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Carathéodory metric on some generalized Teichmüller spaces

Xinlong Dong and Sudeb Mitra

In memory of Professor Clifford J. Earle

Abstract. We study the Carathéodory metric on some generalized Teichmüller spaces. Our paper is especially inspired by the papers by Earle (1974) and Miyachi (2006). Earle (1974) showed that the Carathéodory metric is complete on any Teichmüller space. Miyachi (2006) extended this result for asymptotic Teichmüller spaces. We study the completeness of the Carathéodory metric on product Teichmüller spaces and on the Teichmüller space of a closed set in the Riemann sphere.

Eräiden yleistettyjen Teichmüllerin avaruuksien Carathéodoryn metriikka

Tiivistelmä. Tutkimme eräiden yleistettyjen Teichmüllerin avaruuksien Carathéodoryn metriikkaa. Tarkasteluamme inspiroivat erityisesti Earlen (1974) ja Miyachin (2006) työt. Earle (1974) osoitti, että jokaisen Teichmüllerin avaruuden Carathéodoryn metriikka on täydellinen. Miyachi (2006) yleisti tämän tuloksen asymptoottisiin Teichmüllerin avaruuksiin. Me tutkimme Teichmüllerin tuloavaruuksien ja Riemannin pallon suljetun joukon Teichmüllerin tuloavaruuden Carathéodoryn metriikan täydellisyyttä.

1. Introduction

The study of Kobayashi and Carathéodory metrics on Teichmüller spaces is an important topic. An important theorem of Royden states that the Teichmüller and Kobayashi metrics coincide for finite dimensional Teichmüller spaces; see [25]. Royden's theorem was extended to all Teichmüller spaces by Gardiner; see [7, Chapter 14]. Subsequently, using holomorphic motions, an easy proof was given in the paper [5].

The question of Carathéodory metric on Teichmüller spaces was studied in the important paper [1]. In that paper, using Bers embedding, Earle showed that the Carathéodory metric is complete on Teichmüller spaces. In that same paper, Earle asked the question whether the Carathéodory metric coincides with the Teichmüller metric on Teichmüller spaces. In the fundamental paper [19], Marković proved that for any closed surface of genus $g \geq 2$, the answer is negative.

In the paper [23], Miyachi extended Earle's result to asymptotic Teichmüller spaces. Some other important papers on Kobayashi and Carathéodory metrics and their relationship with Teichmüller theory are [15, 16, 17, 25, 27]. Other comprehensive papers on Schwarz's lemma and Kobayashi and Carathéodory pseudometrics are [4, 9].

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Our present paper is particularly inspired by the techniques used in the papers [1, 23]. We prove that, for a large class of Teichmüller spaces, that is, the *product Teichmüller space*, and the *Teichmüller space of a closed set in the sphere*, the Carathéodory metric is complete. These Teichmüller spaces were first studied by Lieb in his Cornell University doctoral dissertation (see [18]). Subsequently, they have been extensively studied and used in several papers. They have an intimate relationship with holomorphic motions, tame quasiconformal motions, and with some problems in geometric function theory. For applications to holomorphic motions, see the papers [20, 21, 22], and also the expository paper [11]. For applications to some problems in geometric function theory, see the paper [6]. For applications to continuous motions and geometric function theory, see the paper [12]. A recent application to tame quasiconformal motions is the paper [13].

Our paper is arranged as follows. In §2, we state the two main theorems of our paper. In §3, we summarize the definitions and some important properties of Kobayashi and Carathéodory metrics on complex manifolds. In §4, we give the precise definition of product Teichmüller spaces and note some properties that will be useful in our paper. In §5, we prove the first main theorem of our paper. The crucial step is Theorem 1 (in §5), where we prove an estimate for Carathéodory and Kobayashi metrics on Teichmüller spaces. In §6, we define the Teichmüller space of a closed set in the Riemann sphere, and note some properties that are relevant to our paper. In §7, we prove the second main theorem of our paper.

