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Horofunction compactifications of symmetric cones under Finsler distances

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Abstract. In this paper we consider symmetric cones equipped with invariant Finsler distances, namely the Thompson distance and the Hilbert distance. We give a complete characterisation of the horofunctions of the symmetric cone A_+° under the Thompson distance and establish a correspondence between the horofunction compactification of A_+° and the horofunction compactification of the normed space in the tangent bundle. More precisely, we show that the exponential map extends as a homeomorphism between the horofunction compactification of the normed space in the tangent bundle, which is a JB-algebra, and the horofunction compactification of A_+°. Analogues results are established for the Hilbert distance on the projective symmetric cone PA_+°. The analysis yields a concrete description of the horofunction compactifications of these spaces in terms of the facial structure of the closed unit ball of the dual norm of the norm in the tangent space.

Finslerin etäisyyksillä varustettujen symmetristen kartioiden rajafunktiokompaktisoinnit

Tiivistelmä. Tässä työssä tarkastelemme symmetrisiä kartioita, jotka on varustettu invarianteilla Finslerin etäisyyksillä, nimittäin Thompsonin etäisyydellä ja Hilbertin etäisyydellä. Kuvailemme täydellisesti Thompsonin etäisyydellä varustetun symmetrisen kartion A_+° rajafunktiot ja esitämme vastaavuuden kartion A_+° rajafunktiokompaktisoinnin ja tangenttikimpun normiavaruuden rajafunktiokompaktisoinnin välillä. Tarkemmin sanottuna osoitamme, että eksponenttikuvaus voidaan jatkaa homeomorfismiksi näiden rajafunktiokompaktisointien välille, joista jälkimmäisenä mainittu on JB-algebra. Vastaavia tuloksia saadaan Hilbertin etäisyydellä varustetulle projektiiviselle symmetriselle kartiolle PA_+°. Tämän analyysin kautta näiden avaruuksien rajafunktiokompaktisoinnit voidaan konkreettisesti kuvailla tangenttiavaruuden normia vastaavan duaalinormin suljetun yksikköpallon pintarakenteen avulla.

1. Introduction

A basic concept in metric geometry is the horofunction compactification of an unbounded metric space (M,d). It uses a modified version of the Kuratowski embedding, $\iota\colon M\to C(M,b)$, to embed the metric space M into the space $C(M,b)$ of real-valued continuous functions on M that vanish at a fixed basepoint $b\in M$ and is given by $\iota(y)(\cdot)=d(\cdot,y)-d(b,y)$. By equipping $C(M,b)$ with the topology of uniform convergence on compact sets, the closure of $\{\iota(y)\colon y\in M\}$, denoted $\overline{M}^h$, is compact. The origins of this idea go back to Gromov [5] and is called the *horofunction (or metric) compactification*, and the elements in the boundary are called *horofunctions.*

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The construction does not depend in an essential way on the basepoint, as different basepoints yield compact spaces that are homeomorphic to each other.

On the horofunction boundary $\partial\overline{M}^h = \overline{M}^h \setminus M$ there is an equivalence relation, where $h \simeq h'$ if

$$\sup_{x\in M} |h(x) - h'(x)| < \infty.$$

The objective of this paper is to identify the horofunction compactification of finite dimensional symmetric cones under certain invariant Finsler metrics and provide a detailed description of its global topology and the geometry of the equivalence classes in $\partial\overline{M}^h$.

Symmetric cones can be used to realise various classes of (Riemannian) symmetric spaces of non-compact type. Recall [13] that the interior, A_+°, of a cone A_+ in a finite dimensional real vector space A is called a *symmetric cone* if

(1) there exists an inner-product $(\cdot \mid \cdot)$ on A such that A_+ is *self-dual*, i.e.,

$$A_+ = A_+^* = \{y \in A\colon (y \mid x) \geq 0 \text{ for all } x \in A_+\}.$$

(2) A_+° is *homogeneous*, i.e., the automorphism group $\mathrm{G}(A_+) = \{g \in \mathrm{GL}(A)\colon g(A_+) = A_+\}$ acts transitively on A_+°.

A symmetric cone can be equipped with an $\mathrm{G}(A_+)$-invariant Riemannian metric that turns it into a symmetric space, see [13, Section I.4]. They also support $\mathrm{G}(A_+)$-invariant Finsler metrics, which turn them into Finsler symmetric spaces. In particular, every symmetric cone can be identified as the interior of the cone of squares in a finite dimensional formally real Jordan algebra A by the Koecher-Vinberg theorem [13]. The formally real Jordan algebra A with unit u can be equipped with the *spectral norm*:

$$\|x\|_u = \inf\{\mu > 0\colon -\mu u \leq x \leq \mu u\} = \max\{|\lambda|\colon \lambda \in \sigma(x)\},$$

where $x \leq y$ if $y-x \in A_+$, and $\sigma(x)$ is the spectrum of x. In this way A becomes a JB-algebra [3]. One can use the spectral norm to put an $\mathrm{G}(A_+)$-invariant Finsler metric F on the tangent bundle TA_+° by letting $F(x,w) = \|U_{x^{-1/2}}w\|_u$, where $U_z\colon A \to A$ is the quadratic representation of z. The Finsler distance between x and y in A_+° is the infimum of lengths,

$$L(\gamma) = \int_0^1 F(\gamma(t), \gamma'(t))\,\mathrm{d}t,$$

over all piecewise C^1-smooth paths $\gamma\colon [0,1] \to A_+^\circ$ with $\gamma(0) = x$ and $\gamma(1) = y$. In fact, this distance is known [33, 36] to coincide with the *Thompson distance* d_T on A_+°, and can be expressed as

$$d_T(x,y) = \max\{|\mu|\colon \mu \in \sigma(\log U_{y^{-1/2}}x)\} = \|\log U_{x^{-1/2}}y\|_u \quad \text{for all } x, y \in A_+^\circ.$$

It should be noted that in this setting F does not satisfy the smoothness and strong convexity conditions commonly used in the theory of Riemann–Finsler manifolds [6], and the metric space (A_+°, d_T) is not uniquely geodesic.

In this setting the Riemannian symmetries $S_x\colon A_+^\circ \to A_+^\circ$, which are given by $S_x(y) = U_x y^{-1}$, are global d_T-isometries (see [31]) that make A_+° a Finsler symmetric space. Moreover, the automorphism group $\mathrm{G}(A_+) = \{T \in \mathrm{GL}(A)\colon T(A_+) = A_+\}$ is a subgroup of the isometry group of (A_+°, d_T).

A prime example is the symmetric cone of $n\times n$ strictly positive definite Hermitian matrices $\Pi_n(\mathbb{C})$, which can be identified with the symmetric space $\mathrm{GL}_n(\mathbb{C})/\mathrm{U}_n$ by letting $\mathrm{GL}_n(\mathbb{C})$ act on $\Pi_n(\mathbb{C})$ by $a \mapsto mam^*$ for $m \in \mathrm{GL}_n(\mathbb{C})$. In this case the

Jordan product is given by $a \bullet b = (ab + ba)/2$, the unit u is the identity matrix, and the quadratic representation of a is given by $U_a\colon b \mapsto aba$. So, the Finsler metric in the tangent space at a satisfies $F(a, m) = \|a^{-1/2}ma^{-1/2}\|_u = \max\{|\lambda|\colon \lambda \in \sigma(a^{-1/2}ma^{-1/2})\}$, and the Thompson distance is given by

$$d_T(a, b) = \|\log a^{-1/2}ba^{-1/2}\|_u = \max\{|\lambda|\colon \lambda \in \sigma(\log a^{-1/2}ba^{-1/2})\}.$$

We will exploit the Jordan algebra structure of symmetric cones A_+° (which will be recalled in Section 2) to give an explicit description of the horofunctions of (A_+°, d_T) and identify the equivalence classes of $\simeq$ in the horofunction boundary in Section 3. We shall see that for these spaces each horofunction is a Busemann point, i.e., the limit of an almost geodesic. The tangent space at the unit $u \in A_+^\circ$ with its Finsler metric is the finite dimensional JB-algebra $(A, \|\cdot\|_u)$. In [30, Section 4] the horofunction compactification of $(A, \|\cdot\|_u)$ was determined and it was shown that there exists a homeomorphism from $\overline{A}^h$ onto to the closed unit ball in the dual space of $(A, \|\cdot\|_u)$ that maps each equivalence class in the horofunction boundary onto the relative interior of a boundary face of the dual unit ball. In addition, we show in Section 4 that the (Riemannian) exponential map $\exp_u\colon A \to A_+^\circ$ extends as a homeomorphism between the horofunction compactifications of $(A, \|\cdot\|_u)$ and (A_+°, d_T) preserving the equivalence relation. As a consequence we obtain a concrete realisation of the horofunction compactification of (A_+°, d_T) as the closed unit ball in the dual space of $(A, \|\cdot\|_u)$, where the equivalence classes in the horofunction boundary correspond to the relative open boundary faces of the dual unit ball.

Various symmetric spaces with a projective structure, such as $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n$ and $\mathrm{SL}_n(\mathbb{C})/\mathrm{SU}_n$, can also be studied using symmetric cones and the theory of Jordan algebras. For this reason we analyse projective symmetric cones $PA_+^\circ = A_+^\circ/\mathbb{R}_{>0}$, so $x \sim \lambda x$ for $x \in A_+^\circ$ and $\lambda > 0$. We can identify this space with the set of points x in A_+° with $\det x = 1$, that is,

$$PA_+^\circ = \{x \in A_+^\circ\colon \det x = 1\}.$$

In this case the group of projective automorphisms $\mathrm{PG}(A_+) = \{g \in \mathrm{PGL}(A)\colon g(PA_+) = PA_+\}$ acts transitively on PA_+°. For example, the space $P\Pi_n(\mathbb{C})$ corresponds to the symmetric space $\mathrm{SL}_n(\mathbb{C})/\mathrm{SU}_n$.

The tangent space at the unit $u \in PA_+^\circ$ is given by $T_uPA_+^\circ = \{w \in A\colon \mathrm{tr}\, w = 0\}$, as $(D_w \det)(x) = (\det x)\mathrm{tr}(U_{x^{-1/2}}w)$ for $x \in A_+^\circ$, see [13, p. 53]. For simplicity, we shall write T_x to denote the tangent space at $x \in PA_+^\circ$, hence we have that $T_x = U_{x^{1/2}}(T_u)$.

There is a natural $\mathrm{PG}(A_+)$-invariant Finsler metric H on the tangent bundle. For the unit $u \in PA_+^\circ$ and $w \in T_u$, let

$$H(u, w) = \inf\{\lambda\colon w \le \lambda u\} - \sup\{\lambda\colon \lambda u \le w\} = \max \sigma(w) - \min \sigma(w) = \operatorname{diam} \sigma(w).$$

Note that $H(u, \cdot)$ is a norm on T_u, which is called the *variation norm* and will be denoted by $|\cdot|_u$. For $x \in PA_+^\circ$ and $w \in T_x$, let $H(x, w) = |U_{x^{-1/2}}w|_u$.

In this case the Finsler distance d_H on PA_+° coincides with the *Hilbert distance* on PA_+°, see [36] or [30, Proposition 5.3], and satisfies

$$d_H(x, y) = \max \sigma(\log U_{x^{-1/2}}y) - \min \sigma(\log U_{x^{-1/2}}y) = \operatorname{diam} \sigma(\log U_{x^{-1/2}}y)$$

for all $x, y \in PA_+^\circ$. For example, for the space $P\Pi_n(\mathbb{C})$ we have that $d_H(a, b) = \operatorname{diam} \sigma(\log b^{-1/2}ba^{-1/2})$.

The space PA_+° is a Finsler symmetric space, where the symmetry $S_x\colon y \mapsto U_xy^{-1}$ for $y \in PA_+^\circ$, is a global d_H-isometry, see [31]. Note that S_x maps PA_+° into itself, as $\det U_xy^{-1} = (\det x)^2(\det y)^{-1}$, see [13, Proposition III.4.2].

The horofunctions of (PA_+°, d_H) were determined in [28] and it was shown in [30, Section 5] that there exists a homeomorphism from the horofunction compactification of (PA_+°, d_H) onto the closed unit ball in the dual space of $(T_u, |\cdot|_u)$ that maps each equivalence class in the horofunction boundary onto a boundary face of the dual ball. In Sections 5 and 6 we will give a explicit description of the horofunctions of the normed space $(T_u, |\cdot|_u)$, which will allow us to show that the exponential map $\exp_u\colon T_u \mapsto PA_+^\circ$ extends as a homeomorphism preserving the equivalence classes in the horofunction boundary. As result we obtain a correspondence between the horofunction compactifications of (PA_+°, d_H) and $(T_u, |\cdot|_u)$, and find that each one has an explicit realisation as the closed dual unit ball of $(T_u, |\cdot|_u)$.

The results in this paper are motivated by work of Kapovich and Leeb [25] and subsequent works by, Ji and Schilling [23, 24], Haettel, Schilling, Walsh and Wienhard [20], Schilling [39], and the author and Power [30]. In [25, Question 6.18] the question was raised if the horofunction compactification of a finite dimensional normed space is naturally homeomorphic to the closed unit ball of the dual normed space. Our results show that this is the case for the normed spaces $(T_u, |\cdot|_u)$. More generally, it is interesting to understand when there exists a homeomorphism between the horofunction compactification of a Finsler symmetric space and the closed unit ball of the dual norm of the norm in the tangent space, where the equivalence classes of $\simeq$ are mapped onto the relative interiors of the boundary faces of the dual ball. It is also interesting to determine when there exists a homeomorphism between the horofunction compactification of a Finsler symmetric space and the horofunction compactification of the normed space in the tangent bundle at the basepoint, which preserves the relation $\simeq$. The results in this paper show that both questions have a positive answer for the Finsler symmetric spaces considered here.

