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# A fibered Tukia theorem for nilpotent Lie groups

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**Abstract.** We establish a Tukia-type theorem for uniform quasiconformal groups of a Carnot group. More generally we establish a fiber bundle version (or foliated version) of Tukia theorem for uniform quasiconformal groups of a nilpotent Lie group whose Lie algebra admits a diagonalizable derivation with positive eigenvalues. These results have applications to quasi-isometric rigidity of solvable groups [DFX].

**Lien nollanjuuriryhmien säikeittäinen Tukian lause**

**Tiivistelmä.** Todistamme Tukian-tyyppisen lauseen Carnot'n ryhmän tasaisesti kvasikonformisille ryhmille. Yleisemmin osoitamme, että Tukian lauseen säiekimppuversio pätee sellaisen Lien nollanjuuriryhmän tasaisesti kvasikonformisille ryhmille, jonka Lien algebralla on lävistäjämuodon kanssa yhtäpitävä derivaatta ja tämän ominaisarvot ovat positiivisia. Näillä tuloksilla on sovelluksia ratkeavien ryhmien kvasi-isometriseen jäykkyyteen [DFX].

## 1. Introduction

In this paper we study uniform quasiconformal groups of simply connected nilpotent Lie groups. The nilpotent Lie groups considered in this paper are those whose Lie algebras admit a diagonalizable derivation with positive eigenvalues. We start with the special case of Carnot groups.

Let $N$ be a Carnot group equipped with a left invariant Carnot–Carathéodory metric $d_{CC}$. Let $\hat{N} = N \cup \{\infty\}$ be the one-point compactification of $N$. A homeomorphism $f\colon \hat{N} \to \hat{N}$ is $K$-quasiconformal for some $K \geq 1$ if $f\colon (N\backslash\{f^{-1}(\infty)\}, d_{CC}) \to (N\backslash\{f(\infty)\}, d_{CC})$ is $K$-quasiconformal. A 1-quasiconformal map is also called conformal. A group $G$ of homeomorphisms of $\hat{N}$ is *uniformly quasiconformal* if there is some $K \geq 1$ such that every $g \in G$ is $K$-quasiconformal. If $G'$ is a conformal group of $\hat{N}$ and $f$ is a self quasiconformal map of $\hat{N}$, then $fG'f^{-1}$ is a uniform quasiconformal group of $\hat{N}$. A natural question is when a uniform quasiconformal group of $\hat{N}$ arises this way.

Let $G$ be a group of homeomorphisms of $\hat{N}$. Then $G$ also acts diagonally on the space of distinct triples

$$T(\hat{N}) = \Big\{(x, y, z)\colon x, y, z \in \hat{N},\ x \neq y,\ y \neq z,\ z \neq x\Big\}$$

of $\hat{N}$. Our first result is the following.

**Theorem 1.1.** *Let $N$ be a Carnot group. There is a left invariant Carnot–Carathéodory metric $d_0$ on $N$ with the following property. Let $G$ be a uniform quasiconformal group of $\hat{N}$. If the action of $G$ on the space of distinct triples of $\hat{N}$*

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*is co-compact, then there is some quasiconformal map $f\colon \hat{N} \to \hat{N}$ such that $fGf^{-1}$ consists of conformal maps with respect to $d_0$.*

The metric $d_0$ has the largest conformal group in the sense that the conformal group of any left invariant Carnot–Carathéodory metric is conjugated into the conformal group of $d_0$, see Definition 2.2 and Lemma 2.3. In general it is not possible to conjugate a uniform quasiconformal group into the conformal group of an arbitrary left invariant Carnot–Carathéodory metric, see Section 6 for an example.

Theorem 1.1 was first established by Tukia [T86] for $N = \mathbb{R}^n$ ($n \geq 2$) and was later generalized by Chow [Ch96] to the case when $N$ is an Heisenberg group. Before Tukia's result, Sullivan [S78] proved that, when $n = 2$ every uniform quasiconformal group (without the assumption of cocompactness of the induced action on the space of distinct triples) is quasiconformally conjugate to a conformal group.

Tukia's theorem has applications to rigidity of quasi-actions and quasi-isometric rigidity of finitely generated groups. So does Theorem 1.1. A *quasi-action* of a group $\Gamma$ on a metric space $X$ is an assignment $\gamma \mapsto G_\gamma$ where $G_\gamma$ is a self quasi-isometry of $X$ such that

(1) $G_\gamma$ is an $(L, A)$ quasi-isometry where $L$ and $A$ are uniform over all $\gamma \in \Gamma$.
(2) $G_{\gamma\eta}$ and $G_\gamma \circ G_\eta$ are bounded distance apart in the sup norm, uniformly over all $\gamma, \eta \in \Gamma$.
(3) $G_{Id}$ is bounded distance from the identity map on $X$.

A quasi-action is *cobounded* if there is a bounded set $S \subset X$ such that for any $x \in X$ there is $\gamma \in \Gamma$ such that $G_\gamma(x) \in S$.

The standard example of a cobounded quasi-action arises when $\Gamma$ is a group with a left invariant metric (for example, a finitely generated group with a word metric or a Lie group with a left invariant Riemannian metric) and $\phi\colon \Gamma \to X$ is a quasi-isometry with coarse inverse $\bar{\phi}$. Then $\gamma \mapsto \phi \circ L_\gamma \circ \bar{\phi}$ defines a cobounded quasi-action of $\Gamma$ on $X$, where $L_\gamma$ is the left translation of $\Gamma$ by $\gamma$. We note, however, that there exist "non-proper" cobounded quasi-actions and so these do not come from a quasi-isometry between a group and a metric space.

A quasi-action $\{G_\gamma | \gamma \in \Gamma\}$ of $\Gamma$ on a metric space $X$ is quasi-conjugate to a quasi-action $\{G'_\gamma | \gamma \in \Gamma\}$ of $\Gamma$ on another metric space $X'$ if there is a quasi-isometry $f\colon X \to X'$ and a constant $C > 0$ such that $d(G'_\gamma(f(x)), f(G_\gamma(x))) \leq C$ for all $x \in X$ and all $\gamma \in \Gamma$.

The relation between quasi-actions and uniform quasiconformal groups is through negative curvature. A self quasi-isometry of a Gromov hyperbolic space $X$ induces a self quasiconformal map of the Gromov boundary $\partial X$ of $X$ (equipped with a visual metric), and a quasi-action of a group $\Gamma$ on a Gromov hyperbolic space $X$ induces a uniform quasiconformal group action of $\Gamma$ on $\partial X$. A quasi-conjugation between quasi-actions of $\Gamma$ on two Gromov hyperbolic spaces $X$, $X'$ corresponds to a quasiconformal conjugation between the uniform quasiconformal actions of $\Gamma$ on $\partial X$ and $\partial X'$.

The standard Carnot group dilations of $N$ define an action of $\mathbb{R}$ on $N$. Let $S = N \rtimes \mathbb{R}$ be the associated solvable Lie group. Then $S$ with any left invariant Riemannian metric is a Gromov hyperbolic space [H74]. Its boundary is $\partial S \simeq \hat{N}$. Exploiting the above connection between quasi-isometries of $S$ and quasi-conformal maps of $\hat{N}$ we get the following corollary to Theorem 1.1.

**Corollary 1.2.** *Let $N$ be a Carnot group and $S = N \rtimes \mathbb{R}$ be the associated solvable Lie group. There is a left invariant Riemannian metric $g_0$ on $S$ with the following property. Let $G$ be a group that quasi-acts on $S$. If the quasi-action is* *co-bounded, then the quasi-action is quasi-conjugate to an isometric action of $G$ on $(S, g_0)$.*

We next turn to uniform quasiconformal groups on more general nilpotent Lie groups. Let $N$ be a simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}$ and $D$ a derivation of $\mathfrak{n}$. We say $(N, D)$ is a diagonal Heintze pair if $D$ has positive eigenvalues and is diagonalizable over $\mathbb{R}$. Let $(N, D)$ be a diagonal Heintze pair. A distance $d$ on $N$ is called $D$-homogeneous if it is left invariant, induces the manifold topology on $N$ and such that $d(e^{tD}x, e^{tD}y) = e^t d(x, y)$ for all $x, y \in N$ and $t \in \mathbb{R}$, where $\{e^{tD} | t \in \mathbb{R}\}$ denotes the automorphisms of $N$ generated by the derivation $D$. By Theorem 2 of [HSi90], $D$-homogeneous distances exist on $N$. It is easy to see that any two $D$-homogeneous distances on $N$ are biLipschitz equivalent. We will always equip $N$ with a $D$-homogeneous distance. Hence it makes sense to speak of a biLipschitz map of $N$ without specifying the $D$-homogeneous distance.

Let $(N, D)$ be a diagonal Heintze pair. Then there is a sequence of $D$-invariant Lie sub-algebras $\{0\} = \mathfrak{n}_0 \subset \mathfrak{n}_1 \subset \cdots \subset \mathfrak{n}_s = \mathfrak{n}$ with the following properties: each $\mathfrak{n}_{i-1}$ is an ideal of $\mathfrak{n}_i$ with the quotient $\mathfrak{n}_i/\mathfrak{n}_{i-1}$ a Carnot Lie algebra; $D$ induces a derivation $\bar{D}\colon \mathfrak{n}_i/\mathfrak{n}_{i-1} \to \mathfrak{n}_i/\mathfrak{n}_{i-1}$ which is a multiple of the Carnot derivation of $\mathfrak{n}_i/\mathfrak{n}_{i-1}$. See Section 5.1 for more details. Let $N_i$ be the connected Lie subgroup of $N$ with Lie algebra $\mathfrak{n}_i$. Then $N/N_i$ is a homogeneous manifold and the natural map $\pi_i\colon N/N_{i-1} \to N/N_i$ is a fiber bundle with fiber the Carnot group $N_i/N_{i-1}$. We call the sequence of subgroups $0 = N_0 < N_1 < \cdots < N_s = N$ the *preserved subgroup sequence*.

Let $d$ be a $D$-homogeneous distance on $N$. In general $d$ does not induce any metric on the homogeneous space $N/N_i$ when $N_i$ is not normal in $N$. Nonetheless, it induces a metric on the fibers $N_i/N_{i-1}$ of $\pi_i\colon N/N_{i-1} \to N/N_i$. Furthermore, every biLipschitz map $F$ of $N$ permutes the cosets of $N_i$ for each $i$. Hence $F$ induces a map $F_i\colon N/N_i \to N/N_i$ and a bundle map of $\pi_i\colon N/N_{i-1} \to N/N_i$. The restriction of $F_{i-1}$ to the fibers of $\pi_i$ are biLipschitz maps of the Carnot group $N_i/N_{i-1}$ in the following sense. For each $p \in N$, let $F_p = L_{F(p)^{-1}} \circ F \circ L_p$, where $L_x$ denotes the left translation of $N$ by $x$. Notice that the map $(F_p)_{i-1}\colon N/N_{i-1} \to N/N_{i-1}$ satisfies $(F_p)_{i-1}(N_i/N_{i-1}) = N_i/N_{i-1}$. The statement above simply means $(F_p)_{i-1}|_{N_i/N_{i-1}}\colon N_i/N_{i-1} \to N_i/N_{i-1}$ is biLipschitz with respect to any left invariant Carnot–Carathéodory metric on $N_i/N_{i-1}$. See Section 5.1 for more details.

A group $\Gamma$ of homeomorphisms of a metric space $X$ is a *uniform quasisimilarity group* if there is a constant $K \geq 1$ such that each $\gamma \in \Gamma$ satisfies $(C_\gamma/K)d(x, y) \leq d(\gamma(x), \gamma(y)) \leq C_\gamma K d(x, y)$ for some $C_\gamma > 0$ and all $x, y \in X$. A bijection $f\colon X \to X$ is a *similarity* if there is some $L > 0$ such that $d(f(x), f(y)) = Ld(x, y)$ for all $x, y \in X$.

**Theorem 1.3.** *Let $(N, D)$ be a diagonal Heintze pair and $\Gamma$ be a uniform quasisimilarity group of $(N, D)$ that acts cocompactly on the space of distinct pairs of $N$ (or equivalently $\Gamma$ a group that quasi-acts coboundedly on $S = N \rtimes_D \mathbb{R}$). Let $I = \{i | 1 \leq i \leq s,\ \dim(N_i/N_{i-1}) \geq 2\}$. Then there exists a biLipschitz map $F_0\colon N \to N$ and a left invariant Carnot–Carathéodory metric $d_i$ on $N_i/N_{i-1}$ for each $i \in I$ such that for each $p \in N$ and each $g \in F_0\Gamma F_0^{-1}$, the map $(g_p)_{i-1}|_{N_i/N_{i-1}}\colon (N_i/N_{i-1}, d_i) \to (N_i/N_{i-1}, d_i)$ is a similarity.*

Ideally one would like to conjugate the group $\Gamma$ in Theorem 1.3 into a group of similarities of $N$ with respect to some $D$-homogeneous distance. But this question is still open in general. A positive answer was given in [DFX] in the case when the preserved subgroup sequence has only two terms $0 < N_1 < N$. Its proof is much more involved algebraically and uses Theorem 1.3 as a crucial ingredient.

When $s \geq 2$, [CP17] implies that every quasiconformal map of $\hat{N} = N \cup \{\infty\}$ fixes $\infty$ and restricts to a biLipschitz map of $N$. From this it is easy to see that a uniform quasiconformal group of $\hat{N}$ restricts to a uniform quasisimilarity group of $N$. Therefore there is no loss of generality in Theorem 1.3 in considering a uniform quasisimilarity group of $N$ instead of a uniform quasiconformal group of $\hat{N}$.

When $s = 1$, then $N$ is Carnot and depending on $N$ not all quasiconformal maps of $\hat{N}$ are necessarily biLipschitz. In this case, Theorem 1.3 simply asserts that if the quasiconformal group from Theorem 1.1 happens to consist of biLipschitz maps then the conjugating map can be chosen to be biLipschitz.

Theorem 1.3 was proved in [Dy10] in the case when $N$ is a Euclidean group. In the case $N = \mathbb{R}$, this result can be found in the appendix of [FM99] and no additional assumptions other than uniformity are needed on the group.

The case $N = \mathbb{R}$ is used for the last step in the proof of quasi-isometric rigidity of SOL by Eskin–Fisher–Whyte [EFW12], [EFW13], while the cases covered in [Dy10] are used to prove quasi-isometric rigidity of higher rank generalizations of SOL by Peng in [P11a], [P11b]. Similarly Theorem 1.3 played a crucial role in the proof of quasi-isometric rigidity of a class of solvable groups [DFX].

The group $SOL = \mathbb{R}^2 \rtimes \mathbb{R}$ where the action of $\mathbb{R}$ scales by $e^t$ on the first coordinate and by $e^{-t}$ along the second. This action gives rise to two foliations by hyperbolic planes (which we view as $\mathbb{R} \rtimes \mathbb{R}$). More generally a *SOL-like group* is a semi-direct product $(N_1 \times N_2) \rtimes \mathbb{R}$, where $N_i$ is a simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}_i$, and the action of $\mathbb{R}$ on $N_1 \times N_2$ is generated by a derivation $D = (-D_1, D_2)$ of $\mathfrak{n}_1 \times \mathfrak{n}_2$ and $D_i$ is a derivation of $\mathfrak{n}_i$ whose eigenvalues have positive real part. A SOL-like group is foliated by two families of negatively curved solvable Lie groups $N_i \rtimes_{D_i} \mathbb{R}$ (these are called Heintze groups and they are exactly those solvable Lie groups admitting left invariant Riemannian metrics with negative sectional curvature). The quasi-isometric rigidity result proved in [DFX] are for SOL-like groups where $(N_i, D_i)$ is either of Carnot type or has a preserved subgroup sequence of length two. We remark that the quasi-isometric rigidity for non-unimodular SOL-like groups where $(N_i, D_i)$ is of Carnot type is included in Theorem C, [Fe22] and that proof does not use Tukia type theorems.

