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On a result of Bao-Qin Li concerning Dirichlet series and shared values

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Abstract. Li (2011) proved that if two L-functions L_1 and L_2 in the extended Selberg class $\mathcal{S}^\sharp$ satisfy the same functional equation with $a(1) = 1$ and $L_1^{-1}(c_j) = L_2^{-1}(c_j)$ with $j \in \{1, 2\}$ for two distinct finite complex numbers c_1 and c_2, then $L_1 = L_2$. Later on, Gonek–Haan–Ki (2014) proved that if two L-functions L_1 and L_2 in the extended Selberg class $\mathcal{S}^\sharp$ have the positive degrees and $L_1^{-1}(c) = L_2^{-1}(c)$ for a finite non-zero complex number c, then $L_1 = L_2$. This implies that if two L-functions L_1 and L_2 in the extended Selberg class $\mathcal{S}^\sharp$ have the positive degrees and $L_1^{-1}(c_j) = L_2^{-1}(c_j)$ with $j \in \{1, 2\}$ for two distinct finite complex numbers c_1 and c_2, then $L_1 = L_2$. In this paper, we prove that if two L-functions L_1 and L_2 in the extended Selberg class $\mathcal{S}^\sharp$ have the zero degrees and satisfy $L_1^{-1}(c_j) = L_2^{-1}(c_j)$ with $j \in \{1, 2\}$ for two distinct finite complex numbers c_1 and c_2, and if $a_1(1) = a_2(1)$ or $\lim_{r\to+\infty} \frac{T(r,L_2)}{T(r,L_1)} = 1$, then $L_1 = L_2$. The main results obtained in this paper improve Theorem 1 from Li (2011) when the L-functions in the extended Selberg class $\mathcal{S}^\sharp$ have the zero degrees. Some examples are provided to show that the results obtained in this paper, in a sense, are best possible.

Dirichlet'n sarjoja ja jaettuja arvoja koskevasta Bao-Qin Lin tuloksesta

Tiivistelmä. Olkoot L_1 ja L_2 kaksi laajennettuun Selbergin luokkaan $\mathcal{S}^\sharp$ kuuluvaa L-funktiota ja c_1 sekä c_2 kaksi erillistä äärellistä kompleksilukua. Li (2011) todisti, että $L_1 = L_2$, jos L_1 ja L_2 toteuttavat saman funktionaaliyhtälön, jossa $a(1) = 1$ ja $L_1^{-1}(c_j) = L_2^{-1}(c_j)$ molemmilla $j \in \{1, 2\}$. Myöhemmin Gonek, Haan ja Ki (2014) osoittivat, että $L_1 = L_2$, jos L-funktioilla L_1 ja L_2 on positiivinen aste ja $L_1^{-1}(c) = L_2^{-1}(c)$ äärellisellä nollasta poikkeavalla kompleksiluvulla c. Tästä seuraa, että $L_1 = L_2$, jos L-funktioilla L_1 ja L_2 on positiivinen aste ja $L_1^{-1}(c_j) = L_2^{-1}(c_j)$ molemmilla $j \in \{1, 2\}$. Tässä työssä osoitamme, että $L_1 = L_2$, jos L-funktioilla L_1 ja L_2 on aste nolla ja $L_1^{-1}(c_j) = L_2^{-1}(c_j)$ molemmilla $j \in \{1, 2\}$, sekä lisäksi $a_1(1) = a_2(1)$ tai $\lim_{r\to+\infty} \frac{T(r,L_2)}{T(r,L_1)} = 1$. Päätuloksemme parantavat Lin (2011) lausetta 1, kun L-funktioiden aste on nolla. Näytämme esimerkein, että nämä tulokset ovat tietyssä mielessä parhaita mahdollisia.

1. Introduction and main results

L-functions are Dirichlet series with the Riemann zeta function $\zeta(s) = \Sigma_{n=1}^{\infty} \frac{1}{n^s}$ as the prototype, which are important objects in number theory and have been studied extensively (cf. [15]). Throughout the paper, an L-function always means a Dirichlet series $L(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$ of a complex variable $s = \sigma + it$, satisfying the following axioms (cf. [15, p. 111]):

(i) Ramanujan hypothesis: $a(n) \ll n^{\varepsilon}$ for every $\varepsilon > 0$.

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(ii) Analytic continuation: There is a non-negative integer k such that $(s-1)^k L(s)$ is an entire function of finite order.

(iii) Functional equation: L satisfies a functional equation of type $\Lambda_L(s) = \omega\overline{\Lambda_L(1-\overline{s})}$, where $\Lambda_L(s) = L(s)Q^s \prod_{j=1}^{K} \Gamma(\lambda_j s + \nu_j)$ with positive real numbers Q, λ_j and complex numbers ν_j, ω with $\mathrm{Re}(\nu_j) \geq 0$ and $|\omega| = 1$.

The degree d of an L-function L is defined to be $d = 2\sum_{j=1}^{K} \lambda_j$, where K, λ_j are the numbers in the axiom (iii).

The Selberg class $\mathcal{S}$ of L-functions is the set of all Dirichlet series $L(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$ of a complex variable $s = \sigma + it$ with $a(1) = 1$, satisfying the above axioms (i)-(iii) and the following axiom (iv) (cf. [13, 15]):

(iv) Euler product hypothesis: $L(s) = \prod_p \exp\left(\sum_{k=1}^{\infty} \frac{b(p^k)}{p^{ks}}\right)$ with suitable coefficients $b(p^k)$ satisfying $b(p^k) \ll p^{k\theta}$ for some $\theta < 1/2$, where the product is taken over all prime numbers p.

Throughout this paper, all L-functions are assumed to be the L-functions from the extended Selberg class $\mathcal{S}^\sharp$ of those L-functions that are not identically vanishing and only satisfy the axioms (i)–(iii) above. The notion of the extended Selberg class $\mathcal{S}^\sharp$ was originally introduced by Kaczorowski–Perelli [6].

In recent years, value distribution of L-functions has been studied extensively, which can be found, for example in Steuding [15]. Value distribution of L-functions concerns the distribution of zeros of an L-function L and, more generally, the c-points of L, that is to say, the roots of the equation $L(s) = c$, or the points in the pre-image $L^{-1}(c) = \{s \in \mathbb{C}\colon L(s) = c\}$, where and in what follows, s denotes the complex variables in the complex plane $\mathbb{C}$ and c denotes a value in the extended complex plane $\mathbb{C} \cup \{\infty\}$. L-functions can be analytically continued as meromorphic functions in $\mathbb{C}$. Two meromorphic functions f and g in the complex plane are said to share a value $c \in \mathbb{C}\cup\{\infty\}$ IM (ignoring multiplicities) if $f^{-1}(c) = g^{-1}(c)$ as two sets in $\mathbb{C}$. Moreover, f and g are said to share a value c CM (counting multiplicities) if they share the value c IM and if each common root of the equations $f(s) = c$ and $g(s) = c$ has the same multiplicities. In terms of shared values, two non-constant meromorphic functions in the complex plane must be identically equal if they share five distinct values from the extended complex plane IM, and one must be a Möbius transformation of the other one if they share four values from the extended complex plane CM. The numbers "five" and "four" are the best possible, as shown by Nevanlinna (cf. [3, 11, 17]), which are famous theorems due to Nevanlinna and often referred to as Nevanlinna's uniqueness theorems. For a non-constant meromorphic function f, we denote by $\rho(f)$ the order of growth of f, its definition can be found in [8, 18, 17]. For convenience, we give its detailed definition as follows:

Definition 1.1. For a non-constant meromorphic function f, the order of f, denoted as $\rho(f)$, is defined as

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

Throughout this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notations of the Nevanlinna theory of meromorphic functions as explained in [3, 8, 18, 17]. Let f be a non-constant meromorphic function, let k be a positive integer, and let a be a complex value in the extended complex plane. Next we denote by $N_{(k}(r, 1/(f-a)$

the counting function of those a-points of the non-constant meromorphic function f in $|z| < r$, where each point in $N_{(k}(r, 1/(h-a))$ is counted according to its multiplicity, and each point in $N_{(k}(r, 1/(f-a))$ is of multiplicity $\geq k$. We denote by $N_{k)}(r, 1/(f-a)$ the counting function of those a-points of the non-constant meromorphic function f in $|z| < r$, where each point in $N_{k)}(r, 1/(f-a))$ is counted according to its multiplicity, and each point in $N_{k)}(r, 1/(f-a))$ is of multiplicity $\leq k$. We denote by $\overline{N}_{(k}(r, 1/(f-a))$ and $\overline{N}_{k)}(r, 1/(f-a)$ the reduced forms of $N_{(k}(r, 1/(f-a))$ and $N_{k)}(r, 1/(f-a)$ respectively. Here $N_{(k}(r, 1/(f-\infty))$, $N_{k)}(r, 1/(f-\infty))$, $\overline{N}_{(k}(r, 1/(f-\infty))$, $\overline{N}_{k)}(r, 1/(f-\infty))$ mean $N_{(k}(r, f)$, $N_{k)}(r, f)$, $\overline{N}_{(k}(r, f)$, $\overline{N}_{k)}(r, f)$ respectively.