Compactifications of symmetric spaces is a rich subject, which has been studied extensively, see for instance [7, 17]. In recent years Finsler structures have been used to study Satake compactifications of symmetric spaces. In particular, Haettel, Schilling, Walsh and Wienhard [20], see also [39], showed that each generalised Satake compactification can be realised as a horofunction compactification with respect to an invariant polyhedral Finsler metric on the flats. Kapovich and Leeb [25] realised the maximal Satake compactification of a symmetric space of non-compact type as the horofunction compactification with respect to an invariant Finsler metric. Friedland and Freitas [15] showed that the horofunction compactification of the Siegel upper half plane of rank n under the Finsler 1-metric agrees with the bounded symmetric domain compactification, which is a minimal Satake compactification. Greenfield and Ji [16] used invariant Finsler distances to study compactifications of the Teichmüller spaces of flat tori. In particular, they study the horofunction compactification of $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n$ under the Thompson distance, which they refer to as the generalised Hilbert metric. For the Satake and Martin compactifications of symmetric spaces Ji [22] showed that they are homeomorphic to the closed unit ball in the tangent space, see also [27].

As the Hilbert and Thompson distances are important invariant Finsler distances on the symmetric spaces consider here, it would be interesting to know if their horofunction compactifications realise a generalised Satake compactification, and if so, to identify the representation associated to the Satake compactification. At the end of the paper we make a few brief remarks about this problem, but it remains open.

Explicit descriptions of the horofunction compactification of Finsler metric spaces exist only in a limited number of settings. In the case of normed spaces there are results for classical ℓ_p-spaces due to Gutièrrez [18, 19], see also [14]. For normed spaces with a polyhedral unit ball, the horofunction compactification was determined in [9, 23, 26], and it was shown in [23, 24] that it is homeomorphic to the closed dual unit ball and related to projective toric varieties. In [39, Chapter 3] Schilling showed that the horofunction compactification is homeomorphic to the closed dual unit ball for various other classes of normed spaces including l_1-sums of certain normed spaces. The unit ball of the normed space $(T_u, |\cdot|_u)$ is not polyhedral nor strictly convex, and not covered by any of the results in these works. The Busemann points in the horofunction compactification of general normed spaces were studied by Walsh in [40, 43].

Friedland and Freitas [14] determined the horofunction compactification for Finsler p-metrics on $\mathrm{GL}_n(\mathbb{C})/\mathrm{U}_n$ for $1 \leq p < \infty$. The case $p = \infty$ on $\mathrm{GL}_n(\mathbb{C})/\mathrm{U}_n$ corresponds to the Thompson distance on the symmetric cone $\Pi_n(\mathbb{C})$, and is an example of the spaces discussed here. For the Thompson distance and the Hilbert distance on general finite dimensional cones the Busemann points were investigated by Walsh in [41, 42], see also [10]. For symmetric spaces the horofunction compactification with respect to the Riemannian distance is homeomorphic to the Euclidean ball, see [11]. In fact, for CAT(0) spaces it is known that the horofunction compactifications coincides with the visual boundary, see [4, 8].

2. Preliminaries

To fix the terminology and notation we recall the basic concepts and results concerning the horofunction compactification, the Hilbert and Thompson distances on cones, and the theory of symmetric cones.

2.1. Horofunction compactifications. Let (M, d) be a metric space. Fix a *basepoint* $b \in M$, and let $\mathrm{Lip}_1(M, b)$ denote the set of all functions $f\colon M \to \mathbb{R}$ with $f(b) = 0$ and f Lipschitz-1, so $|f(x) - f(y)| \leq d(x, y)$ for all $x, y \in M$. On $\mathrm{Lip}_1(M, b)$ the topology of pointwise convergence coincides with the topology of uniform convergence on compact sets, see [35, Section 46]. Moreover, with this topology $\mathrm{Lip}_1(M, b)$ is compact. To see this we consider $\mathrm{Lip}_1(M, b)$ as a subset of the space of all real-valued functions on M, denoted $\mathbb{R}^M$ and equipped with the pointwise convergence topology. Then it not hard to show that the complement of $\mathrm{Lip}_1(M, b)$ is open in $\mathbb{R}^M$, so $\mathrm{Lip}_1(M, b)$ is closed. Furthermore, for each $f \in \mathrm{Lip}_1(M, b)$ and each $x \in M$ we have that

$$|f(x)| = |f(x) - f(b)| \leq d(x, b),$$

hence $f(x) \in [-d(x, b), d(x, b)]$ for all $x \in M$. As $\prod_{x \in M}[-d(x, b), d(x, b)]$ is a compact subset of $\mathbb{R}^M$ by Tychonoff's theorem, we find that $\mathrm{Lip}_1(M, b)$ is compact.

For $y \in M$ define the real-valued function,

$$h_y(z) = d(z, y) - d(b, y) \quad \text{with } z \in M. \tag{2.1}$$

Then $h_y(b) = 0$ and $|h_y(z) - h_y(w)| = |d(z, y) - d(w, y)| \leq d(z, w)$, hence $h_y \in \mathrm{Lip}_1(M, b)$ for all $y \in M$. Using the fact that $\mathrm{Lip}_1(M, b)$ is compact, one defines the *horofunction compactification* of (M, d) to be the closure of $\{h_y\colon y \in M\}$ in $\mathrm{Lip}_1(M, b)$, which we denote by $\overline{M}^h$. Its elements are called *metric functionals*, and the boundary $\partial\overline{M}^h = \overline{M}^h \setminus \{h_y\colon y \in M\}$ is called the *horofunction boundary*. The metric functionals in $\partial\overline{M}^h$ are called *horofunctions*. The construction does not depend in an essential way on the basepoint $b \in M$, Indeed, if we change b to b', then the horofunction compactifications are homeomorphic via the mapping $h(\cdot) \mapsto h(\cdot) - h(b')$.

If (M,d) is separable, then the topology of pointwise convergence on $\mathrm{Lip}_1(M,b)$ is metrisable, and in that case each horofunction is the limit of a sequence (x_n) in M. If (M,d) is geodesic and proper; i.e., closed balls are compact, then the embedding $\iota\colon y \in M \mapsto h_y \in \mathrm{Lip}_1(M,b)$ is a homeomorphism from M onto $\iota(M)$, and $\overline{M}^h$ is a compactification in the usual topological sense. Recall that a map γ from a (possibly unbounded) interval $I \subseteq \mathbb{R}$ into a metric space (M,d) is called a *geodesic path* if

$$d(\gamma(s),\gamma(t)) = |s-t| \quad \text{for all } s,t \in I.$$

The image, $\gamma(I)$, is called a *geodesic*, and a metric space (M,d) is said to be *geodesic* if for each $x,y \in M$ there exists a geodesic path $\gamma\colon [a,b] \to M$ connecting x and y, i.e, $\gamma(a) = x$ and $\gamma(b) = y$.

The following well-known fact will be useful in the sequel, see [38, Theorem 4.7].

Lemma 2.1. *If (M,d) is a proper geodesic metric space, then $h \in \partial\overline{M}^h$ if and only if there exists a sequence (x_n) in M with $d(b,x_n) \to \infty$ such that (h_{x_n}) converges to $h \in \overline{M}^h$ as $n \to \infty$.*

A path $\gamma\colon T \to M$, where $T \subseteq [0,\infty)$ is unbounded (possibly discrete) and $0 \in T$, is called an *almost geodesic* if for all $\varepsilon > 0$ there exists an $s_0 \geq 0$ such that

$$|d(\gamma(t),\gamma(s)) + d(\gamma(s),\gamma(0)) - t| < \varepsilon \quad \text{for all } s,t \in T \text{ with } t \geq s \geq s_0.$$

The notion of an almost geodesic goes back to Rieffel [38] and was further developed by Walsh and co-workers in [1, 32, 43]. In particular, every almost geodesic yields a horofunction for a proper geodesic metric space [38, Lemma 4.5].

Lemma 2.2. *Let (M,d) be a proper geodesic metric space. If γ is an almost geodesic in M, then*

$$h(z) = \lim_{t\to\infty} d(z,\gamma(t)) - d(b,\gamma(t))$$

exists for all $z \in M$ and $h \in \partial\overline{M}^h$.

A horofunction h in a proper geodesic metric space (M,d), is called a *Busemann point* if it is the limit of an almost geodesic γ in M, and we denote the collection of all Busemann points by $\mathcal{B}_M$. It should be noted that in general not all horofunctions need to be Busemann points.

Suppose that h and h' are horofunctions of a proper geodesic metric space (M,d). Let W_h be the collection of neighbourhoods of h in $\overline{M}^h$. The *detour cost* is defined by

$$H(h,h') = \sup_{W \in W_h} \left(\inf\{d(b,y) + h'(y)\colon y \in M \text{ and } h_y \in W\}\right),$$

and the *detour distance* is given by

$$\delta(h,h') = H(h,h') + H(h',h).$$

We note that $H(h,h') \geq 0$ and could be ∞.

It is known, see for instance [32, Lemma 3.1], that if $\gamma\colon T \to M$ is an almost geodesic converging to a horofunction h, then

$$H(h,h') = \lim_{t\to\infty} d(b,\gamma(t)) + h'(\gamma(t)) \tag{2.2}$$

for all horofunctions h'. It is also known, that on the set of Busemann points $\mathcal{B}_M$ the detour distance is a metric, where points can be at infinite distance from each other, see e.g. [32, Proposition 3.2]. The detour distance yields a partition of $\mathcal{B}_M$ into equivalence classes, called *parts*, where h and h' are equivalent if $\delta(h,h') < \infty$.

The restriction of the equivalence classes of $\simeq$ in the horofunction boundary $\partial\overline{M}^h$ to the set of Busemann points $\mathcal{B}_M$ coincides with the partition of $\mathcal{B}_M$ into parts, as

$$\sup_{x\in M}(h'(x) - h(x)) = H(h,h')$$

for all $h, h' \in \mathcal{B}_M$, see [42, Proposition 4.5].

2.2. Cones and the Hilbert and Thompson distances. Let A be a real vector space. Throughout the paper we will assume that A is finite dimensional. A *cone* A_+ in A is a convex subset such that $\lambda A_+ \subseteq A_+$ for all $\lambda \geq 0$ and $A_+ \cap -A_+ = \{0\}$. A cone A_+ induces a partial ordering $\leq$ on A by $x \leq y$ if $y - x \in A_+$. We write $x < y$ if $x \leq y$ and $x \neq y$. The cone A_+ is said to be *Archimedean* if for each $x \in A$ and $y \in A_+$ with $nx \leq y$ for all $n \geq 1$, we have that $x \leq 0$. A point $u \in A_+$ is called an *order-unit* if for each $x \in A$ there exists $\lambda \geq 0$ such that $-\lambda u \leq x \leq \lambda u$. The triple (A, A_+, u), where A_+ is an Archimedean cone and u is an order-unit, is called an *order-unit space*, see [2, Chapter 1]. An order-unit space can be equipped with the *order-unit norm*,

$$\|x\|_u = \inf\{\lambda \geq 0\colon\ -\lambda u \leq x \leq \lambda u\}.$$

So for each $x \in A$ we have that $-\|x\|_u u \leq x \leq \|x\|_u u$. Moreover, the cone A_+ is closed with respect to the order-unit norm, see [2, Proposition 1.14]. The interior of A_+ will be denoted by A_+°, and is nonempty, as $u \in A_+^\circ$. Indeed, for each $y \in A_+$ with $\|y\|_u \leq 1$ we have that $u + y \in A_+$.

Given $x \in A$ and $y \in A_+$, we say that y *dominates* x if there exist $\alpha, \beta \in \mathbb{R}$ such that $\alpha y \leq x \leq \beta y$. In that case, we let

$$M(x/y) = \inf\{\beta \in \mathbb{R}\colon x \leq \beta y\} \quad \text{and} \quad m(x/y) = \sup\{\alpha \in \mathbb{R}\colon \alpha y \leq x\}.$$

We note that for each $x \in A$ we have that $\|x\|_u = \max\{M(x/u), M(-x/u)\}$. Note that if $w \in A_+^\circ$, then there exists $\delta > 0$ such that $w - \delta u \geq 0$. Thus, $u \leq \delta^{-1}w$, which implies that w is also an order-unit. So, if $w \in A_+^\circ$, then w dominates each $x \in A$. In that case we define

$$|x|_w = M(x/w) - m(x/w) \quad \text{for } x \in A.$$

It can be shown that $|\cdot|_w$ is a semi-norm on A, see [29, Lemma A.1.1], and $|x|_w = 0$ if and only if $x = \lambda w$ for some $\lambda \in \mathbb{R}$. So, $|\cdot|_w$ is a genuine norm on the quotient space $A/\mathbb{R}w$. Furthermore,

$$|x + \lambda w|_w = |x|_w \quad \text{for all } \lambda \in \mathbb{R} \text{ and } x \in V. \tag{2.3}$$

The domination relation yields an equivalence relation on A_+ by $x \sim y$ if y dominates x and x dominates y. So, $x \sim y$ if and only if there exist $0 < \alpha \leq \beta$ such that $\alpha y \leq x \leq \beta y$. The equivalence classes are called *parts* of A_+. The parts of a cone in a finite dimensional order-unit space correspond to the relative interiors of its faces, see [29, Lemma 1.2.2]. (Please note that parts of the cone are not related to parts of the set of Busemann points defined earlier.)

The *Thompson distance* on A_+ is defined as follows:

$$d_T(x,y) = \max\{\log M(x/y), \log M(y/x)\}$$

for all $x \sim y$ with $y \neq 0$, $d_T(0,0) = 0$, and $d_T(x,y) = \infty$ otherwise. The Thompson distance is a metric on each part of A_+, see [29, Chapter 2]. In particular, it is a metric on A_+°.

Likewise, the *Hilbert distance* on A_+ is given by,

$$d_H(x,y) = \log M(x/y) + \log M(y/x)$$

for all $x \sim y$ with $y \neq 0$, $d_H(0,0) = 0$, and $d_H(x,y) = \infty$ otherwise. We note that $d_H(\lambda x, \mu y) = d_H(x,y)$ for all $x, y \in A_+$ and $\lambda, \mu > 0$, hence d_H is not a metric on each part. It is, however, a metric on the parts of PA_+, so in particular on PA_+°, see [29, Proposition 2.1.1].

In the sequel the following fact will be useful. The function $(x,y) \mapsto M(x/y)$ is continuous on $A_+ \times A_+^\circ$, see [28, Lemma 2.2].

2.3. Formally real Jordan algebras. A nice summary of the link between finite dimensional formally real Jordan algebras and Riemannian symmetric spaces can be found in [34, Chapter 0]. We will now recall the basic Jordan theoretic concepts required for the paper and mostly follow the terminology in [13].