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## 2. Preliminaries

**2.1. Carnot groups.** Let $N$ be a Carnot group with Lie algebra $\mathfrak{n} = V_1 \oplus \cdots \oplus V_k$. The exponential map $\exp\colon \mathfrak{n} \to N$ is a diffeomorphism. We will identify $N$ and $\mathfrak{n}$ via the exponential map. For any $t > 0$, the Carnot group dilation $\delta_t\colon \mathfrak{n} \to \mathfrak{n}$ is given by $\delta_t(\sum_{j=1}^k x_j) = \sum_{j=1}^k t^j x_j$, with $x_j \in V_j$. They are similarities w.r.t. any left invariant Carnot–Carathéodory metric $d$: $d(\delta_t(x), \delta_t(y)) = t\, d(x, y)$ for any $x, y \in \mathfrak{n}$. The determinant of $\delta_t\colon \mathfrak{n} \to \mathfrak{n}$ is $t^Q$, where $Q = \sum_{j=1}^k j m_j$ is the homogeneous dimension of $N$.

Let $\{e_{jl}\colon 1 \leq l \leq m_j\}$ be a basis for $V_j$, $1 \leq j \leq k$. Let $n = \dim \mathfrak{n}$. Then the map $\mathbb{R}^n \to N$ given by $(x_{jl}) \to \exp(\sum_{j,l} x_{jl} e_{jl})$ is a diffeomorphism and the push-forward of the Lebesgue measure under this map is a Haar measure on $N$. We shall use the notation $|A|$ for the Haar measure of a subset $A \subset N$. Define a function $\rho\colon \mathfrak{n} \to [0, \infty)$ by

$$\rho\left(\sum_{j,l} x_{jl} e_{jl}\right) = \sum_{j=1}^{k} \sum_{l=1}^{m_j} |x_{jl}|^{\frac{1}{j}}.$$

Then $\rho(\delta_t(x)) = t\,\rho(x)$ for all $x \in \mathfrak{n}$. There is an associated "distance" $d_\rho$ given by $d_\rho(x, y) = \rho((-x) * y)$. It is easy to see that $d_\rho$ is left invariant and satisfies $d_\rho(\delta_t(x), \delta_t(y)) = t\, d_\rho(x, y)$. Hence $d_\rho$ is biLipschitz equivalent with any left invariant Carnot–Carathéodory metric $d$: there exists some constants $L \geq 1$ such that $d(x, y)/L \leq d_\rho(x, y) \leq Ld(x, y)$ for all $x, y \in N$.

Let $V_1$ be equipped with an inner product and we may assume $\{e_{1l}\colon 1 \leq l \leq m_1\}$ is an orthonormal basis for $V_1$. Let $X_{1l}$ be the left invariant vector field on $N$ determined by $e_{1l}$. For any smooth function $u\colon U \to \mathbb{R}$ defined on an open subset of $N$, define the horizontal gradient $\nabla u$ of $u$ by:

$$\nabla u = \sum_{l=1}^{m_1} (X_{1l} u) X_{1l}.$$

The length of $\nabla u$ is:

$$|\nabla u| = \sqrt{\sum_{l=1}^{m_1} (X_{1l} u)^2}.$$

Let $d$ be the left invariant Carnot–Carathéodory metric on $N$ determined by the inner product on $V_1$, let $U, V \subset N$ be open subsets, and $f\colon U \to V$ a homeomorphism. For $x \in U$ and $r > 0$ with $B(x, r) \subset U$, let

$$L_f(x, r) = \sup_{d(y,x) \leq r} d(f(x), f(y)), \quad l_f(x, r) = \inf_{d(y,x) \geq r} d(f(x), f(y)).$$

Define

$$K(f, x) = \limsup_{r \to 0} \frac{L_f(x, r)}{l_f(x, r)}.$$

We call $f$ $K$-quasiconformal for some $K \geq 1$ if $K(f, x) \leq K$ for a.e. $x \in U$.

Let $U, V \subset N$ be open subsets, and $f\colon U \to V$ a quasiconformal map. By [P89], $f$ is Pansu differentiable a.e. and the Pansu differential $Df(x)$ is a graded automorphism for a.e. $x \in U$. Recall that an automorphism $A\colon \mathfrak{n} \to \mathfrak{n}$ is graded if it commutes with $\delta_t$ for all $t > 0$; equivalently $A$ is graded if $A(V_j) = V_j$ for each $1 \leq j \leq k$. Furthermore, $f$ is absolutely continuous w.r.t. the Haar measure, see [P89]. For any $x \in U$, define

$$L_f(x) = \limsup_{y \to x} \frac{d(f(x), f(y))}{d(x, y)}, \quad l_f(x) = \liminf_{y \to x} \frac{d(f(x), f(y))}{d(x, y)}.$$

By Lemma 3.3 in [CC06], for a.e. $x \in U$, we have

$$L_f(x) = \max\{Df(x)X\colon X \in V_1,\ |X| = 1\},$$
$$l_f(x) = \min\{Df(x)(X)\colon X \in V_1,\ |X| = 1\},$$

and $K(f,x) = \frac{L_f(x)}{l_f(x)}$. For any $x \in U$, the volume derivative of $f$ at $x$ is:

$$f'(x) = \lim_{r \to 0} \frac{|f(B(x,r))|}{|B(x,r)|},$$

which exists a.e. and is a.e. finite. By (4.1) in [CC06], $l_f(x)^Q \leq f'(x) \leq L_f(x)^Q$ for a.e. $x \in U$, where $Q$ is the homogeneous dimension of $N$.

**2.2. Homogeneous distances on nilpotent Lie groups.** Let $(N, D)$ be a diagonal Heintze pair. Let $0 < \lambda_1 < \cdots < \lambda_r$ be the distinct eigenvalues of $D$ and $\mathfrak{n} = \oplus_j V_{\lambda_j}$ be the decomposition of $\mathfrak{n}$ into the direct sum of eigenspaces of $D$. An inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{n}$ is called a $D$-inner product if the eigenspaces corresponding to distinct eigenvalues are perpendicular with respect to $\langle \cdot, \cdot \rangle$. By the construction in Theorem 2 of [HSi90], given any $D$-inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{n}$, there is a $D$-homogeneous distance $d$ on $N$ such that $d(0,x) = \langle x, x \rangle^{\frac{1}{2\lambda_j}}$ for $x \in V_{\lambda_j}$.

For computational purposes, we also define a function $\rho$ that is biLipschitz equivalent to a $D$-homogeneous distance $d$. For any $D$-inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{n}$ define a "norm" on $\mathfrak{n}$ by

$$||v|| = \sum_i |v_i|^{\frac{1}{\lambda_i}},$$

where $v = \sum_i v_i$ with $v_i \in V_{\lambda_i}$. Then define $\rho$ by $\rho(x,y) = ||x^{-1} * y||$. We identify $\mathfrak{n}$ and $N$. Clearly $\rho$ is left invariant, induces the manifold topology and satisfies $\rho(e^{tD}x, e^{tD}y) = e^t \rho(x,y)$ for all $x, y \in \mathfrak{n}$ and $t \in \mathbb{R}$. It follows that for any $D$-homogeneous distance $d$ on $\mathfrak{n}$, there is a constant $L \geq 1$ such that $d(x,y)/L \leq \rho(x,y) \leq L \cdot d(x,y)$ for all $x, y \in \mathfrak{n}$. The explicit formula for $\rho$ will make the calculations much easier.

**Lemma 2.1.** *Let $\phi$ be an automorphism of $N$. Then $\phi$ is biLipschitz if and only if $d\phi$ is "layer preserving"; that is, $d\phi(V_{\lambda_j}) = V_{\lambda_j}$ for each $j$.*

*Proof.* First suppose $\phi$ is biLipschitz. Let $0 \neq v \in V_{\lambda_j}$ and write $d\phi(v) = \sum_i x_i$ with $x_i \in V_{\lambda_i}$. Then $d\phi(tv) = \sum_i t x_i$. We have

$$\rho(0, tv) = |v|^{\frac{1}{\lambda_j}} |t|^{\frac{1}{\lambda_j}}$$

and

$$\rho(0, d\phi(tv)) = \sum_i |x_i|^{\frac{1}{\lambda_i}} |t|^{\frac{1}{\lambda_i}}.$$

The biLipschitz condition implies $x_i = 0$ when $i \neq j$ by letting $t \to \infty$ or $t \to 0$.

Conversely assume $d\phi$ is layer preserving. Then there is some constant $C \geq 1$ such that

$$|v|/C \leq |d\phi(v)| \leq C|v| \tag{1}$$

for all $v \in V_{\lambda_j}$, $\forall j$. Now let $v \in \mathfrak{n}$. Write $v = \sum_j v_j$ with $v_j \in V_{\lambda_j}$. Then $d\phi(v) = \sum_j d\phi(v_j)$. We have $\rho(0,v) = \sum_j |v_j|^{\frac{1}{\lambda_j}}$ and $\rho(0, d\phi(v)) = \sum_j |d\phi(v_j)|^{\frac{1}{\lambda_j}}$. Now the claim follows from (1). □

An automorphism $\phi$ of $N$ is called graded if it satisfies the condition in Lemma 2.1. We denote by $\mathrm{Aut}_g(N)$ the group of graded automorphisms of $N$.

For any $D$-homogeneous distance $d$ on $N$, let $\mathrm{Sim}(N,d)$ be the group of similarities of $(N,d)$. By Theorem 1.2 in [KLD17] and Lemma 2.1, $\mathrm{Sim}(N,d)$ has the structure $\mathrm{Sim}(N,d) = N \rtimes (\mathbb{R} \times K)$, where $\mathbb{R}$ acts on $N$ by the automorphisms $\{e^{tD} \mid t \in \mathbb{R}\}$ and $K \subset \mathrm{Aut}_g(N)$ is a compact Lie subgroup. Given two $D$-homogeneous distances

$d_1$, $d_2$ on $N$, although $(N, d_1)$ and $(N, d_2)$ are biLipschitz, a similarity of $(N, d_1)$ in general is not a similarity of $(N, d_2)$ and so their associated similarity groups can be very different; see Section 6 for an example in the case of Carnot groups. The following is a notion of $D$-homogeneous distance with the largest similarity group. It is similar to Definition 0.2 in [GJ19].

**Definition 2.2.** Let $(N, D)$ be a diagonal Heintze pair. A $D$-homogeneous distance $d_0$ on $N$ is maximally symmetric (with respect to similarities) if for any $D$-homogeneous distance $d$ on $N$, there is a biLipschitz automorphism $\phi$ of $N$ such that $\phi \mathrm{Sim}(N, d)\phi^{-1} \subset \mathrm{Sim}(N, d_0)$.

**Lemma 2.3.** *Let $(N, D)$ be a diagonal Heintze pair. Then $N$ admits a maximally symmetric $D$-homogeneous distance.*

*Proof.* Since $N$ is simply connected, $\mathrm{Aut}_g(N)$ can be identified with the group of graded automorphisms $\mathrm{Aut}_g(\mathfrak{n})$ of $\mathfrak{n}$. It is easy to see that $\mathrm{Aut}_g(\mathfrak{n})$ is a real algebraic variety and so has only a finite number of connected components by Whitney's theorem [Wh57]. Therefore $\mathrm{Aut}_g(N)$ is a Lie group with finitely many components.

Let $K_0$ be a maximal compact subgroup of $\mathrm{Aut}_g(N)$. Recall $\mathfrak{n} = \oplus_j V_{\lambda_j}$. For each $j$ let $\langle \cdot, \cdot \rangle_j$ be a $K_0$-invariant inner product on $V_{\lambda_j}$. Let $\langle \cdot, \cdot \rangle$ be the inner product on $\mathfrak{n}$ that agrees with $\langle \cdot, \cdot \rangle_j$ on $V_{\lambda_j}$ such that $V_{\lambda_i}$ and $V_{\lambda_j}$ are perpendicular to each other for $i \neq j$. Let $d$ be a $D$-homogeneous distance on $N$ associated to this inner product. Although $d\phi$ is a linear isometry of $(\mathfrak{n}, \langle , \rangle)$ for any $\phi \in K_0$, it is not clear that $\phi$ is an isometry of $(N, d)$. Let $m$ be the normalized Haar measure on $K_0$. Define a new distance $d_0$ on $N$ by $d_0(x, y) = \int_{K_0} d(k(x), k(y))dm(k)$. Now it is easy to check that $d_0$ is a $K_0$-invariant $D$-homogeneous distance on $N$ associated to $\langle , \rangle$ and $\mathrm{Sim}(N, d_0) = N \rtimes (\mathbb{R} \times K_0)$.

Now let $d$ be an arbitrary $D$-homogeneous distance on $N$. As observed above, $\mathrm{Sim}(N, d) = N \rtimes (\mathbb{R} \times K)$, where $K$ is a compact subgroup of $\mathrm{Aut}_g(N)$. Since $\mathrm{Aut}_g(N)$ has only a finite number of components, there is some $\phi \in \mathrm{Aut}_g(N)$ such that $\phi K \phi^{-1} \subset K_0$. Since $N$ is normal in $N \rtimes \mathrm{Aut}(N)$ and $\phi$ is graded we have $\phi \mathrm{Sim}(N, d)\phi^{-1} \subset \mathrm{Sim}(N, d_0)$. □

Let $G$ be a connected Lie group with a left invariant distance $d$ that induces the manifold topology, and $H$ a closed normal subgroup of $G$. We define a distance on $G/H$ by $\bar{d}(xH, yH) = \inf\{d(xh_1, yh_2) | h_1, h_2 \in H\}$. Then $\bar{d}$ is a left invariant distance on $G/H$ that induces the manifold topology and the quotient map $(G, d) \to (G/H, \bar{d})$ is 1-Lipschitz. Since $H$ is normal, we have $\bar{d}(xH, yH) = d(xh_1, yH) = d(yh_2, xH) = d_H(xH, yH)$ for any $h_1, h_2 \in H$, where $d_H$ denotes the Hausdorff distance. If $F$ is a biLipschitz map of $(G, d)$ that permutes the cosets of $H$, then $F$ induces a biLipschitz map $\bar{F}\colon (G/H, \bar{d}) \to (G/H, \bar{d})$ with the same biLipschitz constant as $F$.

Let $(N, D)$ be a diagonal Heintze pair and $d$ a $D$-homogeneous distance on $N$. Assume $\mathfrak{w}$ is an ideal of $\mathfrak{n}$ such that $D(\mathfrak{w}) \subset \mathfrak{w}$. Then $D$ induces a derivation $\bar{D}$ of $\mathfrak{n}/\mathfrak{w}$ and $(N/W, \bar{D})$ is also a diagonal Heintze pair, where $W$ is the Lie subgroup of $N$ with Lie algebra $\mathfrak{w}$. In this case, the distance $\bar{d}$ on $N/W$ induced by $d$ is a $\bar{D}$-homogeneous distance.

**2.3. Homogeneous manifolds with negative curvature.** Let $N$ be a Carnot group with Lie algebra $\mathfrak{n} = V_1 \oplus \cdots \oplus V_k$. The standard dilations $\delta_t$ of $N$ define an action of $\mathbb{R}$ on $N = \mathfrak{n}$:

$$t \cdot x = \delta_{e^t}(x) \quad \text{for } t \in \mathbb{R} \text{ and } x \in \mathfrak{n}.$$

Let $S = N \rtimes \mathbb{R}$ be the associated semi-direct product. Then $S$ is a solvable Lie group. By [H74] $S$ admits a left invariant Riemannian metric with negative sectional curvature. For $x_0 \in N$, the path $c_{x_0}\colon \mathbb{R} \to S$, $c_{x_0}(t) = (x_0, t)$, is a geodesic in $S$. We call $c_{x_0}$ a vertical geodesic. All vertical geodesics are asymptotic as $t \to \infty$, and so they determine a point $\infty$ in the ideal boundary. If $t \to -\infty$ all vertical geodesics diverge from one another. We call such geodesics downward oriented. Every geodesic ray in $S$ is asymptotic to either an upward oriented vertical geodesic or a downward oriented vertical geodesic. It follows that the ideal boundary $\partial S$ of $S$ can be identified with $\hat{N} = N \cup \{\infty\}$, where points in $N$ correspond to downward oriented vertical geodesics.

