/4}$ when $a \geq \left(\frac{h}{2^m\sqrt{a}}\right)^{2/3}$. The proof of the Proposition 3.12 is complete. □

Proof of Theorem 3.9. The desired estimate follows from Propositions 3.10, 3.11, 3.12. □

4. Dispersive estimates for $|\eta| \leq \epsilon_0\sqrt{a}$

In this section, we prove Theorem 1.5. We first compute the trajectories of the Hamiltonian flow for the operator P. At this frequency localization there is at most one reflection on the boundary. Moreover, we follow the techniques from Sections 2 and 3. It is particularly interesting that at this localization, $\mathcal{G}_{a,\epsilon_0}$ is an oscillatory integral with nondegenerate phase function; this is due to the geometric study of the associated Lagrangian which rules out the swallowtails regime for $|t| \leq 1$ if ϵ_0 is small enough.

4.1. Free space trajectories. Recall that the operator P is given by
$$P(t,x,y,z,\partial_t,\partial_x,\partial_y,\partial_z) = \partial_t^2 - (\partial_x^2 + (1+x)\partial_y^2 + \partial_z^2).$$
Now, we compute the trajectories in the free space for the associated symbol
$$p = \xi^2 + \zeta^2 + (1+x)\eta^2 - \tau^2.$$
To do so, we start at t_0, x_0, y_0, z_0 with ξ_0 close to 0, $\eta_0 = \theta\zeta_0, |\theta| \leq \epsilon_0\sqrt{a}, \zeta_0 \sim 1$, $\tau_0 = 1, \xi_0^2 + (1+x_0)\eta_0^2 + \zeta_0^2 = 1$. The Hamilton Jacobi equation is
$$\dot{x} = 2\xi; \quad \dot{y} = 2\eta(1+x); \quad \dot{z} = 2\zeta; \quad \dot{t} = -2\tau;$$
$$\dot{\xi} = -\eta^2; \quad \dot{\eta} = 0; \quad \dot{\zeta} = 0; \quad \dot{\tau} = 0.$$

This yields
$$\tau(s) = \tau_0; \quad \eta(s) = \eta_0; \quad \zeta(s) = \zeta_0; \quad \xi(s) = \xi_0 - \eta_0^2 s; \quad t(s) = t_0 - 2\tau_0 s;$$
$$z(s) = z_0 + 2\zeta_0 s; \quad x(s) = x_0 + 2\xi_0 s - \eta_0^2 s^2;$$
$$y(s) = y_0 + 2\eta_0\left((1+x_0)s + \xi_0 s^2 - \frac{1}{3}\eta_0^2 s^3\right).$$

In our case, we start at $t_0 = 0$, $x_0 = a$, $y_0 = z_0 = 0$; the system becomes

$$\begin{aligned}
(4.1)\qquad & \tau(s) = \tau_0; \quad \eta(s) = \eta_0; \quad \zeta(s) = \zeta_0; \quad \xi(s) = \xi_0 - \eta_0^2 s; \\
& t(s) = -2\tau_0 s; \quad z(s) = 2\zeta_0 s; \quad x(s) = a + 2\xi_0 s - \eta_0^2 s^2; \\
& y(s) = 2\eta_0 \left((1+a)s + \xi_0 s^2 - \frac{1}{3}\eta_0^2 s^3 \right).
\end{aligned}$$

The Lagrangian $\mathbf{\Lambda}_{a,\epsilon_0} \subset T^*(\mathbb{R}^4_{t,x,y,z})$: we have $\mathbf{\Lambda}_{a,\epsilon_0} \subset \{p = 0\}$ is parametrized by the system (4.1) with parameters $(s, \xi_0, \eta_0, \zeta_0)$ together with $(\xi_0^2 + (1+a)\eta_0^2 + \zeta_0^2)^{1/2} = \tau_0$; s is homogeneous of degree -1. Since $t(s) = -2\tau_0 s$ implies $s = -\frac{t}{2\tau_0}$, we replace it in the system (4.1). Then (4.1) becomes an homogeneous system parametrizing the Lagrangian Λ as follows:

$$\begin{aligned}
& x(t) = a - \frac{\xi_0}{\tau_0}t - \frac{\eta_0^2}{4\tau_0^2}t^2, \quad y(t) = \frac{\eta_0}{\tau_0}\left(-(1+a)t + \frac{\xi_0}{2\tau_0}t^2 + \frac{\eta_0^2}{12\tau_0^2}t^3 \right), \\
& z(t) = -\frac{\zeta_0}{\tau_0}t, \quad \xi(t) = \xi_0 + \frac{\eta_0^2}{2\tau_0}t, \quad \tau(t) = \tau_0 = 1.
\end{aligned}$$