This paper concerns the question of how an L-function is uniquely determined in terms of the pre-images of complex values in the extended complex plane. We refer the reader to the monograph [15] for a detailed discussion on the topic and related works. Concerning the extended Selberg class $S^{\sharp}$, we recall the following two results due to Steuding [15], which actually holds without the Euler product hypothesis:

Theorem 1.2. [15, Theorem 7.11 (i)] *Assume that two L-functions L_1 and L_2 satisfy the axioms (i)–(iii) with $a(1) = 1$. If L_1 and L_2 share a value $c \neq \infty$ CM, then $L_1 = L_2$.*

Theorem 1.3. [15, Theorem 7.11 (ii)] *If two L-functions L_1 and L_2 in the extended Selberg class $\mathcal{S}^{\sharp}$ satisfy the same functional equation with $a(1) = 1$ and $L_1^{-1}(c_j) = L_2^{-1}(c_j)$ with $j \in \{1, 2\}$ for two finite distinct complex numbers c_1 and c_2 such that*

$$\liminf_{T\to\infty} \frac{\tilde{N}^{c_1}_{L_j}(T) + \tilde{N}^{c_2}_{L_j}(T)}{N^{c_1}_{L_j}(T) + N^{c_2}_{L_j}(T)} > \frac{1}{2} + \varepsilon \tag{1.1}$$

for some positive number ε with either $j = 1$ or $j = 2$, then $L_1 = L_2$. Here the term $N^c_L(T)$ denotes the number counted multiplicities of zeros of $L(\sigma + it) - c$ in the rectangle $0 \leq \sigma \leq 1$, $|t| \leq T$, and $\tilde{N}^c_L(T)$ denotes the number of zeros in the rectangle but ignoring multiplicities.

Remark 1.4. In 2016, Hu–Li [5] pointed out that Theorem 1.2 is false when $c = 1$. A counter example was given by Hu–Li [5, Remark 4] as follows: let $L_1(s) = 1 + \frac{2}{4^s}$ and $L_2(s) = 1 + \frac{3}{9^s}$. Then L_1 and L_2 trivially satisfy axioms (i) and (ii). Also, one can check that L_1 satisfies the functional equation

$$2^s L(s) = 2^{1-s}\overline{L(1-\overline{s})},$$

and L_2 satisfies the functional equation

$$3^s L(s) = 3^{1-s}\overline{L(1-\overline{s})}.$$

Thus, L_1 and L_2 also satisfy axiom (iii). It is clear that $L_1 - 1$ and $L_2 - 1$ do not have any zeros and thus satisfy the assumptions of Theorem 1.2 with $c = 1$, but $L_1 \not\equiv L_2$.

In 2011, Li [9] proved the following result by removing the assumption (1.1) of Theorem 1.3:

Theorem 1.5. [9, Theorem 1] *If two L-functions L_1 and L_2 in the extended Selberg class $\mathcal{S}^{\sharp}$ satisfy the same functional equation with $a(1) = 1$ and $L_1^{-1}(c_j) = L_2^{-1}(c_j)$ with $j \in \{1, 2\}$ for two finite distinct complex numbers c_1 and c_2, then $L_1 = L_2$.*

Later on, Ki [7] proved the following result that improved Theorem 1.5:

Theorem 1.6. [7, Theorem 1] *If two L-functions L_1 and L_2 in the extended Selberg class $\mathcal{S}^\sharp$ have the positive degrees and satisfy the same functional equation with $a(1) = 1$ and $L_1^{-1}(c) = L_2^{-1}(c)$ for a finite complex number c, then $L_1 = L_2$. The conclusion need not hold for $c = 0$ or if the functional equation is of degree zero.*

In 2014, Gonek–Haan–Ki [2] dispensed with the assumptions that L_1 and L_2 satisfy the same functional equation and that $a_1(1) = a_2(1)$ in Theorem 1.6, and proved the following result:

Theorem 1.7. [2, Theorem 1] *If two L-functions L_1 and L_2 in the extended Selberg class $\mathcal{S}^\sharp$ have the positive degrees and $L_1^{-1}(c) = L_2^{-1}(c)$ for a finite non-zero complex number c, then $L_1 = L_2$.*

Remark 1.8. It was shown in [7, pp. 2489–2490] that Theorem 1.7 need not hold if the L-functions L_1 and L_2 in Theorem 1.7 have the zero degrees. It was also shown in [7, p. 2489] that Theorem 1.7 need not hold if $c = 0$ and the L-functions L_1 and L_2 in Theorem 1.7 have the positive degrees.

By Theorem 1.7 we deduce the following result:

Theorem 1.9. *If two L-functions L_1 and L_2 in the extended Selberg class $\mathcal{S}^\sharp$ have the positive degrees and $L_1^{-1}(c_j) = L_2^{-1}(c_j)$ with $j \in \{1, 2\}$ for two finite distinct complex numbers c_1 and c_2, then $L_1 = L_2$.*

Based upon Theorem 1.9, one may ask, what can be said about the relationship between two L-functions L_1 and L_2 of zero degree in the extended Selberg class $\mathcal{S}^\sharp$, if $L_1^{-1}(c_j) = L_2^{-1}(c_j)$ with $j \in \{1, 2\}$ for two finite distinct complex numbers c_1 and c_2? In this direction, we will prove the following results:

Theorem 1.10. *If two L-functions L_1 and L_2 in the extended Selberg class $\mathcal{S}^\sharp$ have zero degrees and satisfy $L_1^{-1}(c_j) = L_2^{-1}(c_j)$ with $a_1(1) = a_2(1)$ and $j \in \{1, 2\}$ for two distinct finite complex numbers c_1 and c_2, then $L_1 = L_2$.*

Theorem 1.11. *Suppose that two L-functions L_1 and L_2 in the extended Selberg class $\mathcal{S}^\sharp$ have zero degrees and satisfy $L_1^{-1}(c_j) = L_2^{-1}(c_j)$ with $j \in \{1, 2\}$ for two distinct finite complex numbers c_1 and c_2. If the Nevanlinna's characteristics of L_1 and L_2 satisfy $\lim_{r\to+\infty} \frac{T(r,L_2)}{T(r,L_1)} = 1$, then $L_1 = L_2$.*

Next we follow Steinmetz [14] to introduce the notion of the convex hull of a subset of the complex plane $\mathbb{C}$ as follows: the convex hull of a subset $W \subset \mathbb{C}$, denoted as $\mathrm{co}(W)$, is the intersection of all convex sets containing the set W. If W contains only finitely many elements, then $\mathrm{co}(W)$ is obtained as an intersection of finitely many closed half-planes, and hence $\mathrm{co}(W)$ is either a compact polygon (with a non-empty interior) or a line segment. We denote the perimeter of $\mathrm{co}(W)$ as $C(\mathrm{co}(W))$. If $\mathrm{co}(W)$ is a line segment, then $C(\mathrm{co}(W))$ equals to twice the length of this line segment. Next we let f be defined as

$$f(s) = H_0(s) + H_1(s)e^{\alpha_1 s^q} + H_2(s)e^{\alpha_2 s^q} + \cdots + H_m(s)e^{\alpha_m s^q}, \tag{1.2}$$

where and in what follows, m and q are positive integers, while H_j with $0 \le j \le m$ is either an exponential polynomial of degree less than q or an ordinary polynomial in s and $H_j \not\equiv 0$ with $1 \le j \le m$, and $\alpha_1, \alpha_2, \cdots, \alpha_m$ are m distinct finite non-zero complex numbers. Throughout the rest of the paper, we fix the notations for $W = \{\overline{\alpha}_1, \overline{\alpha}_2, \cdots, \overline{\alpha}_m\}$ and $W_0 = \{0, \overline{\alpha}_1, \overline{\alpha}_2, \cdots, \overline{\alpha}_m\}$.