**2.4. Sphericalization of metrics and measures.** Let $(X, d)$ be an unbounded metric space and $p \in X$ a base point. Let $\infty$ be a point not in $X$ and set $\hat{X} = X \cup \{\infty\}$. The sphericalized metric $\hat{d}_p$ of $d$ relative to the base point $p$ is a metric on $\hat{X}$ satisfying:

$$\frac{d(x,y)}{4(1+d(p,x))(1+d(p,y))} \le \hat{d}_p(x,y) \le \frac{d(x,y)}{(1+d(p,x))(1+d(p,y))} \quad \text{for } x, y \in X,$$

and $\hat{d}_p(x,\infty) = \frac{1}{(1+d(p,x))}$. Furthermore, a proof similar to that of Proposition 4.1 in [BHX08] shows that the identity map $\mathrm{id}\colon (X,d) \to (X,\hat{d}_p)$ is 1-quasiconformal. It follows that a homeomorphism $f\colon (\hat{X},\hat{d}_p) \to (\hat{X},\hat{d}_p)$ is quasiconformal (1-quasiconformal) iff the restriction $f\colon (X\backslash\{f^{-1}(\infty)\}, d) \to (X\backslash\{f(\infty)\}, d)$ is quasiconformal (1-quasiconformal). We shall use this observation when we study quasiconformal maps of $\hat{N}$.

Let $(X, d, \nu)$ be a metric measure space, with $\nu$ a Borel regular measure. The metric measure space $(X, d, \nu)$ is $\alpha$-Alhfors regular for some $\alpha > 0$ if there is some constant $C > 0$ such that $r^\alpha/C \le \nu(B(x,r)) \le Cr^\alpha$ for all balls $B(x,r)$ with radius $0 < r \le \mathrm{diam}(X,d)$. Now let $(X, d, \nu)$ be a $\alpha$-Alhfors regular metric measure space with $(X, d)$ unbounded and $p \in X$ a base point. Li and Shanmugalingam [LS15] defined a measure $\hat{\nu}_p$ on $(\hat{X},\hat{d}_p)$ which is also $\alpha$-regular and showed that $(X, d, \nu)$ supports a $p$-Poincare inequality if and only if $(\hat{X},\hat{d}_p,\hat{\nu}_p)$ supports a $p$-Poincare inequality (see Theorem 1.1 in [LS15]) . Furthermore, the two measures $\nu$ and $\hat{\nu}_p$ are comparable on any bounded subset of $(X, d)$; that is, for any bounded subset $A \subset (X, d)$, there is a constant $C \ge 1$ such that $\nu(E)/C \le \hat{\nu}_p(E) \le C\nu(E)$ for any $E \subset A$. We shall use this in the case of a Carnot group $N$ equipped with a left invariant Carnot–Carathéodory metric and the Lebesgue measure.

## 3. Lemmas on quasiconformal maps of Carnot groups

In this section we collect some results on quasiconformal maps of Carnot groups. These will be used in Section 4.2 for the proof of Theorem 1.1.

Let $N$ be a Carnot group equipped with a left invariant Carnot–Carathéodory metric and $\hat{N} = N \cup \{\infty\}$ its one-point compactification. A ring in $\hat{N}$ is a connected open subset $R \subset \hat{N}$ whose complement has two connected components. We always work with rings $R$ satisfying $\infty \notin \bar{R}$. Let $C_0$ and $C_1$ be the two components of $\partial R$. An admissible function for $R$ is a $C^\infty$ function $u\colon N \to \mathbb{R}$ with $u|_{C_0} = 0$ and $u|_{C_1} = 1$. The conformal capacity of $R$ is:

$$C(R) = \inf_u \int_N |\nabla u|^Q \, dx,$$

where $Q$ is the homogeneous dimension of $N$, $u$ ranges over all admissible functions for $R$, and the integral is with respect to the Lebesgue measure. The above infimum remains the same if we enlarge the class of admissible functions to include all Sobolev functions in $C(\bar{R}) \cap W^{1,Q}(R)$, see Proposition 11 in [KR95], where the proof is valid for all Carnot groups. In particular, if $f\colon \hat{N} \to \hat{N}$ is quasiconformal, $R$ is a ring with $\infty, f^{-1}(\infty) \notin \bar{R}$ and $u\colon N \to \mathbb{R}$ is an admissible function for $f(R)$ as defined above, then $u \circ f$ is admissible for $R$ in the generalized sense and so can be used in the estimate for $C(R)$.

Let $\Gamma$ be a curve family in $N$. A non-negative Borel function $\rho\colon N \to [0, +\infty]$ is an admissible function of $\Gamma$ if $\int_\gamma \rho\, ds \geq 1$ for every locally rectifiable curve $\gamma$ in $\Gamma$. The conformal modulus of $\Gamma$ is:

$$M(\Gamma) = \inf_\rho \int_N \rho^Q\, dx,$$

where $Q$ is the homogeneous dimension of $N$ and $\rho$ varies over all admissible functions of $\Gamma$.

It is a classical result that for any ring $R$ in $\mathbb{R}^n$, the capacity agrees with the modulus, $C(R) = M(\Gamma(R, C_0, C_1))$, where $\Gamma(R, C_0, C_1)$ is the collection of curves in $R$ joining $C_0$ and $C_1$. This result has been generalized to the case of Carnot groups by Markina [M03].

The following result says that a quasiconformal map preserving conformal capacity of rings must be a conformal map.

**Lemma 3.1.** *Let $f\colon \hat{N} \to \hat{N}$ be a quasiconformal map. Suppose $C(R) = C(f(R))$ for any ring $R$ in $\hat{N}$ satisfying $\infty, f^{-1}(\infty) \notin \bar{R}$. Then $f$ is conformal.*

*Proof.* Let $p \in N \backslash \{f^{-1}(\infty)\}$ be a point such that $f$ is Pansu differentiable at $p$ and the Pansu differential $Df(p)$ is a graded automorphism of $N$. Then the maps $\delta_{\frac{1}{t}} \circ L_{f(p)^{-1}} \circ f \circ L_p \circ \delta_t$ converges to $Df(p)$ uniformly on compact subsets as $t \to +0$. Since left translations and the standard Carnot dilations are conformal and so preserve the capacities, all the maps $\delta_{\frac{1}{t}} \circ L_{f(p)^{-1}} \circ f \circ L_p \circ \delta_t$ preserve capacities. Now the continuity of capacities implies the limiting map $Df(p)$ also preserves the capacity. Hence we may assume $f$ is a graded automorphism of $N$ that preserves capacity of rings. By the result of Markina [M03] cited above,

$$M(\Gamma(R, C_0, C_1)) = M(\Gamma(f(R), f(C_0), f(C_1)))$$

for any ring $R$. We need to show that $f$ is a similarity.

The proof below is a modification of the proof of Theorem 36.1 in [V71].

We first construct the rings that we will use. For this we identify $N$ with its Lie algebra $\mathfrak{n} = V_1 \oplus \cdots \oplus V_k$. We fix an inner product on $\mathfrak{n}$ so that the $V_j$'s are perpendicular to each other and that on $V_1$ it agrees with the inner product on $V_1$ that defines the left invariant Carnot–Carathéodory metric on $N$. Denote $m_j = \dim(V_j)$. By the singular value decomposition, there exist orthonormal bases $\{e_{j1}, \cdots, e_{jm_j}\}$ and $\{\tilde{e}_{j1}, \cdots, \tilde{e}_{jm_j}\}$ of $V_j$ and positive numbers $\lambda_{j1} \geq \cdots \geq \lambda_{jm_j}$ such that $f(e_{jl}) = \lambda^j_{jl} \tilde{e}_{jl}$.

Let

$$C_0 = \left\{ \sum_{j,l} x_{jl} e_{jl} \colon x_{11} = 0,\ |x_{1l}| \le 1 \text{ for } l = 2, \cdots, m_1, \right.$$
$$\left. |x_{jl}| \le 1 \text{ for } j \ge 2,\ 1 \le l \le m_j \right\}.$$

We will construct a rectangular box $Y$ that contains $C_0$ in its interior $\mathring{Y}$ and the ring will be $R = \mathring{Y} \backslash C_0$.

**Claim.** *For each pair* $(j,l)$, $2 \le j \le k$, $1 \le l \le m_j$, *there is a polynomial* $P_{jl}(\delta)$ *without constant term satisfying the following property: for* $0 < \delta \ll 1$, *let*

$$Y = \left\{ \sum_{j,l} x_{jl} e_{jl} \colon |x_{11}| \le \frac{\delta}{\lambda_{11}},\ |x_{1l}| \le 1 + \frac{\delta}{\lambda_{1l}} \text{ for } 2 \le l \le m_1, \right.$$
$$\left. |x_{jl}| \le 1 + P_{jl}(\delta) \text{ for } j \ge 2,\ 1 \le l \le m_j \right\},$$

*then* $d(C_0, \partial Y) = \frac{\delta}{\lambda_{11}}$ *and* $d(f(C_0), \partial f(Y)) = \delta$.

We will first assume the claim and finish the proof of the lemma, and then prove the claim. We consider the ring $R = \mathring{Y} \backslash C_0$. Note $\partial R$ has two components: $C_0$ and $C_1 := \partial Y$.

Let $\Gamma_1$ and $\Gamma_2$ be the families of curves defined by:

$$\Gamma_1 = \left\{ x * (0, \frac{\delta}{\lambda_{11}}) e_{11} \colon x \in C_0 \right\}, \quad \Gamma_2 = \left\{ x * (-\frac{\delta}{\lambda_{11}}, 0) e_{11} \colon x \in C_0 \right\}.$$

The claim $d(C_0, \partial Y) = \frac{\delta}{\lambda_{11}}$ and the definition of $Y$ implies $\Gamma_1, \Gamma_2 \subset \Gamma := \Gamma(R, C_0, C_1)$. The curves in $\Gamma_1$ and $\Gamma_2$ are respectively contained in the disjoint Borel sets $C_0 * (0, \frac{\delta}{\lambda_{11}}) e_{11}$ and $C_0 * (-\frac{\delta}{\lambda_{11}}, 0) e_{11}$. This implies $M(\Gamma) \ge M(\Gamma_1) + M(\Gamma_2)$, see Theorem 6.7 in [V71]. The standard calculation in quasiconformal analysis (see 7.2 in [V71]) shows that

$$M(\Gamma_1) = M(\Gamma_2) = \frac{|C_0 * [0, \frac{\delta}{\lambda_{11}}] e_{11}|}{(\frac{\delta}{\lambda_{11}})^Q} = \frac{\mathfrak{L}^{n-1}(C_0) \cdot (\frac{\delta}{\lambda_{11}})}{(\frac{\delta}{\lambda_{11}})^Q},$$

where $\mathfrak{L}^{n-1}(C_0)$ is the $(n-1)$-dimensional Lebesgue measure of $C_0$ and $n = \dim(\mathfrak{n})$. So we have

$$M(\Gamma) \ge M(\Gamma_1) + M(\Gamma_2) = \frac{2\, \mathfrak{L}^{n-1}(C_0)}{(\frac{\delta}{\lambda_{11}})^{Q-1}}. \tag{2}$$

By the claim the minimal distance between the two boundary components $f(C_0)$ and $f(C_1)$ of $f(R)$ is $\delta$. Now by Theorem 7.1 in [V71], with $\tilde{\Gamma} = \Gamma(f(R), f(C_0), f(C_1))$ and $J$ the absolute value of the determinant of $f$,

$$M(\tilde{\Gamma}) \le \frac{|f(R)|}{\delta^Q} = \frac{J \cdot |R|}{\delta^Q} = \frac{J \cdot \mathfrak{L}^{n-1}(F) \cdot 2 \frac{\delta}{\lambda_{11}}}{\delta^Q}, \tag{3}$$

where $F = \{\sum_{j,l} x_{jl} e_{jl} \in Y \colon x_{11} = 0\}$.

Since $M(\Gamma) = M(\tilde{\Gamma})$, (2) and (3) imply

$$\lambda_{11}^Q \le J \cdot \frac{\mathfrak{L}^{n-1}(F)}{\mathfrak{L}^{n-1}(C_0)}.$$

Since $\mathfrak{L}^{n-1}(F)/\mathfrak{L}^{n-1}(C_0) \to 1$ as $\delta \to 0$, we get $\lambda_{11}^Q \le J$. On the other hand, by Lemma 3.3 in [CC06], $f(B(0,1)) \subset B(0, \lambda_{11})$, which implies

$$J \cdot |B(0,1)| = |f(B(0,1))| \le |B(0,\lambda_{11})| = \lambda_{11}^Q |B(0,1)|$$

and so $J \le \lambda_{11}^Q$. Hence $J = \lambda_{11}^Q$. This implies $f$ maps the ball $B(0,1)$ onto the ball $B(0,\lambda_{11})$ and so must be a similarity.

*Proof of the claim.* We only write down the proof for $d(f(C_0), \partial f(Y)) = \delta$, as the proof for $d(C_0, \partial Y) = \frac{\delta}{\lambda_{11}}$ is very similar. Let $P_{jl}$ be a polynomial without constant term and $Y$ be as defined in the claim. By the formula for $f$ we have

$$f(C_0) = \left\{ \sum_{j,l} y_{jl}\tilde{e}_{jl} \colon y_{11} = 0,\ |y_{1l}| \le \lambda_{1l} \text{ for } l = 2, \cdots, m_1,\ |y_{jl}| \le \lambda_{jl} \text{ for } j \ge 2 \right\}$$

and

$$f(Y) = \left\{ \sum_{j,l} y_{jl}\tilde{e}_{jl} \colon |y_{11}| \le \delta,\ |y_{1l}| \le \lambda_{1l} + \delta \text{ for } l = 2, \cdots, m_1, \right.$$
$$\left. |y_{jl}| \le \lambda_{jl} + \lambda_{jl} P_{jl}(\delta) \text{ for } j \ge 2 \right\};$$

First note that $0 \in f(C_0)$, $\delta\tilde{e}_{11} \in \partial f(Y)$ and $d(0, \delta\tilde{e}_{11}) = \delta$. So $d(f(C_0), \partial f(Y)) \le \delta$. It suffices to show for suitable choices of $P_{jl}$ we have $N_\delta(f(C_0)) \subset f(Y)$.

Let $w \in N_\delta(f(C_0))$. Then we can write $w = y * z$ with $y \in f(C_0)$ and $d(0,z) \le \delta$. By the BCH formula we have $w = y + z + \frac{1}{2}[y,z] + \cdots$. Write $w = \sum_{j,l} w_{jl}\tilde{e}_{jl}$, $y = \sum_{j,l} y_{jl}\tilde{e}_{jl}$, $z = \sum_{j,l} z_{jl}\tilde{e}_{jl}$. As $d(0,z) \le \delta$, we have $|z_{jl}| \le \delta^j$. As $y_{11} = 0$ and $|y_{1l}| \le \lambda_{1l}$ for $l \ge 2$ we have $|w_{11}| \le \delta$ and $|w_{1l}| \le \lambda_{1l} + \delta$ for $l \ge 2$. For $j \ge 2$ we have $w_{jl} = y_{jl} + Q_{jl}$, where $Q_{jl}$ is a polynomial of $z_{st}$ and $y_{st}$ with $s \le j$ such that each monomial in $Q_{jl}$ has at least one $z_{st}$ as a factor. As $|y_{st}| \le \lambda_{st}$ and $|z_{s,t}| \le \delta^s$, we see that there is a polynomial $P_{j,l}$ with no constant term such that $|Q_{jl}| \le \lambda_{jl} P_{jl}(\delta)$. This finishes the proof of the claim. □

In Lemmas 3.2, 3.3 and 3.4 we fix an inner product on $\mathfrak{n}$ so that the $V_j$'s are perpendicular to each other; this determines a left invariant Carnot–Carathéodory metric $d_{CC}$ on $N$ and a Haar measure on $N$; the metric on $\hat{N}$ will be the sphericalized metric $\hat{d}_{CC}$ of $d_{CC}$ with respect to the origin and the measure on $\hat{N}$ will be the sphericalized measure $m$ of the Haar measure with respect to the origin. See the end of Section 2 for more details.