By the Koecher–Vinberg theorem [13] there is a one-to-one correspondence between the symmetric cones and the interiors of the cones of squares in finite dimensional formally real Jordan algebras (with unit). Recall that a *real Jordan algebra* is a real vector space A with a commutative bilinear product $(x,y) \in A \times A \mapsto x \bullet y \in A$ satisfying the *Jordan identity*, $x^2 \bullet (x \bullet y) = x \bullet (x^2 \bullet y)$. A Jordan algebra A is said to be *formally real* if $x^2 + y^2 = 0$ implies $x = 0$ and $y = 0$.

The interior A_+° of the cone $A_+ = \{x^2 \colon x \in A\}$ is a symmetric cone, and satisfies $A_+^\circ = \{x^2 \colon x \in A \text{ invertible}\}$, see for instance [13, Proposition III.2.2].

Throughout we will denote the unit in the Jordan algebra by u. As $u \in A_+^\circ$, it is an order-unit for $A_+ = \{x^2 \colon x \in A\}$. We will consider the formally real Jordan algebras A as order-unit spaces (A, A_+, u), where A_+ is the cone of squares and u is the unit, and equip it with the order-unit norm. These normed spaces are precisely the finite dimensional JB-algebras, see [3, Theorem 1.11].

In the sequel the rank of the Jordan algebra A will be denoted by r. In a finite dimensional formally real Jordan algebra A each $x \in A$ has a *spectrum*, $\sigma(x) = \{\lambda \in \mathbb{R} \colon \lambda u - x \text{ is not invertible}\}$.

Recall that $p \in A$ is an *idempotent* if $p^2 = p$. If, in addition, p is non-zero and cannot be written as the sum of two non-zero idempotents, then it is said to be a *primitive* idempotent. The set of all primitive idempotent is denoted $\mathcal{J}_1(A)$ and is known to be a compact set [21]. Two idempotents p and q are said to be orthogonal if $p \bullet q = 0$, which is equivalent to $(p|q) = 0$. According to the spectral theorem [13, Theorem III.1.2], each x has a *spectral decomposition*, $x = \sum_{i=1}^r \lambda_i p_i$, where each p_i is a primitive idempotent, the λ_i's are the eigenvalues of x (including multiplicities), and $p_1, \ldots, p_r$ is a Jordan frame, i.e., the p_i's are mutually orthogonal and $p_1 + \cdots + p_r = u$. So, $\sigma(x) = \{\lambda_1, \ldots, \lambda_r\}$. We write $\operatorname{tr} x = \sum_{i=1}^r \lambda_i$ and $\det x = \prod_{i=1}^r \lambda_i$.

Remark 2.3. Given $x \in A$ with spectral decomposition $x = \sum_{i=1}^r \lambda_i p_i$, the eigenvalues λ_i are unique, but the primitive idempotents p_i in the spectral need not be. If, however, we collect terms with equal non-zero eigenvalue in the sum and write $x = \sum_{i=1}^r \mu_i c_i$, where $\mu_1 > \mu_2 > \ldots > \mu_s$ are non-zero and the c_i's are (not necessarily minimal) pairwise orthogonal idempotents, then the μ_i's and c_i's are unique, see [13, Theorem III.1.1]. We call this decomposition of x the *unique spectral decomposition*. In the sequel we will use the following observation several times. Suppose that $x \in A$ has a spectral decomposition $x = \sum_{i=1}^r \lambda_i p_i$ with $\lambda_1 > \ldots > \lambda_s > 0 = \lambda_{s+1} = \ldots = \lambda_r$ for some $s \leq r$, and $y \in A$ has a spectral decomposition $y = \sum_{i=1}^r \gamma_i q_i$ with $\gamma_1 > \ldots > \gamma_t > 0 = \gamma_{t+1} = \ldots = \gamma_r$ for some $t \leq r$. Now if $x = y$, we know that $s = t$

and $\lambda_i = \gamma_i$ for all i. Moreover, by considering the unique spectral decompositions of x and y we see that $p_1 + \cdots + p_s = q_1 + \cdots + q_s$.

The spectral decomposition gives rise to a functional calculus. Indeed, every real function $f\colon S \to \mathbb{R}$ induces a map on those $x \in A$ with $\sigma(x) \subseteq S$ by $f(x) = \sum_{i=1}^r f(\lambda_i)p_i$. In particular, each invertible element in A has spectrum in $\mathbb{R} \setminus \{0\}$, so the function $t \mapsto t^{-1}$ induces inversion $x^{-1} = \sum_{i=1}^r \lambda_i^{-1} p_i$. We see that A_+° is equal to the set of the elements $x \in A$ with $\sigma(x) \subseteq (0,\infty)$. Furthermore, the exponential function induces a map on A by $e^x = \sum_{i=1}^r e^{\lambda_i} p_i$, which maps A onto A_+°. It is well-known [34, p. 18] that the exponential map coincides with the Riemannian exponential map $\exp_u\colon A \to A_+^\circ$, i.e.,

$$\exp_u(x) = e^x \quad \text{for all } x \in A. \tag{2.4}$$

We write $\Lambda(x) = M(x/u)$. Note that if x has spectral decomposition $x = \sum_{i=1}^r \lambda_i p_i$, then $\lambda u - x = \sum_{i=1}^r (\lambda - \lambda_i)p_i \geq 0$ if and only if $\lambda \geq \lambda_i$ for all i, hence

$$\Lambda(x) = \max\{\lambda\colon \lambda \in \sigma(x)\}.$$

This implies that

$$\|x\|_u = \max\{\Lambda(x), \Lambda(-x)\} = \max\{|\lambda|\colon \lambda \in \sigma(x)\} \quad \text{for all } x \in A.$$

We also note that $\Lambda(x + \mu u) = \Lambda(x) + \mu$ for all $x \in A$ and $\mu \in \mathbb{R}$. Moreover, if $x \leq y$, then $\Lambda(x) \leq \Lambda(y)$.

For $x \in A$ we denote the *quadratic representation* by $U_x\colon A \to A$, which is the linear map, $U_x y = 2x \bullet (x \bullet y) - x^2 \bullet y$. If $x \in A$ is invertible, then U_x is invertible and $U_x(A_+) = A_+$, see [13, Proposition III.2.2].

Given a Jordan frame $p_1, \dots, p_r$ in A and $I \subseteq \{1, \dots, r\}$ nonempty, we write $p_I = \sum_{i \in I} p_i$ and we let $A(p_I) = U_{p_I}(A)$. Recall [13, Theorem IV.1.1] that $A(p_I)$ is the Peirce 1-space of the idempotent p_I, i.e, $A(p_I) = \{x \in A\colon p_I \bullet x = x\}$, which is a subalgebra. Given $z \in A(p_I)$, we write $\Lambda_{A(p_I)}(z)$ to denote the maximal eigenvalue of z in the Jordan subalgebra $A(p_I)$ with unit p_I. So $\Lambda_{A(p_I)}(z) = \inf\{\lambda\colon z \leq \lambda p_I\}$.

3. Thompson distance horofunctions

Let A_+° be a symmetric cone. For $x, y \in A_+^\circ$ we have that $x \leq \lambda y$ if and only if $U_{y^{-1/2}} x \leq \lambda u$. Therefore $\log M(x/y) = \log \max \sigma(U_{y^{-1/2}} x) = \max \sigma(\log U_{y^{-1/2}} x)$. We also have that

$$\begin{aligned}\inf\{\lambda > 0\colon y \leq \lambda x\} &= (\sup\{\mu > 0\colon \mu y \leq x\})^{-1} = (\sup\{\mu > 0\colon \mu u \leq U_{y^{-1/2}} x\})^{-1} \\ &= (\min \sigma(U_{y^{-1/2}} x))^{-1},\end{aligned}$$

so that $\log M(y/x) = \log(\min \sigma(U_{y^{-1/2}} x))^{-1} = -\min \sigma(\log U_{y^{-1/2}} x)$. So the Thompson distance satisfies

$$d_T(x,y) = \max\{\max \sigma(\log U_{y^{-1/2}} x), -\min \sigma(\log U_{y^{-1/2}} x)\} = \|\log U_{y^{-1/2}} x\|_u.$$

The symmetry at $x \in A_+^\circ$ is given by $S_x(y) = U_x y^{-1}$ for $y \in A_+^\circ$. It can be shown that

$$M(x^{-1}/y^{-1}) = M(y/x) \quad \text{for } x, y \in A_+^\circ, \tag{3.1}$$

see [31, p. 1518]. Thus, for each $x \in A_+^\circ$ the symmetry $S_x\colon A_+^\circ \to A_+^\circ$ is a d_T-isometry.

The following observation will be useful when determining the Thompson distance horofunctions for symmetric cones.

Lemma 3.1. *Let $p_1, \dots, p_k$ be orthogonal primitive idempotents in a finite dimensional formally real Jordan algebra A. The restriction of $\exp_u\colon (A, \|\cdot\|_u) \to (A_+^\circ, d_T)$, given by (2.4), to $\mathrm{Span}(\{p_1,\dots,p_k\})$ is an isometry. Moreover, if $x, w \in \mathrm{Span}(\{p_1,\dots,p_k\})$ with $\|w\|_u = 1$ and $\gamma(t) = tw + x$ for $t \in \mathbb{R}$, then $\psi\colon t \mapsto \exp_u(\gamma(t))$ is a geodesic in (A_+°, d_T).*

Proof. For $y, z \in \mathrm{Span}(\{p_1,\dots,p_k\})$ we have that

$$d_T(\exp_u(y), \exp_u(z)) = d_T(e^y, e^z) = \|\log U_{e^{-z/2}} e^y\|_u = \|\log e^{y-z}\|_u = \|y - z\|_u.$$

The second assertion now easily follows, as $d_T(\exp_u(\gamma(t)), \exp_u(\gamma(s))) = \|(t-s)w\|_u = |t - s|$ for all $s, t \in \mathbb{R}$. □

We now give a complete description of the Thompson distance horofunctions on symmetric cones.

Theorem 3.2. *If A_+° is a symmetric cone, then the horofunctions of (A_+°, d_T) are all the functions of the form*

$$h(x) = \max\{\log M(y/x), \log M(z/x^{-1})\} \quad \text{for } x \in A_+^\circ, \tag{3.2}$$

where $y, z \in A_+$ with $y \bullet z = 0$ and $\max\{\|y\|_u, \|z\|_u\} = 1$. Here we use the convention that if y or z is 0, then the corresponding term is omitted from the maximum. Moreover, each horofunction is a Busemann point.

Proof. We first show that each horofunction is of the from (3.2). Let $(y_n) \in A_+^\circ$ be such that $h_{y_n} \to h$ where h is a horofunction. By Lemma 2.1 we know that $d_T(y_n, u) \to \infty$. Let $r_n = e^{d_T(y_n,u)}$ and $z_n = (y_n)^{-1}$ for all n. Then $y_n \leq r_n u$ and $z_n \leq r_n u$. Set $\hat{y}_n = y_n/r_n$ and $\hat{z}_n = z_n/r_n$. By taking a subsequence we may assume that $\hat{y}_n \to y$ and $\hat{z}_n \to z$, as $\|\hat{y}_n\|_u, \|\hat{z}_n\|_u \leq 1$ for all n and A is finite dimensional. As $\hat{y}_n \bullet \hat{z}_n = u/r_n^2 \to 0$, we conclude that $y \bullet z = 0$.

Also note that as $\|y_n\|_u = M(y_n/u)$ and $\|z_n\|_u = M(z_n/u) = M(u/y_n)$, we get that

$$r_n = e^{d_T(y_n,u)} = \max\{M(y_n/u), M(u/y_n)\} = \max\{\|y_n\|_u, \|z_n\|_u\},$$

so that $\max\{\|y\|_u, \|z\|_u\} = \max\{\|\hat{y}_n\|_u, \|\hat{z}_n\|_u\} = 1$. Using (3.1) we find for $x \in A_+^\circ$ that

$$\begin{aligned} h(x) &= \lim_{n\to\infty} \max\{\log M(y_n/x), \log M(x/y_n)\} - \log r_n \\ &= \lim_{n\to\infty} \max\{\log(r_n^{-1} M(y_n/x)), \log(r_n^{-1} M(z_n/x^{-1}))\} \\ &= \lim_{n\to\infty} \max\{\log M(\hat{y}_n/x), \log M(\hat{z}_n/x^{-1})\}. \end{aligned}$$

Note that if $w_n, v \in A_+^\circ$ with $w_n \to 0$, then $M(w_n/v) \to M(0/v) = 0$ by the continuity of the M function [28, Lemma 2.2]. As $h(x) \geq -d_T(x, u)$ for all $x \in A_+^\circ$, we deduce from the previous equality that $h(x) = \max\{\log M(y/x), \log M(z/x^{-1})\}$, where if $y = 0$ or $z = 0$, the corresponding term is omitted from the maximum.

To show that each function of the form (3.2) is a horofunction, let $y, z \in A_+$ with $\max\{\|y\|_u, \|z\|_u\} = 1$ and $y \bullet z = 0$. We will discuss the case where y and z are both non-zero. The other cases can be shown in the same fashion and are left to the reader.

Using the spectral decomposition we can write

$$y = \sum_{i\in I} e^{-\alpha_i} p_i \quad \text{and} \quad z = \sum_{j\in J} e^{-\alpha_j} p_j,$$

where $\min\{\alpha_k : k \in I \cup J\} = 0$, and the p_k's are mutually orthogonal primitive idempotents. Set $p_I = \sum_{i\in I} p_I$ and $p_J = \sum_{j\in J} p_j$. So $p_I \bullet p_J = 0$, and hence orthogonal idempotents in A. Let $v = -\sum_{i\in I}\alpha_i p_i + \sum_{j\in J}\alpha_j p_j$ and $w = p_I - p_J$. Note that $w \in \mathrm{Span}(\{p_k : k \in I \cup J\})$. For $t > 0$ let $\gamma(t) = tw + v$.

From Lemma 3.1 we know that $\psi\colon t \mapsto \exp_u(\gamma(t))$ is a geodesic in A_+°, and for all $t > 0$ sufficiently large we have that

$$d_T(\psi(t), u) = \|\log e^{tw+v}\|_u = \|tw + v\|_u = \max\{|t - \alpha_k| : k \in I \cup J\} = t,$$

as $\min\{\alpha_k : k \in I \cup J\} = 0$. Moreover, $\psi(t)^{-1} = e^{-(tw+v)}$.

