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Relative L^p-cohomology and application to Heintze groups

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Abstract. We introduce the notion of *relative L^p-cohomology* as a quasi-isometry invariant defined for a Gromov-hyperbolic space and a point on its boundary at infinity and reproduce some basic properties of L^p-cohomology in this context. In the case of degree 1 we show a relation between the relative and the classical L^p-cohomology. As an application, we explicitly construct non-zero relative L^p-cohomology classes for a purely real Heintze group of the form $\mathbb{R}^{n-1} \rtimes_\alpha \mathbb{R}$, which gives a way to prove that the eigenvalues of α, up to a scalar multiple, are invariant under quasi-isometries.

Suhteellinen L^p-kohomologia ja sovellus Heintzen ryhmiin

Tiivistelmä. Esittelemme *suhteellisen L^p-kohomologian* käsitteen Gromovin mielessä hyperboliselle avaruudelle ja sen äärettömyydessä sijaitsevalle reunapisteelle määriteltynä kvasi-isometrisenä invarianttina ja näytämme, että eräät L^p-kohomologian perusominaisuudet ovat voimassa myös tässä asetelmassa. Kertaluvun 1 tapauksessa osoitamme yhteyden suhteellisen ja klassisen L^p-kohomologian välillä. Sovelluksena rakennamme täysin reaaliselle Heintzen ryhmälle $\mathbb{R}^{n-1} \rtimes_\alpha \mathbb{R}$ nollasta poikkeavia suhteellisia L^p-kohomologialuokkia, mikä antaa keinon todistaa, että kvasi-isometria säilyttää matriisin α ominaisarvot skalaarikerrointa vaille.

1. Introduction

In the context of some classes of metric spaces, L^p-cohomology is a quasi-isometry invariant with interesting applications to classification problems. This notion is defined, for example, for simplicial complexes [7, 14, 25, 21], Riemannian manifolds [18, 19, 34, 31], discrete and topological groups [3, 4, 5, 6, 12, 30, 35, 36] and more general metric measure spaces [16, 33, 36].

To recall what L^p-cohomology is let us consider a Riemannian manifold M. Given a real number $p \geq 1$, the L^p-norm of a (smooth) differential k-form ω on M is defined by

$$\|\omega\|_{L^p} = \left(\int_M |\omega|_x \, dV(x) \right)^{\frac{1}{p}},$$

where

$$|\omega|_x = \sup \left\{ |\omega_x(v_1, \dots, v_k)| \colon v_i \in T_xM \text{ with } \|v_i\|_x = 1 \text{ for every } i = 1, \dots, k \right\}.$$

Here $\|\cdot\|_x$ is the Riemannian norm in the tangent space T_xM and dV is the Riemannian volume. Then we consider $L^p\Omega^k(M)$ as the space of differential k-forms ω on M with

$$|\omega|_{L^p} = \|\omega\|_{L^p} + \|d\omega\|_{L^p} < +\infty,$$

where d is the usual exterior derivative. Observe that d is continuous with respect to $|\cdot|_{L^p}$.

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Since $(L^p\Omega^k(M), |\cdot|_{L^p})$ is not complete we take its completion $L^pC^k(M)$ and extend continuously the exterior derivative. The *de Rham L^p-cohomolgy* of M is the cohomology of the cochain complex $(L^pC^*(M), d)$, that is, the family of topological vector spaces

$$L^pH^k(M,\xi) = \frac{\text{Ker } d|_{L^pC^k(M)}}{\text{Im } d|_{L^pC^{k-1}(M)}}.$$

It is sometimes convenient to consider the *reduced de Rham L^p-cohomology* of M as the reduced cohomology of $(L^pC^*(M), d)$, that is, the family of Banach spaces

$$L^p\overline{H}^k(M,\xi) = \frac{\text{Ker } d|_{L^pC^k(M)}}{\overline{\text{Im } d|_{L^pC^{k-1}(M)}}}.$$

Observe that $L^p\Omega(M)$ is continuously included in the space of L^p-integrable (not necessarily smooth) k-forms on M up to almost everywhere zero k-forms, denoted by $L^p(M,\Lambda^k)$, which is naturally equipped with the norm $\|\cdot\|_{L^p}$. Since the second space is complete, the elements of $L^pC^k(M)$ can be seen as elements of $L^p(M,\Lambda^k)$. Indeed, an equivalent definition of de Rham L^p-cohomology can be done in terms of L^p-integrable forms and the weak exterior derivative (see for example [20, 24]).

