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Spectral asymptotics for generalized Schrödinger operators

Tan Duc Do* and Le Xuan Truong

Abstract. Let $d \in \{3, 4, 5, \ldots\}$. Consider $L = -\frac{1}{w} \operatorname{div}(A \nabla u) + \mu$ over its maximal domain in $L^2_w(\mathbb{R}^d)$. Under certain conditions on the weight w, the coefficient matrix A and the positive Radon measure μ we obtain upper and lower bounds on $N(\lambda, L)$—the number of eigenvalues of L that are at most $\lambda \geq 1$. Furthermore we show that the eigenfunctions of L corresponding to those eigenvalues are exponentially decaying. In the course of proofs, we develop generalized Poincaré and weighted Young convolution inequalities as the main tools for the analysis.

Yleistettyjen Schrödingerin operaattoreiden spektrin asymptoottiset ominaisuudet

Tiivistelmä. Olkoon $d \in \{3, 4, 5, \ldots\}$. Valitaan operaattorille $L = -\frac{1}{w} \operatorname{div}(A \nabla u) + \mu$ sen suurin mahdollinen määrittelyalue avaruudessa $L^2_w(\mathbb{R}^d)$. Kun teemme tiettyjä oletuksia painosta w, kerroinmatriisista A ja positiivisesta Radonin mitasta μ, saamme ylä- ja alarajoja suureelle $N(\lambda, L)$, joka on operaattorin L niiden ominaisarvojen lukumäärä, joiden suuruus on korkeintaan $\lambda \geq 1$. Lisäksi osoitamme, että näihin ominaisarvoihin liittyvät ominaisfunktiot ovat eksponentiaalisesti vaimenevia. Todistuksen aikana kehitämme Poincarén epäyhtälön ja painotetun Youngin konvoluutioepäyhtälön yleistyksiä, jotka toimivat analyysimme päätyökaluina.

1. Introduction

Shen in [She96] obtained a lower bound and an upper bound on $N(\lambda, H)$—the number of eigenvalues of H that is less than or equal to $\lambda > 0$, where H denotes the Schrödinger operator with magnetic field. The estimate is in the spirit of the classical phase-space volume estimate due to Cwickel–Lieb–Rosenbljum (cf. [Sim79, Theorem 9.3]). Moreover, in the same paper Shen also proved an exponential decay of the eigenfunctions of H. These results are then extended to various settings, namely, a weighted setting in [KS00], a Dunkl–Schrödinger operator in [Hej21], a higher-order Schrödinger operator in [ZT22], magnetic Schrödinger operators with singular potentials and irregular magnetic fields in [Pog21] and Schrödinger operators with potentials satisfying a Kato and doubling condition in [BFS23]. As a whole these together facilitate the existing literature on the behaviors of eigenvalues and eigenfunctions of Schrödinger operators which are certainly of great interest. For more information, the readers may refer to [She96, KS00, Hej21, ZT22, Pog21, BFS23, Fef83, Gur86, Iwa86, Mat91, Sim83, Sol86, Tac90] and references therein.

In this paper we generalize the results in [KS00]. Particularly, we consider degenerate Schrödinger operators with potentials being non-doubling Radon measures in a weighted setting. Our consideration here can be thought of as a local version

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of [BDT20]. Note that the idea of those non-doubling measure potentials in a non-weighted setting were first discussed in [She99].

The main difficulties are to develop corresponding tools in our weighted setting (with the presence of Radon measures). Specifically, we require a generalized Poincaré inequality (Proposition 2.4 below) and a weighted Young convolution inequality (Lemma 4.2 below). These are achieved by invoking the trace inequalities investigated by Sawyer et al.

The precise formulation of our problem is as follows. Let $d \in \{3, 4, 5, \ldots\}$. Consider

$$L = -\frac{1}{w} \operatorname{div}(A(x) \nabla u) + \mu$$

in $L^2_w(\mathbb{R}^d)$. Here w, A and μ are defined as follows.

Conditions on w:

(W1) $w \in A_1$, i.e., there exists a constant $C > 0$ such that

$$\int_B w(x)\, dx \le C\, |B|\, \operatorname*{ess\,inf}_B w$$

for all balls $B \subset \mathbb{R}^d$.

(W2) $w \in RD_\beta$ for some $\beta > 2$, i.e., there exists a constant $C > 0$ such that

$$w(B(x, tr)) \ge C\, t^\beta\, w(B(x, r))$$

for all $t > 1$.

Remark 1.1. It is well-known that (W1) implies the following doubling property (cf. [Gra09, Proposition 9.1.5]).

(D) $w \in D_\alpha$ for some $\alpha > 2$, i.e., there exists a constant $C > 0$ such that

$$w(B(x, tr)) \le C\, t^\alpha\, w(B(x, r))$$

for all $t > 1$.

We will make use of this property frequently later.

Condition on $A = (A_{ij})_{1 \le i,j \le d}$:

(A1) The coefficient matrix A is symmetric with measurable entries. Furthermore there exists a constant $\Lambda \ge 1$ such that

$$\Lambda^{-1}\, w(x)\, |\xi|^2 \le A(x)\xi \cdot \xi \le \Lambda\, w(x)\, |\xi|^2 \tag{1}$$

for a.e. $x \in \mathbb{R}^d$ and for all $\xi \in \mathbb{R}^d$.

Conditions on μ: Set $d\nu = w\, d\mu$.

(M1) There exist constants $C_0 > 0$ and $\delta > 0$ such that

$$\frac{r^2}{w(B(x,r))}\, \nu(B(x,r)) \le C_0 \left(\frac{r}{R}\right)^\delta \frac{R^2}{w(B(x,R))}\, \nu(B(x,R))$$

for all $x \in \mathbb{R}^d$ and $0 < r < R \le 1$.

Assume further that

$$(\delta \wedge 2) + \beta - d > 1,$$

where $\delta \wedge 2 := \min\{\delta, 2\}$.

(M2) There exists a constant $C_1 > 0$ such that
$$\nu(B(x,2r)) \le C_1 \left(\nu(B(x,r)) + \frac{w(B(x,r))}{r^2} \right)$$
for all $x \in \mathbb{R}^d$ and $0 < r \le 1$.

For convenience later we will assume further that $C_0 > 1$ and $C_1 > 2^\alpha$, where α is as in (D).

Remark 1.2. Conditions (M1) and (M2) can be considered as local versions of [BDT20, (4) and (5)] respectively. Specifically, [BDT20, (4) and (5)] requires that μ is a positive Radon measure satisfying the following conditions:

(i) There exist constants $C_0 > 0$ and $\delta > 0$ such that
$$\frac{r^2}{w(B(x,r))} \nu(B(x,r)) \le C_0 \left(\frac{r}{R}\right)^\delta \frac{R^2}{w(B(x,R))} \nu(B(x,R))$$
for all $x \in \mathbb{R}^d$ and $R > r > 0$.

(ii) There exists a constant $C_1 > 0$ such that
$$\nu(B(x,2r)) \le C_1 \left(\nu(B(x,r)) + \frac{w(B(x,r))}{r^2} \right)$$
for all $x \in \mathbb{R}^d$ and $r > 0$.

The operator L can be described precisely via form method. See Subsection 2.2 below.

To formulate our main results the following notion plays a fundamental role. It is convenient to denote
$$d\nu_1 = d\nu + C_1 \, w \, dx,$$
where as before $d\nu = w \, d\mu$ and C_1 is given by (M2). For all $x \in \mathbb{R}^d$ define

$$\rho_w(x,\nu_1) := \frac{1}{m_w(x,\nu_1)} := \sup \left\{ r > 0 \colon \frac{r^2}{w(B(x,r))} \nu_1(B(x,r)) < C_1 \right\} \tag{2}$$

which is called the *critical function*.

For each $\lambda > 0$ denote $N(\lambda, L)$ to be the number of eigenvalues of L which are less than or equal to λ. Our first main result is as follows.

Theorem 1.3. *Suppose* (W1), (W2) *and* (A1) *hold. Let μ be a positive Radon measure on $\mathbb{R}^d$ which satisfies* (M1) *and* (M2). *Suppose further that there exist constants d_1 and d_2 such that*

$$0 < d_1 \le w(B(x,1)) \le d_2 < \infty \tag{3}$$

for all $x \in \mathbb{R}^d$. Then there exist constants $c, C, \kappa_1, \kappa_2 > 0$ such that
$$c \, \lambda^{\beta/2} \, w(\Sigma_1) \le N(\lambda, L) \le C \, \lambda^{\alpha/2} \, w(\Sigma_2)$$
for all $\lambda \ge 1$, where
$$\Sigma_j := \{x \in \mathbb{R}^d \colon m_w(x,\nu_1) \le \kappa_j \sqrt{\lambda}\}, \quad j \in \{1,2\}$$
and α, β are given in (D) *and* (W2).

As a consequence of Theorem 1.3, we obtain a characterization on the discreteness of the spectrum of L. Also see [She96, Corollary 0.11].

Corollary 1.4. *Adopt the assumptions and notation in Theorem 1.3. Then L has a discrete spectrum if and only if*

$$\lim_{|x|\to\infty} m_w(x,\nu_1) = \infty. \tag{4}$$

In particular, if (4) *does not hold then L does not have a compact resolvent.*

Define

$$d(x,y) = \inf_{\gamma} \int_0^1 m_w(\gamma(t),\nu_1)\,|\gamma'(t)|\,dt \tag{5}$$

for each $x, y \in \mathbb{R}^d$, where $\gamma\colon [0,1] \longrightarrow \mathbb{R}^d$ is absolutely continuous with $\gamma(0) = x$ and $\gamma(1) = y$. This is an instance of the Agmon metric defined in [Agm82, (4.8) on p. 55]. It has been employed in establishing the exponential decay of the eigenfunctions of second-order elliptic operators, also known as the Agmon's type estimates. See [Agm82, Chapters 4 and 5]. A motivation for the Agmon metric and the Agmon-type estimates is available in [Ste21, Subsection 1.2].

Next for all $\lambda > 0$ set

$$E_\lambda := \{x \in \mathbb{R}^d \colon m_w(x,\nu_1) \le \sqrt{\lambda}\}$$

and

$$d_\lambda(x) := \inf\{d(x,y) \colon y \in E_\lambda\}$$

for all $x \in \mathbb{R}^d$.

The exponentially decaying property of eigenfunctions of L is stated as follows. The result is in the spirit of [Agm82, Theorems 5.2 and 5.3].