Note that Lemma 3.2 and 3.3 hold more generally for quasisymmetric maps on Alhfors regular metric measure spaces that satisfies a Poincare inequality: their proofs use only the reverse Holder inequality which holds in such generality, see Theorem 7.11 of [HK98].

**Lemma 3.2.** *Let $f\colon (\hat{N}, \hat{d}_{CC}) \to (\hat{N}, \hat{d}_{CC})$ be an $\eta$-quasisymmetric map. Then there exist constants $a, b > 0$ depending only on $N$ and $\eta$ that satisfy the following. For any ball $B_1 \subset \hat{N}$ with center $x$, let $B_2$ be the largest ball in $f(B_1)$ with center*

*$f(x)$. Then for any measurable $E \subset \hat{N}$,*

$$\frac{m(f(E)\cap B_2)}{m(B_2)} \le a\left(\frac{m(E\cap B_1)}{m(B_1)}\right)^b.$$

*Proof.* By [J86], $N$ with the Carnot–Carathéodory metric and the Lebesgue measure supports a 1-Poincare inequality. Then Theorem 1.1 of [LS15] implies $\hat{N}$ with the sphericalized metric and sphericalized measure also supports a 1-Poincare inequality. By Theorem 7.11 in [HK98], there is some $\epsilon > 0$ and a constant $C$ depending only on $N$ and $\eta$ such that

$$\left(\frac{1}{m(B)}\int_B \mu_f^{1+\epsilon}\,dm\right)^{\frac{1}{1+\epsilon}} \le C\left(\frac{1}{m(B)}\int_B \mu_f\,dm\right)$$

for all balls $B$ in $\hat{N}$. Here $\mu_f$ is the volume derivative of $f$ defined by:

$$\mu_f(x) = \lim_{r\to 0}\frac{m(f(B(x,r)))}{m(B(x,r))},$$

which exists and is finite for $m$-a.e. $x \in \hat{N}$. Now for a measurable $E$,

$$\begin{aligned}
m(f(E)\cap B_2) &= \int_{E\cap f^{-1}(B_2)} \mu_f\,dm \le \int_{E\cap B_1}\mu_f\,dm = \int_{B_1}\mu_f\cdot\mathcal{X}_{E\cap B_1}\,dm \\
&\le \left(\int_{B_1}\mu_f^{1+\epsilon}\,dm\right)^{\frac{1}{1+\epsilon}} m(E\cap B_1)^{\frac{\epsilon}{1+\epsilon}} \\
&\le Cm(B_1)^{-\frac{\epsilon}{1+\epsilon}}\left(\int_{B_1}\mu_f\,dm\right) m(E\cap B_1)^{\frac{\epsilon}{1+\epsilon}} \\
&= Cm(B_1)^{-\frac{\epsilon}{1+\epsilon}} m(f(B_1)) m(E\cap B_1)^{\frac{\epsilon}{1+\epsilon}}.
\end{aligned}$$

Hence,

$$\frac{m(f(E)\cap B_2)}{m(f(B_1))} \le C\left(\frac{m(E\cap B_1)}{m(B_1)}\right)^{\frac{\epsilon}{1+\epsilon}}.$$

Since $f$ is $\eta$-quasisymmetric, we have $f(B_1) \subset \eta(1)B_2$, where $\eta(1)B_2$ is the ball in $(\hat{N}, \hat{d}_{CC})$ with the same center as $B_2$ and with radius $\eta(1)$ times that of $B_2$. Now the Alhfors regularity of $\hat{N}$ implies $m(f(B_1)) \le C_1 m(B_2)$ for some constant $C_1 \ge 1$ depending only on $\eta$ and the constant in Alhfors regularity condition. The lemma follows. □

**Lemma 3.3.** *Let $\mathcal{F}$ be a compact family of $K$-quasiconformal maps from $\hat{N}$ to $\hat{N}$. Then there exist positive constants $b'$, $b$, $a'$, $a$ depending only on $\mathcal{F}$ and $N$ such that*

$$a'(m(E))^{b'} \le m(f(E)) \le a(m(E))^b$$

*for all measurable $E \subset \hat{N}$ and all $f \in \mathcal{F}$.*

*Proof.* Since $\mathcal{F}$ is a compact family of $K$-quasiconformal maps, there exists some homeomorphism $\eta\colon [0,\infty) \to [0,\infty)$ such that every $f \in \mathcal{F}$ is $\eta$-quasisymmetric. Let $r_0 = 1/10$. For any $x \in \hat{N}$ and any $f \in \mathcal{F}$, let $r'' = r''(x,f) > 0$ be the largest number such that $B(f(x), r'') \subset f(B(x, r_0))$. Let $r' = r'(x,f) > 0$ be the largest number such that $f(B(x,r')) \subset B(f(x), r'')$. By Lemma 3.2 the following holds for any measurable $E \subset B(x, r')$:

$$\frac{m(f(E))}{m(B(f(x), r''))} \le a \left( \frac{m(E)}{m(B(x, r_0))} \right)^b .$$

Note that $r''$ and $r'$ are continuous functions of $x, f$. Since $\hat{N}$ and $\mathcal{F}$ are compact, we have

$$r'_0 = \inf_{x \in \hat{N}, f \in \mathcal{F}} r'(x, f) > 0.$$

Also set

$$r''_0 = \sup_{x \in \hat{N}, f \in \mathcal{F}} r''(x, f) \le \operatorname{diam}(\hat{N}).$$

Now the space $\hat{N}$ can be covered by a finite number of balls $\{B(x_j, r'_0)\}_{j=1}^k$ with radius $r'_0$. Let $E \subset \hat{N}$ be any measurable subset. Then $f(E) = \bigcup_{j=1}^k f(E \cap B(x_j, r'_0))$. By using the Alhfors regularity of the metric measure space $(\hat{N}, \hat{d}, m)$ we obtain:

$$\begin{aligned} m(f(E)) &\le \sum_{j=1}^k a \left( \frac{m(E \cap B(x_j, r'_0))}{m(B(x_j, r_0))} \right)^b m(B(f(x_j), r''_0)) \\ &\le \sum_{j=1}^k a \frac{C r_0''^Q}{(r_0^Q / C)^b} m(E \cap B(x_j, r'_0))^b \le \frac{kaC r_0''^Q}{(r_0^Q/C)^b} m(E)^b. \end{aligned}$$

This establishes the second inequality in the lemma. The first inequality holds since the family $\mathcal{F}^{-1} = \{f^{-1} | f \in \mathcal{F}\}$ is also a compact family of $K$-quasiconformal maps. □

**Lemma 3.4.** *Let $\{f_j \colon \hat{N} \to \hat{N}\}$ be a sequence of $K$-quasiconformal maps that converge uniformly to a quasiconformal map $f \colon \hat{N} \to \hat{N}$. Suppose for any compact subset $F \subset \hat{N}$ satisfying $\infty, f^{-1}(\infty) \notin F$, and any $\epsilon > 0$,*

$$|\{x \in F \colon K(f_j, x) \ge 1 + \epsilon\}| \to 0 \quad \text{as } j \to \infty.$$

*Then $f$ is conformal.*

*Proof.* We shall show that $f$ satisfies the assumption of Lemma 3.1. Let $R$ be a ring in $\hat{N}$ satisfying $\infty, f^{-1}(\infty) \notin \bar{R}$. Let $C_0$ and $C_1$ be the two components of $\partial R$. For $a > 0$, let

$$\tilde{R}_a = \{z \in f(R) \colon B(z, a) \subset f(R)\}.$$

Then $f_j(R) \supset \tilde{R}_a$ for $j$ sufficiently large (depending on $a$).

Let $\epsilon > 0$. Let $\tilde{u}_a$ be admissible for $\tilde{R}_a$ so that

$$\int_{\tilde{R}_a} |\nabla \tilde{u}_a|^Q \, dx \le C(\tilde{R}_a) + \epsilon.$$

One can assume $\tilde{u}_a$ is smooth and is 0 on a neighborhood of $C_0$ and 1 on a neighborhood of $C_1$. Extend $\tilde{u}_a$ to $\hat{N}$ by continuity and being constant on the two components of $\hat{N} \backslash \tilde{R}_a$. Then clearly

$$\int_{\tilde{R}_a} |\nabla \tilde{u}_a|^Q \, dx = \int_{\tilde{R}} |\nabla \tilde{u}_a|^Q \, dx$$

for $\tilde{R} \supset \tilde{R}_a$.

Let $u_j = \tilde{u}_a \circ f_j$. Then $u_j$ is admissible for $f_j^{-1}(\tilde{R}_a)$ and hence also admissible for $R$. The chain rule (see Lemma 3.7 in [CC06]) implies $Du_j(x) = D\tilde{u}_a(f_j(x)) \circ Df_j(x)$ for a.e. $x \in R$. Hence $Du_j(x)|_{V_1} = D\tilde{u}_a(f_j(x))|_{V_1} \circ Df_j(x)|_{V_1}$ and

$$|Du_j(x)|_{V_1}| \leq |D\tilde{u}_a(f_j(x))|_{V_1}| \cdot |Df_j(x)|_{V_1}|.$$

Notice that $|Du_j(x)|_{V_1}| = |\nabla u_j(x)|$, $|D\tilde{u}_a(f_j(x))|_{V_1}| = |\nabla \tilde{u}_a(f_j(x))|$ and

$$|Df_j(x)|_{V_1}|^Q = L^Q_{f_j}(x) = l^Q_{f_j}(x) K^Q(f_j, x) \leq f_j'(x) K^Q(f_j, x)$$

due to Lemma 3.3 and (4.1) in [CC06]. It follows that

$$|\nabla u_j(x)|^Q \leq K^Q(f_j, x) f_j'(x) |\nabla \tilde{u}_a(f_j(x))|^Q.$$

Set $E_j = \{x \in R\colon K(f_j, x) \geq 1 + \epsilon\}$. The assumption that $\infty, f^{-1}(\infty) \notin \bar{R}$ implies that there are compact subsets $K_1, K_2 \subset N$ with $\bar{R} \subset \mathring{K}_1$ and $\overline{f(R)} \subset \mathring{K}_2$. There is a constant $C \geq 1$ such that $m(E)/C \leq |E| \leq C\, m(E)$ for any $E \subset K_1$ or $E \subset K_2$. By assumption $|E_j| \to 0$ as $j \to +\infty$. Now Lemma 3.3 implies $|f_j(E_j)| \to 0$.

We have

$$\begin{aligned} C(R) &\leq \int_R |\nabla u_j|^Q \, dx \leq \int_R K^Q(f_j, x) f_j'(x) |\nabla \tilde{u}_a(f_j(x))|^Q \, dx \\ &\leq (1+\epsilon)^Q \int_{f_j(R)} |\nabla \tilde{u}_a|^Q \, dx + K^Q \int_{f_j(E_j)} |\nabla \tilde{u}_a|^Q \, dx. \end{aligned}$$

Since $|f_j(E_j)| \to 0$ as $j \to +\infty$, the second term above goes to 0 as $j \to \infty$. Hence

$$C(R) \leq (1+\epsilon)^Q (C(\tilde{R}_a) + \epsilon),$$

and so $C(R) \leq C(\tilde{R}_a)$ for small $a$. We claim that $C(\tilde{R}_a) \to C(f(R))$ as $a \to 0$. Indeed, for any $\delta > 0$ there is a smooth function $v$, admissible for $f(R)$ with value 0 in a neighborhood of $f(C_0)$ and value 1 in a neighborhood of $f(C_1)$, such that

$$\int_{f(R)} |\nabla v|^Q \, dx \leq C(f(R)) + \delta.$$

For small enough $a$, $v$ is also admissible for $\tilde{R}_a$ and

$$\int_{f(R)} |\nabla v|^Q \, dx = \int_{\tilde{R}_a} |\nabla v|^Q \, dx.$$

Since

$$\int_{\tilde{R}_a} |\nabla v|^Q \, dx \geq C(\tilde{R}_a) \geq C(f(R)) \geq \int_{f(R)} |\nabla v|^Q \, dx - \delta$$

and $\delta$ is arbitrary, we have $C(\tilde{R}_a) \to C(f(R))$ as $a \to 0$. Hence $C(R) \leq C(f(R))$.

Finally as $K(f_j^{-1}, f_j(x)) = K(f_j, x)$, Lemma 3.3 implies that $f_j^{-1}$, $f^{-1}$ also satisfy the assumption in the lemma and we conclude that $C(f(R)) \leq C(R)$. Now we have verified the assumption of Lemma 3.1 and so $f$ is conformal. □

## 4. Tukia-type theorem for Carnot groups

In this section we prove Theorem 1.1 and Corollary 1.2.

**4.1. Existence of invariant measurable conformal structure.**In this subsection we show that every uniform quasiconformal group of $\hat{N}$ leaves invariant a measurable conformal structure on $\hat{N}$.

Let $m \geq 2$ be an integer and $X$ the space of symmetric, positive definite real $m \times m$ matrices with determinant 1. The general linear group $GL(m, \mathbb{R})$ acts on $X$ by

$$M[S] = (\det M)^{-\frac{2}{m}} M S M^T$$

for $M \in GL(m, \mathbb{R})$ and $S \in X$, where $M^T$ is the transpose of $M$. Then $SL(m, \mathbb{R}) \subset GL(m, \mathbb{R})$ acts on $X$ transitively, and the stabilizer of $SL(m, \mathbb{R})$ at $I_m$ is $SO(m)$. Hence $X = SL(m, \mathbb{R})/SO(m)$. The homogeneous manifold $X = SL(m, \mathbb{R})/SO(m)$ is a symmetric space of non-compact type (see table V on page 518 of [Hel78]) and so has nonpositive sectional curvature.

The Riemannian distance $d$ on $X$ is invariant under the action of $GL(m, \mathbb{R})$ and satisfies:

$$d(I_m, ODO^T) = \sqrt{a_1^2 + \cdots + a_m^2},$$

where $I_m$ is the identity matrix, $O$ is orthogonal and $D$ is diagonal with diagonal entries $e^{a_1} \geq \cdots \geq e^{a_m}$, see [Ma71, p. 27]. We notice that the dilatation of a non-singular linear map $A := UDV^T \colon \mathbb{R}^m \to \mathbb{R}^m$ (where $D$ is as above and $U, V$ are orthogonal) is given by:

$$K(A) := K(A, 0) = \frac{\max\{|AX| \colon |X| = 1\}}{\min\{|AX| \colon |X| = 1\}} = \frac{e^{a_1}}{e^{a_m}} = e^{a_1 - a_m}.$$

It follows that there is a function $\phi \colon [0, \infty) \to [0, \infty)$ (one may choose $\phi(t) = e^t - 1$) with $\phi(t) \to 0$ as $t \to 0$ such that

$$K(A) \leq 1 + \phi(d(I_m, AA^T)). \tag{4}$$

Let $N$ be a Carnot group with Lie algebra $\mathfrak{n} = V_1 \oplus \cdots \oplus V_r$. Let $V_1$ be equipped with an inner product. Let $m = \dim(V_1)$. By using an orthonormal basis for $V_1$, we can identify the special orthogonal group $SO(V_1)$ associated with the inner product with $SO(m, \mathbb{R})$ and similarly identify $SL(V_1)$ with $SL(m, \mathbb{R})$. Hence

$$X = SL(V_1)/O(V_1).$$

A measurable conformal structure on $\hat{N} = N \cup \{\infty\}$ is an essentially bounded measurable map

$$\mu \colon U \to X$$

defined on a full measure subset $U \subset N$. Note we do not require $\mu$ to be defined everywhere on $\hat{N}$. In particular, we prefer $\mu$ not defined at $\infty$.