Remark 1.12. Theorem 1.5 can be improved by replacing with the assumption "$\lim_{r\to+\infty}\frac{T(r,L_2)}{T(r,L_1)}=1$" instead of the assumption "L_1 and L_2 satisfy the same functional equation" in Theorem 1.5. This can be seen by the following discussion: first of all, by the assumptions of Theorem 1.5, we know that L_1 and L_2 satisfy the same functional equation with $a(1)=1$. If the degree of L_1 and L_2 in Theorem 1.5 satisfy $d_{L_1}=d_{L_2}=d>0$, it follows by Steuding [15, p. 150, Theorem 7.9] that

$$T(r,L_1)=T(r,L_2)+O(r)=\frac{d}{\pi}r\log r+O(r),$$

which implies that $\lim_{r\to+\infty}\frac{T(r,L_2)}{T(r,L_1)}=1$. This together with Theorem 1.11 reveals the conclusion of Theorem 1.5. If the degree of L_1 and L_2 in Theorem 1.5 satisfy $d_{L_1}=d_{L_2}=d=0$, it follows by [6, Theorem 1 (ii)] that there exists some positive integer q_k with $k\in\{1,2\}$ and there exists some complex number ω_k with $|\omega_k|=1$ and $k\in\{1,2\}$ such that

$$L_k(s)=\sum_{n|q_k}\frac{a_k(n)}{n^s},\quad k\in\{1,2\}. \tag{1.3}$$

We rewrite (1.3) into

$$\begin{aligned}L_k(s)&=a_k(1)+\sum_{\substack{n\geq 2\\ n|q_k}}\frac{a_k(n)}{n^s}=:a_k(1)+\sum_{j=1}^{N_k}\frac{a_k(n_{k,j})}{n_{k,j}^s}\\&=a_k(1)+\sum_{j=1}^{N_k}a_k(n_{k,j})e^{-s\log n_{k,j}}\end{aligned} \tag{1.4}$$

with $k\in\{1,2\}$. Here $n_{k,j}$ with $1\leq j\leq N_k$, $j\in\mathbb{Z}$ and $k\in\{1,2\}$ is a positive integer such that $n_{k,j}|q_k$ and $2\leq n_{k,l}<n_{k,l+1}$ with $1\leq l\leq N_k-1$, $l\in\mathbb{Z}$ and $k\in\{1,2\}$. By (1.4) and Lemma 2.1 in Section 2 of this paper we have

$$T(r,L_k)=C(\mathrm{co}(W_{2,0}))\frac{r}{2\pi}+O(\log r)\quad\text{with}\quad k\in\{1,2\}. \tag{1.5}$$

By the notion of the convex hull of a subset of the complex plane $\mathbb{C}$ we have

$$C(\mathrm{co}(W_{k,0}))=2\log n_{k,N_k}\quad\text{with}\quad k\in\{1,2\}, \tag{1.6}$$

where and in what follows,

$$W_{k,0}=\{0,-\log n_{k,1},-\log n_{k,2},\cdots,-\log n_{k,N_k}\}\quad\text{with}\quad k\in\{1,2\}.$$

By substituting (1.6) into (1.5) we have

$$T(r,L_k)=\frac{\log n_{k,N_k}}{\pi}r+O(\log r)\quad\text{with}\quad k\in\{1,2\}. \tag{1.7}$$

By the assumption of Theorem 1.5 we suppose that L_k with $k\in\{1,2\}$ satisfies the functional equation with $a(1)=1$ in the axiom (iii) of the definition of the L-function. Then,

$$\Lambda_{L_k}(s)=\omega\overline{\Lambda_{L_k}(1-\overline{s})}\quad\text{with}\quad k\in\{1,2\}. \tag{1.8}$$

By (1.4) and (1.8) we have

$$\lim_{\mathrm{Re}(s)\to-\infty}\frac{a_2(n_{2,N_2})e^{-s\log n_{2,N_2}}}{a_1(n_{1,N_1})e^{-s\log n_{1,N_1}}}=\lim_{\mathrm{Re}(s)\to-\infty}\frac{a_2(1)+\sum_{j=1}^{N_2}a_2(n_{2,j})e^{-s\log n_{2,j}}}{a_1(1)+\sum_{j=1}^{N_1}a_1(n_{1,j})e^{-s\log n_{1,j}}}$$

$$=\lim_{\mathrm{Re}(s)\to-\infty}\frac{L_2(s)}{L_1(s)}=\lim_{\mathrm{Re}(s)\to-\infty}\frac{\overline{L_2(1-\overline{s})}}{\overline{L_1(1-\overline{s})}}=\lim_{\mathrm{Re}(s)\to-\infty}\overline{\left(\frac{L_2(1-\overline{s})}{L_1(1-\overline{s})}\right)}=\frac{a_2(1)}{a_1(1)}=1,$$

which implies that $\log n_{2,N_2}=\log n_{1,N_1}$. Combining this with (1.7), we have

$$T(r,L_1)=T(r,L_2)+O(\log r)=\frac{\log n_{1,N_1}}{\pi}r+O(\log r),$$

which implies that $\lim_{r\to+\infty}\frac{T(r,L_2)}{T(r,L_1)}=1$. This together with Theorem 1.11 reveals the conclusion of Theorem 1.5.

2. Preliminaries

In this section, we will introduce some important results that are used to prove the main results. First of all, we recall the following results due to Heittokangas–Wen [4]:

Lemma 2.1. [4, Theorem 3.1] *Let f be an exponential polynomial in the normalized form (1.2), where we suppose that $\rho(H_j)\le q-p$ with $0\le j\le m$ and $j\in\mathbb{Z}$ for some positive integer p such that $1\le p\le q$. Then*

$$T(r,f)=C(\mathrm{co}(W_0))\frac{r^q}{2\pi}+O(r^{q-p}+\log r).$$

Lemma 2.2. [4, Theorem 3.2] *Let f be an exponential polynomial in the normalized form (1.2), where we suppose that $\rho(H_j)\le q-p$ with $0\le j\le m$ and $j\in\mathbb{Z}$ for some positive integer p such that $1\le p\le q$. Then, one of the following cases can occur:*

(i) *If $H_0=0$, then*

$$N\left(r,\frac{1}{f}\right)=C(\mathrm{co}(W))\frac{r^q}{2\pi}+O\left(r^{q-p}+\log r\right).$$

(ii) *If $H_0\not\equiv 0$, then*

$$m\left(r,\frac{1}{f}\right)=O\left(r^{q-p}+\log r\right)\quad \textit{and}\quad N\left(r,\frac{1}{f}\right)=C(\mathrm{co}(W_0))\frac{r^q}{2\pi}+O\left(r^{q-p}+\log r\right).$$

We also need the following result that was proved by Ritt [12]:

Lemma 2.3. [12, p. 681, Ritt's theorem] *Assume that g and h are exponential sums of the forms $g(s)=\sum_{j=1}^{m}a_je^{\mu_js}$ and $h(s)=\sum_{k=1}^{n}b_ke^{\nu_ks}$, where $a_j\in\mathbb{C}\setminus\{0\}$ with $1\le j\le m$ and $j\in\mathbb{Z}$, and $b_k\in\mathbb{C}\setminus\{0\}$ with $1\le k\le n$ and $j\in\mathbb{Z}$ are non-zero constants, while μ_1, μ_2, $\cdots$, μ_m are m distinct finite complex constants, and ν_1, ν_2, $\cdots$, ν_n are n distinct finite complex constants. If g/h is an entire function, then there exist p distinct finite complex constants γ_1, γ_2, $\cdots$, γ_p such that*

$$\frac{g(s)}{h(s)}=\sum_{l=1}^{p}c_le^{\gamma_ls},$$

where c_1, c_2, $\cdots$, c_p are complex constants such that $\sum_{l=1}^{p}|c_p|>0$.