Theorem 1.5. *Suppose* (W1), (W2) *and* (A1) *hold. Let μ be a positive Radon measure on $\mathbb{R}^d$ which satisfies* (M1) *and* (M2). *Assume further that A consists of $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$-entries and there exists a constant $\Xi > 0$ such that*

$$\text{(A2)} \qquad |(\nabla A)(x)\,\xi\cdot\xi| \le \Xi\, w(x)\,|\xi|^2$$

for a.e. $x \in \mathbb{R}^d$ and for all $\xi \in \mathbb{R}^d$, where $(\nabla A)_{ij} := \partial_i A_{ij}$ for all $i, j \in \{1,\dots,d\}$. Let u be an eigenfunction of L whose corresponding eigenvalue is $\lambda \ge 1$. Then for all sufficiently small $\epsilon > 0$ there exist constants $C, C_\epsilon > 0$ such that

$$|u(x)| \le C_\epsilon\, \lambda^{\alpha/4}\, e^{-\epsilon\, d_{C\lambda}(x)}\, \|u\|_{L^2_w(\mathbb{R}^d)}$$

for all $x \in \mathbb{R}^d$.

We prove Theorem 1.3 and Corollary 1.4 in Section 3. Theorem 1.5 is proved in Section 4. Before that, we collect essential background on the critical function (2) and the distance function (5) in Section 2.

Notation. Throughout the paper the following set of notation is used without mentioning. Set $\mathbb{N} = \{0,1,2,3,\dots\}$ and $\mathbb{N}^* = \{1,2,3,\dots\}$. Given an index $j \in \mathbb{N}$ and a ball $B = B(x,r)$, we let $2^jB = B(x,2^jr)$, $U_0(B) = B$ and $U_j(B) = 2^jB \setminus 2^{j-1}B$ if $j \ge 1$. For all $a, b \in \mathbb{R}$, $a \wedge b = \min\{a,b\}$ and $a \vee b = \max\{a,b\}$. For all ball $B \subset \mathbb{R}^d$ we write $w(B) := \int_B w$. The constants C and c are always assumed to be positive and independent of the main parameters whose values change from line to line. For any two functions f and g, we write $f \lesssim g$ and $f \sim g$ to mean $f \le Cg$ and $cg \le f \le Cg$ respectively. Given an index $p \in [1,\infty)$, the conjugate index of p is denoted by p'. We write $L^2(\mathbb{R}^d)$ to mean the space of square-integrable function with respect to the Lebesgue measure. In a weighted setting of Lebesgue spaces we will use the notation $L^2_w(\mathbb{R}^d) = L^2(\mathbb{R}^d, dw)$, where $dw := w\,dx$ and dx is the Lebesgue measure on $\mathbb{R}^d$. In case of a measure μ other than the Lebesgue measure being used, the corresponding notation will be $L^2(\mathbb{R}^d, d\mu)$ and $L^2_w(\mathbb{R}^d, d\mu)$.

Throughout assumptions. In the whole paper we always assume (W1), (W2) and (A1) hold. Also μ is a positive Radon measure on $\mathbb{R}^d$ which satisfies (M1) and (M2).

2. Preliminaries

This section collects basic facts about the critical functions and some estimates on the distance functions.

2.1. Critical functions. This section presents crucial properties of critical functions as well as the generalized Poincaré and Fefferman–Phong inequalities.

Recall that we set

$$d\nu_1 = d\nu + C_1\, dw,$$

where $d\nu = w\, d\mu$ and C_1 is given by (M2). For this measure ν_1, the estimates (M1) and (M2) read as follows.

(M1') There exist constants $C_0 > 0$ and $\delta > 0$ such that

$$\frac{r^2}{w(B(x,r))}\, \nu_1(B(x,r)) \le C_0 \left(\frac{r}{R}\right)^{\delta\wedge 2} \frac{R^2}{w(B(x,R))}\, \nu_1(B(x,R))$$

for all $x \in \mathbb{R}^d$ and $0 < r < R \le 1$.

(M2') There exists a constant $C_1 > 0$ such that

$$\nu_1(B(x,2r)) \le C_1 \left(\nu_1(B(x,r)) + \frac{w(B(x,r))}{r^2}\right)$$

for all $x \in \mathbb{R}^d$ and $0 < r \le 1$.

The constants C_0 and C_1 in (M1') and (M2') are exactly the same as those in (M1) and (M2), keeping in mind that we chose $C_0 > 1$ and $C_1 > 2^\alpha$ previously.

Proposition 2.1. *The following properties hold for the critical function.*

(i) $\rho_w(x,\nu_1) \in (0,1]$ *for all* $x \in \mathbb{R}^d$.

(ii) *For all* $x \in \mathbb{R}^d$, *one has*

$$(2^{\beta-2} - 1)\, \frac{w(B(x,r))}{r^2} \le \nu_1(B(x,r)) \le C_1\, \frac{w(B(x,r))}{r^2},$$

where $r = \rho_w(x,\nu_1)$.

(iii) $\rho_w(x,\nu_1) \sim \rho_w(y,\nu_1)$ *if* $|x-y| < \rho_w(x,\nu_1)$.

(iv) *There exists a constant* $k_0 > 0$ *such that*

$$\frac{m_w(y,\nu_1)}{\big(1 + |x-y|\, m_w(y,\nu_1)\big)^{k_0/(k_0+1)}} \lesssim m_w(x,\nu_1) \lesssim m_w(y,\nu_1)\, \big(1 + |x-y|\, m_w(y,\nu_1)\big)^{k_0}$$

for all $|x-y| \le 1$.

Proof. Let $x, y \in \mathbb{R}^d$.

(i) It follows from (M1') that

$$\lim_{r\to 0} \frac{r^2}{w(B(x,r))}\, \nu_1(B(x,r)) = 0.$$

Also $\rho_w(x,\nu_1) \le \rho_w(x,C_1) = 1$. These imply $\rho_w(x,\nu_1) \in (0,1]$.

In the remaining part of the proof set $r = \rho_w(x, \nu_1)$ and $R = \rho_w(y, \nu_1)$. By (i) we know that $r, R \in (0, 1]$.

(ii) By definition we have

$$\nu_1(B(x,r)) = \lim_{t\to r^-} \nu_1(B(x,t)) \le C_1 \frac{w(B(x,r))}{r^2}.$$

Also

$$\begin{aligned} 2^{\beta-2} C_1 \frac{w(B(x,r))}{r^2} &\le C_1 \frac{w(B(x,2r))}{4r^2} \le \nu_1(B(x,2r)) \\ &= \nu(B(x,2r)) + C_1 w(B(x,2r)) \\ &\le C_1 \left(\nu(B(x,r)) + \frac{w(B(x,r))}{r^2}\right) + 2^\alpha C_1 w(B(x,r)), \end{aligned}$$

where we used (W2) in the first step, the definition of ρ_w in the second step as well as (M2) and (D) in the last step. Note that we chose $C_1 > 2^\alpha$, whence

$$\nu_1(B(x,r)) \ge (2^{\beta-2} - 1) \frac{w(B(x,r))}{r^2}.$$

(iii) Suppose that $|x - y| < r$. Then $B(y, r) \subset B(x, 2r)$ and $B(x, r) \subset B(y, 2r)$. Using (M2') and (ii) we obtain

$$\nu_1(B(x,2r)) \le C_1 \left[\nu_1(B(x,r)) + \frac{w(B(x,r))}{r^2}\right] \le 2C_1 \frac{w(B(x,r))}{r^2}. \tag{6}$$

Consequently it follows from (M1') that

$$\begin{aligned} \frac{(tr)^2}{w(B(y,tr))} \nu_1(B(y,tr)) &\lesssim t^{\delta\wedge 2} \frac{r^2}{w(B(y,r))} \nu_1(B(y,r)) \\ &\lesssim t^{\delta\wedge 2} \frac{r^2}{w(B(x,r))} \nu_1(B(x,2r)) \\ &\lesssim t^{\delta\wedge 2} < C_1, \end{aligned}$$

where t is chosen to be sufficiently small. Therefore $R \ge tr$ by definition, where we recall that $R = \rho_w(y, \nu_1)$. Note that this in turn implies $|x - y| \lesssim R$. By swapping the roles of x and y in the above argument, we then obtain $R \lesssim r$.

(iv) Suppose $|x - y| \le 1$. The case $|x - y| < R$ is clear from (iii). So we assume that $|x - y| \ge R$. Let $j \in \mathbb{N}^*$ be such that $2^{j-1} R \le |x - y| < 2^j R$. Then

$$B(y,R) \subset B(x, 2|x-y|), \quad B(x,R) \subset B(y, 2|x-y|) \quad \text{and} \quad B(y, |x-y|) \subset B(y, 2^j R).$$

Consequently, it follows from (M2') that

$$\begin{aligned} \nu_1(B(y, 2|x-y|)) &< C_1 \left[\nu_1(B(y,|x-y|)) + \frac{w(B(y,|x-y|))}{|x-y|^2}\right] \\ &\le C_1 \left[\nu_1(B(y,2^j R)) + \frac{w(B(y,2^j R))}{(2^{j-1}R)^2}\right]. \end{aligned}$$

We estimate the two terms on the right-hand side as follows. First using the doubling property of w we obtain

$$\frac{w(B(y,2^j R))}{(2^{j-1}R)^2} \le 2^{j\alpha} \frac{w(B(y,R))}{(2^{j-1}R)^2} = 4 \times 2^{j(\alpha-2)} \frac{w(B(y,R))}{R^2}.$$

Secondly, an iterating application of (M2') gives

$$\begin{aligned}
&\nu_1(B(y,2^jR)) \le C_1\,\nu_1(B(y,2^{j-1}R)) + C_1\,\frac{w(B(y,2^{j-1}R))}{(2^{j-1}R)^2}\\
&\le C_1^2\,\nu_1(B(y,2^{j-2}R)) + C_1^2\,\frac{w(B(y,2^{j-2}R))}{(2^{j-2}R)^2} + C_1\,2^{(j-1)\,(\alpha-2)}\,\frac{w(B(y,R))}{R^2}\\
&\le \dots\\
&\le C_1^j\,\nu_1(B(y,R)) + \big(C_1^j + C_1^{j-1}\,2^{\alpha-2} + \ldots + C_1\,2^{(j-1)\,(\alpha-2)}\big)\,\frac{w(B(y,R))}{R^2}\\
&\le 2C_1^{j+1}\,\frac{w(B(y,R))}{R^2} + \big(C_1^j + C_1^{j-1}\,2^{\alpha-2} + \ldots + C_1\,2^{(j-1)\,(\alpha-2)}\big)\,\frac{w(B(y,R))}{R^2},
\end{aligned}$$

where we used (6) in the last step. Consequently, we may infer that

$$\nu_1(B(y,2|x-y|)) \le 4C_1^2\,\big(C_1+2^{\alpha-2}\big)^j\,\frac{w(B(y,R))}{R^2}. \tag{7}$$