For a quasiconformal map $f \colon \hat{N} \to \hat{N}$ and a measurable conformal structure $\mu$ on $\hat{N}$, the pull-back measurable conformal structure $f^*\mu$ is defined by:

$$\begin{aligned} f^*\mu(x) &= (Df(x)|_{V_1})^T[\mu(f(x))] \\ &= (\det(Df(x)|_{V_1}))^{-\frac{2}{m}} (Df(x)|_{V_1})^T \mu(f(x)) (Df(x)|_{V_1}), \quad \text{for a.e. } x \in N. \end{aligned}$$

Here we are using the fact that a quasiconformal map is Pansu differentiable a.e. This is similar to the definition of pull-back of a Riemannian metric under a smooth map. It is easy to check that $(g \circ f)^*\mu = f^*(g^*\mu)$, where $f, g \colon \hat{N} \to \hat{N}$ are quasiconformal maps and $\mu$ is a measurable conformal structure on $\hat{N}$.

A quasiconformal map $f$ is called conformal with respect to the measurable conformal structure $\mu$ if $f^*\mu = \mu$; in other words, $\mu(x) = (Df(x)|_{V_1})^T[\mu(f(x))]$ for a.e. $x \in N$.

**Proposition 4.1.** *Let $G$ be a uniform quasiconformal group of $\hat{N}$. Then there is a measurable conformal structure on $\hat{N}$ such that every $g \in G$ is conformal with respect to $\mu$.*

*Proof.* We first assume $G$ is countable. Since $G$ is countable and for each $g \in G$, the Pansu differential $Dg(x)$ exists and is a graded automorphism for a.e. $x \in N$, there is a measurable, $G$-invariant subset $U$ of full measure such that $Dg(x)$ exists and is a graded automorphism for all $g \in G$ and at all $x \in U$. Let $\mu_0$ be the left invariant conformal structure on $N$ associated with an inner product on $V_1$ (so $\mu_0 \colon N \to X$ is a constant map). For each $x \in U$, set $M_x = \{g^*\mu_0(x) \colon g \in G\}$. Since $G$ is a uniform quasiconformal group, $M_x$ is a bounded subset of $X$. The assignment $x \mapsto M_x$ is $G$-invariant: for any $f \in G$,

$$f^* M_{f(x)} = f^*\{g^*\mu_0(f(x)) \colon g \in G\} = \{(g \circ f)^*\mu_0(x) \colon g \in G\} = M_x.$$

Since $M_x$ is a bounded subset of the non-positively curved symmetric space $X$, there exists a unique circumcenter $P(M_x)$ in $X$ for the subset $M_x$, see for example [BH, p. 179]. Define $\mu \colon U \to X$ by $\mu(x) = P(M_x)$. Since the assignment $x \mapsto M_x$ is $G$-invariant, $\mu$ is also $G$-invariant.

It remains to show $\mu$ is measurable. Enumerate $G = \{g_0, g_1, \cdots\}$, and let $M(x,j) = \{g_i^*\mu_0 \colon i \leq j\}$ and $\mu_j(x) = P(M(x,j))$. Now, $Y \mapsto P(Y)$ is continuous with respect to the Hausdorff metric (see [T86, p. 334]). This implies $\mu_j$ is measurable. Since $\mu_j$ converges to $\mu$ point-wise, $\mu$ is also measurable. This completes the proof when $G$ is countable.

Now let $G$ be a general uniform quasiconformal group of $\hat{N}$. Let $G_0 \subset G$ be a countable subgroup of $G$ that is dense in the topology of uniform convergence. By the above argument, there is a $G_0$-invariant measurable conformal structure $\mu$. It follows that $\mu$ is also $G$-invariant as a uniform limit of a sequence of $\mu$-conformal maps is $\mu$-conformal. This follows from an analogue of Theorem D in [T86] in the setting of Carnot groups. The proof of Theorem D in [T86] is valid for Carnot groups after some small modifications: to blow up maps in the Carnot group setting one needs to conjugate using Carnot dilations. □

Let $G_0$ be a countable dense subgroup of $G$ as above. If the induced action of $G$ on the space of distinct triples of $\hat{N}$ is cocompact, then the same is true for the induced action of $G_0$. Another way to handle the case of an uncountable group $G$ is to first run the argument from the next subsection to conjugate $G_0$ into the conformal group, then the same map also conjugates $G$ into the conformal group since the limits of conformal maps are conformal maps.

**4.2. Conjugating into the conformal group.** In this subsection we will prove Theorem 1.1 and Corollary 1.2.

A map $\mu \colon U \to Y$, where $U \subset N$ is open and $Y$ is a metric space, is called approximately continuous at $x_0 \in U$ if for any $\epsilon > 0$, the set $\mu^{-1}(B(\mu(x_0), \epsilon))$ has density 1 at $x_0$; that is, if

$$\frac{|B(x_0, r) \cap \mu^{-1}(B(\mu(x_0), \epsilon))|}{|B(x_0, r)|} \to 1$$

as $r \to 0$. By Theorem 2.9.13 in [F69], if $Y$ is separable and $\mu$ is measurable, then $\mu$ is approximately continuous a.e.

Let $N$ be a Carnot group and $S = N \rtimes \mathbb{R}$ the negatively curved homogeneous manifold associated with $N$. There is a natural map $\pi : T(\partial S) \to S$, that assigns to each distinct triple $(\xi_1, \xi_2, \xi_3) \in T(\partial S)$ the center of the triple. To be more precise, $\pi(\xi_1, \xi_2, \xi_3)$ is defined to be the orthogonal projection of $\xi_3$ onto the complete geodesic $\xi_1\xi_2$. We observe that for any compact $C \subset S$, the set $\pi^{-1}(C)$ is compact in $T(\partial S)$.

Let $G$ be a group of homeomorphisms of $\partial S$. Then $G$ also acts diagonally on $T(\partial S)$: $g(\xi_1, \xi_2, \xi_3) = (g(\xi_1), g(\xi_2), g(\xi_3))$. Recall that a point $\xi \in \partial S$ is said to be a *radial limit point* of $G$ if there exists a sequence of elements $\{h_j\}_{j=1}^{\infty}$ of $G$ with the following property: for any triple $T = (\xi_1, \xi_2, \xi_3) \in T(\partial S)$, and any complete geodesic $\gamma$ asymptotic to $\xi$, there exists a constant $C > 0$ with $\pi(h_j(T)) \to \xi$ and $d(\pi(h_j(T)), \gamma) \leq C$.

We recall that each inner product on $V_1$ determines a left invariant Carnot–Carathéodory metric on $N$.

**Theorem 4.2.** *Let $G$ be a uniform quasiconformal group of $\hat{N}$ and $\mu$ a $G$-invariant measurable conformal structure on $\hat{N}$. Suppose there is a point $p \in N \subset \hat{N}$ such that $\mu$ is approximately continuous at $p$ and $p$ is also a radial limit point for $G$. Then there exists a quasiconformal map $f\colon \hat{N} \to \hat{N}$ and an inner product on $V_1$ such that $fGf^{-1}$ consists of conformal maps with respect to the left invariant Carnot–Carathéodory metric determined by this inner product.*

*Proof.* The left translation action of $S = N \rtimes \mathbb{R}$ on itself is by isometries. The elements in $\mathbb{R}$ translate the vertical geodesic above the origin $0$ in $N$ and the boundary homeomorphisms induced by them are the standard Carnot group dilations of $N$. We shall use $\tilde{\delta}_t$ to denote the isometry of $S$ that induces the standard Carnot group dilation $\delta_t$.

We may assume $p = 0$ is the origin of $N$. Fix a triple $T \in T(\partial S)$ and let $\gamma$ be the vertical geodesic above $0$. Since $0$ is a radial limit point of $G$, there exists a sequence of elements $\{h_j\}_{j=1}^{\infty}$ of $G$ and a constant $C > 0$ with $\pi(h_j(T)) \to 0$ and $d(\pi(h_j(T)), \gamma) \leq C$. Fix a point $x_0 \in \gamma$. For each $j$ there is some $s_j > 0$ with $s_j \to +\infty$ as $j \to \infty$ such that $d(\tilde{\delta}_{s_j} \circ \pi \circ h_j(T), x_0) \leq C$. Since $\tilde{\delta}_{s_j}$ is an isometry of $S$, we have $\tilde{\delta}_{s_j} \circ \pi = \pi \circ \delta_{s_j}$. Hence $d(\pi \circ \delta_{s_j} \circ h_j(T), x_0) \leq C$ and so the set $\{\delta_{s_j} \circ h_j(T)\}_{j=1}^{\infty}$ lies in the compact subset $\pi^{-1}\bar{B}(x_0, C)$. Hence there is a constant $a > 0$ such that if $T = (x_1, x_2, x_3)$ is the distinct triple, then $d(\delta_{s_j} \circ h_j(x_k), \delta_{s_j} \circ h_j(x_l)) \geq a$ for all $j$ and all $1 \leq k \neq l \leq 3$. It follows that the family $\{\delta_{s_j} \circ h_j\}_{j=1}^{\infty}$ of $K$-quasiconformal maps is precompact. Define $f_j\colon \hat{N} \to \hat{N}$ by $f_j = \delta_{s_j} \circ h_j$. By passing to a subsequence, we may assume $f_j$ converges uniformly to a $K$-quasiconformal map $f\colon \hat{N} \to \hat{N}$. We shall show that for any $g \in G$, $fgf^{-1}$ is conformal with respect to the left invariant Carnot–Carathéodory metric determined by $\mu(0)$.

Let $g \in G$ and denote $\tilde{g} = fgf^{-1}$, $\tilde{g}_j = f_j g f_j^{-1}$. Let $\mu_j = (f_j^{-1})^*\mu$. Since $\mu$ is $G$-invariant, $\tilde{g}_j$ is conformal with respect to $\mu_j$:

$$\tilde{g}_j^*\mu_j = (f_j^{-1})^* g^* f_j^* (f_j^{-1})^*\mu = (f_j^{-1})^* g^*\mu = (f_j^{-1})^*\mu = \mu_j.$$

Note $\mu_j = (f_j^{-1})^*\mu = (\delta_{s_j}^{-1})^*(h_j^{-1})^*\mu = (\delta_{s_j}^{-1})^*\mu$. So for $q \in N$,

$$\mu_j(q) = (\delta_{s_j}^{-1})^*\mu(q) = (D\delta_{s_j}^{-1}(q)|_{V_1})^T[\mu(\delta_{s_j}^{-1}(q))] = \mu(\delta_{s_j}^{-1}(q))$$

since $D\delta_t(q)|_{V_1}$ is an Euclidean dilation.

Let $F \subset \hat{N}$ be a compact subset such that $\infty, \tilde{g}^{-1}(\infty) \notin F$. There is a compact subset $F_0$ of $N$ such that $F \subset F_0$, $\tilde{g}_j(F) \subset F_0$ for all sufficiently large $j$. Since $\mu$ is approximately continuous at $0$ and $s_j \to \infty$, the equality $\mu_j(q) = \mu(\delta_{s_j}^{-1}(q))$

implies that for any $\epsilon > 0$ there are subsets $A_j \subset F_0$ with $|A_j| \to 0$ as $j \to \infty$ and $d(\mu_j(x), \mu(0)) \le \epsilon$ for $x \in F_0 \backslash A_j$.

The maps $\tilde{g}_j^{-1}$ and $\tilde{g}^{-1}$ form a compact family of $K$-quasiconformal maps. By Lemma 3.3, we have $|\tilde{g}_j^{-1}(A_j)| \to 0$ as $j \to \infty$. Set $B_j = A_j \cup \tilde{g}_j^{-1}(A_j)$. Now we have $|B_j| \to 0$ as $j \to \infty$ and $d(\mu_j(x), \mu(0)) \le \epsilon$ and $d(\mu_j(\tilde{g}_j(x)), \mu(0)) \le \epsilon$ for $x \in F \backslash B_j$. Since $\tilde{g}_j$ is $\mu_j$-conformal, we have $\mu_j(x) = (D\tilde{g}_j(x)|_{V_1})^T[\mu_j(\tilde{g}_j(x))]$ for a.e. $x$. Now

$$d(\mu_j(x), (D\tilde{g}_j(x)|_{V_1})^T[\mu(0)]) = d(\mu_j(\tilde{g}_j(x)), \mu(0)) \le \epsilon$$

for a.e. $x \in F \backslash B_j$. Combining this with $d(\mu_j(x), \mu(0)) \le \epsilon$ we get

$$d(\mu(0), (D\tilde{g}_j(x)|_{V_1})^T[\mu(0)]) \le 2\epsilon$$

for a.e. $x \in F \backslash B_j$. By Lemma 3.3 in [CC06] and (4) we have $K(\tilde{g}_j, x) = K(D\tilde{g}_j(x)|_{V_1}) \le 1 + \phi(2\epsilon)$ for a.e. $x \in F \backslash B_j$. This implies the assumption of Lemma 3.4 is satisfied and so $\tilde{g}$ is conformal. □

*Proof of Theorem 1.1.* Let $G_0$ be a countable dense subgroup of $G$. By Proposition 4.1, there exists a $G_0$-invariant measurable conformal structure $\mu$ on $\hat{N}$. Since the action of $G_0$ on $T(\partial S)$ is co-compact, every point in $\partial S$ is a radial limit point. By Theorem 2.9.13 in [F69], $\mu$ is approximately continuous at a.e. point in $\hat{N}$. By Theorem 4.2, there exists a quasiconformal map $f$ of $\hat{N}$ such that $fG_0f^{-1}$ is a conformal group of $\hat{N}$ with respect to some left invariant Carnot–Carathéodory metric $d_{CC}$ on $N$. Since $fG_0f^{-1}$ is dense in $fGf^{-1}$ and the limits of conformal maps are conformal, we conclude that $fGf^{-1}$ is also a conformal group with respect to $d_{CC}$.

For any left invariant Carnot–Carathéodory metric $d$ on $N$, let $\mathrm{Conf}(\hat{N}, d)$ be the group of conformal maps of $\hat{N}$ with respect to $d$. We next show that there is a fixed left invariant Carnot–Carathéodory metric $d_0$ on $N$ such that for any left invariant Carnot–Carathéodory metric $d$ on $N$, the group $\mathrm{Conf}(\hat{N}, d)$ can be conjugated into $\mathrm{Conf}(\hat{N}, d_0)$ by a graded automorphism of $N$. For this we use the result of Cowling and Ottazzi [CO15] on conformal maps of Carnot groups. By Theorem 4.1 of [CO15] there are two cases depending on whether $N$ is the Iwasawa $N$ group of a real rank-one simple Lie group.

Let $d$ be a left invariant Carnot–Carathéodory metric on $N$. First assume $N$ is the Iwasawa $N$ group of a real rank-one simple Lie group $I$. We may assume $I = \mathrm{Isom}(X)$ is the isometry group of a rank one symmetric space $X$ of noncompact type. Then the ideal boundary $\partial X$ of $X$ can be identified with $N \cup \{\infty\}$. In this case, Theorem 4.1 of [CO15] states that the action of each $g \in \mathrm{Conf}(\hat{N}, d)$ on $\hat{N}$ agrees with the action of some $\phi(g) \in I$. This $\phi(g)$ is clearly unique and the map $\mathrm{Conf}(\hat{N}, d) \to I, g \mapsto \phi(g)$ defines an injective homomorphism. It is known that in this case there is a left invariant Carnot–Carathéodory metric $d_0$ on $N = \partial S \backslash \{\infty\}$ such that $I = \mathrm{Conf}(\hat{N}, d_0)$. Such a left invariant Carnot–Carathéodory metric $d_0$ is described in Section 9 of [P89]. Lemma 9.6 there implies $I \subset \mathrm{Conf}(\hat{N}, d_0)$ and Proposition 11.5 there states $\mathrm{Conf}(\hat{N}, d_0) \subset I$. Hence the statement holds in this case.