The trajectories hit the boundary when $x(t) = 0$; that is,

$$\frac{\eta_0^2}{4}t^2 + t\xi_0 - a = 0.$$

This yields the time t_* when $x(t_*) = 0$:

$$t_*\xi_0 = a - \frac{\zeta_0^2\theta^2}{4}t_*^2 \sim a.$$

Our goal is to prove that at this frequency localization, the trajectories hit the boundary only once for a given fixed time $0 < t \leq 1$. To do this, suppose that the trajectory hits the boundary at $(x = 0, y_*, z_*, \xi_*, \eta_*, \zeta_0)$, which is given by the system (4.1). More precisely, $\xi_* = -(\xi_0 + \frac{\eta_0^2}{2}t_*)$ and we get

$$\xi(s) = \xi_* - \eta_0^2 s, \quad x(s) = 2\xi_* s - \eta_0^2 s^2, \quad t(s) = t_* - 2s.$$

Now, we assume that the trajectory, issuing from the point $(x = 0, y_*, z_*, \xi_*, \eta_*, \zeta_0)$, hits the boundary; that is, $x(t) = 0$, then $t\eta_0^2 = 2\xi_*$. This yields

$$\begin{aligned}
& t\theta^2\zeta_0^2 = -2\left(\xi_0 + \frac{\theta^2\zeta_0^2}{2}t_* \right) = -2\left(\xi_0 + \frac{\theta^2\zeta_0^2}{2}\Big(a - \frac{\theta^2\zeta_0^2}{4}t_*^2 \Big)/\xi_0 \right), \\
& |t\theta^2\zeta_0^2| \geq 4\sqrt{\frac{a\theta^2}{2}} \ \text{ implies } \ |t| \geq \frac{4\sqrt{a/2}}{\zeta_0^2|\theta|} \geq \frac{1}{\epsilon_0} \gg 1.
\end{aligned}$$

Therefore, we can only see at most one reflection on the boundary of the cylinder for $0 < t \leq 1$ at this frequency location.

4.2. Dispersive estimates for $|\eta| \leq \epsilon_0\sqrt{a}$. In this subsection, we are interested in obtaining dispersive estimates for $\mathcal{G}_{a,\epsilon_0}$. The main result of this section is the following.

Theorem 4.1. (Theorem 1.5) *There exists C such that for every $h \in]0,1]$, every $t \in [h,1]$, the following holds:*

$$(4.2)\qquad \|\mathcal{G}_{a,\epsilon_0}(t,x,y,z)\|_{L^\infty(x\leq a)} \leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2} \min\left\{ \left(\frac{h}{t}\right)^{1/2}, \sqrt{a}|\log(a)| \right\}.$$

We start as in Section 3. Recall that we have

$$\mathcal{G}_{a,\epsilon_0}(t,x,y,z) = \frac{1}{4\pi^2 h^2}\sum_{k\geq 1}\int e^{\frac{i}{h}\Phi_k}\sigma_k \,d\eta\,d\zeta, \tag{4.3}$$

where the phase Φ_k and the function σ_k are defined by

$$\Phi_k = y\eta + z\zeta + t(\eta^2+\zeta^2+\omega_k h^{2/3}\eta^{4/3})^{1/2},$$
$$\sigma_k = \psi_2(\eta/\sqrt{a})e_k(x,\eta/h)e_k(a,\eta/h)\chi_0(\zeta^2+\eta^2)\chi_1(\omega_k h^{2/3}\eta^{4/3})(1-\chi_1)(\varepsilon\omega_k),$$

with $\psi_2 \in C_0^\infty(]-2\epsilon_0, 2\epsilon_0[)$ equal to 1 on $[-\epsilon_0,\epsilon_0]$. We still use the notation $\mu^2 = \eta^2 + \omega_k h^{2/3}\eta^{4/3}$. Let $\chi_4 \in C_0^\infty]-1,1[$ with $\chi_4 = 1$ on $[-1/2,1/2]$. The following lemma (for $|\eta|\leq \epsilon_0\sqrt{a}$) is a refinement of Lemma 3.1.