Next using (M1') we have

$$\begin{aligned}
&\frac{(tR)^2}{w(B(x,tR))}\,\nu_1(B(x,tR)) \le C_0\,t^{\delta\wedge 2}\,\frac{R^2}{w(B(x,R))}\,\nu_1(B(x,R))\\
&\le C_0\,t^{\delta\wedge 2}\left(\frac{2|x-y|}{R}\right)^{\alpha}\frac{R^2}{w(B(x,2|x-y|))}\,\nu_1(B(y,2|x-y|))\\
&< C_0\,t^{\delta\wedge 2}\,2^{(j+1)\,\alpha}\,\frac{R^2}{w(B(x,2|x-y|))}\,\nu_1(B(y,2|x-y|))\\
&\le C_0\,t^{\delta\wedge 2}\,2^{(j+1)\,\alpha}\,\frac{R^2}{w(B(y,R))}\left(4C_1^2\,\big(C_1+2^{\alpha-2}\big)^j\,\frac{w(B(y,R))}{R^2}\right)\\
&< 2^{2\alpha}\,C_0\,C_1^2\,\big(2^{\alpha}\,C_1+2^{2\alpha-2}\big)^j\,t^{\delta\wedge 2} < C_1,
\end{aligned}$$

where we used (7) in the fourth step and then chose

$$t = \left(\frac{1}{2^{2\alpha}\,C_0\,C_1^2\,\Big[1+(2^{\alpha}\,C_1+2^{2\alpha-2})^j\Big]}\right)^{1/(\delta\wedge 2)}$$

in the last step. So the definition of ρ_w gives $r \ge tR$ or equivalently

$$\begin{aligned}
m_w(x,\nu_1) &\le \frac{m_w(y,\nu_1)}{t} \sim m_w(y,\nu_1)\,\Big[1+\big(2^{\alpha}\,C_1+2^{2\alpha-2}\big)^j\Big]^{1/(\delta\wedge 2)}\\
&\le m_w(y,\nu_1)\,(1+2^j)^{k_0}\\
&\sim m_w(y,\nu_1)\,\big(1+|x-y|\,m_w(y,\nu_1)\big)^{k_0}, \qquad (8)
\end{aligned}$$

where

$$k_0 := \big[1 \vee \log_2\big(2^{\alpha}\,C_1+2^{2\alpha-2}\big)\big]^{1/(\delta\wedge 2)}.$$

For the remaining inequality, using (8) we obtain that

$$1+|x-y|\,m_w(x,\nu_1) \lesssim \big(1+|x-y|\,m_w(y,\nu_1)\big)^{k_0+1}.$$

With this in mind we apply (8) once more to obtain

$$m_w(y,\nu_1) \gtrsim m_w(x,\nu_1)\,\big(1+|x-y|\,m_w(x,\nu_1)\big)^{-k_0/(k_0+1)}.$$

The proof is complete. □

The next covering result is immediate from Lemma 2.1, cf. [She96, Proposition 2.1], [KS00, Lemma 10] and also [BDT20, Lemma 2.4].

Lemma 2.2. *There exist a sequence $(x_j)_{j\in\mathbb{N}} \subset \mathbb{R}^d$ and a family of functions $(\psi_j)_{j\in\mathbb{N}}$ such that the following hold.*

(i) $\bigcup_{j\in\mathbb{N}} B(x_j,\rho_j) = \mathbb{R}^d$, *where* $\rho_j = \rho(x_j,\nu_1)$ *for all* $j \in \mathbb{N}$.
(ii) *For all* $\tau \geq 1$ *there exist constants* $C, \zeta_0 > 0$ *such that*

$$\sum_{j\in\mathbb{N}} \mathbb{1}_{B(x_j,\tau\rho_j)} \leq C\,\tau^{\zeta_0}.$$

(iii) $\operatorname{supp}\psi_j \subset B(x_j,\rho_j)$ *and* $0 \leq \psi_j \leq 1$.
(iv) $|\nabla\psi_j(x)| \lesssim 1/\rho_j$ *for all* $x, y \in \mathbb{R}^d$.
(v) $\sum_{j\in\mathbb{N}} \psi_j = 1$.

Our next step is to derive the so-called Fefferman–Phong inequalities. To this end, we first prove some preliminary results.

Lemma 2.3. *The following statements hold.*

(i) *There exists a constant* $C > 0$ *such that*

$$\int_{B(x,R)} \frac{1}{|x-y|^{d-1}}\, d\nu_1(y) \leq C\,\frac{R}{|B(x,R)|}\,\nu_1(B)$$

for all balls $B = B(x,R) \subset \mathbb{R}^d$ *with* $R \in (0,1]$.

(ii) *There exists a constant* $C > 0$ *such that*

$$\int_{B(x,R)} \frac{1}{|x-y|^{d-1}}\, dy \leq C\,R$$

for all balls $B = B(x,R) \subset \mathbb{R}^d$.

Proof. Statement (ii) is a special case of (i). Therefore, we prove (i) only.
Using (M1') and the reverse doubling property of w we obtain

$$\begin{aligned} r^2\,\nu_1(B(x,r)) &\leq C_0 \left(\frac{r}{R}\right)^{\delta\wedge 2} R^2\,\frac{w(B(x,r))}{w(B(x,R))}\,\nu_1(B(x,R)) \\ &\lesssim C_0 \left(\frac{r}{R}\right)^{\delta\wedge 2} R^2 \left(\frac{r}{R}\right)^{\beta} \nu_1(B(x,R)) \end{aligned}$$

or equivalently

$$\frac{r^2}{|B(x,r)|}\,\nu_1(B(x,r)) \lesssim C_0 \left(\frac{r}{R}\right)^{\delta'} \frac{R^2}{|B(x,R)|}\,\nu_1(B(x,R)) \tag{9}$$

for all $x \in \mathbb{R}^d$ and $0 < r < R \leq 1$, where

$$\delta' := (\delta\wedge 2) + \beta - d > 1$$

by the assumption (M1).

Therefore, we infer from (M1') and (M2') that

$$\begin{aligned}\int_{B(x,R)} \frac{1}{|x-y|^{d-1}}\, d\nu_1(y) &= \sum_{j\in -\mathbb{N}} \int_{S_j(B)} \frac{1}{|x-y|^{d-1}}\, d\nu_1(y)\\ &\le \sum_{j\in -\mathbb{N}} \frac{1}{(2^{j-1}R)^{d-1}}\, \nu_1(2^j B) = 2^{d-1} \sum_{j\in -\mathbb{N}} \frac{2^j R}{|2^j B|}\, \nu_1(2^j B)\\ &\le 2^{d-1} C_0 \frac{R}{|B|}\, \nu_1(B) \sum_{j\in -\mathbb{N}} 2^{j(\delta'-1)} \le C \frac{R}{|B|}\, \nu_1(B)\end{aligned}$$

since $\delta' > 1$. □

The next result is a Poincaré-type inequality adapted to the measures dw and $d\nu_1$ in this paper, which can be considered as a weighted version of [She99, Lemma 0.14].

Proposition 2.4. *There exists a constant $C > 0$ such that*

$$\int_B \int_B |u(x) - u(y)|^2\, dw(x)\, d\nu_1(y) \le C\, r_B^2\, \nu_1(B) \int_B |\nabla u(x)|^2\, dw(x)$$

for all balls $B = B(x_B, r_B)$ with $r_B \le 1$ and $u \in C^1(\overline{B})$.

Proof. For ease of technicality, we prove the statement with balls replaced by cubes. The new statement reads as follows: There exists a constant $C > 0$ such that

$$\int_Q \int_Q |u(x) - u(y)|^2\, dw(x)\, d\nu_1(y) \le C\, r_Q^2\, \nu_1(Q) \int_Q |\nabla u(x)|^2\, dw(x)$$

for all cubes $Q = Q(x_Q, r_Q) \subset B(x_Q, 1)$ and $u \in C^1(Q)$. Here we employ the notation $Q(x_Q, r_Q)$ to mean a closed cube centered at x_Q whose side length is r_Q.

Let $Q = Q(x_Q, r_Q) \subset B(x_Q, 1)$ be a cube and $u \in C^1(Q)$. We have

$$\begin{aligned}&\int_Q \int_Q |u(x) - u(y)|^2\, dw(x)\, d\nu_1(y)\\ &\le \int_Q \int_Q |u(x) - u_{Q,0}|^2\, dw(x)\, d\nu_1(y) + \int_Q \int_Q |u(y) - u_{Q,0}|^2\, dw(x)\, d\nu_1(y)\\ &= \nu_1(Q) \int_Q |u(x) - u_{Q,0}|^2\, dw(x) + w(Q) \int_Q |u(y) - u_{Q,0}|^2\, d\nu_1(y)\\ &=: I + II,\end{aligned}$$

where

$$u_{Q,0} := \frac{1}{|Q|} \int_Q u(x)\, dx.$$

Next we estimate each term separately.

Term I: Recall from [FKS82, Lemma 1.4 and Theorem 1.2] that

$$|u(x) - u_{Q,0}| \lesssim \int_Q \frac{|\nabla u(z)|}{|x-z|^{d-1}}\, dz \tag{10}$$

for all $x \in Q$ and

$$\int_Q \left(\int_Q \frac{|\nabla u(z)|}{|x-z|^{d-1}}\, dz \right)^2 dw(x) \lesssim r_Q^2\, \|\nabla u\|^2_{L^2_w(Q)}$$

respectively.

It follows that

$$I \lesssim \qquad \nu_1(Q) \int_Q \left(\int_Q \frac{|\nabla u(z)|}{|x-z|^{d-1}}\, dz \right)^2 dw(x) \lesssim \nu_1(Q)\, r_Q^2\, \|\nabla u\|^2_{L^2_w(Q)}.$$

Term II: First observe that (9) holds if we replace a ball $B(x,r)$ with a closed cube $Q(x,r) \subset B(x,1)$ for each $x \in \mathbb{R}^d$. That is,

$$\frac{r^2}{|Q(x,r)|}\, \nu_1(Q(x,r)) \le C_0 \left(\frac{r}{R}\right)^{\delta'} \frac{R^2}{|Q(x,R)|}\, \nu_1(Q(x,R)) \tag{11}$$

for all cubes $Q(x,r) \subset Q(x,R) \subset B(x,1)$ (cf. [She99, Proof of Lemma 2.24]).

Secondly, by virtue of [SWZ96, Theorem 1.3] (also cf. [VW95, Theorem A]) we deduce the boundedness result

$$\int_Q \left(\int_Q \frac{|f(z)|}{|x-z|^{d-1}}\, dz \right)^2 d\zeta(x) \lesssim \int_Q |f(x)|^2\, dx$$

for all $f \in L^2(Q)$, provided that the measure ζ satisfies

$$\int_A \left(\int_A \frac{d\zeta(x)}{|x-y|^{d-1}} \right)^2 dy \lesssim \zeta(A) \tag{12}$$

and

$$\int_A \left(\int_A \frac{dx}{|x-y|^{d-1}} \right)^2 d\zeta(y) \lesssim |A| \tag{13}$$

for all cubes $A \subset Q$.