Next we assume $N$ is not the Iwasawa $N$ group of a real rank-one simple Lie group. In this case, Theorem 4.1 of [CO15] states that for each $g \in \mathrm{Conf}(\hat{N}, d)$, we have $g(N) = N$ and $g|_N \colon (N, d) \to (N, d)$ is a similarity. In other words, we have $\mathrm{Conf}(\hat{N}, d) = \mathrm{Sim}(N, d)$. Now we finish the proof by applying Lemma 2.3. □

*Proof of Corollary 1.2.* Let $N$ be a Carnot group and $S = N \rtimes \mathbb{R}$ be the associated solvable Lie group. Let $G$ be a group that quasi-acts co-boundedly on $S$. This induces a uniform quasiconformal action of $G$ on $\partial S = \hat{N}$ such that the induced action on the space of distinct triples is co-compact. By Theorem 1.1 there is a fixed (independent of $G$) left invariant Carnot–Carathéodory metric $d_0$ on $N$ such that $G$ is quasiconformally conjugate to a subgroup of $\mathrm{Conf}(\hat{N}, d_0)$. It now suffices to show that there is a left invariant Riemannian metric $g_0$ on $S$ such that every map in $\mathrm{Conf}(\hat{N}, d_0)$ is the boundary map of some isometry of $(S, g_0)$. Again we will use Theorem 4.1 of [CO15].

First assume $N$ is the Iwasawa $N$ group of a real rank-one simple Lie group $I$. In this case, as observed in the proof of Theorem 1.1, $\mathrm{Conf}(\hat{N}, d_0)$ injects into $I$, which is the isometry group of a left invariant Riemannian metric $g_0$ on $S$. Next we assume $N$ is not the Iwasawa $N$ group of a real rank-one simple Lie group. In this case, each $g \in \mathrm{Conf}(\hat{N}, d_0)$ has the form $g|_N = L_n \circ \delta_{t_g} \circ A_g$ for some $n \in N$, $t_g > 0$, where $L_n$ is left translation by $n \in N$, $\delta_t$ $(t > 0)$ is a Carnot dilation, and $A_g \colon (N, d_0) \to (N, d_0)$ is an isometry and also a graded automorphism of $N$. Let $\mathfrak{n} = V_1 \oplus \cdots \oplus V_k$ be the Carnot grading of $\mathfrak{n}$. Then the map $\mathrm{Conf}(\hat{N}, d_0) \to GL(V_j), g \mapsto A_g|_{V_j}$ is a group homomorphism whose image has compact closure. It follows that there is some $\mathrm{Conf}(\hat{N}, d_0)$-invariant inner product $<,>_j$ on $V_j$. Now we equip $\mathfrak{s} = \mathfrak{n} \rtimes \mathbb{R}$ with the inner product $<,>$ satisfying: (1) $<,>$ agrees with $<,>_j$ on $V_j$; (2) the subspaces $V_j$, $1 \leq j \leq k$ and $\{0\} \times \mathbb{R} \subset \mathfrak{n} \rtimes \mathbb{R}$ are all perpendicular to each other with respect to $<,>$. For each $g \in \mathrm{Conf}(\hat{N}, d_0)$, define a map $\phi(g) \colon S \to S$ by $\phi(g)(x, t) = L_{(n, t_g)}(A_g x, t)$, where $L_{(n,t_g)}$ is the left translation of $S$ by $(n, t_g) \in S$. Then $\phi(g)$ is an isometry of $(S, g_0)$ with its induced boundary map equal to $g|_N = L_n \circ \delta_{t_g} \circ A_g$ and $\phi \colon G \to \mathrm{Isom}(S, g_0)$ is a group homomorphism, where $g_0$ is the left invariant Riemannian metric on $S$ determined by $<,>$.

Finally we recall that a self quasiconformal map of $\partial S = \hat{N}$ extends to a quasi-isometry of $S$. Combining this with the preceding two paragraphs we conclude that the original quasi-action of $G$ on $S$ is quasi-conjugate to an isometric action of $G$ on $(S, g_0)$. $\square$

## 5. A fibered Tukia theorem for diagonal Heintze pairs

In this section we prove a fiber bundle version of Tukia's Theorem (Theorem 1.3) in the spirit of [Dy10] for diagonal Heintze pairs. We first explain that such a group $N$ admits an iterated fibration structure with Carnot group fibers and that each self biLipschitz map of $N$ induces bundle maps. The fiber Tukia theorem states that after a biLipschitz conjugation the induced maps between the fibers are similarities.

**5.1. BiLipschitz maps of a diagonal Heintze pair.** Before we prove the fiber Tukia theorem, we first look at individual biLipschitz maps of a diagonal Heintze pair $(N, D)$.

Let $(N, D)$ be a diagonal Heintze pair and $0 < \lambda_1 < \cdots < \lambda_r$ the distinct eigenvalues of $D$. Let $\mathfrak{n} = \oplus_j V_{\lambda_j}$ be the decomposition of the Lie algebra $\mathfrak{n}$ of $N$ into eigenspaces. If $V \subset \mathfrak{n}$ is a linear subspace such that $D(V) \subset V$ then $V$ is graded, that is, $V = \oplus_j (V \cap V_{\lambda_j})$. It follows that if $V \subset V'$ are two linear subspaces of $\mathfrak{n}$ such that $D(V) \subset V$, $D(V') \subset V'$, then $V'/V = \oplus_j (V' \cap V_{\lambda_j})/(V \cap V_{\lambda_j})$ and $D$ induces a linear map on $V'/V$ and acts on $(V' \cap V_{\lambda_j})/(V \cap V_{\lambda_j})$ by multiplication by $\lambda_j$.

Now let $\mathfrak{h}_1$ be the Lie sub-algebra of $\mathfrak{n}$ generated by $V_{\lambda_1}$. We say $(N, D)$ is of Carnot type if $\mathfrak{h}_1 = \mathfrak{n}$. In general, inductively define $\mathfrak{h}_i = N(\mathfrak{h}_{i-1})$ for $i \geq 2$, where for a Lie sub-algebra $\mathfrak{h} \subset \mathfrak{n}$, $N(\mathfrak{h}) = \{X \in \mathfrak{n} \mid [X,Y] \in \mathfrak{h}, \forall Y \in \mathfrak{h}\}$ denotes the normalizer of $\mathfrak{h}$ in $\mathfrak{n}$. Then there is some integer $m \geq 1$ such that $\mathfrak{h}_m = \mathfrak{n}$. We next explain how to refine the sequence $0 < \mathfrak{h}_1 < \cdots < \mathfrak{h}_m$. It is not hard to see that $D(\mathfrak{h}_1) \subset \mathfrak{h}_1$, and by using the definition of derivation we obtain $D(\mathfrak{h}_i) \subset \mathfrak{h}_i$ for all $i$. Hence $D$ induces a diagonal derivation $\bar{D}$ of $\mathfrak{h}_i/\mathfrak{h}_{i-1}$ with positive eigenvalues. Then for every $i$, if $(\mathfrak{h}_i/\mathfrak{h}_{i-1}, \overline{D})$ is of non-Carnot type, we can repeat the above process: first take the Lie sub-algebra generated by the eigenspace of the smallest eigenvalue of $\overline{D}$, then take the normalizers. In this way we obtain a sequence of Lie sub-algebras of $\mathfrak{h}_i/\mathfrak{h}_{i-1}$: $0 = \bar{\mathfrak{h}}_{i,0} < \bar{\mathfrak{h}}_{i,1} < \cdots < \bar{\mathfrak{h}}_{i,m_i} = \mathfrak{h}_i/\mathfrak{h}_{i-1}$. Let $q_i\colon \mathfrak{h}_i \to \mathfrak{h}_i/\mathfrak{h}_{i-1}$ be the quotient map and set $\mathfrak{h}_{i,j} = q_i^{-1}(\bar{\mathfrak{h}}_{i,j})$. The refinement of the sequence $0 < \mathfrak{h}_1 < \cdots < \mathfrak{h}_m$ is obtained as follows: for those $i$ such that $(\mathfrak{h}_i/\mathfrak{h}_{i-1}, \overline{D})$ is of non-Carnot type, we insert between $\mathfrak{h}_{i-1}$ and $\mathfrak{h}_i$ the sequence $\mathfrak{h}_{i,1} < \cdots < \mathfrak{h}_{i,m_i-1}$. This refinement process can be further repeated and eventually we must stop since $\mathfrak{n}$ is finite dimensional. At the end, we obtain a sequence of Lie sub-algebras $0 = \mathfrak{n}_0 < \mathfrak{n}_1 < \cdots < \mathfrak{n}_s = \mathfrak{n}$ such that $\mathfrak{n}_{i-1}$ is an ideal of $\mathfrak{n}_i$ for every $i$, $D(\mathfrak{n}_i) \subset \mathfrak{n}_i$, and each $(\mathfrak{n}_i/\mathfrak{n}_{i-1}, \bar{D})$ is of Carnot type (that is, $\mathfrak{n}_i/\mathfrak{n}_{i-1}$ is a Carnot algebra and $D$ induces a derivation $\bar{D}$ of $\mathfrak{n}_i/\mathfrak{n}_{i-1}$ that is a multiple of a Carnot derivation). We remark that when we talk about Carnot algebra here we include the case of abelian Lie algebra. In other words, $(\mathfrak{n}_i/\mathfrak{n}_{i-1}, \overline{D})$ is of Carnot type when $\mathfrak{n}_i/\mathfrak{n}_{i-1}$ is some $\mathbb{R}^n$ and $\bar{D}$ is a standard Euclidean dilation. Let $N_i$ be the connected Lie subgroup of $N$ with Lie algebra $\mathfrak{n}_i$. Then $N_{i-1}$ is normal in $N_i$ and $N_i/N_{i-1}$ is a Carnot group.

**Definition 5.1.** Given a diagonal Heintze pair $(N, D)$, we call $1 = N_0 < N_1 < \cdots < N_s = N$ the sequence of subgroups defined by the process above the *preserved subgroups sequence.*

Let $d$ be a $D$-homogeneous distance on $N$. The restriction of $d$ on $N_i$ is a $D|_{\mathfrak{n}_i}$-homogeneous distance on $N_i$. As $N_{i-1} \triangleleft N_i$, $d$ induces a distance $\bar{d}$ on $N_i/N_{i-1}$, see end of Section 2.2. By the preceding paragraph, this distance $\bar{d}$ is a $\bar{D}$-homogeneous distance and hence is biLipschitz with $\bar{d}_{CC}^{\frac{1}{\lambda}}$, where $\bar{d}_{CC}$ is a Carnot–Carathéodory metric on $N_i/N_{i-1}$ and $\lambda > 0$ is the smallest eigenvalue of $\bar{D}$. For each $i$ we have a fibration $\pi_i\colon N/N_{i-1} \to N/N_i$ with fiber $N_i/N_{i-1}$. The distance $d$ does not induce any distance on $N/N_i$ when $N_i$ is not normal in $N$. However, as indicated above, $d$ still induces a distance $\bar{d}$ on the fibers $N_i/N_{i-1}$.

The following result in particular applies to biLipschitz maps of $N$ (when $\mathfrak{h}_1 \neq \mathfrak{n}$).

**Theorem 5.2.** [CP17] *Suppose $(N, D)$ is not of Carnot type. Then every quasisymmetric map $F\colon N \to N$ permutes the left cosets of $N_1$ and is biLipschitz.*

By Lemma 3.2, [CPS17] (see also Lemma 4.4, [LX15] in the Carnot case), for a connected Lie subgroup $H \subset N$, two left cosets $g_1H$, $g_2H$ are at finite Hausdorff distance from each other if and only if $g_1H$, $g_2H$ lie in the same left coset of $N(H)$, the normalizer of $H$ in $N$. It follows that a biLipschitz map $F\colon N \to N$ permutes the cosets of $N_i$ for every $i$. As a consequence, $F$ induces a map $F_i\colon N/N_i \to N/N_i$ and a bundle map of the fibration $\pi_i\colon N/N_{i-1} \to N/N_i$. In general we cannot talk about the metric property of these maps as $d$ does not induce a metric on $N/N_i$ when $N_i$ is not normal in $N$. However, the restriction of $F$ to cosets of $N_i$ yields a biLipschitz map of $N_i$, that is, $F_p|_{N_i} = (L_{F(p)^{-1}} \circ F \circ L_p)|_{N_i}\colon N_i \to N_i$ is biLipschitz for every $p \in N$. As $F_p|_{N_i}$ also permutes the cosets of $N_{i-1}$, it induces a biLipschitz map $F_{i,p} : N_i/N_{i-1} \to N_i/N_{i-1}$. In other words, the restrictions of $F_{i-1}$ to the fibers of the fibration $\pi_i\colon N/N_{i-1} \to N/N_i$ are biLipschitz maps between the fibers. Observe that as $N_{i-1} \lhd N_i$, we have $F_{i,p} = F_{i,q}$ when $q = pw$ for some $w \in N_{i-1}$. Because of this, we may define $F_{i,h}$ for $h \in N/N_{i-1}$ by $F_{i,h} = F_{i,p}$ for any $p \in N$ satisfying $h = pN_{i-1}$.

**Remark 5.3.** Note that for $q = pw$ with $w \in N_i - N_{i-1}$, we have $F_{i,q}(N_{i-1}) \neq F_{i,p}(wN_{i-1})$ however the Pansu differentials satisfy $DF_{i,p}(wN_{i-1}) = DF_{i,q}(0)$ by the definition of Pansu differential.

For convenience, we introduce the following terminology.

**Definition 5.4.** Let $(N, D)$ be a diagonal Heintze pair with preserved subgroups sequence $1 = N_0 < N_1 < \cdots < N_s = N$. Let $F\colon N \to N$ be a biLipschitz map. We say $F$ is an $i$-similarity for some $1 \leq i \leq s$ if there exists a Carnot–Carathéodory metric $\bar{d}_i$ on $N_i/N_{i-1}$ such that $F_{i,p}\colon (N_i/N_{i-1}, \bar{d}_i) \to (N_i/N_{i-1}, \bar{d}_i)$ is a similarity for any $p \in N$. The map $F$ is a fiber similarity if it is an $i$-similarity for each $i$. A uniform quasisimilarity group $\Gamma$ of $N$ is a fiber similarity group if there exist Carnot–Carathéodory metrics $\bar{d}_i$ on $N_i/N_{i-1}$ such that each $\gamma \in \Gamma$ is a fiber similarity with respect to these metrics.

**5.2. Invariant measures on homogeneous spaces.** Let $N$ be a simply connected nilpotent Lie group and $H$ a closed connected Lie subgroup. Then the homogeneous space $N/H$ admits a $N$-invariant measure. By Theorem 1.2.12 and its proof in [CG90], the product of a Haar measure $m_H$ on $H$ and an invariant measure $m_{N/H}$ on $N/H$ is a Haar measure on $N$, in the following sense. There is a smooth submanifold $K \subset N$ and a measure $m'$ on $K$ with the following properties: (1) the map $P\colon K \times H \to N$, $(k, h) \mapsto kh$, is a diffeomorphism and $P_*(m' \times m_H)$ is a Haar measure on $N$; (2) $(\pi \circ \iota)_*(m') = m_{N/H}$, where $\iota\colon K \subset N$ is the inclusion and $\pi\colon N \to N/H$ is the quotient map.