Remark 2.4. Under the assumptions of Lemma 2.3, if γ_l with $1 \le l \le p$ and $l \in \mathbb{Z}$ in the conclusion of Lemma 2.3 satisfies $\gamma_l = 0$ with $1 \le l \le p$ and $l \in \mathbb{Z}$, then g/h reduces to a finite non-zero constant.

The following result is due to Markushevich [10]:

Lemma 2.5. [10] *Let $Q(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, where n is a positive integer and $a_n = |a_n| e^{i\theta_n}$ with $|a_n| > 0$ and $\theta_n \in [0, 2\pi)$. For any given positive number ε satisfying $0 < \varepsilon < \frac{\pi}{4n}$, we consider the following $2n$ angles:*

$$S_j : -\frac{\theta_n}{n} + (2j-1)\frac{\pi}{2n} + \varepsilon < \theta < -\frac{\theta_n}{n} + (2j+1)\frac{\pi}{2n} - \varepsilon,$$

where j is an integer satisfying $0 \le j \le 2n-1$. Then there exists a positive number $R = R(\varepsilon)$ such that for $|z| = r > R$, $\mathrm{Re}(Q(z)) > |a_n|(1-\varepsilon) r^n \sin(n\varepsilon)$ if $z \in S_j$ where j is even, while $\mathrm{Re}(Q(z)) < -|a_n|(1-\varepsilon) r^n \sin(n\varepsilon)$ if $z \in S_j$ where j is odd.

The following result is due to Yang–Yi [17]:

Lemma 2.6. [17, Theorem 1.62] *Let $f_1, f_2, \cdots, f_n$ be non-constant meromorphic functions, and let $f_{n+1} \not\equiv 0$ be a meromorphic function such that $\sum_{j=1}^{n+1} f_j = 1$. Suppose that there exists a subset $\mathrm{I} \subseteq \mathbb{R}^+$ with linear measure $\mathrm{mes}\, I = \infty$ such that*

$$\sum_{k=1}^{n+1} N\left(r, \frac{1}{f_k}\right) + n \sum_{\substack{k=1 \\ k \ne j}}^{n+1} \overline{N}(r, f_k) < (\mu + o(1)) T(r, f_j), \quad j = 1, 2, \cdots, n,$$

as $r \in \mathrm{I}$ and $r \to \infty$, where μ is a real number satisfying $0 \le \mu < 1$. Then $f_{n+1} = 1$.

3. Proof of Theorem 1.11

Suppose that $L_1 \not\equiv L_2$. By the assumption of Theorem 1.11 we have

$$d_1 = d_2 = 0, \tag{3.1}$$

where and in what follows, d_k denotes the degree of the L-function L_k with $k \in \{1, 2\}$. Then, it follows by (3.1) and [6, Theorem 1 (ii)] that there exists some positive integer q_k with $k \in \{1, 2\}$ and there exists some complex number ω_k with $|\omega_k| = 1$ and $k \in \{1, 2\}$ such that (1.3) holds. We rewrite (1.3) into (1.4) with $k \in \{1, 2\}$. Here $n_{k,j}$ with $1 \le j \le N_k$, $j \in \mathbb{Z}$ and $k \in \{1, 2\}$ is a positive integer such that $n_{k,j} | q_k$ and $2 \le n_{k,l} < n_{k,l+1}$ with $1 \le l \le N_k - 1$, $l \in \mathbb{Z}$ and $k \in \{1, 2\}$. Next, in the same manner as in Remark 1.12, we deduce by (1.4) and Lemma 2.1 that (1.5)–(1.7) hold. By (1.7) and the assumption $\lim_{r \to +\infty} \frac{T(r, L_2)}{T(r, L_1)} = 1$ we deduce

$$n_{1,N_1} = n_{2,N_2}. \tag{3.2}$$

Since c_1 and c_2 are two distinct finite values, we can see that one of c_1 and c_2, say c_2, satisfies

$$a_2(1) - c_2 \ne 0. \tag{3.3}$$

By (3.3), Lemma 2.1 and Lemma 2.2 (ii) with $p = q = 1$ we deduce

$$\begin{aligned} T(r, L_2) &= N\left(r, \frac{1}{L_2 - c_2}\right) + O(\log r) \\ &= N\left(r, \frac{1}{a_2(1) - c_2 + \sum_{j=1}^{N_2} a_2(n_{2,j}) e^{-s \log n_{2,j}}}\right) + O(\log r) \\ &= C(\mathrm{co}(W_{2,0})) \frac{r}{2\pi} + O(\log r). \end{aligned} \tag{3.4}$$

By the notion of the convex hull of a subset of the complex plane $\mathbb{C}$ we have

$$C(\mathrm{co}(W_{k,0})) = 2 \log n_{k,N_k} \quad \text{with} \quad k \in \{1, 2\}, \tag{3.5}$$

where and in what follows,

$$W_{k,0} = \{0, -\log n_{k,1}, -\log n_{k,2}, \cdots, -\log n_{k,N_k}\} \quad \text{with} \quad k \in \{1, 2\}.$$

By (3.2) and (3.5) we have

$$C(\mathrm{co}(W_{1,0})) = 2 \log n_{1,N_1} = 2 \log n_{2,N_2} = C(\mathrm{co}(W_{2,0})). \tag{3.6}$$

By (1.4), the second fundamental theorem and the assumption that L_1 and L_2 share c_1 and c_2 IM, we deduce

$$\begin{aligned} T(r, L_2) &\le \overline{N}\left(r, \frac{1}{L_2 - c_1}\right) + \overline{N}\left(r, \frac{1}{L_2 - c_2}\right) - N_0\left(r, \frac{1}{L_2'}\right) + O(\log r) \\ &= \overline{N}\left(r, \frac{1}{L_2 - c_1}\right) + \overline{N}\left(r, \frac{1}{L_2 - c_2}\right) - N_0\left(r, \frac{1}{L_2'}\right) + O(\log r) \\ &\le \overline{N}\left(r, \frac{1}{L_1 - L_2}\right) - N_0\left(r, \frac{1}{L_2'}\right) + O(\log r) \\ &\le N\left(r, \frac{1}{L_1 - L_2}\right) - N_0\left(r, \frac{1}{L_2'}\right) + O(\log r) \\ &= N\left(r, \frac{1}{a_1(1) - a_2(1) + \sum_{j=1}^{N_1} a_1(n_{1,j}) e^{-s \log n_{1,j}} - \sum_{j=1}^{N_2} a_2(n_{2,j}) e^{-s \log n_{2,j}}}\right) \\ &\quad - N_0\left(r, \frac{1}{L_2'}\right) + O(\log r), \end{aligned} \tag{3.7}$$

where and in what follows, $N_0\left(r, \frac{1}{L_2'}\right)$ denotes the counting function of those zeros of L_2' in the open disc $|s| < r$, that are neither zeros of $L_2 - c_1$ nor zeros of $L_2 - c_2$.

By (3.6), Lemma 2.2 and the notion of the convex hull of a subset of the complex plane $\mathbb{C}$ we deduce

$$\begin{aligned} &N\left(r, \frac{1}{a_1(1) - a_2(1) + \sum_{j=1}^{N_1} a_1(n_{1,j}) e^{-s \log n_{1,j}} - \sum_{j=1}^{N_2} a_2(n_{2,j}) e^{-s \log n_{2,j}}}\right) + O(\log r) \\ &\le \max\left\{C(\mathrm{co}(W_{1,0})), C(\mathrm{co}(W_{2,0}))\right\} \frac{r}{2\pi} + O(\log r) \\ &= C(\mathrm{co}(W_{2,0})) \frac{r}{2\pi} + O(\log r). \end{aligned} \tag{3.8}$$