The above construction gives quasi-isometry invariants; that is, the existence of a quasi-isometry between two Riemannian manifolds M and N (under some assumptions) implies the existence of isomorphisms between $L^pH^k(M)$ and $L^pH^k(N)$ and between $L^p\overline{H}^k(M)$ and $L^p\overline{H}^k(N)$ for every $p \geq 1$ and $k \in \mathbb{N}$ (see [7, 21, 25]). Therefore, one can distinguish two Riemannian manifolds up to quasi-isometries by computing their L^p-cohomologies.

As we mention above, L^p-cohomolgy can be defined for other classes of metric spaces. In all contexts it gives quasi-isometry invariants. There are also isomorphism theorems that relates different versions.

In this work, we define a variation of the L^p-cohomology defined for a Gromov-hyperbolic space and a point on its boundary at infinity. We call it *relative L^p-cohomology.* It has the advantage of simplifying the computation in some cases, as we will see.

Let M be a Gromov-hyperbolic complete Riemannian manifold and fix a point ξ on its boundary at infinity ∂M. For $p \geq 1$ and $k \in \mathbb{N}$ we consider $L^pC^k(M,\xi)$ the subspace of $L^pC^k(M)$ consisting of all elements that vanish (almost everywhere) on a neighborhood of ξ in $\overline{M} = M \cup \partial M$. It is clear that $d(L^pC^k(M,\xi)) \subset L^pC^{k+1}(M,\xi)$ for every k, then we take the family of spaces

$$L^pH^k(M,\xi) = \frac{\text{Ker } d|_{L^pC^k(M,\xi)}}{\text{Im } d|_{L^pC^{k-1}(M,\xi)}}$$

endowed with the topology induced by $|\cdot|_{L^p}$. These are the spaces of *relative de Rham L^p-cohomology* of the pair (M,ξ).

Taking the reduced cohomology of the complex $(L^pC^*(M,\xi), d)$ seems to have no advantages because the spaces $L^pC^k(M,\xi)$ are not, in general, Banach spaces, so we do not pay attention to it. However, some properties of the reduced L^p-cohomology can be reproduced in this context using arguments given in this work.

We also define a version of relative L^p-cohomology for simplicial complexes. To this end suppose that X is a finite-dimensional simplicial complex with a length distance such that there exist a constant $C \geq 0$ and a function $N\colon [0,+\infty) \to \mathbb{N}$ satisfying the following properties:

(a) All simplices in X have diameter smaller than C.
(b) Every ball of radius r intersects at most $N(r)$ simplices.

In this case we say that X has *bounded geometry.* The set of k-simplices of X will be denoted by $X^{(k)}$.

The *(reduced) simplicial ℓ^p-cohomolgy* of X is the (reduced) cohomology of the $(\ell^p(X^{(*)}), \delta)$, where $\delta\colon \ell^p(X^{(k)}) \to \ell^p(X^{(k+1)})$ is the usual coboundary operator. It is not difficult to see that it is well-defined and continuous using properties (a) and (b).

If X is Gromov-hyperbolic and $\xi \in \partial X$, we consider $\ell^p(X^{(k)}, \xi)$ the subspace of $\ell^k(X^{(k)})$ consisting of all k-cochains that vanish on a neighborhood of ξ in $\overline{X}$. (We say that $\theta \in \ell^p(X^{(k)})$ is zero or vanishes on $U \subset \overline{X}$ if $\theta(\sigma) = 0$ for every k-simplex $\sigma \subset U$.) In this case we also have $\delta(\ell^p(X^{(k)}, \xi)) \subset \ell^p(X^{(k+1)}, \xi)$, then the *relative simplicial ℓ^p-cohomology of the pair* (X, ξ) is defined as the family of topological vector spaces

$$\ell^p H^k(X, \xi) = \frac{\operatorname{Ker} \delta|_{\ell^p(X^{(k)},\xi)}}{\operatorname{Im} \delta|_{\ell^p(X^{(k-1)},\xi)}}.$$

We prove the following result:

Theorem 1.1. *Let X and Y be two uniformly contractible Gromov-hyperbolic simplicial complexes with bounded geometry and ξ a fixed point in ∂X. If $F\colon X \to Y$ is a quasi-isometry, then for every $p \geq 1$ and $k \in \mathbb{N}$ there is an isomorphism of topological vector spaces between $\ell^p H^k(X, \xi)$ and $\ell^p H^k(Y, F(\xi))$.*

A metric space is *uniformly contractible* if there is a function $\phi\colon [0, +\infty) \to [0, +\infty)$ such that every ball $B(x, r) = \{x' \in X : |x' - x| < r\}$ is contractible in the ball $B(x, \phi(r))$.