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2. Statements of the main theorems

Throughout this paper, we will use $\mathbb{C}$ for the complex plane, $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ for the Riemann sphere, and $\Delta = \{z \in \mathbb{C} \colon |z| < 1\}$ for the open unit disk.

We state the main theorems of this paper. For each i in the index set I, let X_i be a hyperbolic Riemann surface. Let X be the disjoint union $\coprod_{i \in I} X_i$, and let $\mathrm{Teich}(X)$ denote its *product Teichmüller space*; the precise definition is given in §4.2.

Theorem A. *The Carathéodory metric on* $\mathrm{Teich}(X)$ *is complete.*

Let E be a closed subset of $\widehat{\mathbb{C}}$ that contains the points 0, 1, and ∞. Let $T(E)$ denote its Teichmüller space; see §6.1 for the precise definition.

Theorem B. *The Carathéodory metric on* $T(E)$ *is complete.*

3. Kobayashi and Carathéodory metrics

In this section, we summarize the definitions and some basic properties of the Kobayashi and Carathéodory pseudometrics. Let ρ denote the Poincaré metric on Δ. We have:

$$\rho(z_1, z_2) = \tanh^{-1} \left| \frac{z_1 - z_2}{1 - \bar{z}_1 z_2} \right|.$$

3.1. Kobayashi pseudometric. Let M be a complex manifold. The *Kobayashi pseudometric* K_M is defined as follows: Given two points p, $q \in M$, we choose points $p = p_0, p_1, \cdots, p_{k-1}, p_k = q$ of M, points $a_1, \cdots, a_k, b_1, \cdots, b_k$ of Δ, and holomorphic maps $f_1, \cdots, f_k$ of Δ into M such that $f_i(a_i) = p_{i-1}$ and $f_i(b_i) = p_i$ for $i = 1, \cdots, k$. For each choice of points and maps thus made, consider the number

$$\rho(a_1, b_1) + \cdots + \rho(a_k, b_k).$$

Then, $K_M(p,q)$ is the infimum of the numbers obtained in this manner for all possible choices.

We note some important properties of K_M. For proofs, we refer to [14, Chapter 4, Section 1].

Proposition 1. *Let M and N be two complex manifolds and let $f\colon M \to N$ be a holomorphic map. Then*

$$K_M(p,q) \geq K_N(f(p), f(q)) \quad \text{for } p, q \in M.$$

Corollary 1. *Every biholomorphic map $f\colon M \to N$ is an isometry; which means:*

$$K_M(p,q) = K_N(f(p), f(q)) \quad \text{for } p, q \in M.$$

Proposition 2. *For the open unit disk Δ, K_Δ coincides with the Poincaré metric ρ.*

The following fact is stated in [9, §3]. For the sake of completeness, we include a proof here.

Proposition 3. *Let $B_r(a)$ be the open ball of radius r and center a in a complex Banach space X. Then*

$$K_{B_r(a)}(a,x) = \tanh^{-1}\left(\frac{\|x-a\|}{r}\right)$$

for all x in $B_r(a)$.

Proof. Let $x \in B_r(a)$. We may assume that $x \neq a$. Define a holomorphic map $f\colon \Delta \to B_r(a)$ as follows:

$$f(t) = a + \frac{rt(x-a)}{\|x-a\|}.$$

Note that

$$f\left(\frac{\|x-a\|}{r}\right) = x.$$

By Proposition 1, we have

$$K_{B_r(a)}\left(f(0), f\left(\frac{\|x-a\|}{r}\right)\right) \leq K_\Delta\left(0, \frac{\|x-a\|}{r}\right)$$

which gives

$$K_{B_r(a)}(a,x) \leq \tanh^{-1}\left(\frac{\|x-a\|}{r}\right). \tag{3.1}$$

Next, by Hahn–Banach theorem there exists a linear functional $\ell \in X^*$ with $\|\ell\| = 1$ and $\ell(x-a) = \|x-a\|$. Define a holomorphic map $g\colon B_r(a) \to \Delta$ given by