By (3.4), (3.7) and (3.8) we have

$$\begin{aligned}
&C(\mathrm{co}(W_{2,0}))\frac{r}{2\pi}+O(\log r)\\
&=N\left(r,\frac{1}{a_2(1)-c_2+\sum_{j=1}^{N_2}a_2(n_{2,j})e^{-s\log n_{2,j}}}\right)+O(\log r)\\
&=N\left(r,\frac{1}{L_2-c_2}\right)+O(\log r)=T(r,L_2)+O(\log r)\\
&\le N\left(r,\frac{1}{L_2-c_1}\right)+N\left(r,\frac{1}{L_2-c_2}\right)-N\left(r,\frac{1}{L_2'}\right)+O(\log r)\\
&=N\left(r,\frac{1}{L_2-c_1}\right)+N\left(r,\frac{1}{L_2-c_2}\right)-N\left(r,\frac{1}{L_2'}\right)+O(\log r)\\
&=\overline{N}\left(r,\frac{1}{L_2-c_1}\right)+\overline{N}\left(r,\frac{1}{L_2-c_2}\right)-N_0\left(r,\frac{1}{L_2'}\right)+O(\log r) \qquad (3.9)\\
&=\overline{N}\left(r,\frac{1}{L_2-c_1}\right)+\overline{N}\left(r,\frac{1}{L_2-c_2}\right)-N_0\left(r,\frac{1}{L_2'}\right)+O(\log r)\\
&\le\overline{N}\left(r,\frac{1}{L_1-L_2}\right)-N_0\left(r,\frac{1}{L_2'}\right)+O(\log r)\\
&\le N\left(r,\frac{1}{L_1-L_2}\right)-N_0\left(r,\frac{1}{L_2'}\right)+O(\log r)\\
&=N\left(r,\frac{1}{a_1(1)-a_2(1)+\sum_{j=1}^{N_1}a_1(n_{1,j})e^{-s\log n_{1,j}}-\sum_{j=1}^{N_2}a_2(n_{2,j})e^{-s\log n_{2,j}}}\right)\\
&\quad-N_0\left(r,\frac{1}{L_2'}\right)+O(\log r)\\
&\le N\left(r,\frac{1}{a_1(1)-a_2(1)+\sum_{j=1}^{N_1}a_1(n_{1,j})e^{-s\log n_{1,j}}-\sum_{j=1}^{N_2}a_2(n_{2,j})e^{-s\log n_{2,j}}}\right)\\
&\quad+O(\log r)\le C(\mathrm{co}(W_{2,0})\frac{r}{2\pi}+O(\log r).
\end{aligned}$$

By (3.9) we obtain that all the inequalities in (3.9) are equalities that are equal to $C(\mathrm{co}(W_{2,0})\frac{r}{2\pi}+O(\log r)$. Combining this with the assumption that L_1 and L_2 share c_1 and c_2 IM, we deduce

$$N_0\left(r,\frac{1}{L_2'}\right)+N_{(2}\left(r,\frac{1}{L_1-L_2}\right)=O(\log r), \qquad (3.10)$$

$$\begin{aligned}
T(r,L_2)&=N\left(r,\frac{1}{a_1(1)-a_2(1)+\sum_{j=1}^{N_1}a_1(n_{1,j})e^{-s\log n_{1,j}}-\sum_{j=1}^{N_2}a_2(n_{2,j})e^{-s\log n_{2,j}}}\right)\\
&\quad+O(\log r)\\
&=N\left(r,\frac{1}{L_1-L_2}\right)+O(\log r)=\overline{N}_{1)}\left(r,\frac{1}{L_1-L_2}\right)+O(\log r) \qquad (3.11)\\
&=\overline{N}\left(r,\frac{1}{L_1-L_2}\right)+O(\log r)=C(\mathrm{co}(W_{2,0}))\frac{r}{2\pi}+O(\log r),
\end{aligned}$$

$$
\begin{aligned}
T(r,L_2) &= \overline{N}\left(r,\frac{1}{L_1-c_1}\right)+\overline{N}\left(r,\frac{1}{L_1-c_2}\right)+O(\log r)\\
&= \overline{N}\left(r,\frac{1}{L_2-c_1}\right)+\overline{N}\left(r,\frac{1}{L_2-c_2}\right)+O(\log r)\\
&= C(\mathrm{co}(W_{2,0}))\frac{r}{2\pi}+O(\log r)
\end{aligned}
\tag{3.12}
$$

and

$$
\begin{aligned}
T(r,L_2) &= N\left(r,\frac{1}{L_2-c_1}\right)+N\left(r,\frac{1}{L_2-c_2}\right)-N\left(r,\frac{1}{L_2'}\right)+O(\log r)\\
&= N\left(r,\frac{1}{L_2-c_2}\right)+O(\log r)=C(\mathrm{co}(W_{2,0}))\frac{r}{2\pi}+O(\log r).
\end{aligned}
\tag{3.13}
$$

By (3.2), (3.11), Lemma 2.2 and the notion of the convex hull of a subset of the complex plane $\mathbb{C}$ we deduce

$$a_1(1)-a_2(1)\neq 0 \tag{3.14}$$

and

$$a_1(n_{1,N_1})-a_2(n_{2,N_2})\neq 0 \quad \text{with} \quad n_{1,N_1}=n_{2,N_2}. \tag{3.15}$$

On the other hand, by (3.10)–(3.12) we deduce

$$N\left(r,\frac{(L_2-c_1)(L_2-c_2)}{L_2'(L_1-L_2)}\right)=O(\log r). \tag{3.16}$$

By the assumption that L_1 and L_2 share c_1 and c_2 IM, we deduce that $\frac{L_2'(L_1-L_2)}{(L_2-c_1)(L_2-c_2)}$ is an entire function. Combining this with (1.4), Lemma 2.3 and Remark 2.4, we deduce that there exist p_2 distinct finite complex constants $\gamma_{2,1}$, $\gamma_{2,2}$, $\cdots$, γ_{2,p_2} such that

$$\frac{L_2'(s)(L_1(s)-L_2(s))}{(L_2(s)-c_1)(L_2(s)-c_2)}=\sum_{l=1}^{p_2}c_{2,l}e^{\gamma_{2,l}s} \quad \text{for each } s\in\mathbb{C}, \tag{3.17}$$

where $c_{2,1}$, $c_{2,2}$, $\cdots$, c_{2,p_2} are complex constants such that $\sum_{l=1}^{p_2}|c_{2,l}|>0$. By (3.16), (3.17) and Lemma 2.2 we deduce that $p_2=1$ and $\sum_{l=1}^{p_2}c_{2,l}e^{\gamma_{2,l}s}$ reduces to $c_{2,1}e^{\gamma_{2,1}s}$. Therefore, (3.17) can be rewritten into

$$\frac{L_2'(s)(L_1(s)-L_2(s))}{(L_2(s)-c_1)(L_2(s)-c_2)}=c_{2,1}e^{\gamma_{2,1}s} \quad \text{for each } s\in\mathbb{C}. \tag{3.18}$$

Next we prove that $c_{2,1}e^{\gamma_{2,1}s}$ reduces to some finite non-zero constant $\tilde{c}_2$, and (3.18) can be rewritten into

$$\frac{L_2'(s)(L_1(s)-L_2(s))}{(L_2(s)-c_1)(L_2(s)-c_2)}=\tilde{c}_2 \quad \text{identically for each } s\in\mathbb{C}. \tag{3.19}$$

Next we prove (3.19): suppose that (3.19) is not valid. Then it follows by (3.18) that $\gamma_{2,1}\neq 0$. Therefore, we have

$$\gamma_{2,1}=|\gamma_{2,1}|e^{i\theta_{2,1}}, \tag{3.20}$$

where $\theta_{2,1}\in[0,2\pi)$. For any given positive number ε satisfying $0<\varepsilon<\frac{\pi}{4}$, we consider the following two angles:

$$S_{2,j}\colon -\theta_{2,1}+(2j-1)\frac{\pi}{2}+\varepsilon<\theta<-\theta_{2,1}+(2j+1)\frac{\pi}{2}-\varepsilon, \tag{3.21}$$

where $j = 0, 1$. Then, by (3.20), (3.21) and Lemma 2.5 we can see that there exists a positive number $R = R(\varepsilon)$ depending only upon ε such that for $|s| = r > R$, we have

$$\begin{aligned} &\operatorname{Re}(\gamma_{2,1}s) > |\gamma_{2,1}|(1-\varepsilon)r\sin\varepsilon, \quad \text{if} \\ &s \in S_{2,0} =: \left\{s\colon -\theta_{2,1} - \frac{\pi}{2} + \varepsilon < \arg s < -\theta_{2,1} + \frac{\pi}{2} - \varepsilon\right\}, \end{aligned} \tag{3.22}$$

while

$$\begin{aligned} &\operatorname{Re}(\gamma_{2,1}s) < -|\gamma_{2,1}|(1-\varepsilon)r\sin\varepsilon, \quad \text{if} \\ &s \in S_{2,1} =: \left\{s\colon -\theta_{2,1} + \frac{\pi}{2} + \varepsilon < \arg s < -\theta_{2,1} + \frac{3\pi}{2} - \varepsilon\right\}. \end{aligned} \tag{3.23}$$