Thus,

$$\lim_{t\to\infty} e^{-t}\psi(t) = \lim_{t\to\infty} \sum_{i\in I} e^{-\alpha_i} p_i + \sum_{j\in J} e^{-2t+\alpha_j} p_j + e^{-t}(u - p_I - p_J) = y$$

and

$$\lim_{t\to\infty} e^{-t}\psi(t)^{-1} = \lim_{t\to\infty} \sum_{i\in I} e^{-2t+\alpha_i} p_i + \sum_{j\in J} e^{-\alpha_j} p_j + e^{-t}(u - p_I - p_J) = z.$$

Now using (3.1) we deduce for each $x \in A_+^\circ$ that

$$\begin{aligned}\lim_{t\to\infty} d_T(x, \psi(t)) - d_T(\psi(t), u) &= \lim_{t\to\infty} \max\{\log M(\psi(t)/x), \log M(x/\psi(t))\} - t\\ &= \lim_{t\to\infty} \max\{\log M(e^{-t}\psi(t)/x), \log M(e^{-t}\psi(t)^{-1}/x^{-1})\}\\ &= \max\{\log M(y/x), \log M(z/x)\},\end{aligned}$$

as $v \in A_+ \mapsto M(v/w)$ is continuous for all $w \in A_+^\circ$ by [28, Lemma 2.2]. We conclude that the function of the form (3.2) is a horofunction. In fact, it is a Busemann point, as $t \mapsto \psi(t)$ is a geodesic. □

In the case of $\Pi_n(\mathbb{C})$ we get that the horofunctions are the functions $h\colon \Pi_n(\mathbb{C}) \to \mathbb{R}$ of the form:

$$h(x) = \max\{\log\max\sigma(x^{-1/2}ax^{-1/2}), \log\max\sigma(x^{1/2}bx^{1/2})\},$$

where a and b are positive semi-definite, with $\max\{\max\sigma(a), \max\sigma(b)\} = 1$ and $1/2(ab + ba) = 0$.

Remark 3.3. We see from the proof of Theorem 3.2 that each horofunction of (A_+°, d_T) is obtained as the limit of a geodesic in the span of a Jordan frame. In fact, if h is a horofunction given by $h(x) = \max\{\log M(y/x), \log M(z/x^{-1})\}$, where $y = \sum_{i\in I} e^{-\alpha_i} p_i$, $z = \sum_{j\in J} e^{-\alpha_j} p_j$, then $\psi\colon t \mapsto \exp_u(\gamma(t))$, with

$$\gamma(t) = t\left(\sum_{i\in I} p_i - \sum_{j\in J} p_j\right) - \sum_{i\in I}\alpha_i p_i + \sum_{j\in J}\alpha_j p_j$$

for $t > 0$, is a geodesic A_+° converging to h. The subspace $\mathrm{Span}(\{p_k : k \in I \cup J\}) \cap A_+^\circ$ is a totally geodesic flat subspace (with respect to the Riemannian metric). The fact that each horofunction of (A_+°, d_T) arises as a limit of a sequence in such a subspace is in agreement with [20, Lemma 4.4].

As all horofunctions are Busemann points, the equivalence classes of $\simeq$ coincide with the parts. In the remainder of this section we identify the parts and the detour distance.

The following observation will be useful. If (V_i, C_i, u_i), $i = 1, 2$, are order-unit spaces, then the product space $V_1 \oplus V_2$ is an order unit space with cone $C_1 \times C_2$ and order unit $u = (u_1, u_2)$. Moreover, for $x = (x_1, x_2), y = (y_1, y_2) \in C_1 \times C_2$ we have that

$$M(x/y) = \max\{M(x_1/y_1), M(x_2/y_2)\}.$$

Indeed, if $x \le \lambda y$, then $\lambda y - x \in C_1 \times C_2$, and hence $\lambda y_1 - x_1 \in C_1$ and $\lambda y_2 - x_2 \in C_2$. So $\lambda \ge \max\{M(x_1/y_1), M(x_2/y_2)\}$. On the hand, if $x_1 \le \mu y_1$ and $x_2 \le \mu y_2$, then $\mu y - x \in C_1 \times C_2$, so that $\mu \ge M(x/y)$.

Theorem 3.4. *Suppose that h and h' are horofunctions in $\overline{A_+^\circ}^h$, which by Theorem 3.2 can be given by*

$$h(x) = \max\{\log M(y/x), \log M(z/x^{-1})\} \text{ and}$$
$$h'(x) = \max\{\log M(y'/x), \log M(z'/x^{-1})\} \quad \text{for } x \in A_+^\circ,$$

where $y, z, y', z' \in A_+$ with $y \bullet z = 0$, $y' \bullet z' = 0$, and $\max\{\|y\|_u, \|z\|_u\} = 1 = \max\{\|y'\|_u, \|z'\|_u\}$. Let $y = \sum_{i\in I} e^{-\alpha_i} p_i$, $z = \sum_{j\in J} e^{-\alpha_j} p_j$, and set $p_I = \sum_{i\in I} p_i$ if $y \neq 0$, and $p_J = \sum_{j\in J} p_j$ if $z \neq 0$. Then h and h' are in the same part if and only if $y \sim y'$ and $z \sim z'$. Moreover, in that case,

$$\delta(h, h') = d_H((y, z), (y', z')), \quad \text{where } (y, z), (y', z') \in U_{p_I}(A) \oplus U_{p_J}(A)$$

and d_H is the Hilbert distance on the product cone $U_{p_I}(A)_+ \times U_{p_J}(A)_+$. Here, if y or z is 0, we omit the corresponding term in the sum $U_{p_I}(A) \oplus U_{p_J}(A)$.

Proof. We use (2.2) to determine the detour cost $H(h, h')$. Note that as $\|y\|_u = 1$ or $\|z\|_u = 1$, we have that $\min\{\alpha_k : k \in I \cup J\} = 0$. Let $v = -\sum_{i\in I} \alpha_i p_i + \sum_{j\in J} \alpha_j p_j$ and $w = p_I - p_J$, so $w \in \mathrm{Span}(\{p_k : k \in I \cup J\})$. For $t > 0$ let $\gamma(t) = tw + v$. From Lemma 3.1 we know that $\psi : t \mapsto \exp_u(\gamma(t))$ is a geodesic in A_+°, and for all $t > 0$ sufficiently large we have that

$$d_T(\psi(t), u) = \|\log e^{tw+v}\|_u = \|tw + v\|_u = \max\{|t - \alpha_k| : k \in I \cup J\} = t,$$

as $\min\{\alpha_k : k \in I \cup J\} = 0$. Moreover, $\psi(t)^{-1} = e^{-(tw+v)}$ and $h_{\psi(t)} \to h$ by Remark 3.3.

As $e^{-t}\psi(t) = \sum_{i\in I} e^{-\alpha_i} p_i + \sum_{j\in J} e^{-2t+\alpha_j} p_j + e^{-t}(u - p_I - p_J)$, we have that $e^{-t}\psi(t) \to y$ and

$$e^{-t}\psi(t) \le e^{-s}\psi(s) \quad \text{for all } 0 \le s \le t.$$

Likewise, $e^{-t}\psi(t)^{-1} \to z$ and $e^{-t}\psi(t)^{-1} \le e^{-s}\psi(s)^{-1}$ for all $0 \le s \le t$.

It follows from [30, Lemma 5.3] that if y dominates y' and z dominates z', then

$$\begin{aligned} H(h, h') &= \lim_{t\to\infty} d_T(\psi(t), u) + h'(\psi(t)) \\ &= \lim_{t\to\infty} t + \max\{\log M(y'/\psi(t)), \log M(z'/\psi(t)^{-1})\} \\ &= \lim_{t\to\infty} \max\{\log M(y'/e^{-t}\psi(t)), \log M(z'/e^{-t}\psi(t)^{-1})\} \\ &= \max\{\log M(y'/y), \log M(z'/z)\}, \end{aligned}$$

and otherwise, $H(h, h') = \infty$. Here, if $y' = 0$ or $z' = 0$, the corresponding term is omitted from the maximum.

Interchanging the roles of h and h' gives

$$\begin{aligned} \delta(h, h') &= \max\{\log M(y'/y), \log M(z'/z)\} + \max\{\log M(y/y'), \log M(z/z')\} \\ &= d_H((y, z), (y', z')) \end{aligned}$$

if $y \sim y'$ and $z \sim z'$, and $\delta(h, h') = \infty$ otherwise. □

4. Extension of the exponential map $\exp_u A \to A_+^\circ$

To define the extension of the exponential map we need to recall the description of the horofunctions in $\overline{A}^h$ from [30, Theorem 4.2]. It was shown there that the horofunctions in $\overline{A}^h$ are precisely the functions $g\colon A \to \mathbb{R}$ of the form,

$$g(v) = \max\left\{\Lambda_{A(p_I)}(-U_{p_I}v - \sum_{i\in I}\alpha_i p_i), \Lambda_{A(p_J)}(U_{p_J}v - \sum_{j\in J}\alpha_j p_j)\right\}, \tag{4.1}$$

where $p_1, \dots, p_r \in A$ is a Jordan frame, $I, J \subseteq \{1, \dots, r\}$ are disjoint and $I \cup J$ is nonempty, $p_I = \sum_{i\in I} p_i$, $p_J = \sum_{j\in J} p_j$, and $\alpha \in \mathbb{R}^{I\cup J}$ with $\min\{\alpha_k \colon k \in I \cup J\} = 0$. Here the convention is that if I or J is empty, the corresponding term in the maximum is omitted.

Definition 4.1. The *exponential map* $\exp_u\colon \overline{A}^h \to \overline{A_+^\circ}^h$ is defined by, $\exp_u(v) = e^v$ for $v \in A$, and for $g \in \partial\overline{A}^h$ given by (4.1) we let $\exp_u(g) = h$, where

$$h(x) = \max\{\log M(y/x), \log M(z/x^{-1})\} \quad \text{for } x \in A_+^\circ, \tag{4.2}$$

with $y = \sum_{i\in I} e^{-\alpha_i} p_i$ and $z = \sum_{j\in J} e^{-\alpha_j} p_j$.

Note that $y, z \in A_+$, with $\max\{\|y\|_u, \|z\|_u\} = 1$, as $\min\{\alpha_k \colon k \in I \cup J\} = 0$, and $y \bullet z = 0$. So, $\exp_u(g)$ is a horofunction by Theorem 3.2. Moreover, the extension is well defined. To show this we use the following observation.

Lemma 4.2. *Suppose $x, y \in A$ have spectral decompositions $x = \sum_{i\in I}\alpha_i p_i$ and $y = \sum_{j\in J}\beta_j q_j$. If $\sum_{i\in I} p_i = \sum_{j\in J} q_j$ and $x = y$, then $\sum_{i\in I} e^{-\alpha_i}p_i = \sum_{j\in J} e^{-\beta_j} q_j$.*

Proof. Let $p_I = \sum_{i\in I} p_i$ and $q_J = \sum_{j\in J} q_j$, so $p_I = q_J$. Since $\sum_{i\in I}\alpha_i p_i = \sum_{j\in J}\beta_j q_j$, we have that $(u - p_I) + \sum_{i\in I} e^{-\alpha_i}p_i = e^{-x} = e^{-y} = (u - q_J) + \sum_{j\in J} e^{-\beta_j} q_j$ by the functional calculus, hence $\sum_{i\in I} e^{-\alpha_i}p_i = \sum_{j\in J} e^{-\beta_j} q_j$. □

Now to see that the extension is well-defined assume that g in (4.1) is represented differently as

$$g(v) = \max\left\{\Lambda_{A(q_{I'})}(-U_{q_{I'}}v - \sum_{i\in I'}\beta_i q_i), \Lambda_{A(q_{J'})}(U_{q_{J'}}v - \sum_{j\in J'}\beta_j q_j)\right\},$$

It follows from [30, Theorem 4.3] and the fact that $\delta(g, g) = 0$ that $p_I = q_{I'}$, $p_J = q_{J'}$, and

$$\sum_{i\in I}\alpha_i p_i + \sum_{j\in J}\alpha_j p_j = \sum_{i\in I'}\beta_i q_i + \sum_{j\in J'}\beta_j q_j,$$

as $\min\{\alpha_m \colon m \in I \cup J\} = 0 = \min\{\beta_m \colon m \in I' \cup J'\}$. So,

$$\sum_{i\in I}\alpha_i p_i = U_{p_I}\left(\sum_{i\in I}\alpha_i p_i + \sum_{j\in J}\alpha_j p_j\right) = U_{q_{I'}}\left(\sum_{i\in I'}\beta_i q_i + \sum_{j\in J'}\beta_j q_j\right) = \sum_{i\in I'}\beta_i q_i$$

and

$$\sum_{j\in J}\alpha_j p_j = U_{p_J}\left(\sum_{j\in J}\alpha_i p_i + \sum_{j\in J}\alpha_j p_j\right) = U_{q_{J'}}\left(\sum_{i\in I'}\beta_i q_i + \sum_{j\in J'}\beta_j q_j\right) = \sum_{j\in J'}\beta_j q_j.$$

Using Lemma 4.2 we conclude that $\sum_{i\in I} e^{-\alpha_i}p_i = \sum_{i\in I'} e^{-\beta_i}q_i$ and $\sum_{j\in J} e^{-\alpha_j}p_j = \sum_{j\in J'} e^{-\beta_j}q_j$, and hence the extension is well defined.

We will establish the following result.

Theorem 4.3. *Let A_+° be a symmetric cone in a finite dimensional formally real Jordan algebra A. The map $\exp_u\colon \overline{A}^h \to \overline{A_+^\circ}^h$ is a homeomorphism which maps each part in the horofunction boundary of A onto a part of the horofunction boundary of A_+°.*

Before we prove this theorem we give some preliminary results.