Lemma 4.2. *Let*

$$\mathcal{J} = \frac{1}{4\pi^2 h^2}\sum_{k\geq 1}\int e^{\frac{i}{h}\Phi_k}\chi_4\Big(\frac{t\mu^2}{h}\Big)\sigma_k\,d\eta\,d\zeta.$$

There exists C such that

$$|\mathcal{J}| \leq Ch^{-3}\Big(\frac{h}{t}\Big)^{1/2}\min\Big\{\Big(\frac{h}{t}\Big)^{1/2}, \sqrt{a}\Big\}.$$

Proof. As in Lemma 3.1, and taking in account the cutoff $\psi_2(\eta/\sqrt{a})$, we get

$$|\mathcal{J}| \leq Ch^{-3}\Big(\frac{h}{t}\Big)\int_{-1}^{1}(1-x^2)^{1/2}\psi_2\big(x\sqrt{h/(ta)}\big)\,dx,$$

and the result follows from

$$\Big(\frac{h}{t}\Big)^{1/2}\int_{-1}^{1}(1-x^2)^{1/2}\psi_2\big(x\ \sqrt{h/(ta)}\big)\,dx \leq \min\Big\{\Big(\frac{h}{t}\Big)^{1/2},\sqrt{a}\Big\}. \qquad \square$$

By the proof of Lemma 3.3, in the case $\sqrt{a}\leq Mh$, we get the estimate

$$\Big|\mathcal{G}_{a,\epsilon_0}\Big| \leq C_M h^{-3}\Big(\frac{h}{t}\Big)^{1/2}\sqrt{a}|\log(\sqrt{a})|,$$

hence we may assume in what follows that $h^* = h/\sqrt{a}$ is a small parameter.

Using Lemmas 4.2 and 3.2, we are now reduced to the study of

$$J_{a,\epsilon_0} = \frac{1}{4\pi^2 h^2}\Big(\frac{h}{t}\Big)^{1/2}\sum_{k\geq 1}\int e^{\frac{i}{h}(y\eta + t\mu(1-\tilde{z}^2)^{1/2})}\tilde{\sigma}(\omega_k)(\eta/h)^{2/3}\frac{2\pi}{L'(\omega_k)}\frac{\psi_2(\eta/\sqrt{a})}{\mu}\,d\eta, \tag{4.4}$$

with $\tilde{\sigma}(\omega)$ defined by

$$\tilde{\sigma} = \sigma_0(z^*,\eta,\mu^2;\lambda)(1-\chi_4(\lambda))\chi_1(\omega h^{2/3}\eta^{4/3})(1-\chi_1)(\varepsilon\omega)$$
$$\times Ai((\eta/h)^{2/3}x-\omega)Ai((\eta/h)^{2/3}a-\omega),$$

where $\lambda = t\mu^2/h$. By Airy–Poisson summation formula, we have $J_{a,\epsilon_0} = \sum_{N\in\mathbb{Z}} J_N$ with

$$J_N = \frac{1}{4\pi^2 h^2}\Big(\frac{h}{t}\Big)^{1/2}\int e^{\frac{i}{h}(y\eta + t\mu(1-\tilde{z}^2)^{1/2})}\tilde{\sigma}(\omega)(\eta/h)^{2/3}\frac{\psi_2(\eta/\sqrt{a})}{\mu}e^{-iNL(\omega)}\,d\omega\,d\eta. \tag{4.5}$$