In view of (11) we may choose

$$d\zeta = \frac{|2Q|}{r_Q^2\, \nu_1(2Q)}\, d\nu_1,$$

where we recall that $Q = Q(x_Q, r_Q)$. Then ζ satisfies (12) and (13) due to Lemma 2.3. Explicitly we have

$$\int_Q \left(\int_Q \frac{|f(z)|}{|x-z|^{d-1}}\, dz \right)^2 d\nu_1(x) \lesssim \frac{r_Q^2}{|2Q|}\, \nu_1(2Q) \int_Q |f(x)|^2\, dx \tag{14}$$

for all $f \in L^2(Q)$.

Consequently,

$$\begin{aligned} II &\lesssim w(Q) \int_Q \left(\int_Q \frac{|\nabla u(z)|}{|x-z|^{d-1}}\, dz \right)^2 d\nu_1(x) \lesssim w(Q)\, \frac{r_Q^2}{|2Q|}\, \nu_1(2Q) \int_Q |\nabla u(x)|^2\, dx \\ &\lesssim (\operatorname*{ess\,inf}_Q w)\, r_Q^2\, \nu_1(2Q) \int_Q |\nabla u(x)|^2\, dx \le r_Q^2\, \nu_1(2Q) \int_Q |\nabla u(x)|^2\, dw(x), \end{aligned}$$

where we used the assumption that $w \in A_1$ in the third step.

Combining the estimates for I and II together, we arrive at the claim. □

Hereafter denote

$$W^{1,2}_{w,\mathrm{loc}}(\mathbb{R}^d) := \left\{ u \in L^2_{w,\mathrm{loc}}(\mathbb{R}^d) \colon \partial_j u \in L^2_{w,\mathrm{loc}}(\mathbb{R}^d) \text{ for all } j \in \{1, \ldots, d\} \right\}.$$

The following estimates are often called Fefferman–Phong inequalities.

Proposition 2.5. *Let $u \in W^{1,2}_{w,\mathrm{loc}}(\mathbb{R}^d)$ be such that $\nabla u \in L^2_w(\mathbb{R}^d)$. Then the following hold.*

(a) *If* $u \in L^2(\mathbb{R}^d, d\nu_1)$ *then* $m_w(\cdot,\nu_1)\,u \in L^2_w(\mathbb{R}^d)$ *and*

$$\int_{\mathbb{R}^d} |u|^2\, m_w(\cdot,\nu_1)^2\, dw \lesssim \int_{\mathbb{R}^d} |\nabla u|^2\, dw + \int_{\mathbb{R}^d} |u|^2\, d\nu_1.$$

(b) *If* $m_w(\cdot,\nu_1)\,u \in L^2_w(\mathbb{R}^d)$ *then* $u \in L^2(\mathbb{R}^d, d\nu_1)$ *and*

$$\int_{\mathbb{R}^d} |u|^2\, d\nu_1 \lesssim \int_{\mathbb{R}^d} |\nabla u|^2\, dw + \int_{\mathbb{R}^d} |u|^2\, m_w(\cdot,\nu_1)^2\, dw.$$

Proof. We prove (a) only. The proof for (b) is similar.

Let $x_0 \in \mathbb{R}^d$ and $r_0 = \rho_w(x_0,\nu_1)$. Set $B = B(x_0,r_0)$. It is easy to verify that Proposition 2.4 remains valid when balls are used in place of cubes in the statement. Also by density (cf. [Tur00, Theorem 2.1.4]), Proposition 2.4 holds for all $u \in W^{1,2}_w(B)$. It follows that

$$\begin{aligned}\nu_1(B) \int_B |u(x)|^2\, dw(x) &\lesssim \int_B \int_B |u(x)-u(y)|^2\, dw(x)\, d\nu_1(y) + w(B) \int_B |u(y)|^2\, d\nu_1(y)\\ &\lesssim r_0^2\, \nu_1(B) \int_B |\nabla u(x)|^2\, dw(x) + w(B) \int_B |u(x)|^2\, d\nu_1(x)\\ &\lesssim r_0^2\, \nu_1(B) \left(\int_B |\nabla u(x)|^2\, dw(x) + \int_B |u(x)|^2\, d\nu_1(x) \right),\end{aligned}$$

where we used Proposition 2.1(ii) in the last step. Hence

$$\frac{1}{r_0^2} \int_B |u|^2\, dw \lesssim \int_B |\nabla u|^2\, dw + \int_B |u|^2\, d\nu_1. \tag{15}$$

Equivalently,

$$\int_{\frac12 B} |u|^2\, d\nu_1 \lesssim \int_B |\nabla u|^2\, dw + \int_B |u|^2\, m_w(\cdot,\nu_1)^2\, dw,$$

as $m_w(x,\nu_1) \sim 1/r_0$ for all $x \in B$ by Proposition 2.1(iii).

Integrating both sides with respect to x_0 on $\mathbb{R}^d$, keeping in mind that for each $x \in B$ one has

$$\int_{|x-x_0|<\rho_w(x_0,\nu_1)} dx_0 \sim \int_{|x-x_0|<\rho_w(x,\nu_1)} dx_0 \sim m_w(x,\nu_1)^{-d}$$

and then applying Fubini's theorem, we arrive at the conclusion. □

As a direct consequence of Proposition 2.5, we obtain the following.

Corollary 2.6. *Let*

$$H := \left\{ u \in W^{1,2}_{w,\mathrm{loc}}(\mathbb{R}^d) \colon \nabla u \in L^2_w(\mathbb{R}^d) \text{ and } m_w(\cdot,\nu_1)\, u \in L^2_w(\mathbb{R}^d) \right\} \tag{16}$$

be equipped with the norm

$$\|u\|_H^2 = \int_{\mathbb{R}^d} |\nabla u|^2\, dw + \int_{\mathbb{R}^d} m_w(\cdot,\nu_1)^2\, |u|^2\, dw$$

and

$$H' := \left\{ u \in W^{1,2}_{w,\mathrm{loc}}(\mathbb{R}^d) \colon \nabla u \in L^2_w(\mathbb{R}^d) \text{ and } u \in L^2(\mathbb{R}^d, d\nu_1) \right\}$$

be equipped with the norm

$$\|u\|_{H'}^2 = \int_{\mathbb{R}^d} |\nabla u|^2\, dw + \int_{\mathbb{R}^d} |u|^2\, d\nu_1.$$

Then $H = H'$ *with equivalent norms.*

Moreover, H *is a Hilbert space (with respect to the induced inner product).*

2.2. Form and its associated operator. Consider the quadratic form

$$\mathfrak{a}(u,v) = \int_{\mathbb{R}^d} A\,\nabla u\cdot\nabla v\,dx + \int_{\mathbb{R}^d} u\,v\,d\nu$$

on the domain

$$D(\mathfrak{a}) = \big\{u \in W^{1,2}_w(\mathbb{R}^d)\colon u \in L^2(\mathbb{R}^d, d\nu)\big\},$$

where $d\nu = w\,d\mu$, the space H is given by (16) and

$$W^{1,2}_w(\mathbb{R}^d) := \big\{u \in L^2_w(\mathbb{R}^d)\colon \partial_j u \in L^2_w(\mathbb{R}^d) \text{ for all } j \in \{1,\ldots,d\}\big\}.$$

We endow $D(\mathfrak{a})$ with the graph norm

$$\|u\|_{D(\mathfrak{a})} = \Big(\mathfrak{a}(u,u) + (1+C_1)\,\|u\|^2_{L^2_w(\mathbb{R}^d)}\Big)^{1/2}$$

for all $u \in D(\mathfrak{a})$. It follows from (A1) and Corollary 2.6 that

$$\|u\|^2_{D(\mathfrak{a})} \sim \int_{\mathbb{R}^d} |\nabla u|^2\,dw + \int_{\mathbb{R}^d} |u|^2\,m_w(\cdot,\nu_1)^2\,dw + \int_{\mathbb{R}^d} |u|^2\,dw \tag{17}$$

for all $u \in D(\mathfrak{a})$.

It is easy to see that $\mathfrak{a}$ is positive and symmetric. In addition $\mathfrak{a}$ is also densely defined and closed. Specifically $C_c^\infty(\mathbb{R}^d)$ is a core for $\mathfrak{a}$. The details are presented in [BDT20, Subsection 3.1].

Hence there exists a unique self-adjoint operator

$$Lu := -\frac{1}{w}\operatorname{div}(A\,\nabla u) + \mu\,u$$

on the domain

$$D(L) = \{u \in D(\mathfrak{a})\colon Lu \in L^2_w(\mathbb{R}^d)\}$$

such that

$$\mathfrak{a}(u,v) = \langle Lu, v\rangle_{L^2_w(\mathbb{R}^d)}$$

for all $u \in D(L)$ and $v \in D(\mathfrak{a})$.

2.3. Distance function. Recall that for each $x, y \in \mathbb{R}^d$ we define

$$d(x,y) = \inf_\gamma \int_0^1 m_w(\gamma(t),\nu_1)\,|\gamma'(t)|\,dt,$$

where $\gamma\colon [0,1] \longrightarrow \mathbb{R}^d$ is absolutely continuous with $\gamma(0) = x$ and $\gamma(1) = y$.

In this section, we collect certain properties of the distance function $d(\cdot,\cdot)$ to be used in the proofs of the main theorems. With Subsection 2.1 in mind, these readily follow from the arguments performed in [She96] and [She99]. For the sake of clarity, we specify the detailed references in the proof of each statement.

For each $\lambda > 0$ set

$$d_\lambda(x) = \inf\{d(x,y)\colon y \in E_\lambda\},$$

where

$$E_\lambda = \{x \in \mathbb{R}^d\colon m_w(x,\nu_1) \le \sqrt{\lambda}\}.$$

Lemma 2.7. *One has the following.*

(i) *There exists a constant $C > 0$ such that*

$$d(x,y) \le C\,\big(1 + |x-y|\,m_w(x,\nu_1)\big)^{k_0}$$

for all $x, y \in \mathbb{R}^d$.

(ii) *There exists a constant* $C > 0$ *such that*

$$d(x,y) \geq C \left(1 + |x-y|\, m_w(x,\nu_1)\right)^{1/(k_0+1)}$$

for all $|x-y| \geq 2\rho_w(x,\nu_1)$.

Here k_0 *is taken from Proposition 2.1(iv).*

Proof. This follows from [She99, Proof of Theorem 3.11 and Remark 3.21] with obvious modifications. □

Lemma 2.8. *There exist constants* $k_1 > 0$ *and* $C > 0$ *such that*

$$m_w(y,\nu_1) \leq C\, m_w(x,\nu_1) \left(1 + d(x,y)\right)^{k_1}$$

for all $x, y \in \mathbb{R}^d$.