Lemma 4.4. *The map $\exp_u\colon \overline{A}^h \to \overline{A_+^\circ}^h$ is a bijection, which maps A onto A_+°, and $\partial\overline{A}^h$ onto $\partial\overline{A_+^\circ}^h$.*

Proof. Clearly $\exp_u$ is a bijection from A onto A_+°. Moreover, it follows from Theorem 3.2 and [30, Theorem 4.2] that $\exp_u$ maps $\partial\overline{A}^h$ onto $\partial\overline{A_+^\circ}$. To show that the extension is injective on $\partial\overline{A}^h$ suppose that $h = \exp_u(g) = \exp_u(g') = h'$, where g is as in (4.1) and

$$g'(v) = \max\left\{\Lambda_{A(q_{I'})}(-U_{q_{I'}}v - \sum_{i\in I'}\beta_i q_i), \Lambda_{A(q_{J'})}(U_{q_{J'}}v - \sum_{j\in J'}\beta_j q_j)\right\}, \tag{4.3}$$

where $q_1,\ldots,q_r \in A$ is a Jordan frame, $I', J' \subseteq \{1,\ldots,r\}$ are disjoint and $I'\cup J'$ is nonempty, and $\beta \in \mathbb{R}^{I\cup J}$ with $\min\{\beta_k\colon k\in I'\cup J'\} = 0$.

By definition of the extension, we have that

$$h(x) = \max\{\log M(y/x), \log M(z/x^{-1})\},$$

with

$$y = \sum_{i\in I} e^{-\alpha_i} p_i \quad\text{and}\quad z = \sum_{j\in J} e^{-\alpha_j} p_j,$$

and

$$h'(x) = \max\{\log M(y'/x), \log M(z'/x^{-1})\},$$

with

$$y' = \sum_{i\in I'} e^{-\beta_i} q_i \quad\text{and}\quad z' = \sum_{j\in J'} e^{-\beta_j} q_j.$$

As $h = h'$ we know that $\delta(h,h') = 0$, and hence $y = y'$ and $z = z'$ by Theorem 3.4. Now using Remark 2.3 we find that $p_I = q_{I'}$ and $p_J = q_{J'}$. Moreover, $-\sum_{i\in I}\alpha_i p_i = \log(y+u-p_I) = \log(y'+u-q_{I'}) = -\sum_{i\in I'}\beta_i q_i$ and $-\sum_{j\in J}\alpha_j p_j = \log(z+u-p_J) = \log(z'+u-q_{J'}) = -\sum_{j\in J'}\beta_j q_j$ using the functional calculus. This implies that $g = g'$, and hence $\exp_u$ is injective, which completes the proof. □

Clearly $\exp_u$ is continuous on A. To establish the continuity on all of $\overline{A}^h$ we prove two lemmas.

Lemma 4.5. *If (w_n) in A converges to $g \in \partial\overline{A}^h$, then $(\exp_u(w_n))$ converges to $\exp_u(g)$.*

Proof. To prove the statement we show that each subsequence of $(\exp_u(w_n))$ has a convergent subsequence with limit $\exp_u(g)$. So let $(\exp_u(w_{n_k}))$ be a subsequence and let g be given by (4.1), so that $h = \exp_u(g)$ is given by (4.2). As g is a horofunction, we know by Lemma 2.1 that $\|w_{n_k}\|_u \to \infty$. For $k \geq 1$ write $r_{n_k} = \|w_{n_k}\|_u$ and let $w_{n_k} = \sum_{i=1}^r \lambda_{i,n_k} q_{i,n_k}$ be the spectral decomposition of w_{n_k}. After taking a subsequence, may assume that

(1) There exists $s \in \{1,\ldots,r\}$ such that $r_{n_k} = |\lambda_{s,n_k}|$ for all $k \geq 1$.
(2) There exist $I_+ \subseteq \{1,\ldots,r\}$ such that for each $k \geq 1$ we have $\lambda_{i,n_k} > 0$ if and only if $i \in I_+$.

(3) $q_{i,n_k} \to q_i$ for all $i \in \{1, \dots, r\}$.

The third property follows from the fact that the set of primitive idempotents is compact, see [21].

Now let $\beta_{i,n_k} = r_{n_k} - \lambda_{i,n_k}$ for all $i \in I_+$ and $\beta_{i,n_k} = r_{n_k} + \lambda_{i,n_k}$ for all $i \notin I_+$. So $\beta_{i,n_k} \geq 0$ for all i, and $\beta_{s,n_k} = 0$ for all k. By taking a further subsequence we may also assume that $\beta_{i,n_k} \to \beta_i \in [0, \infty]$ for all i. Let $I' = \{i \in I_+ \colon \beta_i < \infty\}$ and $J' = \{j \notin I_+ \colon \beta_j < \infty\}$. Note that $s \in I' \cup J'$ and hence the union is nonempty. Moreover, I' and J' are disjoint. It now follows from [30, Lemma 4.7] that $g_{w_{n_k}} \to g'$, where

$$g'(v) = \max\left\{\Lambda_{A(q_{I'})}(-U_{q_{I'}}v - \sum_{i\in I'} \beta_i q_i), \Lambda_{A(q_{J'})}(U_{q_{J'}}v - \sum_{j\in J'} \beta_j q_j)\right\},$$

with $q_{I'} = \sum_{i\in I'} q_i$ and $q_{J'} = \sum_{j\in J'} q_j$.

As (w_n) converges to g, we find that $g = g'$, and hence $\delta(g, g') = 0$. It now follows from [30, Theorem 4.3] that $p_I = q_{I'}$, $p_J = q_{J'}$, and

$$\sum_{i\in I} \alpha_i p_i + \sum_{j\in J} \alpha_j p_j = \sum_{i\in I'} \beta_i q_i + \sum_{j\in J'} \beta_j q_j,$$

as $\min_{i\in I\cup J} \alpha_i = 0 = \min_{i\in I'\cup J'} \beta_i$. This implies that

$$\sum_{i\in I} \alpha_i p_i = U_{p_I}\left(\sum_{i\in I} \alpha_i p_i + \sum_{j\in J} \alpha_j p_j\right) = U_{q_{I'}}\left(\sum_{i\in I'} \beta_i q_i + \sum_{j\in J'} \beta_j q_j\right) = \sum_{i\in I'} \beta_i q_i.$$

Likewise, we have $\sum_{j\in J} \alpha_j p_j = \sum_{j\in J'} \beta_j q_j$. From Lemma 4.2 we now deduce that $y = \sum_{i\in I} e^{-\alpha_i} p_i = \sum_{i\in I'} e^{-\beta_i} q_i$ and $z = \sum_{j\in J} e^{-\alpha_j} p_j = \sum_{j\in j'} e^{-\beta_j} q_j$.

Note that

$$\lim_{k\to\infty} e^{-r_{n_k}} \exp_u(w_{n_k}) = \lim_{k\to\infty} \sum_{i=1}^{r} e^{-(r_{n_k} - \lambda_{i,n_k})} q_{i,n_k} = \sum_{i\in I'} e^{-\beta_i} q_i = y$$

and

$$\lim_{k\to\infty} e^{-r_{n_k}} \exp_u(-w_{n_k}) = \lim_{k\to\infty} \sum_{i=1}^{r} e^{-(r_{n_k} + \lambda_{i,n_k})} q_{i,n_k} = \sum_{j\in J'} e^{-\beta_j} q_j = z.$$

It now follows from the continuity of the M function, see [28, Lemma 2.2], that

$$\begin{aligned}
&\lim_{k\to\infty} h_{\exp_u(w_{n_k})}(x) = \lim_{k\to\infty} d_T(x, \exp_u(w_{n_k})) - d_T(u, \exp_u(w_{n_k}))\\
&= \lim_{k\to\infty} \max\{\log M(\exp_u(w_{n_k})/x), \log M(\exp_u(-w_{n_k})/x^{-1})\} - \log e^{r_{n_k}}\\
&= \lim_{k\to\infty} \max\{\log M(e^{-r_{n_k}} \exp_u(w_{n_k})/x), \log M(e^{-r_{n_k}} \exp_u(-w_{n_k})/x^{-1})\}\\
&= \max\{\log M(y/x), \log M(z/x^{-1})\},
\end{aligned}$$

which shows that $(\exp_u(w_{n_k}))$ converges to $h = \exp_u(g)$, and hence the proof is complete. □

Next we show continuity of $\exp_u$ in the horofunction boundary.

Lemma 4.6. *If (g_n) in $\partial\overline{A}^h$ converges to a horofunction g, then $(\exp_u(g_n))$ converges to $\exp_u(g)$.*

Proof. Let (g_n) be a sequence in $\partial\overline{A}^h$ converging to g, where g is given by (4.1). So $\exp_u(g) = h$, where is h given by (4.2). To show the lemma, we prove that each subsequence of $(\exp_u(g_n))$ has a convergent subsequence with limit h. Let $(\exp_u(g_{n_k}))$ be a subsequence. By [30, Theorem 4.2] we can write for $k \geq 1$,

$$g_{n_k}(v) = \max\left\{\Lambda_{A(q_{I_k,k})}(-U_{q_{I_k,k}}v - \sum_{i\in I_k}\beta_{i,k}q_{i,k}), \Lambda_{A(q_{J_k,k})}(U_{q_{J_k,k}}v - \sum_{j\in J_k}\beta_{j,k}q_{j,k})\right\},$$

where the $q_{i,k}$ and $q_{j,k}$ are orthogonal primitive idempotents, $I_k, J_k \subseteq \{1,\dots,r\}$ are disjoint with $I_k \cup J_k$ nonempty, $\min\{\beta_{m,k}\colon m \in I_k \cup J_k\} = 0$, $q_{I_k,k} = \sum_{i\in I_k} q_{i,k}$ and $q_{J_k,k} = \sum_{j\in J_k} q_{j,k}$.

The approach will be similar to the one taken in the proof of the previous lemma. After taking subsequences we may assume that:

(1) There exist $I_0, J_0 \subseteq \{1,\dots,r\}$ such that $I_0 = I_k$ and $J_0 = J_k$ for all k.
(2) There exists $s \in I_0 \cup J_0$ such that $\beta_{s,k} = 0$ for all k.
(3) $\beta_{m,k} \to \beta_m \in [0,\infty]$ and $q_{m,k} \to q_m$ for all $m \in I_0 \cup J_0$.

Now let $I' = \{i \in I_0\colon \beta_i < \infty\}$ and $J' = \{j \in J_0\colon \beta_j < \infty\}$, and note that $s \in I' \cup J'$.

Next we show that

$$\lim_{k\to\infty} g_{n_k}(v) = \max\left\{\Lambda_{A(q_{I'})}(-U_{q_{I'}}v - \sum_{i\in I'}\beta_i q_i), \Lambda_{A(q_{J'})}(U_{q_{J'}}v - \sum_{j\in J'}\beta_j q_j)\right\} \tag{4.4}$$

where the term is omitted if the corresponding set I' or J' is empty. Here $q_{I'} = \sum_{i\in I'} q_i$ and $q_{J'} = \sum_{j\in J'} q_j$.

First let us assume that $I_0 \neq \emptyset$ and $I' = \emptyset$. Note that

$$-U_{q_{I_0,k}}v \leq \|v\|_u U_{q_{I_0,k}}u = \|v\|_u q_{I_0,k},$$

as $-v \leq \|v\|_u u$ and $U_{q_{I_0,k}}(A_+) \subseteq A_+$. It follows that

$$-U_{q_{I_0,k}}v - \sum_{i\in I_0}\beta_{i,k}q_{i,k} \leq \sum_{i\in I_0}(\|v\|_u - \beta_{i,k})q_{i,k},$$

so that

$$\Lambda_{A(q_{I_0,k})}\left(-U_{q_{I_0,k}}v - \sum_{i\in I_0}\beta_{i,k}q_{i,k}\right) \leq \Lambda_{A(q_{I_0,k})}\left(\sum_{i\in I_0}(\|v\|_u - \beta_{i,k})q_{i,k}\right) \leq \max_{i\in I_0}(\|v\|_u - \beta_{i,k}).$$

The right-hand side diverges to $-\infty$ as $k \to \infty$, since I' is empty. We also know that for each horofunction $\bar{g}$ in $\overline{A}^h$ we have that $\bar{g}(v) \geq -\|v\|_u$. So, if $I_0 \neq \emptyset$ and $I' = \emptyset$, then $s \in J'$ and for each $v \in A$ we have that

$$g_{n_k}(v) = \Lambda_{A(q_{J_0,k})}\left(U_{q_{J_0,k}}v - \sum_{j\in J_0}\beta_{j,k}q_{j,k}\right)$$

for all k large. In the same way we get that if $J_0 \neq \emptyset$ and $J' = \emptyset$, then

$$g_{n_k}(v) = \Lambda_{A(q_{I_0,k})}\left(-U_{q_{I_0,k}}v - \sum_{i\in I_0}\beta_{i,k}q_{i,k}\right)$$

for all k large.

On the other hand, if $I' \neq \emptyset$, then by [30, Lemma 4.7] we know that

$$\Lambda_{A(q_{I_0,k})}\left(-U_{q_{I_0,k}}v - \sum_{i\in I_0}\beta_{i,k}q_{i,k}\right) \to \Lambda_{A(q_{I'})}\left(-U_{q_{I'}}v - \sum_{i\in I'}\beta_i q_i\right),$$

and, similarly, if $J' \neq \emptyset$, we have

$$\Lambda_{A(q_{J_0,k})}\left(U_{q_{J_0,k}}v - \sum_{j\in J_0}\beta_{j,k}q_{j,k}\right) \to \Lambda_{A(q_{J'})}\left(U_{q_{J'}}v - \sum_{j\in J'}\beta_j q_j\right).$$

Note that if $I_0 = \emptyset$, then $s \in J'$. Likewise, if $J_0 = \emptyset$, then $s \in I'$. It follows that in each of the cases (4.4) holds.

Let $g'\colon A \to \mathbb{R}$ be given by

$$g'(v) = \max\left\{\Lambda_{A(q_{I'})}\left(-U_{q_{I'}}v - \sum_{i\in I'}\beta_i q_i\right), \Lambda_{A(q_{J'})}\left(U_{q_{J'}}v - \sum_{j\in J'}\beta_j q_j\right)\right\}.$$

Then by [30, Theorem 4.2] we know that $g' \in \partial\overline{A}^h$. As $g_n \to g$, we find that $g = g'$ and hence $\delta(g, g') = 0$.