By the preceding paragraph, we know that it is sufficient to prove an estimate on $J_{-1}+J_0+J_1$. We will focus on J_1, since J_{-1} is similar and J_0 is simpler since it is the free wave. One has J_1 equal to:

$$J_1 = \frac{(h/t)^{1/2}}{(2\pi)^4 h^2}\, h^{-4/3} \int e^{\frac{i}{h}\phi_1} |\eta|^{2/3} \underline{\chi}(\omega,\eta,\mu^2,\lambda,h) \frac{\psi_2(\eta/\sqrt{a})}{\mu}\, ds\, d\sigma\, d\eta\, d\omega, \tag{4.6}$$

with the phase function

$$\phi_1 = y\eta + t\mu(1-\tilde{z}^2)^{1/2} + \frac{s^3}{3} + s(|\eta|^{2/3}x - \omega h^{2/3}) + \frac{\sigma^3}{3} + \sigma(|\eta|^{2/3}a - \omega h^{2/3}) - hL(\omega),$$

and symbol

$$\underline{\chi}(\omega,\eta,\mu^2,\lambda,h) = \sigma_0(z^*,\eta,\mu^2;\lambda)(1-\chi_4(\lambda))\chi_1(\omega h^{2/3}\eta^{4/3})(1-\chi_1)(\varepsilon\omega).$$

Recall that

$$L(\omega) = \frac{4}{3}\omega^{3/2} - B(\omega^{3/2}),\ \text{for } \omega \geq 1,$$

with

$$B(\omega) \sim_{1/\omega} \sum_{j\geq 1} b_j \omega^{-j},\ \ b_j \in \mathbb{R},\ \ b_1 > 0.$$

Lemma 4.3. *Let L be as in Section 2.2,*

$$L(\omega) = \pi + i\log\left(\frac{A_-(\omega)}{A_+(\omega)}\right).$$

Then for all $\omega \geq 0$, we have

$$L'(\omega) \geq 2\omega^{1/2}.$$

This lemma is useful in the geometric study of the canonical set and the Lagrangian submanifold associated to the phase function of J_1.

Proof of Theorem 4.1. To study J_1 in (4.6), we restrict the integral to $\eta > 0$ and we first make the change of variables $\omega = h^{-2/3}\eta^{2/3}\omega^*, s = \eta^{1/3}s^*, \sigma = \eta^{1/3}\sigma^*$, and we obtain, since $\mu = \eta(1+\omega^*)^{1/2}$

$$J_1 = \frac{(h/t)^{1/2}}{(2\pi h)^4} \int e^{\frac{i\eta}{h}(y+\tilde{\phi}_1)}\ \eta\ (1+\omega^*)^{-1/2} \underline{\chi}\psi_2(\eta/\sqrt{a})\, ds^*\, d\sigma^*\, d\omega^*\, d\eta, \tag{4.7}$$

with the phase function $\tilde{\phi}_1$ equal to

$$\begin{aligned}\tilde{\phi}_1 = t(1-\tilde{z}^2)^{1/2}(1+\omega^*)^{1/2} + \frac{{s^*}^3}{3} + s^*(x-\omega^*)\\ + \frac{{\sigma^*}^3}{3} + \sigma^*(a-\omega^*) - \frac{h}{\eta}L(\eta^{2/3}h^{-2/3}\omega^*).\end{aligned} \tag{4.8}$$

We have

$$\partial_{s^*}\tilde{\phi}_1 = {s^*}^2 + x - \omega^*, \quad \partial_{\sigma^*}\tilde{\phi}_1 = {\sigma^*}^2 + a - \omega^*,$$
$$\partial_{\omega^*}\tilde{\phi}_1 = \frac{t(1-\tilde{z}^2)^{1/2}(1+\omega^*)^{-1/2}}{2} - (s^*+\sigma^*) - \frac{h^{1/3}}{\eta^{1/3}}L'(\eta^{2/3}h^{-2/3}\omega^*).$$

Therefore, at a stationary point in s^*, σ^*, ω^* of $\tilde{\phi}_1$, we must have, using Lemma 4.3, $|s^*| \leq \sqrt{\omega^*}$ and $|\sigma^*| \leq \sqrt{(\omega^*-a)}$:

$$t(1-\tilde{z}^2)^{1/2}(1+\omega^*)^{-1/2} \geq 2\big(\sqrt{\omega^*} - \sqrt{(\omega^*-a)}\big).$$