Proof. Note that our $m_w(\cdot,\nu_1)$ acquires all the properties as those in [She96, Lemma 1.5]. Hence the claim follows by analogous arguments used in [She96, Lemma 4.18]. □

Let $\{x_j\}_{j\in\mathbb{N}}$ and $\{\psi_j\}_{j\in\mathbb{N}}$ be as in Lemma 2.2. In what follows set

$$\phi_\lambda(x) = \sum_{j\in\mathbb{N}} d_\lambda(x_j)\, \psi_j(x).$$

Note that $0 \leq \phi_\lambda \in C^\infty(\mathbb{R}^d)$. Furthermore ϕ_λ also approximates d_λ as given in the next lemma.

Lemma 2.9. *For all* $y \in \mathbb{R}^d$ *there exists a constant* $C > 0$ *such that*

$$|\phi_\lambda(x) - d_\lambda(x)| \leq C$$

and

$$|\partial^\xi \phi_\lambda(x)| \leq C\, m_w(x,\nu_1)^{|\xi|}$$

for all $x \in \mathbb{R}^d$ *and* $\xi \in \mathbb{N}^d$ *such that* $|\xi| \leq 2$.

Proof. This follows from [She96, Lemma 4.6] with obvious modifications. □

Next we regularize ϕ_λ. Let $F \in C^\infty(0,\infty)$ be such that $F(t) = t$ if $t \in (0,1/2)$, $F(t) = 0$ if $t \geq 2$ and $0 \leq F(t) \leq t$ for all $t \geq 0$. For all $j \in \mathbb{N}$ and $x \in \mathbb{R}^d$ set

$$\phi_{\lambda,j}(x) = j\, F\left(\frac{\phi_\lambda(x)}{j}\right). \tag{18}$$

Note that $0 \leq \phi_{\lambda,j} \in C^\infty(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$.

The following properties are clear from the construction of $\{\phi_{\lambda,j}\}_{j\in\mathbb{N}}$.

Lemma 2.10. *The sequence* $\{\phi_{\lambda,j}\}_{j\in\mathbb{N}}$ *satisfies*

(i) $\phi_{\lambda,j}(x) \leq \phi_\lambda(x)$,
(ii) $\lim_{j\to\infty} \phi_{\lambda,j}(x,y) = \phi_\lambda(x)$ *and*
(iii) $|D^\xi \phi_{\lambda,j}(x,y)| \lesssim m_w(x,\nu_1)^{|\xi|}$

for all $x \in \mathbb{R}^d$ *and* $\xi \in \mathbb{N}^d$ *such that* $|\xi| \leq 2$.

3. Eigenvalue asymptotics

This section is devoted to the proofs of Theorem 1.3 and Corollary 1.4. To begin with, recall the Poincaré's inequality from [FKS82, Theorem 1.5].

Lemma 3.1. *Let $B \subset \mathbb{R}^d$ be a ball with radius $r > 0$. Then there exists a constant $C > 0$ such that*

$$\int_B |u - u_B|^2 \, dw \leq C \, r^2 \int_B |\nabla u|^2 \, dw$$

for all $u \in C^1(\overline{B})$, where $u_B := \frac{1}{w(B)} \int_B u \, dw$.

Proof of Theorem 1.3. The proof outline follows the arguments used in [She96] closely.

Let $\lambda \geq 1$. We divide the proof into two steps.

Step 1: We derive the lower bound of $N(\lambda, L)$. Let

$$E_\lambda := \{x \in \mathbb{R}^d \colon m_w(x, \nu_1) \leq \sqrt{\lambda}\}.$$

Let $\{Q_j\}_{j \in \mathbb{N}}$ be a sequence of closed cubes which tessellate $\mathbb{R}^d$ in the manner that their sides are parallel to the coordinate axes and the side lengths are identically $1/\sqrt{\lambda}$. Moreover, the interiors $\mathring{Q}_j$'s of these cubes are required to be disjoint.

Let κ be the number of such cubes whose intersection with E_λ is non-empty. Without loss of generality we may assume that those cubes are $Q_j := Q_j(x_j, 1/\sqrt{\lambda})$ with $j \in \{1, \ldots, \kappa\}$. Then

$$w(E_\lambda) = \sum_{j=1}^{\kappa} w(E_\lambda \cap Q_j) \leq \sum_{j=1}^{\kappa} w(Q_j) \lesssim d_2 \, \kappa \, \lambda^{-\beta/2},$$

where the last step follows from

$$w(Q(x, 1/\sqrt{\lambda})) \lesssim \lambda^{-\beta/2} \, w(Q(x, 1))$$

due to (RH) and $w(Q(x, 1)) \leq d_2$ for all $x \in \mathbb{R}^d$ by assumption.

Next we will show that there exists a constant κ-dimensional subspace M of $L^2_w(\mathbb{R}^d)$ with the property that there exists a constant $C_2 > 0$ such that

$$\int_{\mathbb{R}^d} A\nabla u \cdot \nabla u \, dx + \int_{\mathbb{R}^d} u^2 \, d\nu \leq C_2 \lambda \int_{\mathbb{R}^d} u^2 \, dw \tag{19}$$

for all $u \in M$.

We note that this estimate together with the min-max principle immediately imply

$$N(C_2 \lambda, L) \geq \kappa \gtrsim \frac{1}{d_2} \, \lambda^{\beta/2} \, w(E_\lambda)$$

which is the required lower bound.

To this aim, let $\eta \in C_c^\infty(\mathring{Q}(0, 1))$ be such that $\eta|_{Q(0,1/2)} = 1$. Set

$$\eta_{j,\lambda}(\cdot) = \lambda^{\beta/4} \, \eta(\sqrt{\lambda}(x - x_j)), \quad j \in \{1, \ldots, \kappa\}.$$

Define M to be the space spanned by $\{\eta_{j,\lambda}\}_{j=1}^{\kappa}$. Since the interiors $\mathring{Q}_1, \ldots, \mathring{Q}_\kappa$ are disjoint, M forms a κ-dimensional subspace of $L^2_w(\mathbb{R}^d)$.

Next let $j \in \{1, \ldots, \kappa\}$ and set $r_j = \rho_w(x_j, \nu_1)$. Denote

$$\xi = \sup_{x \in \mathbb{R}^d} \big(|\nabla \eta(x)| + |\eta(x)| \big).$$

Then

$$\int_{\mathbb{R}^d} A\nabla \eta_{j,\lambda} \cdot \nabla \eta_{j,\lambda} \, dx \leq \Lambda \, \xi^2 \, \lambda^{1+\beta/2} \, w(Q_j),$$

where we used (A1).

Also

$$\int_{\mathbb{R}^d} \eta_{j,\lambda}^2 \, d\nu \le \xi^2 \, \lambda^{\beta/2} \, \nu(Q_j) = \xi^2 \, \lambda^{1+\beta/2} \, w(Q_j) \, \frac{(1/\sqrt{\lambda})^2}{w(Q_j)} \, \nu(Q_j)$$
$$\lesssim \xi^2 \, \lambda^{1+\beta/2} \, w(Q_j) \left(\frac{1}{\sqrt{\lambda} \, r_j} \right)^{\delta} \frac{r_j^2}{w(Q(x_j, r_j))} \, \nu(Q(x_j, r_j))$$
$$\lesssim \xi^2 \, \lambda^{1+\beta/2} \, w(Q_j),$$

where the last step follows from Proposition 2.1(ii) and the fact that $1/\sqrt{\lambda} \lesssim r_j$ due to $Q_j \cap E_\lambda \neq \emptyset$.

Consequently

$$\int_{\mathbb{R}^d} A \nabla \eta_{j,\lambda} \cdot \nabla \eta_{j,\lambda} \, dx + \int_{\mathbb{R}^d} \eta_{j,\lambda}^2 \, d\nu \lesssim \xi^2 \, \lambda^{1+\beta/2} \, w(Q_j).$$

On the other hand

$$\int_{\mathbb{R}^d} \eta_{j,\lambda}^2 \, dw \ge \int_{\frac{1}{2} Q_j} \eta_{j,\lambda}^2 \, dw = \lambda^{\beta/2} \, w\Big(\frac{1}{2} Q_j \Big).$$

Now we use the doubling property of w to obtain (19).

Step 2: We derive the upper bound of $N(\lambda, L)$.

Again using the min-max principle it suffices to construct a subspace M of $L_w^2(\mathbb{R}^d)$ with the following properties: There exist constants $C_3, C_4, C_5 > 0$ such that

$$\dim M \le C_3 \, \lambda^{\alpha/2} \, w(E_\lambda) \tag{20}$$

and

$$\int_{\mathbb{R}^d} |\nabla u|^2 \, dw + \int_{\mathbb{R}^d} u^2 \, d\nu \ge C_4 \, \lambda \int_{\mathbb{R}^d} u \, dw \tag{21}$$

for all $u \in M^\perp$ and $\lambda \ge C_5$, where $M^\perp$ denotes the subspace perpendicular to M.

Fix $u \in H$, where H is given by (16). Let $(\rho_j)_{j \in \mathbb{N}}$ and $(\psi_j)_{j \in \mathbb{N}}$ be as in Lemma 2.2. Set $B_j = B(x_j, \rho_j)$ for each $j \in \mathbb{N}$.

Then

$$\sum_{j \in \mathbb{N}} \left(\int_{B_j} |\nabla (u \, \psi_j)|^2 \, dw + \int_{B_j} (u \, \psi_j)^2 \, d\nu_1 + \int_{B_j} (u \, \psi_j)^2 \, dw \right)$$
$$\lesssim \sum_{j \in \mathbb{N}} \left(\int_{B_j} |\nabla u|^2 \, \psi_j^2 \, dw + \int_{B_j} u^2 \, \frac{1}{\rho_j^2} \, dw + \int_{B_j} (u \, \psi_j)^2 \, d\nu_1 + \int_{B_j} (u \, \psi_j)^2 \, dw \right)$$
$$\lesssim \int_{\mathbb{R}^d} |\nabla u|^2 \, \psi_j^2 \, dw + \int_{\mathbb{R}^d} u^2 \, \frac{1}{\rho_j^2} \, dw + \int_{\mathbb{R}^d} (u \, \psi_j)^2 \, d\nu_1 + \int_{\mathbb{R}^d} (u \, \psi_j)^2 \, dw$$
$$\lesssim \int_{\mathbb{R}^d} |\nabla u|^2 \, \psi_j^2 \, dw + \int_{\mathbb{R}^d} (u \, \psi_j)^2 \, d\nu_1 + \int_{\mathbb{R}^d} (u \, \psi_j)^2 \, dw$$
$$\lesssim \int_{\mathbb{R}^d} |\nabla u|^2 \, \psi_j^2 \, dw + \int_{\mathbb{R}^d} (u \, \psi_j)^2 \, d\nu + \int_{\mathbb{R}^d} (u \, \psi_j)^2 \, dw \tag{22}$$

for each $j \in \mathbb{N}$, where we used Propositions 2.2(iv,ii) and 2.5(a) in the first, second and third steps respectively. Here and in what follows we recall that $d\nu_1 := d\nu + C_1 \, dw$.