$$g(y) = \frac{\ell(y-a)}{r}.$$

Again, by Proposition 1, we have

$$K_\Delta(g(x), g(a)) \leq K_{B_r(a)}(x,a).$$

It immediately follows that

$$K_{B_r(a)}(a,x) \geq \tanh^{-1}\left(\frac{\|x-a\|}{r}\right). \tag{3.2}$$

Combining inequalities (3.1) and (3.2) we get the required result. □

3.2. Carathéodory pseudometric. Let M be a complex manifold. The *Carathéodory pseudometric* C_M is defined as follows:

$$C_M(p,q) = \sup_f \rho(f(p), f(q)) \quad \text{for } p, q \in M,$$

where the supremum is taken with respect to the family of holomorphic maps $f\colon M \to \Delta$.

We note some important properties of C_M. For proofs, we refer to [14, Chapter 4, Section 2].

Proposition 4. *Let M and N be two complex manifolds and let $f\colon M \to N$ be a holomorphic map. Then*

$$C_M(p,q) \geq C_N(f(p), f(q)) \quad \text{for } p, q \in M.$$

Corollary 2. *Every biholomorphic map $f\colon M \to N$ is an isometry; which means:*

$$C_M(p,q) = C_N(f(p), f(q)) \quad \text{for } p, q \in M.$$

Proposition 5. *For the open unit disk Δ, C_Δ coincides with the Poincaré metric ρ.*

Proposition 6. *If M and M' are complex manifolds with complete Carathéodory metric, so is $M \times M'$.*

Proposition 7. *Let $B_r(a)$ be the open ball of radius r and center a in a complex Banach space X. Then*

$$C_{B_r(a)}(a,x) = \tanh^{-1}\left(\frac{\|x-a\|}{r}\right)$$

for all x in $B_r(a)$.

See [1, Lemma 2]. The proof is similar to the proof of Proposition 3.

Corollary 3. *The Carathéodory metric induces the standard topology on $B_r(a)$.*

See [1, Corollary of Lemma 2].

4. Some properties of product Teichmüller spaces

We study some basic properties of product Teichmüller spaces. The details are given in [6, Sections 7.1 to 7.8]. For standard facts on classical Teichmüller spaces, the reader is referred to the standard references [2, 3, 7, 8, 10, 24].

4.1. Some complex Banach spaces. Let I be an index set. For full generality, in this section we will assume that I is uncountable. For each i in the index set I, let X_i be a hyperbolic Riemann surface. Let X be the disjoint union $\coprod_{i\in I} X_i$. We introduce the following important Banach spaces:

By definition, a Beltrami form on X is a tensor μ whose restriction to each X_i is a bounded measurable Beltrami form μ_i on X_i with L^∞ norm less than some finite constant independent of i in I. We define

$$\|\mu\| = \sup\{\|\mu_i\|_\infty \colon i \in I\}.$$

We denote the Banach space of Beltrami forms on X by $\mathrm{Belt}(X)$ and we denote the open unit ball of $\mathrm{Belt}(X)$ by $M(X)$. The basepoint of $M(X)$ is its center 0.

Let $\pi\colon \Delta \to R$ be a holomorphic universal covering of the hyperbolic Riemann surface R. Every holomorphic quadratic differential ψ on R lifts to a holomorphic quadratic differential $\widetilde{\psi}(z)\,dz^2$ on Δ. We say that ψ is bounded if its Nehari norm

$$\|\psi\|_N = \sup\{\|\widetilde{\psi}(z)\|(1-|z|^2)^2 \colon z \in \Delta\}$$

is finite.

For each i in I let X_i^* be the conjugate Riemann surface of X_i, and let X^* be the disjoint union of the X_i^*. Let ψ be a holomorphic quadratic differential on X^*. We say that ψ is bounded if its restriction ψ_i to X_i^* is a bounded holomorphic quadratic differential for each i and its Nehari norm

$$\|\psi\|_N = \sup\{\|\psi_i\|_N \colon i \in I\}$$

is finite. We denote the complex Banach space of bounded holomorphic quadratic differentials on X^* by $B(X^*)$.