Since $(1-\tilde z^2)\sim \mu=\eta(1+\omega^*)^{1/2}$, $t\leq 1$ and $\eta\leq \epsilon_0\sqrt{a}$ we obtain

$$\epsilon_0\sqrt{a}\geq \epsilon_0 t\sqrt{a}\geq 2\big(\sqrt{\omega^*}-\sqrt{(\omega^*-a)}\big),$$

and therefore, we may assume $\omega^*>Ma$ with M large if ϵ_0 is small. This proves that the swallowtail in the first reflection appears after a time $t>1$. Hence we are reduced to study what happen before the first occurence of a swallowtail. This case corresponds to a regime where there are no swallowtails and no cusps. We are reduce to estimate the oscillatory integral J:

$$J=\frac{(h/t)^{1/2}}{(2\pi h)^4}\int e^{\frac{i\eta}{h}(y+\tilde\phi_1)}\ \eta\ (1+\omega^*)^{-1/2}\underline{\chi}\psi_2(\eta/\sqrt{a})\kappa(\omega^*/(Ma))\,ds^*\,d\sigma^*\,d\omega^*\,d\eta, \tag{4.9}$$

where $\kappa\in C^\infty(]1/2,\infty[),0\leq\kappa\leq 1$, and κ equal to 1 on $[1,\infty[$. We then re-perform the $ds^*\,d\sigma^*$ integration using the definition of the Airy function, and we make the change of variables $\eta=\sqrt{a}\tilde\eta$ and $\omega^*=a\tilde\omega$. As in Proposition 3.11, we get with $\Lambda^*=a^{3/2}/h^*=a^2/h$,

$$\begin{aligned}
J&=\frac{a}{(2\pi)^4h^3}\Big(\frac{h}{t}\Big)^{1/2}\int e^{i\Lambda^*Y\tilde\eta}\ \tilde J\ \psi_2(\tilde\eta)\,d\tilde\eta,\\
\tilde J&=\sum_{\pm,\pm}\int e^{i\Lambda^*\tilde\eta\Phi_{\pm,\pm}}\ \Theta_{\pm,\pm}\ \kappa(\tilde\omega/M)\ (1+a\tilde\omega)^{-1/2}\,d\tilde\omega,\\
\Phi_{\pm,\pm}&=\tilde T\gamma_a(\tilde\omega)\pm\frac{2}{3}(\tilde\omega-X)^{3/2}\pm\frac{2}{3}(\tilde\omega-1)^{3/2}-\frac{4}{3}\tilde\omega^{3/2}+\frac{1}{\Lambda^*\tilde\eta}B(\Lambda^*\tilde\omega^{3/2}\tilde\eta),\\
\Theta_{\pm,\pm}&=(\tilde\omega-1)^{-1/4}(\tilde\omega-X)^{-1/4}\Psi_\pm(\Lambda^{*2/3}\tilde\eta^{2/3}(\tilde\omega-1))\Psi_\pm(\Lambda^{*2/3}\tilde\eta^{2/3}(\tilde\omega-X))\underline{\chi},
\end{aligned}$$

where $\Psi_\pm(\vartheta)\in C^\infty([0,+\infty[)$ are classical symbols of degree 0 in $\vartheta\to+\infty$. Therefore, it remains to prove

$$\Big|\int e^{i\Lambda^*Y\tilde\eta}\,a\tilde J\,\psi_2(\tilde\eta)\,d\tilde\eta\Big|\leq C\min\Big\{\Big(\frac{h}{t}\Big)^{1/2},\sqrt{a}|\log(a)|\Big\}. \tag{4.10}$$

Since on the support of f one has $\tilde\omega\leq\frac{1}{a^2\tilde\eta^2}$, we get

$$|\tilde J|\leq C\int_1^{\frac{1}{a^2\tilde\eta^2}}\tilde\omega^{-1/2}(1+a\tilde\omega)^{-1/2}\,d\tilde\omega\leq Ca^{-1/2}|\log(a\tilde\eta^2)|, \tag{4.11}$$

and this implies

$$\Big|\int e^{i\Lambda^*Y\tilde\eta}\,a\tilde J\,\psi_2(\tilde\eta)\,d\tilde\eta\Big|\leq C\sqrt{a}|\log(a)|.$$