Now we consider two cases.

Case 1: Let $j \in \mathbb{N}$ be such that $B_j \cap E_\lambda^C \neq \emptyset$. Then

$$\sup_{x \in B_j} m_w(x, \nu_1) \geq \sqrt{\lambda}$$

by the definition of E_λ, whence we also have

$$\inf_{x \in B_j} m_w(x, \nu_1) \geq \sqrt{\lambda}$$

due to Proposition 2.1(iii). As a consequence,

$$\begin{aligned}\int_{B_j} |\nabla(u\,\psi_j)|^2\,dw + \int_{B_j} (u\,\psi_j)^2\,d\nu_1 + \int_{B_j} (u\,\psi_j)^2\,dw &\gtrsim \int_{B_j} m_w(x,\nu_1)^2\,(u\,\psi_j)^2\,dw \\ &\gtrsim \lambda \int_{B_j} (u\,\psi_j)^2\,dw,\end{aligned}$$

where we used Proposition 2.5(a) in the first step.

Case 2: Let $j \in \mathbb{N}$ be such that $B_j \subset E_\lambda$. Let $S_j = Q(x_j, 2\rho_j)$ be the closed cube centered at x_j with sidelength $2\rho_j$ whose sides are parallel to the coordinate axes. Then $B_j \subset S_j$. Now divide S_j into n_j sub-cubes $\{S_j^k\}_{k=1}^{n_j}$ with sidelength comparable to $1/\sqrt{\lambda}$ and pairwise disjoint interiors $\mathring{S}_j^k$'s.

Next define M to be the subspace spanned by $\mathcal{F} := \{\psi_i\,\mathbb{1}_{\mathring{S}_i^k} \colon B_i \subset E_\lambda\}$ and choose $u \in M^\perp$, i.e.

$$\int_{\mathbb{R}^d} u\,f\,dw = 0$$

for all $f \in \mathcal{F}$.

We have

$$\begin{aligned}\lambda \int_{B_j} (u\psi_j)^2\,dw &= \lambda \sum_{k=1}^{n_j} \int_{S_j^k} (u\psi_j)^2\,dw \lesssim \lambda \sum_{k=1}^{n_j} \frac{1}{\lambda} \int_{S_j^k} |\nabla(u\psi_j)|^2\,dw = \int_{B_j} |\nabla(u\psi_j)|^2\,dw \\ &\leq \int_{B_j} |\nabla(u\,\psi_j)|^2\,dw + \int_{B_j} (u\,\psi_j)^2\,d\nu_1 + \int_{B_j} (u\,\psi_j)^2\,dw,\end{aligned}$$

where we used Lemma 3.1 in the second step.

By summing over $j \in \mathbb{N}$ and referring to (22) together with Proposition 2.2(ii) we conclude that (21) always holds in either case.

It remains to show (20). To this end we note that

$$w(B_j) \gtrsim n_j\,w(S_j^k) \geq n_j\,(1/\sqrt{\lambda})^\alpha\,w(Q(x_j,1)) \geq d_1\,n_j\,\lambda^{-\alpha/2},$$

where we used (D) in the second step and (3) in the third step.

As a result we obtain

$$\dim M \lesssim \sum_{B_j \subset E_\lambda} \lambda^{\alpha/2}\,w(B_j) \lesssim \lambda^{\alpha/2}\,w(E_\lambda).$$

This finishes our proof. □

We end this section with the proof of Corollary 1.4.

Proof of Corollary 1.4. Recall the sets Σ_1 and Σ_2 given in Theorem 1.3. In what follows, we signify the dependence of these two sets on the parameter $\lambda > 0$ by writing

$$\Sigma_j = \Sigma_j^\lambda, \quad j \in \{1, 2\}.$$

($\Longleftarrow$) Suppose (4) holds. Then

$$w(\Sigma_2^\lambda) < \infty$$

for all $\lambda > 0$. In view of Theorem 1.3,

$$N(\lambda, L) < \infty.$$

This implies L has a discrete spectrum.

($\Longrightarrow$) Suppose (4) does not hold. That is,

$$\lim_{|x|\to\infty} m_w(x, \nu_1) \neq \infty.$$

Then there exist a sequence $\{x_k\}_{k\in\mathbb{N}}$ and a constant $M > 1$ such that

$$\begin{cases} \lim_{k\to\infty} |x_k| = \infty, \\ |x_k - x_l| \geq 1 \quad \text{for all } k \neq l, \\ m_w(x_k, \nu_1) < M \quad \text{for all } k \in \mathbb{N}. \end{cases}$$

Using Proposition 2.1(iii), we deduce that

$$\bigcup_{k\in\mathbb{N}} B\Big(x_k, \frac{1}{M}\Big) \subset \Sigma_1^{c_0 M}$$

for some constant $c_0 > 0$ independent of $\{x_k\}_{k\in\mathbb{N}}$. Note that our choice of $\{x_k\}_{k\in\mathbb{N}}$ guarantees that the balls $B(x_k, \frac{1}{M})$ are disjoint. Moreover, the condition (3) and the doubling property (D) give

$$w\Big(B\Big(x_k, \frac{1}{M}\Big)\Big) \geq C\, M^{-\alpha}\, w(B(x_k, 1)) > C\, M^{-\alpha}\, d_1$$

for all $k \in \mathbb{N}$ and for some constant $C > 0$ independent of k. Hence

$$w(\Sigma_1^{c_0 M}) = \infty.$$

As such L does not have a discrete spectrum due to Theorem 1.3. □

4. Exponential decay of eigenfunctions

In this section we show the exponential decay of certain eigenfunctions of L, which is the content of Theorem 1.5.

We need the following Caccioppoli-type inequality.

Lemma 4.1. *Let $\lambda > 0$ and $u \in D(L)$ be such that $Lu = \lambda\, u$. Then there exists a constant $C > 0$ such that*

$$\int_B |\nabla u|^2\, dw \leq C \left(\frac{1}{R^2} \int_{2B} |u|^2\, dw + \lambda \int_{2B} |u|^2\, dw \right)$$

for all balls $B = B(x, R)$.

Proof. Let $\eta \in C_c^\infty(B)$ be such that

$$\eta \geq 0, \quad \eta|_{\sigma B} = 1 \quad \text{and} \quad |\nabla \eta| \leq \frac{1}{(1-\sigma)\, R}.$$

Using $\eta^2 u$ as a test function, we have

$$\begin{aligned}\Lambda^{-1}\int_{\sigma B}\eta^2\,|\nabla u|^2\,dw+\int_{\sigma B}\eta^2\,u^2\,d\nu &\le \int_B \eta^2\,A\,\nabla u\cdot\nabla u\,dw+\int_B\eta^2\,u^2\,d\nu\\
&= -2\int_B \eta\,u\,A\,\nabla u\cdot\nabla\eta\,dw+\lambda\int_B\eta^2\,u^2\,dw\\
&\le 2\Lambda\int_B \eta\,\nabla u\cdot u\,\nabla\eta\,dw+\lambda\int_B u^2\,dw\\
&\le \epsilon\int_B\eta^2\,|\nabla u|^2\,dw+\frac{\Lambda^2}{\epsilon}\int_B u^2\,|\nabla\eta|^2\,dw+\lambda\int_B u^2\,dw\\
&\le \epsilon\int_B\eta^2\,|\nabla u|^2\,dw+\frac{\Lambda^2}{\epsilon\,(1-\sigma)^2\,R^2}\int_B u^2\,dw+\lambda\int_B u^2\,dw\end{aligned}$$

for all $\epsilon > 0$.

Choosing a sufficiently small ϵ in the above inequality justifies our claim. □

The following result can be considered as a weighted Young convolution inequality.

Lemma 4.2. *Let $p, q, s \in (1, \infty)$ be such that*

$$\frac{1}{q}=\frac{1}{s}+\frac{1}{p}-1 \quad \text{and} \quad (\alpha-2)\,s<\alpha.$$

Then there exists a constant $C > 0$ such that

$$\left(\int_B\left(\int_B\frac{|x-y|^2}{w(B(x,|x-y|))}\,|u(y)|\,dw(y)\right)^q dw(x)\right)^{1/q}\le C\,\frac{R^2}{w(B)^{1/s'}}\,\|u\|_{L^p_w(B)} \tag{23}$$

for all balls $B \subset \mathbb{R}^d$ and $u \in L^p_w(B)$, where s' denotes the conjugate index of s.

Proof. We can rewrite (23) into the form

$$\left(\int_B\left(\int_B\frac{|x-y|^2}{w(B(x,|x-y|))}\,|u(y)|\,dw(y)\right)^q d\zeta(x)\right)^{1/q}\le C\,\|u\|_{L^p_w(B)}, \tag{24}$$

where

$$d\zeta=\left(\frac{w(B)^{s-1}}{r_B^{2s}}\right)^{q/s}dw=\left(\frac{w(B)^{1/s'}}{r_B^2}\right)^q dw$$

for all balls $B \subset \mathbb{R}^d$ and $u \in L^p_w(B)$. For ease of technicality, we will prove (24) with balls replaced by cubes.

Let $Q \subset \mathbb{R}^d$ be a cube. In view of [SWZ96, Theorem 1.3] (also cf. [VW95, Theorem A]), (24) follows if one has

$$\left(\int_A\left(\int_A\frac{|x-y|^2}{w(Q(x,|x-y|))}\,dw(y)\right)^q d\zeta(x)\right)^{1/q}\lesssim w(A)^{1/p} \tag{25}$$

and

$$\left(\int_A\left(\int_A\frac{|x-y|^2}{w(Q(y,|x-y|))}\,d\zeta(y)\right)^{p'} dw(x)\right)^{1/p'}\lesssim \zeta(A)^{1/q'} \tag{26}$$

for all cubes $A \subset Q$, where p' and q' are conjugate indices of p and q respectively.