Suppose that M is a complete Gromov-hyperbolic Riemannian manifold admitting a triangulation X_M with bounded geometry such that every simplex is uniformly biLipschitz diffeomorphic to the standard Euclidean simplex of the same dimension. The existence of such triangulation is guaranteed if, for example, M has bounded geometry, that is, it has bounded curvature and positive injectivity radius (see [1, Theorem 1.14]).

For each vertex v of X_M we define its *open star* $U(v)$ as the interior of the union of all simplices containing v. Observe that

$$\mathcal{U} = \{U(v) \colon v \in X_M^{(0)}\}$$

is an open covering of M satisfying that every nonempty intersection $U(v_1) \cap \cdots \cap U(v_k)$ is biLipschitz diffeomorphic to the unit ball in $\mathbb{R}^n$ with uniform Lipschitz constant. The simplicial complex X_M can be seen as the nerve of $\mathcal{U}$, that is, we can identify $X_M^{(k)}$ with the set

$$\{(U_1, \ldots, U_k) \colon U_1 \cap \cdots \cap U_k \neq \emptyset \text{ and } U_1, \ldots U_k \in \mathcal{U}\}.$$

Under the above considerations, we prove the following result:

Theorem 1.2. *Let M and X_M be as above and $\xi \in \partial M = \partial X_M$, then for every $p \geq 1$ and $k \in \mathbb{N}$ the spaces $L^p H^k(M, \xi)$ and $\ell^p H^k(X_M, \xi)$ are isomorphic.*

The method we use to prove Theorem 1.2 is a bicomplex argument as the one used, for example, in [8, Theorem 8.9], [16] and [33]. Similar results are also obtained via the construction of Sullivan complexes and Whitney forms, as is done in [13] and [21].

Combining Theorem 1.1 with Theorem 1.2 we obtain the quasi-isometry invariance of the de Rham relative L^p-cohomology:

Theorem 1.3. *Let M and N be two complete, uniformly contractible and Gromov-hyperbolic Riemannian manifolds with bounded geometry, and ξ a fixed point in ∂M. If $F\colon M \to N$ is a quasi-isometry, then for every $p \geq 1$ and $k \in \mathbb{N}$ there is an isomorphism of topological vector spaces between $L^pH^k(M,\xi)$ and $L^pH^k(N,F(\xi))$.*

In the case of degree 1 we use the identification between L^p-cohomology and Besov spaces given in [7] to prove the following result:

Theorem 1.4. *Let M be a complete Gromov-hyperbolic Riemannian manifold with bounded geometry and $\xi \in \partial M$. Then for every $p \geq 1$ there exists a continuous embedding $L^pH^1(M,\xi) \hookrightarrow L^pH^1(M)$. Moreover, if $p > \mathrm{Cdim}_{AR}(\partial M)$, the image of this embedding is dense.*

In the previous theorem, $\mathrm{Cdim}_{AR}(\partial M)$ denotes the Ahlfors regular conformal dimension of ∂M, that is, the infimum of Hausdorff dimensions of all Ahlfors regular metrics that are quasi-symmetry equivalent to any visual metric on ∂M (see for example [29, Chapter 7] for more details). We assume that the infimum of the empty set is $+\infty$.

It is possible to construct examples for which the relative L^p-cohomology depends on the point on the boundary. For example one can consider X the space that results from gluing $\mathbb{H}^n$ and $[0,+\infty)$ by identifying 0 with any point of $\mathbb{H}^n$ with a convenient simplicial structure. If $\xi_0 \in \partial X$ is the point represented by the geodesic ray $[0,+\infty)$ one can see that $\dim(\ell^pH^1(X,\xi_0)) = 1$, while $\dim(\ell^pH^1(X,\xi)) = \infty$ if $\xi \neq \xi_0$. This says, for example, that every self quasi-isometry of X must fix the point ξ_0. It is an interesting question whether relative L^p-cohomology can be used to distinguish special points on the boundary at infinity of a Gromov-hyperbolic space.