4.2. Product Teichmüller space. For each $i \in I$, let $\mathrm{Teich}(X_i)$ be the Teichmüller space of the Riemann surface X_i, let 0_i be the basepoint of $\mathrm{Teich}(X_i)$ and let d_i be the Teichmüller metric on $\mathrm{Teich}(X_i)$. By definition, the Teichmüller space $\mathrm{Teich}(X)$ is the set of functions t on I such that $t(i)$ is in $\mathrm{Teich}(X_i)$ for each i and the set of numbers $\{d_i(0_i, t(i)) \colon i \in I\}$ is bounded. As usual, we shall write t_i for $t(i)$. The basepoint of $\mathrm{Teich}(X)$ is the function t such that $t_i = 0_i$ for each i; we shall denote it by 0_X.

The Teichmüller metric on $\mathrm{Teich}(X)$ is defined by

$$d_T(s,t) = \sup\{d_i(s_i, t_i) \colon i \in I\},$$

for s and t in $\mathrm{Teich}(X)$. Since each metric d_i is complete, the metric d_T on $\mathrm{Teich}(X)$ is also complete.

Lemma 1. *The Teichmüller metric on* $\mathrm{Teich}(X)$ *is the same as its Kobayashi metric.*

See [6, Proposition 7.28].

For each $i \in I$, let Φ_i be the usual projection of $M(X_i)$ onto $\mathrm{Teich}(X_i)$; see, for example, [2, 3, 10, 24] for standard facts on the classical Teichmüller spaces. By definition $d_i(0_i, t_i) = \inf\{\rho(0, \|\mu_i\|) \colon \mu_i \in M(X_i) \text{ and } \Phi_i(\mu_i) = t_i\}$ for each $t \in \mathrm{Teich}(X)$, so if $\mu \in M(X)$ then $d_i(0_i, \Phi(\mu_i)) \le \rho(0, \|\mu\|)$ for all i. We can therefore define the *standard projection* $\Phi\colon M(X) \to \mathrm{Teich}(X)$ by the formula

$$\Phi(\mu)_i = \Phi_i(\mu_i), \quad \mu \in M(X) \quad \text{and} \quad i \in I.$$

It is easy to see that the map Φ is surjective.

Definition 1. For each $i \in I$ let $\mathcal{B}_i\colon M(X_i) \to B(X_i^*)$ be the classical Bers projection (see [2, 3, 10, 24]). The *generalized Bers projection* $\mathcal{B}\colon M(X) \to B(X^*)$ is defined by the formula $\mathcal{B}(\mu)_i = \mathcal{B}_i(\mu_i)$, i in I and μ in $M(X)$.

Definition 2. For each $i \in I$, let $\alpha_i\colon B(X_i^*) \to L^\infty(X_i)$ be the classical Ahlfors–Weill map (see [1, 2, 8, 10, 24]). The *generalized Ahlfors–Weill map* $\alpha\colon B(X^*) \to \mathrm{Belt}(X)$ is defined by the formula $\alpha(\psi)_i = \alpha_i(\psi_i)$, i in I and ψ in $B(X^*)$.

Proposition 8. *The generalized Bers projection* $\mathcal{B}\colon M(X) \to B(X^*)$ *is a holomorphic split submersion with the following properties:*

(i) $\mathcal{B}(0) = 0$ *and* $\|\mathcal{B}(\mu)\|_N \le 6$ *for all* μ *in* $M(X)$*;*

(ii) *for all* μ *and* ν *in* $M(X)$, $\mathcal{B}(\mu) = \mathcal{B}(\nu)$ *if and only if* $\Phi(\mu) = \Phi(\nu)$*;*

(iii) *if* $\psi \in B(X^*)$ *and* $\|\psi\|_{B(X^*)} < 2$, *then* $\mathcal{B}(\alpha(\psi)) = \psi$.