Let $A = Q(x_A, r_A) \subset Q$ be a sub-cube. Observe that for all $x \in A$ there holds

$$A\subset B(x_A,r_A)\subset B(x,2r_A)$$

and

$$
\begin{aligned}
&\int_A \left(\frac{|x-y|^2}{w(B(x,|x-y|))}\right)^s dw(y) \\
&< \int_{B(x_A,r_A)} \left(\frac{|x-y|^2}{w(B(x,|x-y|))}\right)^s dw(y) \\
&< \int_{B(x,2r_A)} \left(\frac{|x-y|^2}{w(B(x,|x-y|))}\right)^s dw(y) \\
&= \sum_{j\in -\mathbb{N}} \int_{S_j(B(x,2r_A))} \left(\frac{|x-y|^2}{w(B(x,|x-y|))}\right)^s dw(y) \\
&\le \sum_{j\in -\mathbb{N}} \frac{(2^{j+1}\, r_A)^{2s}}{w(2^j\, B(x,r_A))^s}\, w(2^{j+1}\, B(x,r_A)) \lesssim \sum_{j\in -\mathbb{N}} \frac{(2^{j+1}\, r_A)^{2s}}{w(2^j\, B(x,r_A))^{s-1}} \\
&\lesssim \sum_{j\in -\mathbb{N}} \frac{(2^j\, r_A)^{2s}}{2^{j\,\alpha\,(s-1)}\, w(B(x,r_A))^{s-1}} = \frac{r_A^{2s}}{w(B(x,r_A))^{s-1}} \sum_{j\in -\mathbb{N}} 2^{j\,[\alpha-(\alpha-2)\,s]} \\
&\sim \frac{r_A^{2s}}{w(B(x,r_A))^{s-1}} \le \frac{r_A^{2s}}{w(A)^{s-1}},
\end{aligned} \tag{27}
$$

where we used the reverse doubling and doubling properties of w in the fifth and sixth steps respectively.

Hence to verify (25) we note that

$$
\begin{aligned}
&\left(\int_A \left(\int_A \frac{|x-y|^2}{w(Q(x,|x-y|))}\, dw(y)\right)^q d\zeta(x)\right)^{1/q} \\
&= \frac{w(A)^{1/s'}}{r_A^2} \left(\int_A \left(\int_A \frac{|x-y|^2}{w(Q(x,|x-y|))}\, dw(y)\right)^q dw(x)\right)^{1/q} \\
&\le \frac{w(A)^{1/s'}}{r_A^2}\, w(A)^{1/s'} \left(\int_A \left(\int_A \frac{|x-y|^{2s}}{w(Q(x,|x-y|))^s}\, dw(y)\right)^{q/s} dw(x)\right)^{1/q} \\
&\lesssim \frac{w(A)^{1/s'}}{r_A^2}\, w(A)^{1/s'}\, \frac{r_A^2}{w(A)^{1/s'}}\, w(A)^{1/q} = w(A)^{1/p},
\end{aligned}
$$

where we used (27) in the second-to-last step. That is, (25) is fulfilled.

Concerning (26) one has

$$
\begin{aligned}
&\left(\int_A \left(\int_A \frac{|x-y|^2}{w(Q(y,|x-y|))}\, d\zeta(y)\right)^{p'} dw(x)\right)^{1/p'} \\
&= \left(\frac{w(A)^{1/s'}}{r_A^2}\right)^q \left(\int_A \left(\int_A \frac{|x-y|^2}{w(Q(y,|x-y|))}\, dw(y)\right)^{p'} dw(x)\right)^{1/p'} \\
&\lesssim \left(\frac{w(A)^{1/s'}}{r_A^2}\right)^q \left(\int_A \left(\int_A \frac{|x-y|^2}{w(Q(y,2|x-y|))}\, dw(y)\right)^{p'} dw(x)\right)^{1/p'} \\
&\le \left(\frac{w(A)^{1/s'}}{r_A^2}\right)^q \left(\int_A \left(\int_A \frac{|x-y|^2}{w(Q(x,|x-y|))}\, dw(y)\right)^{p'} dw(x)\right)^{1/p'}
\end{aligned}
$$

$$\leq \left(\frac{w(A)^{1/s'}}{r_A^2}\right)^q w(A)^{1/s'} \left(\int_A \left(\int_A \frac{|x-y|^{2s}}{w(Q(y,|x-y|))^s}\, dw(y)\right)^{p'/s} dw(x)\right)^{1/p'}$$

$$\lesssim \left(\frac{w(A)^{1/s'}}{r_A^2}\right)^q w(A)^{1/s'} \frac{r_A^2}{w(A)^{1/s'}}\, w(A)^{1/p'} = \zeta(A)^{1/q'},$$

where we used (27) in the second-to-last step.

The proof is complete. □

Next we derive a sub-harmonic estimate for an eigenfunction of L.

Lemma 4.3. *Let $\lambda \geq 1$ and $u \in D(L)$ be such that $Lu = \lambda\, u$. Then there exists a constant $C > 0$ such that*

$$u(x_0) \leq C \left(\frac{1}{w(B)} \int_B |u|^2\, dw\right)^{1/2}$$

for all balls $B = B(x_0, R)$ such that $\lambda\, R^2 \leq 1$.

Proof. Let $B = B(x_0, R)$ be a ball such that $\lambda\, R^2 \leq 1$. Let $\eta \in C_c^\infty(\frac{1}{2}B)$ satisfy

$$\eta \leq 1, \quad \eta|_{\frac{1}{4}B} = 1, \quad \text{and} \quad |\nabla \eta| \lesssim \frac{1}{R}.$$

Then

$$|L_0\eta| = \left|\frac{1}{w}\sum_{j=1}^d \partial_j \left(\sum_{k=1}^d A_{jk}\, \partial_j \eta\right)\right| = \left|\frac{1}{w}\sum_{j=1}^d \left(\sum_{k=1}^d \left[(\partial_j A_{jk})\, \partial_j \eta + A_{jk}\, \partial_j^2 \eta\right]\right)\right|$$

$$\lesssim \frac{1}{R} + \frac{1}{R^2} \lesssim \frac{1}{R^2},$$

where we used the assumptions (A1), (A2) and the fact that $R \leq 1$ in the last two steps respectively.

Observe that

$$L(u\,\eta) = \eta\, Lu + -\frac{2}{w}\, A\nabla u \cdot \nabla \eta + u\, L_0 \eta = \lambda\, u\, \eta + -\frac{2}{w}\, A\nabla u \cdot \nabla\eta + u\, L_0\eta.$$

It follows that

$$|(u\,\eta)(x)| = \left|\left(L^{-1}\, L(u\,\eta)\right)(x)\right|$$

$$\leq \left|\left(L^{-1}\left(\lambda\, u\, \eta + -\frac{2}{w}\, A\nabla u \cdot \nabla \eta + u\, L_0\eta\right)\right)(x)\right|$$

$$\lesssim \int_{\mathbb{R}^d} \frac{|x-y|^2}{w(B(x,|x-y|))} \left(\lambda\, |u\,\eta| + 2\Lambda\, |\nabla u|\, |\nabla \eta| + |u\, L_0 \eta|\right) dw(y),$$

where we used [BDT20, Theorem 1.1] and (A1) in the last step.

Consequently, if we take $x \in \frac{1}{8}B$ then

$$\begin{aligned}
|u(x)| &\lesssim \lambda \int_{\frac{1}{2}B} \frac{|x-y|^2}{w(B(x,|x-y|))}\,|u(y)|\,dw(y) \\
&\quad + \frac{2\Lambda\,R}{w(B(x,\frac{R}{8}))}\int_{\frac{1}{2}B}|\nabla u(y)|\,dw(y) + \frac{1}{w(B(x,\frac{R}{8}))}\int_{\frac{1}{2}B}|u(y)|\,dw(y) \\
&\lesssim \lambda \int_{\frac{1}{2}B} \frac{|x-y|^2}{w(B(x,|x-y|))}\,|u(y)|\,dw(y) \\
&\quad + \frac{R}{w(B)}\int_{\frac{1}{2}B}|\nabla u(y)|\,dw(y) + \frac{1}{w(B)}\int_{\frac{1}{2}B}|u(y)|\,dw(y) \\
&\lesssim \lambda \int_{\frac{1}{2}B} \frac{|x-y|^2}{w(B(x,|x-y|))}\,|u(y)|\,dw(y) + \left(\frac{1}{w(B)}\int_{\frac{1}{2}B}|u(y)|^2\,dw(y)\right)^{1/2}, \qquad (28)
\end{aligned}$$

where we used the estimate

$$w(B(x,R)) \lesssim 8^\alpha\, w(B(x,\frac{R}{8})) \quad \text{and} \quad \frac{1}{4}B \subset B(x,R)$$

in the second step as well as Lemma 4.1, the fact that $\lambda\,R^2 \leq 1$ and Hölder's inequality in the third step.

At this stage, let $p, q, s \in (1,\infty)$ satisfy

$$\frac{1}{q} = \frac{1}{s} + \frac{1}{p} - 1 \quad \text{and} \quad (\alpha-2)\,s < \alpha. \qquad (29)$$

Then we infer from (28) and Lemma 4.2 that

$$\begin{aligned}
\|u\|_{L^q_w(\frac{1}{8}B)} &\lesssim \frac{\lambda\,R^2}{w(B)^{1/s'}}\left(\int_{\frac{1}{2}B}|u(y)|^p\,dw(y)\right)^{1/p} \\
&\quad + w(B)^{1/q}\left(\frac{1}{w(B)}\int_{\frac{1}{2}B}|u(y)|^2\,dw(y)\right)^{1/2} \\
&\lesssim \frac{1}{w(B)^{1/s'}}\left(\int_{\frac{1}{2}B}|u(y)|^p\,dw(y)\right)^{1/p} \\
&\quad + w(B)^{1/q}\left(\frac{1}{w(B)}\int_{\frac{1}{2}B}|u(y)|^2\,dw(y)\right)^{1/2},
\end{aligned}$$

where s' is the conjugate index of s and we used (27) together with $\lambda\,R^2 \leq 1$ in the last two steps respectively.

In view of (29) we can restate the last display as

$$\left(\frac{1}{w(B)}\int_{\frac{1}{8}B}|u|^q\,dw\right)^{1/q} \lesssim \left(\frac{1}{w(B)}\int_{\frac{1}{2}B}|u|^p\,dw\right)^{1/p} + \left(\frac{1}{w(B)}\int_{B}|u|^2\,dw\right)^{1/2}$$

for all $2 \leq p \leq q \leq \infty$ with $\frac{1}{q} - \frac{1}{p} > -\frac{2}{d}$. Hence by first fixing a sufficiently large p such that $\frac{1}{p} - \frac{2}{d} < 0$ and then sending $q \longrightarrow \infty$, we arrive at the claim. □

Before proving Theorem 1.5, we require two more estimates involving the distance function discussed in Subsection 2.3.

Lemma 4.4. *Let $\lambda \geq 1$ and $u \in D(L)$ be such that $Lu = \lambda u$. Then for all sufficiently small $\epsilon > 0$ there exists a constant $C > 0$ such that*

$$\|e^{\epsilon\, d_{C\lambda}}\, u\|_{L^2_w(\mathbb{R}^d)} \leq C\, \|u\|_{L^2_w(\mathbb{R}^d)}$$

Proof. Let $\epsilon > 0$ and $T > 1$ be constants which will be chosen later. Set $\phi = \phi_{T\lambda,j}$, where $\phi_{T\lambda,j}$ is given by (18).