By (1.4) with $k = 2$ we deduce

$$L_2'(s) = -\sum_{j=1}^{N_2} \frac{a_2(n_{2,j})\log n_{2,j}}{n_{2,j}^s}, \tag{3.24}$$

By (1.4) and (3.24) we deduce

$$\begin{aligned} &\frac{L_2'(s)(L_1(s) - L_2(s))}{(L_2(s) - c_1)(L_2(s) - c_2)} \\ &= -\frac{\left(\sum_{j=1}^{N_2} \frac{a_2(n_{2,j})\log n_{2,j}}{n_{2,j}^s}\right)\left(a_1(1) - a_2(1) + \sum_{j=1}^{N_1} \frac{a_1(n_{1,j})}{n_{1,j}^s} - \sum_{j=1}^{N_2} \frac{a_2(n_{2,j})}{n_{2,j}^s}\right)}{\left(a_2(1) - c_1 + \sum_{j=1}^{N_2} \frac{a_2(n_{2,j})}{n_{2,j}^s}\right)\left(a_2(1) - c_2 + \sum_{j=1}^{N_2} \frac{a_2(n_{2,j})}{n_{2,j}^s}\right)} \end{aligned} \tag{3.25}$$

for each $s \in \mathbb{C}$.

Suppose that $S_{2,0} \cap \{s\colon \operatorname{Re}(s) > 0\}$ is not an empty set. Then $S_{2,0} \cap \{s\colon \operatorname{Re}(s) > 0\}$ is a domain of an angle. Combining this with (3.22), we can see that there exist two real numbers $\vartheta_1, \vartheta_2 \in \left(-\theta_{2,1} - \frac{\pi}{2} + \varepsilon, -\theta_{2,1} + \frac{\pi}{2} - \varepsilon\right)$ satisfying $\vartheta_1 < \vartheta_2$ such that

$$\begin{aligned} &\{s\colon \vartheta_1 \le \arg s \le \vartheta_2\} \subset S_{2,0} \cap \{s\colon \operatorname{Re}(s) > 0\} \quad \text{and} \\ &\operatorname{Re}(\gamma_{2,1}s) > |\gamma_{2,1}|(1-\varepsilon)r\sin\varepsilon \end{aligned} \tag{3.26}$$

for each $s \in \{s\colon \vartheta_1 \le \arg s \le \vartheta_2\}$. Next we set

$$S_1 = \{s\colon \vartheta_1 \le \arg s \le \vartheta_2\}. \tag{3.27}$$

By (3.3), (3.14), (3.15), (3.18), (3.26), (3.27) we deduce

$$\begin{aligned} +\infty &= \lim_{\substack{|s|\to+\infty \\ s\in S_1}} |c_{2,1}|e^{|\gamma_{2,1}|(1-\varepsilon)r\sin\varepsilon} \le \lim_{\substack{|s|\to+\infty \\ s\in S_1}} |c_{2,1}||e^{\operatorname{Re}(\gamma_{2,1}s)} = \lim_{\substack{|s|\to+\infty \\ s\in S_1}} |c_{2,1}|\,|e^{\gamma_{2,1}s}| \\ &= \lim_{\substack{|s|\to+\infty \\ s\in S_1}} \left|\frac{L_2'(s)(L_1(s) - L_2(s))}{(L_2(s) - c_1)(L_2(s) - c_2)}\right| \\ &= \begin{cases} \dfrac{|a_1(1) - a_2(1)|\log n_{2,1}}{|a_2(1) - c_2|}, & \text{when } a_2(1) - c_1 = 0, \\ 0, & \text{when } a_2(1) - c_1 \ne 0. \end{cases} \end{aligned}$$

This is a contradiction.

Suppose that $S_{2,0} \cap \{s\colon \operatorname{Re}(s) > 0\}$ is an empty set. Then $S_{2,1} \cap \{s\colon \operatorname{Re}(s) > 0\}$ is not an empty set such that $S_{2,1} \cap \{s\colon \operatorname{Re}(s) > 0\}$ is a domain of an angle. Combining this with (3.23), we can see that there exist two real numbers $\vartheta_3, \vartheta_4 \in (-\theta_{2,1} + \frac{\pi}{2} + \varepsilon, -\theta_{2,1} + \frac{3\pi}{2} - \varepsilon)$ satisfying $\vartheta_3 < \vartheta_4$ such that

$$\begin{aligned} &\{s\colon \vartheta_3 \le \arg s \le \vartheta_4\} \subset S_{2,1} \cap \{s\colon \operatorname{Re}(s) > 0\} \quad \text{and} \\ &\operatorname{Re}(\gamma_{2,1} s) < -|c_{2,1}|(1-\varepsilon) r \sin\varepsilon \end{aligned} \tag{3.28}$$

for each $\theta \in [\vartheta_3, \vartheta_4]$. Next we set

$$S_2 = \{s\colon \vartheta_3 \le \arg s \le \vartheta_4\}. \tag{3.29}$$

Suppose that $a_2(1) - c_1 = 0$. By (3.3), (3.14), (3.25), (3.28) and (3.29) we have

$$\begin{aligned} 0 < \frac{|a_1(1) - a_2(1)| \log n_{2,1}}{|a_2(1) - c_2|} &= \lim_{\substack{|s|\to+\infty \\ s\in S_2}} \left| \frac{L_2'(s)(L_1(s) - L_2(s))}{(L_2(s) - c_1)(L_2(s) - c_2)} \right| \\ = \lim_{\substack{|s|\to+\infty \\ s\in S_2}} |c_{2,1}|\, |e^{\gamma_{2,1} s}| &= \lim_{\substack{|s|\to+\infty \\ s\in S_2}} |c_{2,1}| e^{\operatorname{Re}(\gamma_{2,1} s)} \le \lim_{\substack{|s|\to+\infty \\ s\in S_2}} |c_{2,1}| e^{-|\gamma_{2,1}|(1-\varepsilon) r \sin\varepsilon} = 0, \end{aligned}$$

which is a contradiction. Therefore, we have

$$a_2(1) - c_1 \ne 0. \tag{3.30}$$

By (3.3), (3.13), (3.30), Lemma 2.1, Lemma 2.2 (ii) and (1.4) for $k = 2$, we deduce

$$\begin{aligned} T(r, L_2) &= N\left(r, \frac{1}{L_2 - c_1}\right) + O(\log r) = N\left(r, \frac{1}{L_2 - c_2}\right) + O(\log r) \\ &= C(\operatorname{co}(W_{2,0}))\frac{r}{2\pi} + O(\log r) \end{aligned} \tag{3.31}$$

and

$$N\left(r, \frac{1}{L_2'}\right) = C(\operatorname{co}(W_2))\frac{r}{2\pi} + O(\log r), \tag{3.32}$$

where and in what follows,

$$W_k = \{-\log n_{k,1}, -\log n_{k,2}, \cdots, -\log n_{k,N_k}\} \quad \text{and} \quad k \in \{1, 2\}.$$

By (3.13), (3.31) and (3.32) we deduce

$$C(\operatorname{co}(W_{2,0}))\frac{r}{2\pi} = C(\operatorname{co}(W_2))\frac{r}{2\pi} + O(\log r),$$

and so we have $C(\operatorname{co}(W_{2,0})) = C(\operatorname{co}(W_2))$. Combining this with (3.31) and (1.4) for $k = 2$, we deduce $a_2(1) - c_1 = a_2(1) - c_2 = 0$. Therefore, we have $c_1 = c_2$. This is impossible. The formula (3.19) is thus completely proved.