It now follows from [30, Theorem 4.3] that $p_I = q_{I'}$, $p_J = q_{J'}$, and

$$\sum_{i\in I}\alpha_i p_i + \sum_{j\in J}\alpha_j p_j = \sum_{i\in I'}\beta_i q_i + \sum_{j\in J'}\beta_j q_j,$$

as $\min\{\alpha_m\colon m \in I\cup J\} = 0 = \min\{\beta_m\colon m \in I'\cup J'\}$. This implies that

$$\sum_{i\in I}\alpha_i p_i = U_{p_I}\left(\sum_{i\in I}\alpha_i p_i + \sum_{j\in J}\alpha_j p_j\right) = U_{q_{i'}}\left(\sum_{i\in I'}\beta_i q_i + \sum_{j\in J'}\beta_j q_j\right) = \sum_{i\in I'}\beta_i q_i$$

and

$$\sum_{j\in J}\alpha_j p_j = U_{p_J}\left(\sum_{j\in J}\alpha_i p_i + \sum_{j\in J}\alpha_j p_j\right) = U_{q_{J'}}\left(\sum_{i\in I'}\beta_i q_i + \sum_{j\in J'}\beta_j q_j\right) = \sum_{j\in J'}\beta_j q_j.$$

Using Lemma 4.2 we conclude that

$$\sum_{i\in I}e^{-\alpha_i}p_i = \sum_{i\in I'}e^{-\beta_i}q_i \quad\text{and}\quad \sum_{j\in J}e^{-\alpha_j}p_j = \sum_{j\in J'}e^{-\beta_j}q_j,$$

So, if we let $\bar{y}_k = \sum_{i\in I_0}e^{-\beta_{i,k}}q_{i,k}$ and $\bar{z}_k = \sum_{j\in J_0}e^{-\beta_{j,k}}q_{j,k}$, then

$$\lim_{k\to\infty}\bar{y}_k = \sum_{i\in I'}e^{-\beta_i}q_i = y \quad\text{and}\quad \lim_{k\to\infty}\bar{z}_k = \sum_{j\in J'}e^{-\beta_j}q_j = z.$$

Using the continuity of the M function [28, Lemma 2.2] we now get that

$$\begin{aligned}\lim_{k\to\infty}\exp_u(g_{n_k})(x) &= \lim_{k\to\infty}\max\{\log M(\bar{y}_k/x), \log M(\bar{z}_k/x^{-1})\}\\ &= \max\{\log M(y/x), \log M(z/x^{-1})\} = h(x),\end{aligned}$$

which completes the proof. □

To complete the proof Theorem 4.3 the following concepts are useful. For $x, z \in A$ we let $[x, z] = \{y \in A\colon x \leq y \leq z\}$, which is called an *order-interval*. Given $y \in A_+$ we let

$$\mathrm{face}(y) = \{x \in A_+\colon x \leq \lambda y \text{ for some } \lambda \geq 0\}.$$

Note that $y \sim y'$ if and only if $\mathrm{face}(y) = \mathrm{face}(y')$. In a Euclidean Jordan algebra A every idempotent p satisfies

$$\mathrm{face}(p) \cap [0, u] = [0, p],$$

by [3, Lemma 1.39].

Let us now prove Theorem 4.3.

Proof of Theorem 4.3. It follows from Lemma 4.4, 4.5, and 4.6 that $\exp_u : \overline{A}^h \to \overline{A_+^\circ}^h$ is a continuous bijection between the compact spaces $\overline{A}^h$ and $\overline{A_+^\circ}^h$. As $\overline{A_+^\circ}^h$ is Hausdorff, we conclude that $\exp_u$ is a homeomorphism. We know from Theorem 3.4 that if h and h' are horofunctions in $\overline{A_+^\circ}^h$, where

$$h(x) = \max\{\log M(y/x), \log M(z/x^{-1})\} \quad \text{and}$$
$$h'(x) = \max\{\log M(y'/x), \log M(z'/x^{-1})\} \quad \text{for } x \in A_+^\circ,$$

then h and h' are in the same part if and only if $y \sim y'$ and $z \sim z'$. Consider the spectral decompositions: $y = \sum_{i\in I} e^{-\alpha_i} p_i$, $y' = \sum_{i\in I'} e^{-\beta_i} q_i$, $z = \sum_{j\in J} e^{-\alpha_j} p_j$, and $z' = \sum_{j\in j'} e^{-\beta_j} q_j$. If $y \sim y'$, then $p_I \sim q_{I'}$, where $p_I = \sum_{i\in I} p_i$ and $q_{I'} = \sum_{i\in I'} q_i$. Note that $p_I \sim q_{I'}$ implies $\mathrm{face}(p_I) = \mathrm{face}(q_{I'})$. As $\mathrm{face}(p_I) \cap [0, u] = [0, p_I]$ by [3, Lemma 1.39], we get that $p_I = q_{I'}$. So $y \sim y'$ implies that $p_I = q_{I'}$. Conversely, if $p_I = q_{I'}$, then $y \sim p_I \sim q_{I'} \sim y'$. Thus, $y \sim y'$ if and only if $p_I = q_{I'}$. Likewise, $z \sim z'$ if and only if $p_J = q_{J'}$. Now using [30, Theorem 4.3], we conclude that $\exp_u$ maps parts onto parts. □

Remark 4.7. It was shown in [30, Section 4] that there exists a homeomorphism from the horofunction compactification of $(A, \|\cdot\|_u)$ onto the closed unit ball in the dual space of $(A, \|\cdot\|_u)$, which maps each part of the horofunction boundary onto a relative open boundary face of the ball. The dual space $(A^*, \|\cdot\|_u^*)$ is a *base-norm space*, see [2, Theorem 1.19]. That is to say, it is an ordered normed vector space with cone $A_+^* = \{\varphi \in A^* : \varphi(x) \geq 0 \text{ for all } x \in A_+\}$, $A_+^* - A_+^* = A^*$, and the unit ball of the norm is given by

$$B_1^* = \mathrm{conv}(S(A) \cup -S(A)),$$

where $S(A) = \{\varphi \in A_+^* : \varphi(u) = 1\}$ is the *state space* of A.

If we identify the finite dimensional formally real Jordan algebra A with A^* using the inner-product $(x|y) = \mathrm{tr}(x \bullet y)$, we get that $A_+^* = A_+$, as A_+° is a symmetric cone (see [13, Proposition III.4.1]) and $S(A) = \{w \in A_+ : (u|w) = 1\}$. It was shown in [12, Theorem 4.4] that the (closed) boundary faces of the dual ball $B_1^* = \mathrm{conv}(S(A) \cup -S(A)) \subset A$ are precisely the sets of the form,

$$F_{p,q} = \mathrm{conv}((U_p(A) \cap S(A)) \cup (U_q(A) \cap -S(A))), \tag{4.5}$$

where p and q are orthogonal idempotents in A.

5. Variation norm horofunctions

Let A be a finite dimensional formally real Jordan algebra and $T_u = \{w \in A : \mathrm{tr}\, w = 0\}$, which is the tangent space of $PA_+^\circ = \{x \in A_+^\circ : \det x = 1\}$ at the unit u. Consider the variation norm,

$$|w|_u = M(w/u) - m(w/u) = \mathrm{diam}\, \sigma(w)$$

on T_u. In this section we determine the horofunction compactification of the normed space $(T_u, |\cdot|_u)$.

We start by giving the general form of the horofunctions.

Proposition 5.1. *If* $g\colon T_u \to \mathbb{R}$ *is a horofunction of* $\overline{T_u}^h$*, then there exist a Jordan frame* $p_1, \dots, p_r \in A$*,* $I, J \subseteq \{1, \dots, r\}$ *disjoint and nonempty, and* $\alpha \in \mathbb{R}^{I \cup J}$ *with* $\min_{i \in I} \alpha_i = 0 = \min_{j \in J} \alpha_j$ *such that*

$$g(v) = \Lambda_{A(p_I)}\left(-U_{p_I} v - \sum_{i \in I} \alpha_i p_i\right) + \Lambda_{A(p_J)}\left(U_{p_J} v - \sum_{j \in J} \alpha_j p_j\right) \tag{5.1}$$

for $v \in T_u$*, where* $p_I = \sum_{i \in I} p_i$ *and* $p_J = \sum_{j \in J} p_j$*.*

Proof. Suppose that (w_n) in T_u is such that $g_{w_n} \to g \in \partial \overline{T_u}^h$. Then by Lemma 2.1 we know that $|w_n|_u = \operatorname{diam} \sigma(w_n) = \Lambda(w_n) + \Lambda(-w_n) \to \infty$. Let $z_n = w_n - \frac{1}{2}(\Lambda(w_n) - \Lambda(-w_n))u \in A$. Note that for each $v \in A$ we have that $|v - w_n|_u = |v - z_n|_u$ by (2.3). Moreover, by construction, $\Lambda(z_n) = \frac{1}{2}(\Lambda(w_n) + \Lambda(-w_n)) = \Lambda(-z_n)$. Let $r_n = \Lambda(z_n)$.

Using the spectral decomposition we write $z_n = \sum_{i=1}^r \mu_{i,n} p_{i,n}$. After taking subsequences we may assume:

(1) There exists $I_+ \subseteq \{1, \dots, r\}$ such that for each $n \geq 1$ we have that $\mu_{i,n} > 0$ if and only if $i \in I_+$.
(2) $p_{i,n} \to p_i$ for all $i \in \{1, \dots, r\}$.

For $i \in I_+$ let $\alpha_{i,n} = r_n - \mu_{i,n}$, and set $\alpha_{i,n} = r_n + \mu_{i,n}$ for $i \notin I_+$. So, $\alpha_{i,n} \geq 0$ for all i. Taking a further subsequence we may assume that $\alpha_{i,n} \to \alpha_i \in [0, \infty]$ for all i. Let $I = \{i \in I_+ \colon \alpha_i < \infty\}$ and $J = \{j \notin I_+ \colon \alpha_j < \infty\}$.

As $\Lambda(z_n) = \Lambda(-z_n) = r_n$, we know that

$$\min_{i \in I_+} \alpha_{i,n} = 0 = \min_{j \notin I_+} \alpha_{j,n},$$

and hence I and J are both nonempty.

It now follows from [30, Lemma 4.7] that

$$\Lambda(-v + z_n - r_n u) \to \Lambda_{A(p_I)}\left(-U_{p_I} v - \sum_{i \in I} \alpha_i p_i\right)$$

and

$$\Lambda(v - z_n - r_n u) \to \Lambda_{A(p_J)}\left(U_{p_J} v - \sum_{j \in J} \alpha_j p_j\right).$$

Thus,

$$\begin{aligned}
\lim_{n\to\infty} g_{w_n}(v) &= \lim_{n\to\infty} |v - w_n|_u - |w_n|_u = \lim_{n\to\infty} |v - z_n|_u - |z_n|_u \\
&= \lim_{n\to\infty} \Lambda(-v + z_n) + \Lambda(v - z_n) - 2r_n \\
&= \lim_{n\to\infty} \Lambda(-v + z_n - r_n u) + \Lambda(v - z_n - r_n u) \\
&= \Lambda_{A(p_I)}\left(-U_{p_I} v - \sum_{i \in I} \alpha_i p_i\right) + \Lambda_{A(p_J)}\left(U_{p_J} v - \sum_{j \in J} \alpha_j p_j\right)
\end{aligned}$$

for all $v \in T_u$, which completes the proof. □

The next proposition shows that each function of the form (5.1) is indeed a horofunction. In fact, we shall see that it is a Busemann point in $\partial \overline{T_u}^h$.

Proposition 5.2. *Let $p_1,\dots,p_r\in A$ be a Jordan frame, $I,J\subseteq\{1,\dots,r\}$ disjoint and nonempty, and $\alpha\in\mathbb{R}^{I\cup J}$ be such that $\min_{i\in I}\alpha_i=0=\min_{j\in J}\alpha_j$. If*

$$\zeta=\sum_{i\in I}-\alpha_i p_i+\sum_{j\in J}\alpha_j p_j \quad\text{and}\quad \omega=p_I-p_J, \tag{5.2}$$

then for $\xi_t=t\omega+\zeta-\frac{1}{r}\mathrm{tr}(t\omega+\zeta)u\in T_u$ with $t>0$ we have that $g_{\xi_t}\to g$, where g is given by (5.1), and hence g is a Busemann point.

Proof. For $t>0$, $\omega_t=t\omega+\zeta\in A$, and note that $\Lambda(\omega_t)=\Lambda(-\omega_t)=t$ for all $t>0$ large. Then by [30, Lemma 4.7] we get that

$$\begin{aligned}\lim_{t\to\infty}\Lambda(-v+\omega_t-tu)&=\lim_{t\to\infty}\Lambda\left(-v-\sum_{i\in I}\alpha_i p_i+\sum_{j\in J}(-2t+\alpha_j)p_j+\sum_{k\notin I\cup J}-tp_k\right)\\&=\Lambda_{A(p_I)}\left(-U_{p_I}v-\sum_{i\in I}\alpha_i p_i\right).\end{aligned}$$

Likewise,

$$\lim_{t\to\infty}\Lambda(v-\omega_t-tu)=\Lambda_{A(p_J)}\left(U_{p_J}v-\sum_{j\in J}\alpha_j p_j\right).$$

Thus, for $v\in T_u$ we have that

$$\begin{aligned}\lim_{t\to\infty}g_{\xi_t}(v)&=\lim_{t\to\infty}|v-\xi_t|_u-|\xi_t|_u=\lim_{t\to\infty}|v-\omega_t|_u-|\omega_t|_u\\&=\lim_{t\to\infty}\Lambda(-v+\xi_t)+\Lambda(v-\xi_t)-2t\\&=\lim_{t\to\infty}\Lambda(-v+\xi_t-tu)+\Lambda(v-\xi_t-tu)\\&=\Lambda_{A(p_I)}\left(-U_{p_I}v-\sum_{i\in I}\alpha_i p_i\right)+\Lambda_{A(p_J)}\left(U_{p_J}v-\sum_{j\in J}\alpha_j p_j\right),\end{aligned}$$

which shows that $g_{\xi_t}\to g$. As $t\mapsto\xi_t$ is a straight-line geodesic, we find that g is Busemann point. □

By combining Propositions 5.1 and 5.2 we get the following description of the horofunctions in $\overline{T_u}^h$.