Statements (i), (ii), and (iii) follow immediately from the corresponding statements in the classical case (see [2, 3, 8, 10, 24]). The fact that $\mathcal{B}$ is a holomorphic split submersion is proved in [6, Proposition 7.3].

Corollary 4. *There is a unique complex Banach manifold structure on* $\mathrm{Teich}(X)$ *that has the following properties:*

(i) *the map* $\Phi\colon M(X) \to \mathrm{Teich}(X)$ *is a holomorphic split submersion;*
(ii) *the map* $\widehat{\mathcal{B}}\colon \mathrm{Teich}(X) \to \mathcal{B}(M(X))$ *such that* $\widehat{\mathcal{B}} \circ \Phi = \mathcal{B}$ *is biholomorphic,*
(iii) *if* $t \in \mathrm{Teich}(X)$ *and* $\|\widehat{\mathcal{B}}(t)\|_{B(X^*)} < 2$, *then* $\Phi(\alpha(\widehat{\mathcal{B}}(t))) = t$.

See [6, Corollary 7.4].

Definition 3. The biholomorphic map $\widehat{\mathcal{B}}$ is called the *generalized Bers embedding* of $\mathrm{Teich}(X)$ in $B(X^*)$.

Definition 4. The *generalized Ahlfors–Weill section of* $\mathcal{B}$ is the restriction of the map α to the set of ψ in $B(X^*)$ with $\|\psi\|_{B(X^*)} < 2$.

4.3. Changing the basepoint. For each $i \in I$, let h_i be a K-quasiconformal mapping of X_i onto a hyperbolic Riemann surface Y_i, with K independent of i. Let $Y = \coprod Y_i$ be the disjoint union. Each h_i induces a biholomorphic map h_i^* of $\mathrm{Teich}(X_i)$ onto $\mathrm{Teich}(Y_i)$.

Proposition 9. *There is a unique biholomorphic map* h *of* $\mathrm{Teich}(X)$ *onto* $\mathrm{Teich}(Y)$ *such that* $h^*(t)_i = h_i^*(t_i)$ *for all* t *in* $\mathrm{Teich}(X)$ *and* i *in* I. *Furthermore, if* μ *is the point in* $M(X)$ *such that* μ_i *is the Beltrami coefficient of* h_i, *then* h^* *maps the point* $\Phi(\mu)$ *in* $\mathrm{Teich}(X)$ *to the basepoint* 0_Y *of* $\mathrm{Teich}(Y)$*; here* $\Phi\colon M(X) \to \mathrm{Teich}(X)$ *is the standard projection.*

For a proof, see [6, Proposition 7.9].

Remark 1. For any μ in $M(X)$ and any i in I there are a Riemann surface Y_i and a quasiconformal mapping of X_i onto Y_i whose Beltrami coefficient is μ_i. Therefore each point $\Phi(\mu)$ in $\mathrm{Teich}(X)$ can be mapped to the basepoint of some $\mathrm{Teich}(Y)$ by some biholomorphic map h^*.

5. Proof of Theorem A

For each i in the index set I, let X_i be a hyperbolic Riemann surface. Let X be the disjoint union $\coprod_{i\in I} X_i$. Let $\mathrm{Teich}(X)$ be the product Teichmüller space discussed in §4. Let C_T and K_T respectively denote the Carathéodory and Kobayashi metrics on $\mathrm{Teich}(X)$.

Let 0 denote the origin of the complex Banach space $B(X^*)$ in §4.1. To simplify notations, let $B_2(0)$ denote the ball of radius 2 centered at the origin in $B(X^*)$, and let $B_6(0)$ denote the ball of radius 6 centered at the origin in $B(X^*)$. Let $C_{B_6(0)}$ denote the Carathéodory metric on $B_6(0)$, and let $K_{B_2(0)}$ denote the Kobayashi metric on $B_2(0)$.

Theorem 1. $\tanh C_T(x, y) \le \tanh K_T(x, y) \le 3 \tanh C_T(x, y)$ *for all* x, y *in* $\mathrm{Teich}(X)$.