Next, we use the lines from (3.2) to (3.19) and the assumption that L_1 and L_2 share c_1 and c_2 IM to deduce

$$\frac{L_1'(L_1 - L_2)}{(L_1 - c_1)(L_1 - c_2)} = \tilde{c}_1, \tag{3.33}$$

where $\tilde{c}_1$ is some finite non-zero constant. By (3.19) and (3.33) we have

$$\frac{L_1'}{(L_1 - c_1)(L_1 - c_2)} = \frac{\tilde{c} L_2'}{(L_2 - c_1)(L_2 - c_2)}, \tag{3.34}$$

where $\tilde{c} = \tilde{c}_1/\tilde{c}_2$.

Since L_1 and L_2 share c_1 and c_2 IM, we deduce by (3.11) and (3.12) that

$$
\begin{aligned}
T(r,L_2) &= \overline{N}_{1)}\left(r,\frac{1}{L_1-L_2}\right)+O(\log r)=\sum_{j=1}^{2}\sum_{m=2}^{\infty}\overline{N}_{(1,m)}\left(r,c_j,L_1,L_2\right)\\
&\quad+\sum_{j=1}^{2}\overline{N}_{(1,1)}\left(r,c_j,L_1,L_2\right)+\sum_{j=1}^{2}\sum_{m=2}^{\infty}\overline{N}_{(m,1)}\left(r,c_j,L_1,L_2\right)+O(\log r)\\
&=C(\mathrm{co}(W_{2,0}))\frac{r}{2\pi}+O(\log r),
\end{aligned}
\tag{3.35}
$$

where and in what follows, $\overline{N}_{(1,m)}\left(r,c_j,L_1,L_2\right)$ with $j\in\{1,2\}$ denotes the reduced counting function of those common zeros of L_1-c_j and L_2-c_j with $j\in\{1,2\}$ in $|s|<r$, such that each such common zero of L_1-c_j and L_2-c_j with $j\in\{1,2\}$ is a simple zero of L_1-c_j, and a zero of L_2-c_j of multiplicity m, and $\overline{N}_{(m,1)}\left(r,c_j,L_1,L_2\right)$ with $j\in\{1,2\}$ denotes the reduced counting function of those common zeros of L_1-c_j and L_2-c_j with $j\in\{1,2\}$ in $|s|<r$, such that each such common zero of L_1-c_j and L_2-c_j with $j\in\{1,2\}$ is a simple zero of L_2-c_j, and a zero of L_1-c_j of multiplicity m, while $\overline{N}_{(1,1)}\left(r,c_j,L_1,L_2\right)$ with $j\in\{1,2\}$ denotes the reduced counting function of the common simple zeros of L_1-c_j and L_2-c_j in $|s|<r$. Based upon (3.35), we consider the following three cases:

Case 1. Suppose that

$$
\overline{N}_{(1,1)}(r,c_1,L_1,L_2)+\overline{N}_{(1,1)}(r,c_2,L_1,L_2)\neq o(T(r,L_k))\quad\text{with}\quad k\in\{1,2\}. \tag{3.36}
$$

By (3.36), without loss of generality we suppose that there exists some common simple zero $z_0\in\mathbb{C}$ of L_1-c_1 and L_2-c_1. We substitute $L_1(z_0)=L_2(z_0)=c_1$ into (3.34), and then we deduce $\tilde{c}=1$. Therefore, (3.34) can be rewritten into

$$
\frac{L_1'}{(L_1-c_1)(L_1-c_2)}=\frac{L_2'}{(L_2-c_1)(L_2-c_2)}. \tag{3.37}
$$

By (3.37) and the assumption that L_1 and L_2 share c_1 and c_2 IM we deduce that L_1 and L_2 share c_1 and c_2 CM. This together with (3.10) gives

$$
N_{(2}\left(r,\frac{1}{L_k-c_1}\right)+N_{(2}\left(r,\frac{1}{L_k-c_2}\right)\leq N_{(2}\left(r,\frac{1}{L_1-L_2}\right)=O(\log r) \tag{3.38}
$$

with $k\in\{1,2\}$. By (3.38) we can see that the first equality of (3.4) can be rewritten into

$$
T(r,L_2)=\overline{N}_{1)}\left(r,\frac{1}{L_2-c_2}\right)+O(\log r)=C(\mathrm{co}(W_{2,0}))\frac{r}{2\pi}+O(\log r). \tag{3.39}
$$

By (3.38) we also see that (3.12) can be rewritten into

$$
\begin{aligned}
T(r,L_2)= &\ \overline{N}_{1)}\left(r,\frac{1}{L_1-c_1}\right)+\overline{N}_{1)}\left(r,\frac{1}{L_1-c_2}\right)+O(\log r)\\
= &\ \overline{N}_{1)}\left(r,\frac{1}{L_2-c_1}\right)+\overline{N}_{1)}\left(r,\frac{1}{L_2-c_2}\right)+O(\log r)\\
&=C(\mathrm{co}(W_{2,0}))\frac{r}{2\pi}+O(\log r).
\end{aligned}
\tag{3.40}
$$

By (3.38)–(3.40) and the assumption that L_1 and L_2 share c_1 and c_2 IM we deduce

$$
N\left(r,\frac{1}{L_k-c_1}\right)=O(\log r)\quad\text{with}\quad k\in\{1,2\}. \tag{3.41}
$$

Since L_1 and L_2 are transcendental entire functions satisfying (1.4), we have by (3.41) that

$$L_1 = c_1 + P_1 e^{\beta_1} \quad \text{and} \quad L_2 = c_1 + P_2 e^{\beta_2}, \tag{3.42}$$

where P_1 and P_2 are polynomials such that $P_k \not\equiv 0$ with $k \in \{1,2\}$, and β_k with $k \in \{1,2\}$ is a non-constant entire functions. Moreover, by (3.2), (3.42) and Definition 1.1 we deduce that β_k with $k \in \{1,2\}$ is a non-constant polynomial of degree $\deg(\beta_k) = 1$ with $k \in \{1,2\}$. Therefore

$$\beta_k(s) = A_k s + B_k \quad \text{with} \quad k \in \{1,2\}, \tag{3.43}$$

where and in what follows, A_k and B_k with $A_k \neq 0$ and $k \in \{1,2\}$ are complex constants.

By substituting (1.4) and (3.43) into (3.42) we have

$$a_k(1) + \sum_{j=1}^{N_k} a_k(n_{k,j}) e^{-s\log n_{k,j}} = c_1 + P_k(s) e^{A_k s + B_k} \quad \text{with } k \in \{1,2\} \text{ for each } s \in \mathbb{C}. \tag{3.44}$$

We rewrite (3.44) into

$$\sum_{j=1}^{N_k} a_k(n_{k,j}) e^{-s\log n_{k,j}} - P_k(s) e^{A_k s + B_k} = c_1 - a_k(1) \quad \text{with } k \in \{1,2\} \text{ for each } s \in \mathbb{C}. \tag{3.45}$$

By (3.45) and Lemma 2.6 we deduce $c_1 = a_k(1)$. Therefore, (3.45) can be rewritten into

$$\sum_{j=1}^{N_k} a_k(n_{k,j}) e^{-s\log n_{k,j}} - P_k(s) e^{A_k s + B_k} = 0 \quad \text{with } k \in \{1,2\} \text{ for each } s \in \mathbb{C}. \tag{3.46}$$

We rewrite (3.46) into

$$\sum_{j=1}^{N_k} \frac{a_k(n_{k,j})}{P_k(s)} e^{-(A_k + \log n_{k,j})s - B_k} = 1 \quad \text{with } k \in \{1,2\} \text{ for each } s \in \mathbb{C}. \tag{3.47}$$

We consider the following two subcases:

Subcase 1.1. Suppose that $N_k \geq 2$ with $k \in \{1,2\}$. First of all, by the obtained result that $n_{k,j}$ with $1 \leq j \leq N_k$, $j \in \mathbb{Z}$ and $k \in \{1,2\}$ is a positive integer such that $n_{k,j} | q_k$ and $2 \leq n_{k,l} < n_{k,l+1}$ with $1 \leq l \leq N_k - 1$, $l \in \mathbb{Z}$ and $k \in \{1,2\}$, we deduce by Lemma 2.6 that there exists one and only one term on the left hand side of (3.47), say $\frac{a_k(n_{k,1})}{P_k(s)} e^{-s(A_k + \log n_{k,1})s - B_k}$, is a constant, such that

$$\frac{a_k(n_{k,1})}{P_k(s)} e^{-(A_k + \log n_{k,1})s - B_k} = 1$$

and

$$\sum_{j=2}^{N_k} \frac{a_k(n_{k,j})}{P_k(s)} e^{-(A_k + \log n_{k,j})s - B_k} = 0 \quad \text{with } k \in \{1,2\} \text{ for each } s \in \mathbb{C}. \tag{3.48}$$

By (3.48) we deduce that the positive integer N_k with $k \in \{1,2\}$ satisfies $N_k \geq 3$ with $k \in \{1,2\}$, such that

$$\sum_{j=3}^{N_k} \frac{a_k(n_{k,j})}{a_k(n_{k,2})} e^{(\log n_{k,2} - \log n_{k,j})s} = -1 \quad \text{with } k \in \{1,2\} \text{ for each } s \in \mathbb{C}. \tag{3.49}$$

By (3.49), Lemma 2.6 and the obtained result $N_k \geq 3$ we get a contradiction.