Theorem 5.3. *The horofunctions of $\overline{T_u}^h$ are precisely the functions $g\colon T_u\to\mathbb{R}$ of the form (5.1), and each horofunction is a Busemann point.*

It follows that the equivalence classes of $\simeq$ on $\partial\overline{T_u}^h=\mathcal{B}_{T_u}$ coincides with the parts. Thus, to understand the geometry of the equivalence classes we can analyse the parts of $\partial\overline{T_u}^h$.

Proposition 5.4. *Let $g,g'\in\partial\overline{T_u}^h$ be two horofunctions, where g is given by (5.1) and*

$$g'(v)=\Lambda_{A(q_{I'})}\left(-U_{q_{I'}}v-\sum_{i\in I'}\beta_i q_i\right)+\Lambda_{A(q_{J'})}\left(U_{q_{J'}}v-\sum_{j\in J'}\beta_j q_j\right)\quad\text{for } v\in T_u. \tag{5.3}$$

If $p_I=q_{I'}$ and $p_J=q_{J'}$, then g and g' are in the same part and

$$\delta(g,g')=\big(\Lambda_{A(p_I)}(a_I-b_{I'})+\Lambda_{A(p_I)}(b_{I'}-a_I)\big)+\big(\Lambda_{A(p_J)}(a_J-b_{J'})+\Lambda_{A(p_J)}(b_{J'}-a_J)\big),$$

where $a_I=\sum_{i\in I}\alpha_i p_i$, $a_J=\sum_{j\in J}\alpha_j p_j$, $b_{I'}=\sum_{i\in I'}\beta_i q_i$ and $b_{J'}=\sum_{j\in J'}\beta_j q_j$.

Proof. Let ζ, ω and ξ_t be as in Proposition 5.2. Then for all $t>0$ large we have that

$$\begin{aligned}|\xi_t|_u + g'(\xi_t) &= |\xi_t|_u + \Lambda_{A(q_{I'})}(-U_{q_{I'}}\xi_t - b_{I'}) + \Lambda_{A(q_{J'})}(U_{q_{J'}}\xi_t - b_{J'})\\ &= 2t + \Lambda_{A(p_I)}(-U_{p_I}\xi_t - b_{I'}) + \Lambda_{A(p_J)}(U_{p_J}\xi_t - b_{J'})\\ &= \Lambda_{A(p_I)}(tp_I - U_{p_I}\xi_t - b_{I'}) + \Lambda_{A(p_J)}(tp_J + U_{p_J}\xi_t - b_{J'})\\ &= \Lambda_{A(p_I)}(\tfrac{1}{r}\mathrm{tr}(t\omega+\zeta)p_I + a_I - b_{I'}) + \Lambda_{A(p_J)}(-\tfrac{1}{r}\mathrm{tr}(t\omega+\zeta)p_J + a_J - b_{J'})\\ &= \Lambda_{A(p_I)}(a_I - b_{I'}) + \Lambda_{A(p_J)}(a_J - b_{J'}).\end{aligned}$$

From (2.2) and Proposition 5.2, we conclude that $H(g,g') = \Lambda_{A(p_I)}(a_I - b_{I'}) + \Lambda_{A(p_J)}(a_J - b_{J'})$. Interchanging the roles of g and g' gives $H(g',g) = \Lambda_{A(p_J)}(a_J - b_{J'}) + \Lambda_{A(p_J)}(b_{J'} - a_J)$, which completes the proof. □

The condition in Proposition 5.4 characterises the parts in the horofunction boundary as the next proposition shows.

Proposition 5.5. *If g and g' are horofunctions in $\overline{T_u}^h$ given by (5.1) and (5.3), respectively, then g and g' are in the same part if and only if $p_I = q_{I'}$ and $p_J = q_{J'}$.*

Proof. By Proposition 5.4 it remains to show that $\delta(g,g') = \infty$ if $p_I \neq q_{I'}$ or $p_J \neq q_{J'}$. Suppose that $p_I \neq q_{I'}$. Then $p_I \not\leq q_{I'}$ or $q_{I'} \not\leq p_I$. Suppose that $p_I \not\leq q_{I'}$. Let ζ, ω and ξ_t be as in Proposition 5.2. We will show that $H(g,g') = \infty$ in this case.

For all $t > 0$ large we have that

$$\begin{aligned}|\xi_t|_u + g'(\xi_t) &= |\xi_t|_u + \Lambda_{A(q_{I'})}(-U_{q_{I'}}\xi_t - b_{I'}) + \Lambda_{A(q_{J'})}(U_{q_{J'}}\xi_t - b_{J'})\\ &= 2t + \Lambda_{A(q_{I'})}(-U_{q_{I'}}\xi_t - b_{I'}) + \Lambda_{A(q_{J'})}(U_{p_J}\xi_t - b_{J'})\\ &= \Lambda_{A(q_{I'})}(tq_{I'} - U_{q_{I'}}\xi_t - b_{I'}) + \Lambda_{A(q_{J'})}(tq_{J'} + U_{q_{J'}}\xi_t - b_{J'})\\ &= \Lambda_{A(q_{I'})}(tq_{I'} - U_{q_{I'}}(t\omega+\zeta) - b_{I'}) + \Lambda_{A(q_{J'})}(tq_{J'} + U_{q_{J'}}(t\omega+\zeta) - b_{J'}).\end{aligned}$$

Note that $t\omega + \zeta \leq tp_I$ for all $t > 0$ large. So, $U_{q_{I'}}(t\omega+\zeta) \leq tU_{q_{I'}}p_I$ for all $t>0$ large. This implies that

$$\Lambda_{A(q_{I'})}(tq_{I'} - U_{q_{I'}}(t\omega+\zeta) - b_{I'}) \geq \Lambda_{A(q_{I'})}(t(q_{I'} - U_{q_{I'}}p_I) - b_{I'}) \to \infty$$

as $t\to\infty$, since $q_{I'} - U_{q_{I'}}p_I > 0$ by [30, Lemma 4.12].

Also note that for all $t > 0$ large we have that $t\omega+\zeta \geq -tp_J$, and hence $U_{q_{J'}}(t\omega+\zeta) \geq -tU_{q_{J'}}p_J$. As $U_{q_{J'}}p_J \leq U_{q_{J'}}u = q_{J'}$, we have $t(q_{J'} - U_{q_{J'}}p_J) \geq 0$ for all $t>0$ large. It follows that

$$\Lambda_{A(q_{J'})}(tq_{J'} + U_{q_{J'}}(t\omega+\zeta) - b_{J'}) \geq \Lambda_{A(q_{J'})}(t(q_{J'} - U_{q_{J'}}p_J) - b_{J'}) \geq \Lambda_{A(q_{J'})}(-b_{J'}) > -\infty$$

for all $t>0$ large. Combining this inequality with the previous one and using (2.2), we conclude that $H(g,g') = \infty$.

For the other cases the result can be shown in the same way. □

6. Extension of the exponential map $\exp_u\colon T_u \to PA_+^\circ$

In this section we show that the exponential map extends as a homeomorphism between the horofunction compactifications and preserves the geometry of the equivalence classes, i.e., the parts. More specifically we show the following result.

Theorem 6.1. *Let A_+° be a symmetric cone in a finite dimensional formally real Jordan algebra A. The map $\exp_u\colon \overline{T_u}^h \to \overline{PA_+^\circ}^h$ is a homeomorphism which* *maps each part in the horofunction boundary of T_u onto a part of the horofunction boundary of PA_+°.*

The proof of Theorem 6.1 follows the same steps as the one taken in the proof of Theorem 4.3. To define the extension we recall the characterisation of the Hilbert distance horofunctions of PA_+° from [30]. The horofunctions of (PA_+°, d_H) are all Busemann points and precisely the functions $h\colon PA_+^\circ \to \mathbb{R}$ of the form

$$h(x) = \log M(y/x) + \log M(z/x^{-1}) \quad \text{for } x \in PA_+^\circ, \tag{6.1}$$

where $y, z \in \partial A_+$ are such that $\|y\|_u = \|z\|_u = 1$ and $y \bullet z = 0$, see [30, Theorem 5.4]. In this case the extension of the exponential map is defined as follows.

Definition 6.2. The *exponential map*, $\exp_u\colon \overline{T_u}^h \to \overline{PA_+^\circ}^h$, is defined by $\exp_u(v) = e^v$ for $v \in T_u$, and for $g \in \partial\overline{T_u}^h$ given by (5.1) we let $\exp_u(g) = h$, where h is given by (6.1) with $y = \sum_{i\in I} e^{-\alpha_i} p_i$ and $z = \sum_{j\in J} e^{-\alpha_j} p_j$.

Note that the extension is well-defined. Indeed, if g given by (5.1) is represented as

$$g(v) = \Lambda_{A(q_{I'})}(-U_{q_{I'}} v - \sum_{i\in I'} \beta_i q_i) + \Lambda_{A(q_{J'})}(U_{q_{J'}} v - \sum_{j\in J'} \beta_j q_j),$$

then, as $\delta(g, g) = 0$, we get by Propositions 5.4 and 5.5 that $p_I = q_{I'}$, $p_J = q_{J'}$, $\sum_{i\in I} \alpha_i p_i = \sum_{i\in I'} \beta_i q_i$ and $\sum_{j\in J} \alpha_j p_j = \sum_{j\in J'} \beta_j q_j$. From Lemma 4.2 we deduce that $\sum_{i\in I} e^{-\alpha_i} p_i = \sum_{i\in I'} e^{-\beta_i} q_i$ and $\sum_{j\in J} e^{-\alpha_j} p_j = \sum_{j\in J'} e^{-\beta_j} q_j$, and hence the extension is well-defined.

We first show that $\exp_u\colon \overline{T_u}^h \to \overline{PA_+^\circ}^h$ is a bijection.

Lemma 6.3. $\exp_u\colon \overline{T_u}^h \to \overline{PA_+^\circ}^h$ *is a bijection which maps T_u onto PA_+°, and $\partial\overline{T_u}^h$ onto $\partial\overline{PA_+^\circ}^h$.*

Proof. As $\det \exp_u(x) = e^{\operatorname{tr} x} = 1$ for $x \in T_u$, we see that $\exp_u$ is a bijection from T_u onto PA_+°. It follows from [30, Theorem 5.4] and Theorem 5.3 that $\exp_u$ maps $\partial\overline{T_u}^h$ onto $\partial\overline{PA_+^\circ}^h$.

To complete the proof it remains to show that if $g, g' \in \partial\overline{T_u}^h$ with $\exp_u(g) = \exp_u(g')$, then $g = g'$. Let g and g' be given by (5.1) and (5.3), respectively. By definition of $\exp_u$ we have that $\exp_u(g) = h$, where

$$h(x) = \log M(y/x) + \log M(z/x^{-1}) \quad \text{for } x \in PA_+^\circ,$$

with $y = \sum_{i\in I} e^{-\alpha_i} p_i$ and $z = \sum_{j\in J} e^{-\alpha_j} p_j$. Likewise, $\exp_u(g') = h'$, where $h'(x) = \log M(y'/x) + \log M(z'/x^{-1})$ for $x \in PA_+^\circ$, with $y' = \sum_{i\in I'} e^{-\beta_i} q_i$ and $z = \sum_{j\in J'} e^{-\beta_j} q_j$.

As $h = h'$, we have that $\delta(h, h') = 0$, hence $y = y'$ and $z = z'$ by [30, Proposition 5.6]. Now using Remark 2.3 we find that $p_I = q_{I'}$ and $p_J = q_{J'}$. Furthermore, $-\sum_{i\in I} \alpha_i p_i = \log(y + u - p_I) = \log(y' + u - q_{I'}) = -\sum_{i\in I'} \beta_i q_i$ and $-\sum_{j\in J} \alpha_j p_j = \log(z + u - p_J) = \log(z' + u - q_{J'}) = -\sum_{j\in J'} \beta_j q_j$. This implies that $g = g'$, which completes the proof. □

The proof of the continuity of the extension of $\exp_u$ is split up into two lemmas.

Lemma 6.4. *If (w_n) in T_u converges to $g \in \partial\overline{T_u}^h$, then $(\exp_u(w_n))$ converges to $\exp_u(g)$.*

Proof. Let (w_n) be a sequence in T_u converging to $g \in \partial\overline{T_u}^h$, where g is given by (5.1). To prove that $(\exp_u(w_n))$ converges to $\exp_u(g) = h$, where h is given by

(6.1), we show that each of its subsequences has a subsequence converging to h. So let $(\exp_u(w_k))$ be a subsequence. As g is a horofunction, it follows from Lemma 2.1 that $|w_k|_u \to \infty$. Set $z_k = w_k - \frac{1}{2}(\Lambda(w_k) - \Lambda(-w_k))u$ and let $r_k = \Lambda(z_k)$. So, $2r_k = \Lambda(z_k) + \Lambda(-z_k) = |z_k|_u = |w_k|_u$.

Using the spectral decomposition we write $z_k = \sum_{i=1}^r \lambda_{i,k} q_{i,k}$. By taking subequences we may assume that there exists $I_+ \subseteq \{1, \ldots, r\}$ such that for all k we have that $\lambda_{i,k} > 0$ if and only if $i \in I_+$. In addition, we may assume that $q_{i,k} \to q_i$ for all i. Let $\beta_{i,k} = r_k - \lambda_{i,k} \geq 0$ for $i \in I_+$, and $\beta_{i,k} = r_k + \lambda_{i,k} \geq 0$ for $i \notin I_+$. By taking further subsequences we may assume that $\beta_{i,k} \to \beta_i \in [0, \infty]$ for all i. Let $I' = \{i \in I_+ \colon \beta_i < \infty\}$ and $J' = \{j \notin I_+ \colon \beta_j < \infty\}$.

Note that $\min_{i \in I'} \beta_i = 0 = \min_{j \in J'} \beta_j$, as $\min_{i \in I_+} \beta_{i,k} = 0 = \min_{i \notin I_+} \beta_{i,k}$ for all k, so I' and J' are both nonempty. It follows from [30, Lemma 4.7] and (2.3) that

$$\begin{aligned}\lim_{k\to\infty} g_{w_k}(v) &= \lim_{k\to\infty} |v - w_k|_u - |w_k|_u = \lim_{k\to\infty} |v - z_k|_u - 2r_k \\ &= \lim_{k\to\infty} \Lambda(-v + z_k - r_k u) + \Lambda(v - z_k - r_k u) \\ &= \Lambda_{A(q_{I'})}\left(-U_{q_{I'}} v - \sum_{i\in I'} \beta_i q_i\right) + \Lambda_{A(q_{J'})}\left(U_{q_{J'}} v - \sum_{j\in J'} \beta_j q_j\right).\end{aligned}$$

If we denote the righthand side by $g'(v)$, we find that $g' \colon T_u \to \mathbb{R}$ is a horofunction by Proposition 5.2.