Proof. Step 1. Let $x = 0$; to simplify notations, we use 0 for the basepoint of $\mathrm{Teich}(X)$.

We know that

$$C_T(0,t) \le K_T(0,t) \quad \text{for all } t \in \mathrm{Teich}(X)$$

and so we have $\tanh C_T(0,t) \le \tanh K_T(0,t)$ for all t in $\mathrm{Teich}(X)$.

By the generalized Bers embedding discussed in Proposition 8 and Corollary 4, we know that $B_2(0) \subset \mathrm{Teich}(X) \subset B_6(0)$. Hence we have $C_T(0,t) \ge C_{B_6(0)}(0,t)$ and therefore,

$$C_T(0,t) \ge \tanh^{-1} \frac{\|t\|}{6} \quad \text{for all } t \in \mathrm{Teich}(X).$$

It follows that

$$6 \tanh C_T(0,t) \ge \|t\| \tag{5.1}$$

for all $t \in \mathrm{Teich}(X)$.

We also have $K_T(0,t) \le K_{B_2(0)}(0,t)$ which implies that

$$K_T(0,t) \le \tanh^{-1} \frac{\|t\|}{2} \quad \text{if } \|t\| < 2.$$

It follows that

$$2 \tanh K_T(0,t) \le \|t\| \quad \text{if } \|t\| < 2.$$

This is also true if $\|t\| > 2$. It follows that

$$2 \tanh K_T(0,t) \le \|t\| \tag{5.2}$$

for all $t \in \mathrm{Teich}(X)$.

Combining (5.1) and (5.2), we get

$$2 \tanh K_T(0,t) \le \|t\| \le 6 \tanh C_T(0,t)$$

for all $t \in \mathrm{Teich}(X)$; hence,

$$\tanh K_T(0,t) \le 3 \tanh C_T(0,t)$$

for all $t \in \mathrm{Teich}(X)$.

Step 2. Let x, y be any points in $\mathrm{Teich}(X)$. By Remark 1, there exists a biholomorphic map h^* on $\mathrm{Teich}(X)$ onto some Teichmüller space $\mathrm{Teich}(Y)$ such that h^* maps the point x in $\mathrm{Teich}(X)$ to the basepoint 0_Y of $\mathrm{Teich}(Y)$. Let $h^*(y) = t \in \mathrm{Teich}(Y)$.

By Step 1, we have

$$\tanh C_T(0_Y,t) \le \tanh K_T(0_Y,t) \le 3 \tanh C_T(0_Y,t) \quad \text{for all } t \in \mathrm{Teich}(Y),$$

where C_T denotes the Carathéodory metric on $\mathrm{Teich}(Y)$ and K_T denotes the Kobayashi metric on $\mathrm{Teich}(Y)$.

Since h^* is biholomorphic, the Kobayashi and Carathéodory metrics on $\mathrm{Teich}(X)$ and $\mathrm{Teich}(Y)$ respectively, are preserved. It follows that

$$\tanh C_T(x,y) \le \tanh K_T(x,y) \le 3 \tanh C_T(x,y)$$

for all x, y in $\mathrm{Teich}(X)$. □

Proof of Theorem A. Let $\{t_n\}$ be a Cauchy sequence in $\mathrm{Teich}(X)$, with respect to the Carathéodory metric C_T.

Let $\epsilon > 0$ be given. Choose

$$\widehat{\epsilon} = \tanh^{-1}\left(\frac{1}{3} \tanh \epsilon\right).$$

Then, for this $\widehat{\epsilon} > 0$, there exists a positive integer N such that for all $m, n > N$, we have $C_T(t_m, t_n) < \widehat{\epsilon}$. Hence, $3 \tanh C_T(t_m, t_n) < 3 \tanh \widehat{\epsilon}$ for all $m, n > N$. It follows from Theorem 1 that

$$\tanh K_T(t_m, t_n) < 3 \tanh \widehat{\epsilon} = 3\big(\frac{1}{3} \tanh \epsilon\big) = \tanh \epsilon$$

for all $m, n > N$.