1.1. Application to Heintze groups. In [27] Heintze characterizes the complete homogeneous Riemannian manifolds of negative curvature. More precisely, he proves that such a manifold is isometric to a Lie group of the form $G = N \rtimes_\alpha \mathbb{R}$ equipped with a left-invariant metric, where N is a connected and simply connected nilpotent Lie group and α is a derivation on the Lie algebra of N whose eigenvalues all have positive real part. They are the so called *Heintze groups*. The product in G is done by

$$(x,t)\cdot(y,s) = (x\cdot\tau_t(y), t+s),$$

where τ_t is the automorphism of N that satisfy $d_e\tau_t = e^{t\alpha}$ (here e denotes the identity in N). Conversely, every Heintze group admits a left-invariant Riemannian metric of negative curvature.

The boundary at infinity of a Heintze group $G = N \rtimes_\alpha \mathbb{R}$ has a particular structure. Indeed, for every $x \in N$ the curve $t \mapsto (x,t)$ is a geodesic on G. All these vertical geodesics are asymptotic to the future and so they define a unique point on the boundary, denoted by ∞ . Moreover, any other point of the boundary can be represented by one of these geodesics to the past. As a consequence we identify the boundary ∂G with $N \cup \{\infty\}$.

All left-invariant metrics on a Lie group are biLipschitz equivalent and hence quasi-isometric. Thus, the structure as Heintze group determines the quasi-isometry class of a homogeneous manifold of negative curvature. The converse is not true in general: every Heintze group is quasi-isometric to a *purely real* one, that is, a

Heintze group $N \rtimes_\alpha \mathbb{R}$ where α has only real eigenvalues (see [11]). On the problem of quasi-isometric classification of purely real Heintze groups we have the following conjecture by Cornulier:

Conjecture 1.5. [11] Two purely real Heintze groups are quasi-isometric if and only if they are isomorphic.

This conjecture remains open in its full generality, however, there are some partial results. For instance, it is true in the case of Heintze groups of Carnot type (see [32]) and for groups of the form $\mathbb{R}^n \rtimes_\alpha \mathbb{R}$ (see [38]).

An interesting example of an application of L^p-cohomolgoy to this problem is the following result by Pansu:

Theorem 1.6. [34] *Let $G_1 = \mathbb{R}^{n-1} \rtimes_{\alpha_1} \mathbb{R}$ and $G_2 = \mathbb{R}^{n-1} \rtimes_{\alpha_2} \mathbb{R}$ be two purely real Heintze groups. If G_1 and G_2 are quasi-isometric, then there exists $\lambda > 0$ such that α_1 and $\lambda\alpha_2$ have the same eigenvalues counted with multiplicity.*

A more general version of Theorem 1.6 is proved in [10] using all previously known machinery [9, 28, 32, 38]. The complete quasi-isometry characterization of purely real Heinze groups of the form $\mathbb{R}^{n-1} \rtimes_\alpha \mathbb{R}$ given by Xie in [38] also implies Theorem 1.6 .

The strategy used by Pansu for proving Theorem 1.6 is to show that $L^pH^k(G_i)$ (with $k \geq 2$ and $i = 1, 2$) is zero for p bigger than a critical exponent $p_k(G_i)$, and that it is not zero if p belongs to an interval of the form $(p_k(G_i) - \epsilon, p_k(G_i))$. The quasi-isometry between G_1 and G_2 implies $p_k(G_1) = p_k(G_2)$ for every k. The result is then obtained using the relation between critical exponents and the eigenvalues of the derivations.

The non-vanishing of the L^p-cohomology involves the explicit construction of non-zero L^p-cohomology classes, which is a difficult problem from a technical point of view. A goal of this work is to show a shortcut at this point via the computation of the relative L^p-cohomology. In particular, we prove the following result:

Theorem 1.7. *Let $G = \mathbb{R}^{n-1} \rtimes_\alpha \mathbb{R}$ be a purely real Heintze group, where α has positive eigenvalues $\lambda_1 \leq \cdots \leq \lambda_{n-1}$. For $k = 1, \ldots, n-1$ we write $w_k = \lambda_1 + \cdots + \lambda_k$. Then*

- *$L^pH^k(G, \infty) = 0$ for $k \geq 2$ and $p > \frac{\mathrm{tr}(\alpha)}{w_{k-1}}$;*
- *$L^pH^k(G, \infty) \neq 0$ for $k \geq 2$ and $\frac{\mathrm{tr}(\alpha)}{w_k} < p \leq \frac{\mathrm{tr}(\alpha)}{w_{k-1}}$; and*
- *$L^pH^1(G, \infty) \neq 0$ for $p > \frac{\mathrm{tr}(\alpha)}{w_1}$.*