Subcase 1.2. Suppose that $N_k = 1$ with $k \in \{1,2\}$. Then, (3.46) can be rewritten into

$$a_k(n_{k,1})e^{-s\log n_{k,1}} = P_k(s)e^{A_k s + B_k} \quad \text{with } k \in \{1,2\} \text{ for each } s \in \mathbb{C}. \tag{3.50}$$

By substituting (3.43) and (3.50) into (3.42) we have

$$L_1(s) = c_1 + a_1(n_{1,1})e^{-s\log n_{1,1}} \quad \text{and} \quad L_2(s) = c_1 + a_2(n_{2,1})e^{-s\log n_{2,1}} \tag{3.51}$$

for each $s \in \mathbb{C}$.

By (3.51) we deduce that each zero of $L_1 - c_2$ and $L_2 - c_2$ is simple zero. Combing this with Lemma 2.1 and Lemma 2.2 (ii) with $p = q = 1$, and the assumption that L_1 and L_2 share c_2 IM we deduce

$$\begin{aligned} T(r, L_k) &= N\left(r, \frac{1}{L_k - c_2}\right) + O(\log r) = \overline{N}\left(r, \frac{1}{L_k - c_2}\right) + O(\log r) \\ &= \frac{\log n_{k,1}}{\pi} r + O(\log r) \quad \text{with} \quad k \in \{1,2\}. \end{aligned} \tag{3.52}$$

By (3.52) and the assumption that L_1 and L_2 share c_2 IM, we have

$$\begin{aligned} T(r, L_1) &= T(r, L_2) + O(\log r) = \overline{N}\left(r, \frac{1}{L_1 - c_2}\right) + O(\log r) \\ &= \overline{N}\left(r, \frac{1}{L_2 - c_2}\right) + O(\log r) = \frac{\log n_{1,1}}{\pi} r + O(\log r) = \frac{\log n_{2,1}}{\pi} r + O(\log r), \end{aligned}$$

which implies that $n_{1,1} = n_{2,1}$. Combining this with (3.51) and the assumption that L_1 and L_2 share c_2 IM, we deduce $a_1(n_{1,1}) = a_2(n_{2,1})$ and $L_1 = L_2$, which contradicts the assumption $L_1 \not\equiv L_2$.

Case 2. Suppose that there exists some positive integer $m \geq 2$ such that

$$\overline{N}_{(1,m)}(r, c_1, L_1, L_2) + \overline{N}_{(1,m)}(r, c_2, L_1, L_2) \neq o(T(r, L_k)) \quad \text{with} \quad k \in \{1,2\}. \tag{3.53}$$

By (3.53) we suppose, without loss of generality, that there exists some point $z_0 \in \mathbb{C}$ such that z_0 is a simple zero of $L_1 - c_1$, and z_0 is a zero of $L_2 - c_1$ of multiplicity m. Then, by substituting $L_1(z_0) = L_2(z_0) = c_1$ into (3.34) we deduce $\tilde{c} = 1/m$. Therefore, (3.34) can be rewritten into

$$\frac{mL_1'}{(L_1 - c_1)(L_1 - c_2)} = \frac{L_2'}{(L_2 - c_1)(L_2 - c_2)}. \tag{3.54}$$

By (3.54) and the assumption that L_1 and l_2 share c_1 and c_2 IM we deduce that each common zero $z_0 \in \mathbb{C}$ of $L_1 - c_j$ and $L_2 - c_j$ with $j \in \{1,2\}$ is a simple zero of $L_1 - c_j$ and a zero of $L_2 - c_j$ of multiplicity m. Combining this with the supposition $m \geq 2$

and the second fundamental theorem, we have

$$
\begin{aligned}
T(r,L_2) &\le \overline{N}\left(r,\frac{1}{L_2-c_1}\right)+\overline{N}\left(r,\frac{1}{L_2-c_2}\right)+O(\log r)\\
&\le \frac{1}{2}N\left(r,\frac{1}{L_2-c_1}\right)+\frac{1}{2}N\left(r,\frac{1}{L_2-c_2}\right)+O(\log r)\\
&\le \frac{1}{2}T\left(r,L_2\right)+\frac{1}{2}T\left(r,L_2\right)+O(\log r)\le T(r,L_2)+O(\log r),
\end{aligned}
\tag{3.55}
$$

which together with (1.4) and Lemma 2.1 gives

$$
\begin{aligned}
T(r,L_2) &= N\left(r,\frac{1}{L_2-c_1}\right)+O(\log r)=N\left(r,\frac{1}{L_2-c_2}\right)+O(\log r)\\
&= N\left(r,\frac{1}{a_2(1)-c_1+\sum_{j=1}^{N_2}a_2(n_{2,j})e^{-s\log n_{2,j}}}\right)+O(\log r)\\
&= N\left(r,\frac{1}{a_2(1)-c_2+\sum_{j=1}^{N_2}a_2(n_{2,j})e^{-s\log n_{2,j}}}\right)+O(\log r)\\
&= C(\mathrm{co}(W_{2,0}))\frac{r}{2\pi}+O(\log r).
\end{aligned}
\tag{3.56}
$$

On the other hand, from (1.4) for $k=2$, the first line of (3.55), the formula (3.56), the above analysis and the supposition that m is a positive integer such that $m\ge 2$, we deduce

$$
\begin{aligned}
T(r,L_2) &\le \overline{N}\left(r,\frac{1}{L_2-c_1}\right)+\overline{N}\left(r,\frac{1}{L_2-c_2}\right)+O(\log r)\le N\left(r,\frac{1}{L_2'}\right)+O(\log r)\\
&= N\left(r,\frac{1}{\sum_{j=1}^{N_2}a_2(n_{2,j})(\log n_{2,j})e^{-s\log n_{2,j}}}\right)+O(\log r)\\
&= C(\mathrm{co}(W_2))\frac{r}{2\pi}+O(\log r)\le C(\mathrm{co}(W_{2,0}))\frac{r}{2\pi}+O(\log r).
\end{aligned}
$$

This together with (3.56) implies that

$$C(\mathrm{co}(W_{2,0}))\frac{r}{2\pi}=C(\mathrm{co}(W_2))\frac{r}{2\pi}+O(\log r),$$

and so we have $W_{2,0}=W_2$ and $a_2(1)-c_1=a_2(1)-c_2=0$. Therefore, we have $c_1=c_2$, which is impossible.

Case 3. Suppose that there exists some positive integer $m\ge 2$ such that

$$\overline{N}_{(m,1)}\left(r,c_1,L_1,L_2\right)+\overline{N}_{(m,1)}\left(r,c_2,L_1,L_2\right)\ne o(T(r,L_k))\quad\text{with}\quad k\in\{1,2\}. \tag{3.57}$$

Next, in the same manner as in Case 2 we deduce by (3.57) that $a_1(1)-c_1=a_1(1)-c_2=0$, and so $c_1=c_2$, this is impossible. Theorem 1.11 is thus completely proved.

4. Proof of Theorem 1.10

Using the lines before the formula (3.14) of the proof of Theorem 1.11, we get (3.14), which contradicts the assumption $a_1(1)-a_2(1)=0$ of Theorem 1.10. This completes the proof of Theorem 1.10.

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