Therefore, $K_T(t_m, t_n) < \epsilon$ for all $m, n > N$. Hence, $\{t_n\}$ is a Cauchy sequence with respect to K_T. By Lemma 1, it follows that $\{t_n\}$ is a Cauchy sequence with respect to the Teichmüller metric d_T. Since d_T is complete, $t_n \to t$ in $\mathrm{Teich}(X)$, and by Lemma 1 again, $t_n \to t$ in $\mathrm{Teich}(X)$ with respect to K_T, and so, we have $K_T(t_n, t) \to 0$.

Let $\widetilde{\epsilon} > 0$ be given. There exists a natural number $\widetilde{N} > 0$ such that $K_T(t_n, t) < \widetilde{\epsilon}$ for all $n > \widetilde{N}$. Hence, $\tanh K_T(t_n, t) < \tanh \widetilde{\epsilon}$ for all $n > \widetilde{N}$. It follows by Theorem 1 that $\tanh C_T(t_n, t) < \tanh \widetilde{\epsilon}$ for all $n > \widetilde{N}$. Therefore, $C_T(t_n, t) < \widetilde{\epsilon}$ for all $n > \widetilde{N}$. It follows that $t_n \to t$ in $\mathrm{Teich}(X)$ with respect to C_T. Hence, the Carathéodory metric on $\mathrm{Teich}(X)$ is complete. □

Remark 2. Let S be a hyperbolic Riemann surface, and let $\mathrm{Teich}(S)$ denote its usual Teichmüller space. In [1], Earle proved that the Carathéodory metric on $\mathrm{Teich}(S)$ is complete. Our method gives an alternative proof of Earle's theorem; in particular the estimate on the right-hand side of Theorem 1 is explicit.

Remark 3. If the index set is finite, then $\mathrm{Teich}(X)$ is simply the (finite) cartesian product of the Teichmüller spaces $\mathrm{Teich}(X_i)$. In this case, Theorem A follows from the main theorem in Earle's paper [1] and Proposition 6.

6. Teichmüller space of a closed set in the Riemann sphere

Recall that a homeomorphism of $\widehat{\mathbb{C}}$ is called *normalized* if it fixes the points 0, 1, and ∞. We use $M(\mathbb{C})$ to denote the open unit ball of the complex Banach space $L^\infty(\mathbb{C})$. Each μ in $M(\mathbb{C})$ is the Beltrami coefficient of a unique normalized quasiconformal homeomorphism w^μ of $\widehat{\mathbb{C}}$ onto itself. The basepoint of $M(\mathbb{C})$ is the zero function.

The Kobayashi metric $K_{M(\mathbb{C})}$ on $M(\mathbb{C})$ is defined by

$$K_{M(\mathbb{C})}(\mu, \nu) = \tanh^{-1} \|(\mu - \nu)(1 - \overline{\mu}\nu)^{-1}\|_\infty$$

for all μ, ν in $M(\mathbb{C})$.

6.1. Teichmüller space of a closed set in the Riemann sphere. Let E be a closed subset of $\widehat{\mathbb{C}}$ that contains the points 0, 1, and ∞.

Definition 5. Two normalized quasiconformal self-mappings f and g of $\widehat{\mathbb{C}}$ are said to be E-equivalent if and only if $f^{-1} \circ g$ is isotopic to the identity rel E. The *Teichmüller space* $T(E)$ is the set of all E-equivalence classes of normalized quasiconformal self-mappings of $\widehat{\mathbb{C}}$. The basepoint of $T(E)$ is the E-equivalence class of the identity map.

We can define the quotient map

$$P_E \colon M(\mathbb{C}) \to T(E)$$

by setting $P_E(\mu)$ equal to the E-equivalence class of w^μ, written as $[w^\mu]_E$. Clearly, P_E maps the basepoint of $M(\mathbb{C})$ to the basepoint of $T(E)$.