If two Heintze groups G_1 and G_2 as above are quasi-isometric, then there exists a quasi-isometry that sends the point $\infty \in \partial G_1$ to $\infty \in \partial G_2$ (see Remark 5.1). Combining this fact with Theorem 1.3 and Theorem 1.7, we easily get Theorem 1.6. Indeed, if $\mathbb{R}^{n-1} \rtimes_{\alpha_1} \mathbb{R}$ and $\mathbb{R}^{n-1} \rtimes_{\alpha_2} \mathbb{R}$ are quasi-isometric, then α_1 and $\frac{tr(\alpha_1)}{tr(\alpha_2)}\alpha_2$ have the same eigenvalues.

If $G = N \rtimes_\alpha \mathbb{R}$ is a Heintze group, then it is known that $L^pH^1(G) = 0$ for every $p \leq \mathrm{tr}(\alpha)/w_1$, and $L^pH^1(G) \neq 0$ for $p > \mathrm{tr}(\alpha)/w_1$ (see for example [31, Theorem 4] or [9, Corollary 1.6]). Assuming that, the first part of Theorem 1.4 allows to conclude that $L^pH^1(G, \xi) = 0$ for every $p \leq \mathrm{tr}(\alpha)/w_1$ and any $\xi \in \partial G$, which completes Theorem 1.7 by taking $\xi = \infty$.

1.2. Notation and conventions. We denote by $|x - y|$ the distance between two points x and y belonging to any metric space. In some cases it is convenient to distinguish between the distance on a Gromov-hyperbolic space and a visual metric on its boundary. For that reason we use the notation $d(x,y)$ in Section 4.

A quasi-isometry of constants $\lambda \geq 1$ and $\epsilon \geq 0$ from a metric space to another is a map $F\colon X \to Y$ such that

(c) $\lambda^{-1}|x-x'| - \epsilon \leq |F(x)-F(x')| \leq \lambda|x-x'| + \epsilon$ for every $x, x' \in X$; and
(d) for every $y \in Y$ there exists $x \in X$ such that $|F(x)-y| \leq \epsilon$.

If $F\colon X \to Y$ is a quasi-isometry between two Gromov-hyperbolic spaces, the induced homeomorphism between their boundaries is denoted also by F. We write $\overline{F}$ to mean a quasi-inverse of F, that is, a quasi-isometry from Y to X such that $F \circ \overline{F}$ and $\overline{F} \circ F$ are at bounded uniform distance from the identity. We refer to [17] for details about quasi-isometries and Gromov-hyperbolic spaces.

By a *measurable k-form*, or simply *k-form*, on a smooth manifold M we mean a function

$$\omega\colon M \to \bigcup_{x \in M} \Lambda^k(T_xM), \quad x \mapsto \omega_x \in \Lambda^k(T_xM)$$

whose coefficients with respect to any parametrization of M are measurable functions. Here $\Lambda^k(T_xM)$ denotes the space of alternating k-linear maps on the tangent space T_xM. If its coefficients are, in addition, smooth functions, we say that ω is a *differential k-form*. The space of differential k-forms on M is denoted by $\Omega^k(M)$.

Given two real functions f and g defined in the same domain we write $f \preceq g$ if there exists a constant $C > 0$ such that $f \leq Cg$, and $f \asymp g$ if $f \preceq g$ and $g \preceq f$.

When we talk about *cochain complexes*, *cochain maps*, *homotopies* and *homotopy equivalences* between cochain complexes, we do it in a continuous sense, that is, all maps involved are continuous.

2. Quasi-isometry invariance

In this section we prove Theorem 1.1 adapting the proof of Theorem 1.1 in [7] to our context.