Next, we have

$$\partial_{\tilde\omega}\tilde\Phi_{\pm,\pm}=\frac{\tilde T}{2}(1+a\tilde\omega)^{-1/2}\pm(\tilde\omega-X)^{1/2}\pm(\tilde\omega-1)^{1/2}-2\tilde\omega^{1/2}+O(\tilde\omega^{-1/2}),$$

and $(1+a\tilde\omega)^{-1/2}\tilde T\sim T\eta\leq\epsilon_0 t\leq 1$. Hence the phases $\Phi_{-,\pm},\Phi_{+,-}$ have no critical points $\tilde\omega\geq M/2$ large, and this implies in particular for their contribution J^* to J the estimate

$$|J^*|\leq C(\Lambda^*\tilde\eta)^{-1/2}=Ch^{1/2}\tilde\eta^{-1/2}a^{-1},$$

which implies

$$\Big|\int e^{i\Lambda^*Y\tilde\eta}\ a\tilde J^*\ \psi_2(\tilde\eta)\,d\tilde\eta\Big|\leq Ch^{1/2}\int\tilde\eta^{-1/2}\psi_2(\tilde\eta)\,d\tilde\eta\leq C(h/t)^{1/2}.$$

For the contribution to J of the phase $\Phi_{+,+}$, we use the same proof as the proof of Proposition 3.11. We thus get the estimate

$$a|\tilde{J}| \leq C(h/t)^{1/2}.$$

This concludes the proof of Theorem 1.5 □

Appendix

Airy function. Let $\vartheta > 0$. The Airy function Ai is defined as follows:

$$Ai(-\vartheta) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i(s^3/3 - s\vartheta)}\, ds.$$

It satisfies the Airy equation

$$Ai''(\vartheta) - \vartheta Ai(\vartheta) = 0 \tag{4.12}$$

Let $\nu = e^{2i\pi/3}$. Obviously, $\vartheta \mapsto Ai(\nu\vartheta)$ is a solution to (4.12). Any two of these three solutions $Ai(\vartheta), Ai(\nu\vartheta), Ai(\nu^2\vartheta)$ yield a basis of solutions to (4.12) and the linear relation between them is $\sum_{j\in\{0,1,2\}} \nu^j Ai(\nu^j\vartheta) = 0$. Then, it follows that $Ai(\vartheta) = -\nu Ai(\nu\vartheta) - \bar{\nu} Ai(\bar{\nu}\vartheta)$, which we rewrite as follows:

$$Ai(-\vartheta) = e^{-i\pi/3} Ai(e^{-i\pi/3}\vartheta) + e^{i\pi/3} Ai(e^{i\pi/3}\vartheta) = A_+(\vartheta) + A_-(\vartheta),$$

where we set $A_\pm(\vartheta) = e^{\mp i\pi/3} Ai(e^{\mp i\pi/3}\vartheta)$. Notice that $A_-(\vartheta) = \overline{A_+(\bar{\vartheta})}$. We also have the following asymptotic expansions

$$A_-(\vartheta) = \frac{1}{2\sqrt{\pi}\vartheta^{1/4}} e^{i\pi/4} e^{-\frac{2}{3}i\vartheta^{3/2}} \exp \Upsilon(\vartheta^{3/2}) = \frac{1}{\vartheta^{1/4}} e^{i\pi/4} e^{-\frac{2}{3}i\vartheta^{3/2}} \Psi_-(\vartheta),$$

with $\exp \Upsilon(\vartheta^{3/2}) \sim_{1/\vartheta} (1 + \sum_{l\geq 1} c_l \vartheta^{-3l/2}) \sim_{1/\vartheta} 2\sqrt{\pi}\Psi_-(\vartheta)$ as $\vartheta \to +\infty$ and the corresponding expansion for A_+, where we define $\Psi_+(\vartheta) = \bar{\Psi}_-(\bar{\vartheta})$. Moreover, we have

$$\frac{A_-(\vartheta)}{A_+(\vartheta)} = ie^{-\frac{4}{3}i\vartheta^{3/2}} e^{iB(\vartheta^{3/2})}, \quad \text{with } iB = \Upsilon - \bar{\Upsilon}.$$

Notice that for $\vartheta \in \mathbb{R}_+, B(\vartheta) \in \mathbb{R}$ and $B(\vartheta) \sim_{1/\vartheta} \sum_{j\geq 1} b_j \vartheta^{-j}$ for $\vartheta \to +\infty$ and $b_1 > 0$.

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Len Meas
Royal University of Phnom Penh
Department of Mathematics
Phnom Penh, Cambodia
meas.len@rupp.edu.kh