**Lemma 5.5.** *Let $(N, D)$ be a diagonal Heintze pair, and $H$ a connected Lie subgroup of $N$. Let $F\colon N \to N$ be a biLipschitz map that permutes the cosets of $H$. Let $\bar{F}\colon N/H \to N/H$ be the induced map of the homogeneous space $N/H$. Then there is a constant $C$ depending only on the biLipschitz constant of $F$ such that for any measurable subset $A \subset N/H$ the inequality $\frac{1}{C} m_{N/H}(A) \leq m_{N/H}(\bar{F}(A)) \leq C m_{N/H}(A)$ holds, where $m_{N/H}$ is an invariant measure on $N/H$.*

*Proof.* Since the Hausdorff measure on $N$ with respect to a $D$-homogeneous distance is a Haar measure and $F$ is biLipschitz, there is a constant depending only on the biLipschitz constant of $F$ such that $\frac{1}{C} m_N(U) \leq m_N(F(U)) \leq C m_N(U)$ for any measurable subset $U \subset N$. Similarly, since the restriction of $F$ to cosets of $H$ is also biLipschitz, we have $\frac{1}{C} m_H(B) \leq m_H(F(B)) \leq C m_H(B)$ for measurable subsets $B$ of cosets of $H$. Now let $A \subset N/H$ be a measurable subset. Let $A' = \pi^{-1}(A) \cap K$, where $K$ is as above. Let $B$ be a bounded open subset of $H$. Set $U = P(A' \times B) \subset N$. Then $m_N(U) = m_H(B) m_{N/H}(A)$. By the above we have

$$\frac{1}{C} m_N(U) \leq m_N(F(U)) \leq C m_N(U). \tag{5}$$

On the other hand,

$$\begin{aligned} m_N(F(U)) &= \int_N 1_{F(U)} \, dm_N = \int_K \Big(\int_{kH} 1_{F(U)} \, dm_H\Big) \, dm'(k) \\ &\leq \int_{\pi^{-1}(\bar{F}(A)) \cap K} C m_H(B) \, dm'(k) = C m_H(B) m_{N/H}(\bar{F}(A)). \end{aligned}$$

Similarly we obtain $m_N(F(U)) \geq \frac{1}{C} m_H(B) m_{N/H}(\bar{F}(A))$. Combining the above inequalities we get $\frac{1}{C^2} m_{N/H}(A) \leq m_{N/H}(\bar{F}(A)) \leq C^2 m_{N/H}(A)$. □

**5.3. Conjugating into a fiber similarity group.** Let $1 \leq i \leq s$ be such that $\dim(N_i/N_{i-1}) \geq 2$. Recall that the quotient $N_i/N_{i-1}$ is a Carnot group. Let $H_i$ be the first layer of $\mathfrak{n}_i/\mathfrak{n}_{i-1}$ and $m_i = \dim(H_i)$. Notice that $m_i \geq 2$.

**Definition 5.6.** A measurable fiber conformal structure on the fibration $\pi_i\colon N/N_{i-1} \to N/N_i$ is an essentially bounded measurable map

$$\mu\colon N/N_{i-1} \to X := SL(m_i)/SO(m_i).$$

So a measurable fiber conformal structure on the fibration $\pi_i\colon N/N_{i-1} \to N/N_i$ can be thought of as a measurable assignment of inner products (up to scalar multiple) to horizontal subspaces of the tangent spaces of the fibers (of the fibration). For a review of $SL(n)/SO(n)$ and its Riemannian distance $d_X$ see Section 4.1.

For $F\colon N \to N$ a biLipschitz map we define the pull-back

$$\begin{aligned} F^*\mu(pN_{i-1}) &= (DF_{i,p}(0)|_{H_i})^T[\mu(F_{i-1}(pN_{i-1}))] \\ &= (\det(DF_{i,p}(0)|_{H_i}))^{-\frac{2}{m_i}} (DF_{i,p}(0)|_{H_i})^T \mu(F_{i-1}(pN_{i-1}))(DF_{i,p}(0)|_{H_i}), \end{aligned}$$

for a.e. $pN_{i-1} \in N/N_{i-1}$. Notice that the pull-back is well-defined as $F_{i,p}$ depends only on the coset $pN_{i-1}$.

**Definition 5.7.** We say that $F$ is *conformal with respect to* $\mu$ if $F^*\mu(pN_{i-1}) = \mu(pN_{i-1})$ for a.e. $pN_{i-1} \in N/N_{i-1}$.

**Proposition 5.8.** *Let $\Gamma$ be a countable uniform quasisimilarity group of $(N,d)$. Let $1 \leq i \leq s$ be such that $\dim(N_i/N_{i-1}) \geq 2$. Then there is a measurable fiber conformal structure $\mu$ on the fibration $\pi_i\colon N/N_{i-1} \to N/N_i$ such that every $\gamma \in \Gamma$ is conformal with respect to $\mu$.*

*Proof.* There is a $\Gamma$-invariant subset $U \subset N/N_{i-1}$ of full measure such that for all $\gamma \in \Gamma$ and all $pN_{i-1} \in U$ the foliated Pansu derivative $D\gamma_{i,p}(0)$ exists and is a graded automorphism of $N_i/N_{i-1}$. Let $\mu_0$ be the $N$-invariant fiber conformal structure on the fibration $\pi_i\colon N/N_{i-1} \to N/N_i$ that is associated with an inner product on $H_i$ (the first layer of $\mathfrak{n}_i/\mathfrak{n}_{i-1}$). For each $pN_{i-1} \in U$, set $M_{pN_{i-1}} = \{\gamma^*\mu_0(pN_{i-1})\colon \gamma \in \Gamma\}$. Since $\Gamma$ is a uniform quasisimilarity group, $M_{pN_{i-1}}$ is a bounded subset of $SL(m_i)/SO(m_i)$. The assignment $pN_{i-1} \mapsto M_{pN_{i-1}}$ is $\Gamma$-invariant. Let $P(M_{pN_{i-1}})$ be the circumcenter of $M_{pN_{i-1}}$ in $SL(m_i)/SO(m_i)$ and define $\mu\colon U \to SL(m_i)/SO(m_i)$ by $\mu(pN_{i-1}) = P(M_{pN_{i-1}})$. Then $\mu$ is $\Gamma$-invariant and measurable (see the proof of Proposition 4.1 for details). □

For the proof of the main result in this section we need a modified version of Lemma 3.4.

**Lemma 5.9.** *Let $\bar{d}_i$ be a left invariant Carnot–Carathéodory metric on $N_i/N_{i-1}$. For a biLipschitz map $G\colon N \to N$ and any $pN_{i-1} \in N/N_{i-1}$, denote by $K(G_{i,p}, 0)$ the dilatation of the map $G_{i,p}\colon (N_i/N_{i-1}, \bar{d}_i) \to (N_i/N_{i-1}, \bar{d}_i)$ at 0. Let $\{F^j\colon N \to N\}$ be a sequence of $(L,C)$-quasisimilarities that converge uniformly on compact subsets to a map $F\colon N \to N$. Suppose for any compact subset $S \subset N/N_{i-1}$ and any $\epsilon > 0$ we have*

$$m_{N/N_{i-1}}(\{pN_{i-1}\colon pN_{i-1} \in S,\ K(F^j_{i,p}, 0) \geq 1+\epsilon\}) \to 0$$

*as $j \to \infty$, where $m_{N/N_{i-1}}$ is an invariant measure on the space $N/N_{i-1}$. Then for every $pN_{i-1} \in N/N_{i-1}$, $F_{i,p}$ is a similarity of $(N_i/N_{i-1}, \bar{d}_i)$.*

*Proof.* We shall show that for almost every fiber of $\pi_i$, the map $F_{i,p}$ and a subsequence of $\{F^j_{i,p}\}_j$ satisfy the assumption of Lemma 3.4 and so $F_{i,p}\colon (N_i/N_{i-1}, \bar{d}_i) \to (N_i/N_{i-1}, \bar{d}_i)$ is conformal. By Theorem 4.1 of [CO15], $F_{i,p}$ is a similarity. Since the limit of similarity maps is a similarity, the same is true for every fiber of $\pi_i$.

By Theorem 1.2.12 and its proof in [CG90], the product of a Haar measure on $N_i/N_{i-1}$ and an invariant measure on $N/N_i$ is an invariant measure on $N/N_{i-1}$, in the following sense. There is a smooth submanifold $Y_i$ of $N/N_{i-1}$ and a measure $m'$ on $Y_i$ with the following properties: (1) the (well-defined) map $P\colon Y_i \times N_i/N_{i-1} \to N/N_{i-1}$, $(yN_{i-1}, xN_{i-1}) \mapsto yxN_{i-1}$, is a diffeomorphism and $P_*(m' \times m_{N_i/N_{i-1}})$ is an invariant measure on $N/N_{i-1}$, where $m_{N_i/N_{i-1}}$ is a Haar measure on $N_i/N_{i-1}$; (2) $(\pi_i \circ \iota)_*(m')$ is an invariant measure on $N/N_i$, where $\iota\colon Y_i \subset N/N_{i-1}$ is the inclusion.

Let $B_n \subset Y_i$ $(n \geq 1)$ be an exhausting sequence of compact subsets of $Y_i$, and $A_n \subset N_i/N_{i-1}$ $(n \geq 1)$ be a sequence of balls in $N_i/N_{i-1}$ centered at the origin 0 with radius going to infinity as $n \to \infty$. Set $C_n = P(B_n \times A_n) \subset N/N_{i-1}$. For integers $n, k, j \geq 1$, let $B_{n,k,j} = \{pN_{i-1} \in C_n \mid K(F^j_{i,p}, 0) \geq 1 + 1/k\}$ and $g_{n,k,j} = 1_{B_{n,k,j}}$ be the characteristic function of $B_{n,k,j}$. Then $g_{n,k,j} \in L^1(C_n)$. By Fubini, for a.e. $h \in B_n$, the function $g^h_{n,k,j}\colon A_n \to [0,1]$ given by $g^h_{n,k,j}(xN_{i-1}) = g_{n,k,j}(P(h, xN_{i-1}))$ is integrable, $q_{n,k,j}(h) := \int_{A_n} g^h_{n,k,j}(xN_{i-1})\, dm_{N_i/N_{i-1}}$ is an integrable function on $B_n$, and $\int_{B_n} q_{n,k,j}(h)\, dm'(h) = \int_{C_n} g_{n,k,j}(pN_{i-1})\, dm_{N/N_{i-1}} = m_{N/N_{i-1}}(B_{n,k,j})$. By the assumption, for fixed $n$ and $k$ we have $m_{N/N_{i-1}}(B_{n,k,j}) \to 0$ as $j \to \infty$, which implies $q_{n,k,j} \to 0$ in $L^1(B_n)$ as $j \to \infty$. This in turn implies that for fixed $n, k$ there is a null set $E_{n,k} \subset B_n$ and a subsequence $\{q_{n,k,j_l}\}_l$ of $\{q_{n,k,j}\}_j$ such that $q_{n,k,j_l}(h) \to 0$ as $l \to \infty$ for every $h \in B_n \backslash E_{n,k}$. Set $E = \bigcup_{k,n} E_{n,k}$. Then $E$ is a null set in $Y_i$.

Let $h \in Y_i \backslash E$. Let $A$ be a compact subset of $N_i/N_{i-1}$ and $\epsilon > 0$. Pick $n, k$ sufficiently large such that $1/k < \epsilon$ and $h \in B_n$, $A \subset A_n$. By the definition of $E$ we have $q_{n,k,j_l}(h) \to 0$ as $l \to \infty$. Due to $DF_{i,p}(xN_{i-1}) = DF_{i,px}(0)$, we notice that

$$\begin{aligned} 0 \leftarrow q_{n,k,j_l}(h) &= m_{N_i/N_{i-1}}(\{x \in A_n | K(F^{j_l}_{i,hx}, 0) \geq 1 + 1/k\}) \\ &= m_{N_i/N_{i-1}}(\{x \in A_n | K(F^{j_l}_{i,h}, x) \geq 1 + 1/k\}) \\ &\geq m_{N_i/N_{i-1}}(\{x \in A | K(F^{j_l}_{i,h}, x) \geq 1 + \epsilon\}) \end{aligned}$$

as $l \to \infty$ and so the assumption of Lemma 3.4 is satisfied by $F^{j_l}_{i,h}$, $l \geq 1$, $F_{i,h}$. □

Let $S = N \rtimes_D \mathbb{R}$ be the negatively curved homogeneous space associated with $(N, D)$. As $\Gamma$ acts cocompactly on the space of distinct pairs of $N$, every point of $N$ is a radial limit point. Such an action induces a cobounded quasi-action on $S$ and vice versa. Recall that $\mu$ is measurable and hence approximately continuous at a.e. $pN_{i-1} \in N/N_{i-1}$.

**Theorem 5.10.** *Assume that* $\dim(N_i/N_{i-1}) \geq 2$. *Let* $\Gamma$ *be a uniform quasisimilarity group of* $N$ *and* $\mu$ *a* $\Gamma$*-invariant measurable fiber conformal structure on the fibration* $\pi_i\colon N/N_{i-1} \to N/N_i$. *Suppose there is a point* $p \in N$ *such that* $\mu$ *is approximately continuous at* $pN_{i-1}$ *and* $p$ *is a radial limit point for* $\Gamma$. *Then there exists a Carnot–Carathéodory metric* $\bar{d}_i$ *on* $N_i/N_{i-1}$ *and a quasisimilarity* $F_0\colon N \to N$ *such that* $F_0 G F_0^{-1}$ *consists of elements* $F$ *where the associated* $F_{i,p}\colon (N_i/N_{i-1}, \bar{d}_i) \to (N_i/N_{i-1}, \bar{d}_i)$ *are similarities for each* $p \in N$.

*Proof.* We may assume $p = 0$, and use $e^{tD}$ to denote the dilation on $N$ and $\tilde{\delta}_t$ to denote the isometry of $S$ that translates along $\{0\} \times \mathbb{R}$ and induces the boundary map $e^{tD}$. Fix a triple $T \in T(\partial S)$ and let $\sigma$ be the vertical geodesic above 0. Since 0 is a radial limit point of $\Gamma$, there exists a sequence of elements $\{h_j\}_{j=1}^{\infty}$ of $\Gamma$ and a constant $C>0$ with $\pi(h_j(T))\to 0$ and $d(\pi(h_j(T)),\sigma)\le C$. Fix a point $x_0\in\sigma$. For each $j$ there is some $s_j>0$ with $s_j\to+\infty$ as $j\to\infty$ such that $d(\tilde{\delta}_{s_j}\circ\pi\circ h_j(T),x_0)\le C$. Since $\tilde{\delta}_{s_j}$ is an isometry of $S$ that induces $e^{tD}$, we have $\tilde{\delta}_{s_j}\circ\pi=\pi\circ e^{tD}$. Hence $d(\pi\circ e^{s_jD}\circ h_j(T),x_0)\le C$ and so the set $\{e^{s_jD}\circ h_j(T)\}_{j=1}^{\infty}$ lies in the compact subset $\pi^{-1}\bar{B}(x_0,C)$. It follows that the family $\{e^{s_jD}\circ h_j\}_{j=1}^{\infty}$ of quasisimilarity maps is precompact. After possibly passing to a subsequence we have that $F_j=e^{s_jD}\circ h_j$ converges to a quasisimilarity $F_0\colon N\to N$ uniformly on compact subsets.

We now show that for any $\gamma\in\Gamma$, the map $F=F_0\gamma F_0^{-1}$ has the property that for all $p\in N$ the associated maps $F_{i,p}$ are similarities of $N_i/N_{i-1}$ with respect to the left invariant Carnot–Carathéodory metric on $N_i/N_{i-1}$ determined by $\mu(0)$.