Observe that every $\theta \in \ell^p(X^{(k)})$ has a natural linear extension $\theta\colon C_k(X) \to \mathbb{R}$, where

$$C_k(X) = \left\{ \sum_{i=1}^m t_i\sigma_i \colon t_1, \ldots, t_m \in \mathbb{R}, \sigma_1, \ldots, \sigma_m \in X^{(k)} \right\}$$

is the space of *k-chains* on X. The *support* of $c = \sum_{i=1}^m t_i\sigma_i$ in $C_k(X)$ (with $t_i \neq 0$ for all $i = 1, \ldots, m$) is $|c| = \{\sigma_1, \ldots, \sigma_m\}$. We also define the *uniform norm* and the *length* of c by

$$\|c\|_\infty = \max\{|t_1|, \ldots, |t_m|\} \text{ and } \ell(c) = m.$$

Lemma 2.1. [7] *A quasi-isometry $F\colon X \to Y$ induces a family of linear maps $c_F\colon C_k(X) \to C_k(Y)$ at bounded uniform distance from F which verify $\partial c_F(\sigma) = c_F(\partial\sigma)$ for every $\sigma \in X^{(k)}$. Moreover, for every $k \in \mathbb{N}$ there exist constants N_k and L_k (depending only on k and the geometric data of X, Y and F) such that $\|c_F(\sigma)\|_\infty \leq N_k$ and $\ell(c_F(\sigma)) \leq L_k$ for every $\sigma \in X^{(k)}$*

Lemma 2.2. [7] *If $G\colon X \to Y$ is another quasi-isometry at bounded uniform distance from F, then there exists an homotopy $h\colon C_k(X) \to C_{k+1}(Y)$ between c_F and c_G. That is,*

$$\begin{cases} h(v) = c_F(v) - c_G(v) & \text{if } v \in X^{(0)}, \\ \partial h(\sigma) + h(\partial\sigma) = c_F(\sigma) - c_G(\sigma) & \text{if } \sigma \in X^{(k)},\ k \geq 1. \end{cases}$$

Furthermore, $\|h(\sigma)\|_\infty$ and $\ell(h(\sigma))$ are uniformly bounded by constants N'_k and L'_k that depend only on the geometric data of X, Y, F and G.

Proof of Theorem 1.1. We consider the *pull-back* of a k-cochain $\theta \in \ell^p(Y^{(k)}, F(\xi))$ as the composition $F^*\theta = \theta \circ c_F$ (it depends on the choice of c_F). Observe first that F^* is well-defined and continuous from $\ell^p(Y^{(k)}, F(\xi))$ to $\ell^p(X^{(k)})$. Indeed, Jensen's inequality allows to prove that for $\theta \in \ell^p(Y^{(k)}, F(\xi))$,

$$\|F^*\theta\|_{L^p}^p = \sum_{\sigma \in X^{(k)}} |\theta(c_F(\sigma))|^p \leq N_k^p L_k^{p-1} \sum_{\sigma \in X^{(k)}} \sum_{\tau \in |c_F(\sigma)|} |\theta(\tau)|^p.$$

Since F is a quasi-isometry and the distance between $c_F(v)$ and $F(v)$ is uniformly bounded for all $v \in X^{(0)}$, we can find a constant C_k such that if $\operatorname{dist}(\sigma_1, \sigma_2) > C_k$, then $c_F(\sigma_1) \cap c_F(\sigma_2) = \emptyset$. Using the bounded geometry of X we have that every $\tau \in Y^{(k)}$ satisfies $\tau \in |c_F(\sigma)|$ for at most $N(C + C_k)$ simplices $\sigma \in X^{(k)}$, where C and N are as in (a) and (b). This implies that

$$\|F^*\theta\|_{L^p}^p \leq N_k^p L_k^{p-1} N(C + C_k) \sum_{\tau \in Y^{(k)}} |\theta(\tau)|^p = N_k^p L_k^{p-1} N(C + C_k) \|\theta\|_{\ell^p}^p.$$

Let us prove that $F^*\theta$ is zero on some neighborhood of ξ for any fixed θ in $\ell^p(Y^{(k)}, F(\xi))$. Assume that θ is zero on V, a neighborhood of $F(\xi)$ in $\overline{Y}$. If $\sigma \in X^{(k)}$ and v is one of its vertices, then

$$\operatorname{dist}_H(c_F(\sigma), F(v)) \leq \operatorname{dist}_H(c_F(\sigma), c_F(v)) + \operatorname{dist}_H(c_F(v), F(v)), \tag{1}$$

where dist_H denotes the Hausdorff distance. By construction of c_F the distance (1) is uniformly bounded by a constant D_k. We define W as the closure in $\overline{Y}$ of the set

$$\{y \in Y\colon\ \operatorname{dist}(y, V^c \cap Y) > D_k\}.$$

Since F is a quasi-isometry, there exists $U \subset \overline{X}$ a neighborhood of ξ such that $F(U) \subset W$. For every k-simplex $\sigma \subset U$ we have $c_F(\sigma) \subset V$ and hence $F^*\theta(\sigma) = 0$. We conclude that $F^*\theta$ vanishes on U.