Since $g_{w_n} \to g$, we conclude that $g = g'$, and hence $\delta(g, g') = 0$. It now follows from Propositions 5.4 and 5.5 that $p_I = q_{I'}$ and $p_J = q_{J'}$. Moreover, $\sum_{i\in I} \alpha_i p_i = \sum_{i\in I'} \beta_i q_i$ and $\sum_{j\in J} \alpha_j p_j = \sum_{j\in J'} \beta_j q_j$. So by Lemma 4.2 we get that $y = \sum_{i\in I} e^{-\alpha_i} p_i = \sum_{i\in I'} e^{-\beta_i} q_i$ and $z = \sum_{j\in J} e^{-\alpha_j} p_j = \sum_{j\in J'} e^{-\beta_j} q_j$.

It follows that

$$e^{-r_k}\exp_u(z_k) = \sum_{i=1}^r e^{-(r_k - \lambda_{i,k})} q_{i,k} \to \sum_{i\in I'} \beta_i q_i = y$$

and

$$e^{-r_k}\exp_u(-z_k) = \sum_{i=1}^r e^{-(r_k + \lambda_{i,k})} q_{i,k} \to \sum_{j\in J'} \beta_j q_j = z.$$

As $\exp_u(a + \lambda u) = e^\lambda \exp_u(a)$ for all $a \in A$ and $\lambda \in \mathbb{R}$, we have that

$$\begin{aligned}\lim_{k\to\infty} h_{\exp_u(w_k)}(x) &= \lim_{k\to\infty} d_H(x, \exp_u(w_k)) - d_H(u, \exp_u(w_k)) \\ &= \lim_{k\to\infty} d_H(x, \exp_u(z_k)) - d_H(u, \exp_{\mathrm{u}}(z_k)) \\ &= \lim_{k\to\infty} \log M(\exp_u(z_k)/x) + \log M(x/\exp_u(z_k)) - \operatorname{diam}\sigma(z_k) \\ &= \lim_{k\to\infty} \log M(\exp_u(z_k)/x) + \log M(\exp_u(-z_k)/x^{-1}) - 2r_k \\ &= \lim_{k\to\infty} \log M(e^{-r_k}\exp_u(z_k)/x) + \log M(e^{-r_k}\exp_u(-z_k)/x^{-1}) \\ &= \log M(y/x) + \log M(z/x^{-1})\end{aligned}$$

for all $x \in PA_+^\circ$ by continuity of the M functions, see [28, Lemma 2.2]. This shows that $(\exp(w_k))$ in PA_+° converges h, and hence the proof is complete. □

Next we establish the continuity in the horofunction boundary.

Lemma 6.5. *If* (g_n) *in* $\partial\overline{T_u}^h$ *converges to* $g \in \partial\overline{T_u}^h$, *then* $(\exp_u(g_n))$ *converges to* $\exp_u(g)$.

Proof. Let (g_n) be a sequence in $\partial\overline{T_u}^h$ converging to g, where g is given by (5.1). So $\exp_u(g) = h$, where is h given by (6.1). We prove that each subsequence of $(\exp_u(g_n))$ has a convergent subsequence with limit h. Let $(\exp_u(g_{n_k}))$ be a subsequence. By Theorem 5.1 we can write for $k \geq 1$,

$$g_{n_k}(v) = \Lambda_{A(q_{I_k,k})}\left(-U_{q_{I_k,k}}v - \sum_{i\in I_k}\beta_{i,k}q_{i,k}\right) + \Lambda_{A(q_{J_k,k})}\left(U_{q_{J_k,k}}v - \sum_{j\in J_k}\beta_{j,k}q_{j,k}\right),$$

where the $q_{i,k}$ and $q_{j,k}$'s are orthogonal primitive idempotents, $I_k, J_k \subseteq \{1,\ldots,r\}$ are nonempty and disjoint, and $\min_{i\in I_k}\beta_{i,k} = 0 = \min_{j\in J_k}\beta_{j,k}$.

After taking subsequences we may assume that

(1) There exist $I_0, J_0 \subseteq \{1,\ldots,r\}$ such that $I_0 = I_k$ and $J_0 = J_k$ for all k.
(2) There exist $i_0 \in I_0$ and $j_0 \in J_0$ such that $\beta_{i_0,k} = 0 = \beta_{j_0,k}$ for all k.
(3) $\beta_{m,k} \to \beta_m \in [0,\infty]$ and $q_{m,k} \to q_m$ for all $m \in I_0 \cup J_0$.

Now let $I' = \{i \in I_0 \colon \beta_i < \infty\}$ and $J' = \{j \in J_0 \colon \beta_j < \infty\}$. So I' and J' are nonempty and $\min_{i\in I'}\beta_i = 0 = \min_{j\in J'}\beta_j$.

As I' is nonempty, it follows from [30, Lemma 4.7] that

$$\Lambda_{A(q_{I_0,k})}\left(-U_{q_{I_0,k}}v - \sum_{i\in I_0}\beta_{i,k}q_{i,k}\right) \to \Lambda_{A(q_{I'})}\left(-U_{q_{I'}}v - \sum_{i\in I'}\beta_i q_i\right).$$

Likewise, as J' is nonempty, we know that

$$\Lambda_{A(q_{J_0,k})}\left(U_{q_{J_0,k}}v - \sum_{j\in J_0}\beta_{j,k}q_{j,k}\right) \to \Lambda_{A(q_{J'})}\left(U_{q_{J'}}v - \sum_{j\in J'}\beta_j q_j\right).$$

Thus,

$$\lim_{k\to\infty} g_{n_k}(v) = \Lambda_{A(q_{I'})}\left(-U_{q_{I'}}v - \sum_{i\in I'}\beta_i q_i\right) + \Lambda_{A(q_{J'})}\left(U_{q_{J'}}v - \sum_{j\in J'}\beta_j q_j\right).$$

So, if we denote the righthand side by $g'(v)$, then $g' \colon A \to \mathbb{R}$ is a horofunction by Proposition 5.2. As $g_n \to g$, we find that $g = g'$ and hence $\delta(g, g') = 0$.

It now follows from Propositions 5.4 and 5.5 that $p_I = q_{I'}$ and $p_J = q_{J'}$. Moreover,

$$\sum_{i\in I}\alpha_i p_i = \sum_{i\in I'}\beta_i q_i \quad \text{and} \quad \sum_{j\in J}\alpha_j p_j = \sum_{j\in J'}\beta_j q_j,$$

So by Lemma 4.2 we get that $y = \sum_{i\in I} e^{-\alpha_i}p_i = \sum_{i\in I'} e^{-\beta_i}q_i$ and $z = \sum_{j\in J} e^{-\alpha_j}p_j = \sum_{j\in J'} e^{-\beta_j}q_j$.

If we now let $\bar{y}_k = \sum_{i\in I_0} e^{-\beta_{i,k}}q_{i,k}$ and $\bar{z}_k = \sum_{j\in J_0} e^{-\beta_{j,k}}q_{j,k}$, then

$$\lim_{k\to\infty}\bar{y}_k = \sum_{i\in I'} e^{-\beta_i}q_i = y \quad \text{and} \quad \lim_{k\to\infty}\bar{z}_k = \sum_{j\in J'} e^{-\beta_j}q_j = z.$$

Therefore, by continuity of the M function [28, Lemma 2.2],

$$\begin{aligned}\lim_{k\to\infty}\exp_u(g_{n_k})(x) &= \lim_{k\to\infty}\log M(\bar{y}_k/x) + \log M(\bar{z}_k/x^{-1})\\ &= \log M(y/x) + \log M(z/x^{-1}) = h(x),\end{aligned}$$

which completes the proof. □

Collecting the results so far we can now easily proof Theorem 6.1.

Proof of Theorem 6.1. It follows from Lemmas 6.3, 6.4 and 6.5 that $\exp_u\colon \overline{T_u}^h \to \overline{PA_+^\circ}^h$ is a continuous bijection. As $\overline{T_u}^h$ is compact and $\overline{PA_+^\circ}^h$ is Hausdorff, we conclude that $\exp_u$ is a homeomorphism.

Suppose that g and g' are horofunctions in the same part of $\overline{T_u}^h$, where g is given by (5.1) and g' is given by (5.3). It follows from Propositions 5.4 and 5.5 that $p_I = q_{I'}$ and $p_J = q_{J'}$. By definition $\exp_u(g) = h$, where h is given by (6.1) with $y = \sum_{i\in I} e^{-\alpha_i} p_i$ and $z = \sum_{j\in J} e^{-\alpha_j} p_j$. Likewise, $\exp_u(g') = h'$, where h' is given by $h'(x) = \log M(y'/x) + \log M(z'/x^{-1})$, with $y' = \sum_{i\in I'} e^{-\beta_i} q_i$ and $z' = \sum_{j\in J'} e^{-\beta_j} q_j$. As $p_I = q_{I'}$ and $p_J = q_{J'}$, we have that $y \sim p_I \sim q_{I'} \sim y'$ and $z \sim p_J \sim q_{J'} \sim z'$, and hence h and h' are in the same part by [30, Proposition 5.6].

Conversely, if h and h' are in the same part, then $y \sim y'$ and $z \sim z'$ by [30, Proposition 5.6], and hence $p_I \sim q_{I'}$ and $p_J \sim q_{J'}$. This implies that $\mathrm{face}(p_I) = \mathrm{face}(q_{I'})$. As $\mathrm{face}(p_I) \cap [0,u] = [0,p_I]$ by [3, Lemma 1.39], we get that $p_I = q_{I'}$. Likewise we have that $p_J = q_{J'}$. So, if h and h' are in the same part, then g and g' are in the same part. This completes the proof. □

Remark 6.6. The horofunction compactification is homeomorphic to the closed unit ball of the dual space $(T_u, |\cdot|_u)^*$. Indeed, it was shown in [30, Section 5] that there exists a homeomorphism from the horofunction compactification of (PA_+°, d_H) to the closed unit ball B_1^* of the dual space $(T_u, |\cdot|_u)^*$, which maps parts onto parts. We know from [30, Section 5.3] that the dual space $(T_u, |\cdot|_u)^*$ is given by $(T_u, \frac{1}{2}\|\cdot\|_u^*)$, where we use the inner-product $(x|y) = \mathrm{tr}(x \bullet y)$ to identify T_u^* with T_u. The unit ball B_1^* satisfies

$$B_1^* = 2\,\mathrm{conv}(S(A) \cup -S(A)) \cap T_u,$$

where $S(A) = \{w \in A_+ \colon (u|w) = 1\}$ is the state space of A. Its (closed) boundary faces are precisely the nonempty sets of the form,

$$A_{p,q} = 2\,\mathrm{conv}((U_p(A) \cap S(A) \cup (U_q(A) \cap -S(A))) \cap T_u,$$

where p and q are orthogonal idempotents by [12, Theorem 4.4].

7. Final remarks

Symmetric cones A_+° and their projective cones PA_+° are examples of Riemannian symmetric spaces $X = G/K$ of non-compact type. The Finsler metrics of the Thompson distance and the Hilbert distance are examples of invariant Finsler metrics, which have been chararcterised by Planche [37]. In [20] it was shown that each generalised Satake compactification of a symmetric space $X = G/K$ of non-compact type can be realised as a horofunction compactification under an invariant Finsler metric, whose restriction to a flat is a (possibly non-symmetric) norm with polyhedral unit ball. In [20, Examples 5.3 and 5.7] the symmetric space $\mathrm{SL}_n(\mathbb{C})/\mathrm{SU}_n$ is considered for $n = 3$ and $n = 4$. This space corresponds to the projective symmetric cone $P\Pi_n(\mathbb{C})$ consisting of $n \times n$ positive definite Hermitian matrices with determinant 1. If we consider the restriction of the unit ball of the Finsler metric $H(I,\cdot) = |\cdot|_I$ to the flat consisting of diagonal matrices in T_I, then for $n = 3$ we get a hexagon, and for $n = 4$ we get a rhombic dodecahedron. These unit balls correspond to the invariant Finsler metrics in [20, Examples 5.3 and 5.7], where the generalised Satake compactification is considered for the adjoint representation of $\mathrm{SL}_n(\mathbb{C})$. So for $n = 3$ and $n = 4$, the Hilbert distance on $\mathrm{SL}_n(\mathbb{C})/\mathrm{SU}_n$ realises the generalised Satake compactification with respect to the adjoint representation.

Question 7.1. Does the horofunction compactification of $\mathrm{SL}_n(\mathbb{C})/\mathrm{SU}_n$ with respect to the Hilbert distance realise the generalised Satake compactification with respect to the adjoint representation? In fact, one could ask if the same true for the Hilbert metric on general projective symmetric cones PA_+°?

For a symmetric cone A_+° a flat in the tangent space at $u \in A_+^\circ$ is given by $\mathrm{Span}(\{p_1,\dots,p_r\})$, where $p_1,\dots,p_r$ is a Jordan frame. The restriction to the flat of the Finsler metric F for the Thompson distance is a polyhedral norm. In fact, if $w \in \mathrm{Span}(\{p_1,\dots,p_r\})$ has spectral decomposition $w = \sum_{i=1}^r \delta_i p_i$, then $F(u,w) = \|w\|_u = \max_i |\delta_i|$. So, the restriction of the unit ball of F to a flat is an r-dimensional hypercube. It would be interesting to investigate the following question.

Question 7.2. Does the horofunction compactification of (A_+°, d_T) realise a generalised Satake compactification of the symmetric space A_+°, and if so, to which representation does it correspond?

In view of the results in this paper it is also natural to investigate the following problem for a general symmetric space $X = G/K$ of non-compact type with an invariant Finsler metric.

Question 7.3. Does the exponential map $\exp_u\colon T_uX \to X$ extend as a homeomorphism between the horofunction compactification of X under the Finsler distance and the horofunction compactification of the tangent space T_uX under the Finsler norm, preserving the equivalence classes in the horofunction boundary?

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