Let $\gamma\in\Gamma$ and denote $\tilde{\gamma}=F_0\gamma F_0^{-1}$, $\tilde{\gamma}_j=F_j\gamma F_j^{-1}$. Let $\mu_j=(F_j^{-1})^*\mu$. Since $\mu$ is $\Gamma$-invariant, $\tilde{\gamma}_j$ is conformal with respect to $\mu_j$. Additionally, $\mu_j=(e^{-s_jD})^*\mu$ so for $q\in N$ and $G=e^{-s_jD}$,

$$\mu_j(qN_{i-1})=G^*\mu(qN_{i-1})=(DG_{i,q}(0)|_{H_i})^T[\mu(G_{i-1}(qN_{i-1}))]=\mu(G_{i-1}(qN_{i-1}))$$

since $DG_{i,q}(0)|_{H_i}$ is a constant multiple of the identity map.

Let $Z\subset N/N_{i-1}$ be a compact subset containing $N_{i-1}\in N/N_{i-1}$. Since $\mu$ is approximately continuous at $N_{i-1}\in N/N_{i-1}$ and $s_j\to\infty$, the equality $\mu_j(qN_{i-1})=\mu(G_{i-1}(qN_{i-1}))$ implies that for any $\epsilon>0$ there are subsets $A_j\subset Z$ with $m_{N/N_{i-1}}(A_j)\to 0$ as $j\to\infty$ and $d_X(\mu_j(qN_{i-1}),\mu(N_{i-1}))\le\epsilon$ for $qN_{i-1}\in Z\backslash A_j$.

The maps $\tilde{\gamma}_j^{-1}$ and $\tilde{\gamma}^{-1}$ are all $(L,C)$-quasisimilarities for fixed $L$ and $C$ and so by Lemma 5.5 we have $m_{N/N_{i-1}}((\tilde{\gamma}_j)_{i-1}^{-1}(A_j))\to 0$ as $j\to\infty$. Set $B_j=A_j\cup(\tilde{\gamma}_j)_{i-1}^{-1}(A_j)$. Now we have $m_{N/N_{i-1}}(B_j)\to 0$ as $j\to\infty$ and $d_X(\mu_j(qN_{i-1}),\mu(N_{i-1}))\le\epsilon$ and $d_X(\mu_j((\tilde{\gamma}_j)_{i-1}(qN_{i-1})),\mu(N_{i-1}))\le\epsilon$ for $qN_{i-1}\in Z\backslash B_j$.

Since $\tilde{\gamma}_j$ is $\mu_j$-conformal, we have $\mu_j(qN_{i-1})=(D(\tilde{\gamma}_j)_{i,q}(0)|_{H_i})^T[\mu_j((\tilde{\gamma}_j)_{i-1}(qN_{i-1}))]$ for a.e. $qN_{i-1}\in N/N_{i-1}$. Now

$$d_X(\mu_j(qN_{i-1}),(D(\tilde{\gamma}_j)_{i,q}(0)|_{H_i})^T[\mu(N_{i-1})])=d_X(\mu_j((\tilde{\gamma}_j)_{i-1}(qN_{i-1})),\mu(N_{i-1}))\le\epsilon$$

for a.e. $qN_{i-1}\in Z\backslash B_j$. Combining this with $d_X(\mu_j(qN_{i-1}),\mu(N_{i-1}))\le\epsilon$ , we get

$$d_X(\mu(N_{i-1}),(D(\tilde{\gamma}_j)_{i,q}(0)|_{H_i})^T[\mu(N_{i-1})])\le 2\epsilon$$

for a.e. $qN_{i-1}\in Z\backslash B_j$. By (4) this implies that the assumption of Lemma 5.9 is satisfied by $\tilde{\gamma}_j$ $(j\ge 1)$, $\tilde{\gamma}$ and so for any $p\in N$, $\tilde{\gamma}_{i,p}$ is a similarity of $N_i/N_{i-1}$ with respect to the left invariant Carnot–Carathéodory metric determined by $\mu(N_{i-1})$. □

*Proof of Theorem 1.3.* We observe that we may assume the group $\Gamma$ is countable: Let $\Gamma_0$ be a countable subgroup of $\Gamma$ that is dense in the topology of uniform convergence on compact subsets; if every element of some conjugate of $\Gamma_0$ is a $i$-fiber similarity then the same is true for $\Gamma$ as the limits of $i$-fiber similarity maps are $i$-fiber similarity maps.

The assumption implies that every point $p\in N$ is a radial limit point of $\Gamma$ and so Theorem 5.10 can be applied. List the elements of $I$ by $i_1<i_2<\cdots<i_k$. We first conjugate $\Gamma$ to get a group $\Gamma_1$ which is a $i_1$-fiber similarity group, then conjugate $\Gamma_1$ to get a group $\Gamma_2=F_0\Gamma_1F_0^{-1}$ which is a $i_2$-fiber similarity group, and so on. We can finish by induction once we observe that $\Gamma_2$ is also a $i_1$-fiber similarity group. For this we just notice that the conjugating map $F_0$ is the limit of a sequence of maps $F_j$ which are compositions of elements of $\Gamma_1$ and the dilations $e^{tD}$ $(t\in\mathbb{R})$ of $N$. Since both $\{e^{tD}|t\in\mathbb{R}\}$ and $\Gamma_1$ are $i_1$-fiber similarity groups, each $F_j$ is a $i_1$-fiber similarity

map. As a consequence, the limiting map $F_0$ is also a $i_1$-fiber similarity map. Hence $\Gamma_2 = F_0\Gamma_1 F_0^{-1}$ is a $i_1$-fiber similarity group. □

## 6. A counterexample

In this section we exhibit an example which shows that in general it is impossible to conjugate a uniform quasiconformal group of $\hat{N}$ into a conformal group with respect to an arbitrary pre-specified Carnot–Carathéodory metric on $N$ when that metric is not maximally symmetric.

Let $N$ be a Carnot group and $d_{CC}$ a left invariant Carnot–Carathéodory metric on $N$. Let $IA(N, d_{CC})$ be the group consisting of graded automorphisms of $N$ that are also isometries with respect to $d_{CC}$. By the main result of [CO15], the group $\mathrm{Conf}(N, d_{CC})$ of conformal maps of $(N, d_{CC})$ is generated by left translations, standard Carnot dilations and $IA(N, d_{CC})$.

**Lemma 6.1.** *Let $N$ be a Carnot group and $d_1, d_2$ two left invariant Carnot–Carathéodory metrics on $N$. Suppose there is a quasiconformal map $f\colon (N, d_1) \to (N, d_2)$ such that $f \cdot \mathrm{Conf}(N, d_1) \cdot f^{-1} \subset \mathrm{Conf}(N, d_2)$, then there is a graded automorphism $h$ of $N$ such that $h \cdot IA(N, d_1) \cdot h^{-1} \subset IA(N, d_2)$.*

*Proof.* Denote by $\phi\colon \mathrm{Conf}(N, d_1) \to \mathrm{Conf}(N, d_2)$ the injective homomorphism given by: $\phi(g_1) = fg_1f^{-1}$. Let $p \in N$ be a point such that $Df(p)$ exists and is a graded automorphism. After pre-composing and post-composing with left translations, we may assume $p = f(p) = 0$ is the origin. Set $h = Df(0)$.

Let $g \in IA(N, d_1)$. Then $g(0) = 0$. It follows that $\phi(g)(0) = 0$ and so $\phi(g)$ is the composition of a standard Carnot dilation and an element of $IA(N, d_2)$. Since $\{g^i\colon i \in \mathbb{N}\}$ has compact closure, the same is true for $\{\phi(g)^i\colon i \in \mathbb{N}\}$. This follows from the fact that conjugation by $f$ is continuous. Alternatively one can argue using the fact that $f$ is quasisymmetric. This implies that $\phi(g) \in IA(N, d_2)$. Being graded automorphisms, both $g$ and $\phi(g)$ commute with Carnot dilations. Now for any $t > 0$,

$$\delta_t f \delta_t^{-1} g (\delta_t f \delta_t^{-1})^{-1} = \delta_t f \delta_t^{-1} g \delta_t f^{-1} \delta_t^{-1} = \delta_t f g f^{-1} \delta_t^{-1} = \delta_t \phi(g) \delta_t^{-1} = \phi(g).$$

Since $\delta_t f \delta_t^{-1}$ converges uniformly on compact subsets to $h$ as $t \to +\infty$, we have $hgh^{-1} = \phi(g) \in IA(N, d_2)$. Since this is true for every $g \in IA(N, d_1)$, the lemma follows. □

We note that for any two left invariant Carnot–Carathéodory metrics $d_1, d_2$ on $N$, the group $Conf(N, d_1)$ is a uniform quasiconformal group of $(N, d_2)$. Furthermore, the action of $\mathrm{Conf}(N, d_1)$ on the space of distinct triples of $(N, d_2)$ is cocompact. To see this, let $S = N \rtimes \mathbb{R}$ where $\mathbb{R}$ acts on $N$ by Carnot dilations. Then $S \subset \mathrm{Conf}(N, d_1)$. Furthermore, when equipped with a left invariant Riemannian metric $S$ is Gromov hyperbolic with $\partial S = N \cup \{\infty\}$ and $S$ acts on itself isometrically and transitively by left translations. It follows that $S$ and hence $Conf(N, d_1)$ acts cocompactly on the space of distinct triples of $N$. If every uniform quasiconformal group of $\hat{N}$ satisfying the assumptions of Theorem 1.1 can be conjugated into the conformal group with respect to $d_2$, then there is a quasiconformal map $f\colon \hat{N} \to \hat{N}$ such that for any $g \in \mathrm{Conf}(N, d_1)$, the map $fgf^{-1}\colon (N \backslash \{fg^{-1}f^{-1}(\infty)\}, d_2) \to (N \backslash \{fgf^{-1}(\infty)\}, d_2)$ is conformal. Suppose $N$ has the property that every quasiconformal map of $\hat{N} = N \cup \{\infty\}$ fixes $\infty$. Then we have $(f|_N)\, \mathrm{Conf}(N, d_1)(f|_N)^{-1} \subset \mathrm{Conf}(N, d_2)$ and by Lemma 6.1 we conclude that $IA(N, d_1)$ injects into $IA(N, d_2)$. Next we exhibit an example where such an injection does not exist.

Let $N = H \times H$ be the direct product of the first Heisenberg group with itself. Theorem 1.1 in [KMX20] implies that every quasiconformal map of $\hat{N} = N \cup \{\infty\}$ fixes $\infty$. The Lie algebra of $N$ can be written as $\mathfrak{n} = \mathfrak{h} \oplus \mathfrak{h} = V_1 \oplus V_2$ with first layer $V_1 = \mathbb{R}^2 \oplus \mathbb{R}^2$, where the first $\mathbb{R}^2$ is the first layer of the first $\mathfrak{h}$ and the second $\mathbb{R}^2$ is the first layer of the second $\mathfrak{h}$. Let $e_1, e_2$ denote the standard basis in the first $\mathbb{R}^2$, and $\tilde{e}_1, \tilde{e}_2$ denote the standard basis in the second $\mathbb{R}^2$. We consider two different left invariant Carnot–Carathéodory metrics $d_1$, $d_2$ on $N$. The metric $d_1$ is determined by the inner product $<,>_1$ on $V_1$ that has $e_1$, $e_2$, $\tilde{e}_1$, $\tilde{e}_2$ as an orthonormal basis. The metric $d_2$ is determined by the inner product $<,>_2$ on $V_1$ that has $e_1$, $e_2$, $\tilde{e}_1$, $\frac{\sqrt{2}}{2}(\tilde{e}_2 - e_1)$ as an orthonormal basis.

**Lemma 6.2.** (1) *$IA(N, d_1)$ is isomorphic to $(O(2) \oplus O(2)) \rtimes \mathbb{Z}_2$, where the generator of $\mathbb{Z}_2$ acts on $O(2) \oplus O(2)$ by $(M_1, M_2) \to (M_2, M_1)$;*

(2) *$IA(N, d_2)$ is isomorphic to $(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2) \rtimes \mathbb{Z}_2$, where the generator of $\mathbb{Z}_2$ acts on $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ by $(a, b, c) \to (b, a, c)$.*

*Proof.* For any element $X$ of a Lie algebra $\mathfrak{n}$, let $r(X)$ be the rank of the linear map

$$ad(X)\colon \mathfrak{n} \to \mathfrak{n}.$$

Then $r(X) = r(A(X))$ for any $x \in \mathfrak{n}$ and any isomorphism $A$ of Lie algebras. Notice that $r(x, y) \geq 1$ for every nonzero element $(x, y) \in V_1 = \mathbb{R}^2 \oplus \mathbb{R}^2$, and furthermore $r(x, y) = 1$ if and only if one of the following happens:

(a) $x = 0$ and $y \neq 0$;

(b) $y = 0$ and $x \neq 0$.

It follows that for any graded isomorphism $A\colon \mathfrak{n} \to \mathfrak{n}$, we have one of the following:

(i) $A(\mathbb{R}^2 \oplus \{0\}) = \mathbb{R}^2 \oplus \{0\}$ and $A(\{0\} \oplus \mathbb{R}^2) = \{0\} \oplus \mathbb{R}^2$;

(ii) $A(\mathbb{R}^2 \oplus \{0\}) = \{0\} \oplus \mathbb{R}^2$ and $A(\{0\} \oplus \mathbb{R}^2) = \mathbb{R}^2 \oplus \{0\}$.

(1) follows easily from the above paragraph.

(2) Now let $A \in IA(N, d_2)$. First assume $A$ satisfies (i). Note the orthogonal complement of $\mathbb{R}^2 \oplus \{0\}$ in $(V_1, <,>_2)$ is $E_1 := \mathbb{R}\tilde{e}_1 \oplus \mathbb{R}(\frac{\sqrt{2}}{2} \cdot (\tilde{e}_2 - e_1))$, and similarly the orthogonal complement of $\{0\} \oplus \mathbb{R}^2$ is $E_2 := \mathbb{R}e_2 \oplus \mathbb{R}(e_1 - \frac{1}{3}\tilde{e}_2)$. Since $A$ preserves orthogonal complement, we have $A(E_1) = E_1$ and $A(E_2) = E_2$. Observe that $E_1 \cap (\{0\} \oplus \mathbb{R}^2) = \mathbb{R}\tilde{e}_1$ and $E_2 \cap (\mathbb{R}^2 \oplus \{0\}) = \mathbb{R}e_2$. It follows that $A(\mathbb{R}\tilde{e}_1) = \mathbb{R}\tilde{e}_1$ and $A(\mathbb{R}e_2) = \mathbb{R}e_2$. Then $A$ also preserves the orthogonal complement of $\mathbb{R}\tilde{e}_1$ in $\{0\} \oplus \mathbb{R}^2$ and that of $\mathbb{R}e_2$ in $\mathbb{R}^2 \oplus \{0\}$. That is, we have $A(\mathbb{R}\tilde{e}_2) = \mathbb{R}\tilde{e}_2$ and $A(\mathbb{R}e_1) = \mathbb{R}e_1$. From this it is easy to see that there are only 8 such isometric graded isomorphisms. They are given by $A(\tilde{e}_1) = \epsilon_1\tilde{e}_1$, $A(e_2) = \epsilon_2 e_2$, $A(e_1) = \epsilon_3 e_1$ and $A(\tilde{e}_2) = \epsilon_3\tilde{e}_2$, where $\epsilon_1, \epsilon_2, \epsilon_3 \in \{1, -1\}$. They form a group $F$ isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

A similar argument shows that there are also 8 isometric graded isomorphisms with respect to $d_2$ that satisfy (ii). They are given by $A(\tilde{e}_1) = \epsilon_1 e_2$, $A(e_2) = \epsilon_2\tilde{e}_1$, $A(e_1) = \frac{1}{\sqrt{3}}\epsilon_3\tilde{e}_2$ and $A(\tilde{e}_2) = \sqrt{3}\epsilon_3 e_1$, where $\epsilon_1, \epsilon_2, \epsilon_3 \in \{1, -1\}$. If we denote by $A_0$ the isometric graded isomorphism corresponding to $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$, then these 8 isomorphisms are simply $A_0 \cdot F$. Now it is easy to see that (2) holds. □

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