The Teichmüller metric $d_{T(E)}$ on $T(E)$ is given by

$$d_{T(E)}(P_E(\mu), t) = \inf\{K_{M(\mathbb{C})}(\mu, \nu) \colon \nu \in M(\mathbb{C}) \text{ and } P_E(\nu) = t\}$$

for all μ in $M(\mathbb{C})$ and t in $T(E)$.

Since E^c is an open subset of $\mathbb{C} \setminus \{0, 1\}$, each of its connected components is a hyperbolic Riemann surface. We index these components X_i by a set I of positive integers, and we form the product Teichmüller space of their disjoint union E^c; let $\mathrm{Teich}(E^c)$ denote this product Teichmüller space. Let $M(E)$ be the open unit ball in $L^\infty(E)$. Then, the product $\mathrm{Teich}(E^c) \times M(E)$ is a complex Banach manifold.

For μ in $L^\infty(\mathbb{C})$, let $\mu|E^c$ and $\mu|E$ be the restrictions of μ to E^c and E respectively. We define the projection map $\widetilde{P}_E$ from $M(\mathbb{C})$ to $\mathrm{Teich}(E^c) \times M(E)$ by the formula

$$\widetilde{P}_E(\mu) = \big(\Phi(\mu|E^c), \mu|E\big) \tag{6.1}$$

for all μ in $M(\mathbb{C})$, where $\Phi \colon M(E^c) \to \mathrm{Teich}(E^c)$ is the standard projection.

We now state "Lieb's isomorphism theorem".

Theorem 2. *For all μ and ν in $M(\mathbb{C})$ we have $P_E(\mu) = P_E(\nu)$ if and only if $\widetilde{P}_E(\mu) = \widetilde{P}_E(\nu)$. Consequently, there is a well defined bijection $\theta \colon T(E) \to \mathrm{Teich}(E^c) \times M(E)$ such that $\theta \circ P_E = \widetilde{P}_E$, and $T(E)$ has a unique complex manifold structure such that P_E is a holomorphic split submersion and the map θ is biholomorphic.*

See [6, §7.10] for a complete proof.

Proposition 10. *The Teichmüller and Kobayashi metrics on $T(E)$ are equal.*

See [6, Proposition 7.30].

7. Proof of Theorem B

By Theorem A, the Carathéodory metric on $\mathrm{Teich}(E^c)$ is complete, and it is well-known that the Carathéodory metric on $M(E)$ is complete. Therefore, by Proposition 6, $\mathrm{Teich}(E^c) \times M(E)$ is also complete. It follows by Theorem 2, that the Carathéodory metric on $T(E)$ is complete. □

Remark 4. If E is a finite set, then $T(E)$ is naturally identified with the classical Teichmüller space $\mathrm{Teich}(\widehat{\mathbb{C}} \setminus E)$. This easily follows from Theorem 2. A direct proof is given in [20, Example 3.1]. Therefore, when E is finite, Theorem B is exactly the main theorem in Earle's paper [1].

Remark 5. If $E = \widehat{\mathbb{C}}$, then $T(E)$ is naturally identified with $M(\mathbb{C})$. In this case, the Carathéodory, Kobayashi, and Poincaré metrics coincide.

Remark 6. If the closed set E has zero area, then $M(E)$ contains only one point, and $\mathrm{Teich}(E^c) \times M(E)$ is isomorphic to $\mathrm{Teich}(E^c)$ and in that case, Theorem B is a special case of Theorem A.

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Xinlong Dong
City University of New York
Kingsborough Community College
Department of Mathematics and
Computer Science
Brooklyn, NY 11235–2398, USA
Xinlong.Dong@kbcc.cuny.edu

Sudeb Mitra
Queens College of
the City University of New York
Department of Mathematics
Flushing, NY 11367–1597, USA
and Graduate Center of
the City University of New York
Department of Mathematics
New York, NY 10016, USA
sudeb.mitra@qc.cuny.edu