Direct calculation gives

$$L(u\, e^{\epsilon\phi}) = e^{\epsilon\phi}\, Lu - \frac{2}{w}\, A\, \nabla u \cdot \nabla e^{\epsilon\phi} + u\, L_0 e^{\epsilon\phi} = \lambda\, u\, e^{\epsilon\phi} - \frac{2}{w}\, A\, \nabla u \cdot \nabla e^{\epsilon\phi} + u\, L_0 e^{\epsilon\phi}.$$

Also note that

$$|\nabla e^{\epsilon\phi}| \lesssim \epsilon\, m_w(\cdot, \nu_1)\, e^{\epsilon\,\phi} \quad \text{and}$$
$$|\Delta e^{\epsilon\phi}| \lesssim \epsilon\, m_w(\cdot, \nu_1)^2\, e^{\epsilon\,\phi},$$

where we used Lemma 2.10(iii). Consequently

$$\begin{aligned} |L_0 e^{\epsilon\phi}| &= \left| \frac{1}{w} \sum_{j=1}^{d} \partial_j \left(\sum_{k=1}^{d} A_{jk}\, \partial_j e^{\epsilon\phi} \right) \right| = \left| \frac{1}{w} \sum_{j=1}^{d} \left(\sum_{k=1}^{d} \left[(\partial_j A_{jk})\, \partial_j e^{\epsilon\phi} + A_{jk}\, \partial_j^2 e^{\epsilon\phi} \right] \right) \right| \\ &\lesssim \epsilon\, m_w(\cdot, \nu_1)\, e^{\epsilon\,\phi} + \epsilon\, m_w(\cdot, \nu_1)^2\, e^{\epsilon\,\phi} \\ &\leq 2\epsilon\, m_w(\cdot, \nu_1)^2\, e^{\epsilon\,\phi}, \end{aligned}$$

where we used the assumptions (A1), (A2) in the third step and the fact that $m_w(\cdot, \nu_1) \geq 1$ in the last step.

Next

$$\begin{aligned} &\left| \int_{\mathbb{R}^d} A\, \nabla u \cdot \nabla e^{\epsilon\phi}\, u\, e^{\epsilon\phi}\, dx \right| \leq \Lambda \int_{\mathbb{R}^d} |\nabla u|\, |\nabla e^{\epsilon\phi}|\, |u|\, e^{\epsilon\phi}\, dw \\ &= \Lambda \int_{\mathbb{R}^d} |\nabla(u\, e^{\epsilon\phi}) - u\, \nabla e^{\epsilon\phi}|\, |\nabla e^{\epsilon\phi}|\, |u|\, dw \\ &\leq \Lambda \int_{\mathbb{R}^d} |\nabla(u\, e^{\epsilon\phi})|\, |\nabla e^{\epsilon\phi}|\, |u|\, dw + \int_{\mathbb{R}^d} |u\, \nabla e^{\epsilon\phi}|^2\, dw \\ &\leq \epsilon \int_{\mathbb{R}^d} |\nabla(u\, e^{\epsilon\phi})|^2\, dw + \left(1 + \frac{\Lambda^2}{4\epsilon}\right) \int_{\mathbb{R}^d} |u\, \nabla e^{\epsilon\phi}|^2\, dw \\ &\lesssim \epsilon \int_{\mathbb{R}^d} |\nabla(u\, e^{\epsilon\phi})|^2\, dw + \left(1 + \frac{\Lambda^2}{4\epsilon}\right) \epsilon^2 \int_{\mathbb{R}^d} |m_w(x, \nu_1)\, u\, e^{\epsilon\phi}|^2\, dw. \end{aligned}$$

With the above estimates in mind, we proceed as follows:

$$\begin{aligned} &\mathfrak{a}(u\, e^{\epsilon\phi}, u\, e^{\epsilon\phi}) = \langle L(u\, e^{\epsilon\phi}), u\, e^{\epsilon\phi} \rangle_{L^2_w(\mathbb{R}^d)} \\ &= \lambda \int_{\mathbb{R}^d} |u\, e^{\epsilon\phi}|^2\, dw - 2 \int_{\mathbb{R}^d} A\, \nabla u \cdot \nabla e^{\epsilon\phi}\, u\, e^{\epsilon\phi}\, dx + \int_{\mathbb{R}^d} u^2\, e^{\epsilon\phi}\, L_0 e^{\epsilon\phi}\, dw \\ &\leq \lambda \int_{\mathbb{R}^d} |u\, e^{\epsilon\phi}|^2\, dw + \epsilon \int_{\mathbb{R}^d} |\nabla(u\, e^{2\epsilon\phi})|^2\, dw + \epsilon \int_{\mathbb{R}^d} |m_w(x, \nu_1)\, u\, e^{\epsilon\phi}|^2\, dw. \end{aligned}$$

On the other hand,

$$\begin{aligned} \mathfrak{a}(u\, e^{\epsilon\phi}, u\, e^{\epsilon\phi}) + C_1 \int_{\mathbb{R}^d} |u\, e^{\epsilon\phi}|^2\, dw &\geq \Lambda^{-1} \int_{\mathbb{R}^d} |\nabla(u\, e^{\epsilon\,\phi})|^2\, dw + \int_{\mathbb{R}^d} |u\, e^{\epsilon\,\phi}|^2\, d\nu_1 \\ &\gtrsim \int_{\mathbb{R}^d} |\nabla(u\, e^{\epsilon\,\phi})|^2 + \int_{\mathbb{R}^d} |m_w(x, \nu_1)\, u\, e^{\epsilon\phi}|^2\, dw, \end{aligned}$$

where we used Proposition 2.5(a) in the second step.

Hence by choosing ϵ sufficiently small and then combining the above two estimates, we obtain

$$\int_{\mathbb{R}^d} |m_w(x,\nu_1)\, u\, e^{\epsilon\phi}|^2\, dw \lesssim \int_{\mathbb{R}^d} |u\, e^{\epsilon\phi}|^2\, dw. \tag{30}$$

Recall that $\phi(x) \lesssim 1$ for all $x \in E_{T\lambda}$. Therefore

$$\begin{aligned}\int_{\mathbb{R}^d} |u\, e^{\epsilon\phi}|^2\, dw &\lesssim \int_{E_{T\lambda}} |u|^2\, dw + \frac{1}{T\lambda} \int_{E^C_{T\lambda}} |m_w(x,\nu_1)\, u\, e^{\epsilon\phi}|^2\, dw \\ &\lesssim \int_{\mathbb{R}^d} |u|^2\, dw + \frac{1}{T} \int_{\mathbb{R}^d} |u\, e^{\epsilon\phi}|^2\, dw,\end{aligned}$$

where the last step follows from (30).

Now we choose T large enough to derive

$$\int_{\mathbb{R}^d} |u\, e^{\epsilon\phi}|^2\, dw \lesssim \int_{\mathbb{R}^d} |u|^2\, dw.$$

Lastly Fatou's lemma gives

$$\int_{\mathbb{R}^d} |u\, e^{\epsilon d_{C\lambda}}|^2\, dw \le \liminf_{j\to\infty} \int_{\mathbb{R}^d} |u\, e^{\epsilon\phi}|^2\, dw \lesssim \int_{\mathbb{R}^d} |u|^2\, dw.$$

This completes our proof. □

Lemma 4.5. *Let $\lambda \ge 1$ and $u \in D(L)$ be such that $Lu = \lambda\, u$. Then for all sufficiently small $\epsilon > 0$ there exists a constant $C > 0$ such that*

$$|u(x)| \le C \left(m_w(x,\nu_1) \vee \sqrt{\lambda}\right)^{\alpha/2} e^{-\epsilon\, d_{C\lambda}(x)}\, \|u\|_{L^2_w(\mathbb{R}^d)}$$

for all $x \in \mathbb{R}^d$.

Proof. Let $x \in \mathbb{R}^d$. We consider two cases.

Case 1: Suppose $x \in E_\lambda$. We have

$$|d_\lambda(y) - d_\lambda(x)| \le d(x,y) \lesssim 1$$

for all $y \in B(x, \sqrt{\lambda})$. Therefore,

$$\begin{aligned}|u(x)| &\lesssim \lambda^{\alpha/4} \int_{B(x,\sqrt{\lambda})} |u(y)|^2\, dw(y) \\ &\lesssim \lambda^{\alpha/4}\, e^{-\epsilon\, d_\lambda(x)} \int_{B(x,\sqrt{\lambda})} |e^{\epsilon\, d_\lambda(y)} u(y)|^2\, dw(y) \\ &\lesssim \lambda^{\alpha/4}\, e^{-\epsilon\, d_\lambda(x)} \|u\|_{L^2_w(\mathbb{R}^d)},\end{aligned}$$

where we used Lemmas 4.3 and 4.4 in the first and third steps respectively.

Case 2: Suppose $x \notin E_\lambda$. Set

$$r = \rho_w(x,\nu_1) < \frac{1}{\sqrt{\lambda}}.$$

Then Lemma 4.3 gives

$$|u(x)| \lesssim r^{-\alpha/2} \left(\int_{B(x,r)} |u(y)|^2\, dy\right)^{1/2} = m_w(x,\nu_1)^{\alpha/2} \left(\int_{B(x,r)} |u(y)|^2\, dy\right)^{1/2}.$$

Now we proceed as in Step 1 to arrive at

$$|u(x)| \lesssim m_w(x,\nu_1)^{\alpha/2}\, e^{-\epsilon\, d_\lambda(x)} \|u\|_{L^2_w(\mathbb{R}^d)}.$$

Combining the two cases together justifies our claim. □

We are now ready to prove Theorem 1.5.

Proof of Theorem 1.5. Let ϵ be sufficiently small so that Lemma 4.5 holds. Let $C > 0$ be as in Lemma 4.5. Let $x \in \mathbb{R}^d$ and $y \in E_{C\lambda}$. Then Lemma 2.7(a) gives

$$m_w(x,\nu_1) \lesssim m_w(y,\nu_1)\,\big(1 + d(x,y)\big)^{k_0} \le C\sqrt{\lambda}\,\big(1 + d(x,y)\big)^{k_0}.$$

Let $C_\epsilon > 0$ be such that

$$\big(1 + d(x,y)\big)^{k_0} \le C\,C_\epsilon\,e^{\frac{\epsilon}{d}\,d_{C\lambda}(x)},$$

whence

$$m_w(x,\nu_1) \le C\,C_\epsilon\,\sqrt{\lambda}\,e^{\frac{\epsilon}{d}\,d_{C\lambda}(x)}.$$

Next we apply Lemma 4.5 to obtain

$$|u(x)| \lesssim m_w(x,\nu_1)^{d/2}\,e^{-\epsilon\,d_{C\lambda}(x)}\,\|u\|_{L^2_w(\mathbb{R}^d)} \le C\,C_\epsilon\,\lambda^{d/4}\,e^{\frac{-\epsilon}{2}\,d_{C\lambda}(x)}\,\|u\|_{L^2_w(\mathbb{R}^d)}$$

as required. □

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