By definition we have $\delta F^* = F^*\delta$, which implies that F^* defines a map in cohomology, denoted by $F^\#\colon \ell^pH^k(Y, F(\xi)) \to \ell^pH^k(X, \xi)$. We have to prove that $F^\#$ is an isomorphism.

Claim: If $F, G : X \to Y$ are two quasi-isometries at bounded uniform distance, then $F^\# = G^\#$.

We need a family of continuous linear maps $H_k\colon \ell^p(Y^{(k)}, F(\xi)) \to \ell^p(X^{(k-1)}, \xi)$ such that

$$\begin{cases} F^*\theta - G^*\theta = H_1\delta\theta & \text{if } \theta \in \ell^p(Y^{(0)}, F(\xi)), \\ F^*\theta - G^*\theta = H_{k+1}\delta\theta + \delta H_k\theta & \text{if } \theta \in \ell^p(Y^{(k)}, F(\xi)),\ k \geq 1 \end{cases}$$

It is easy to verify that both conditions are satisfied by the map

$$H_k\theta\colon X^{(k)} \to \mathbb{R}, \quad H_k\theta(\sigma) = \theta(h(\sigma)),$$

where h is the map given by Lemma 2.2. Using the same argument as for F^* we can prove that $H_k\theta$ is in $\ell^p(X^{(k-1)})$ and H_k is continuous. To see that $H_k\theta$ vanishes on some neighborhood of ξ (and finish the proof of the claim) observe that $h(\sigma)$ have uniformly bounded length, which implies that $\operatorname{dist}_H(c_F(\sigma), h(\sigma))$ is uniformly bounded.

As a consequence of the claim we have that $F^\#$ does not depend on the choice of c_F. Moreover, if $T\colon Y \to Z$ is another quasi-isometry, a possibe choice of the function $c_{T\circ F}$ is the composition $c_T \circ c_F$. In this case $(T\circ F)^* = F^* \circ T^*$ and hence $(T\circ F)^\# = F^\# \circ T^\#$.

Finally, if $\overline{F}\colon Y \to X$ is a quasi-inverse of F, then by the claim $(F\circ \overline{F})^\#$ and $(\overline{F}\circ F)^\#$ are the identity in relative cohomology. Since $(F\circ\overline{F})^\# = \overline{F}^\# \circ F^\#$ and $(\overline{F}\circ F)^\# = F^\# \circ \overline{F}^\#$, the statement follows. □

3. Equivalence between simplicial and de Rham relative L^p-cohomology

Here we prove Theorem 1.2. For that end we use two lemmas that appear in [33]. We give their complete and more detailed proofs.

By a *bicomplex* we mean a family of topological vector spaces $\{C^{k,\ell}\}_{k,\ell\in\mathbb{N}}$ together with two families of continuous linear operators $d' = d'_k : C^{k,\ell} \to C^{k+1,\ell}$ and $d'' = d''_\ell : C^{k,\ell} \to C^{k,\ell+1}$ such that $d'\circ d' = 0$, $d''\circ d'' = 0$ and $d'\circ d'' + d''\circ d' = 0$.

Lemma 3.1. [33, Lemma 5] *Let $(C^{*,*}, d', d'')$ be a bicomplex. Suppose that for every $\ell\in\mathbb{N}$ the complex $(C^{*,\ell}, d')$ retracts to the subcomplex $(E^\ell := \operatorname{Ker} d'|_{C^{0,\ell}} \to 0 \to 0 \to \cdots)$. Then the complex $(D^*, \mathfrak{d})$, defined by*

$$D^m = \bigoplus_{k+\ell=m} C^{k,\ell} \quad \text{and} \quad \mathfrak{d} = d' + d'',$$

is homotopy equivalent to (E^*, d'').

Proof. For every $K\in\mathbb{N}$ let $(C^{*,*}_{[K]}, d', d'')$ be the subcomplex of $(C^{*,*}, d', d'')$ defined by

$$C^{k,\ell}_{[K]} = \begin{cases} C^{k,\ell} & \text{if } k < K, \\ \operatorname{Ker} d'|_{C^{k,\ell}} & \text{if } k = K, \\ 0 & \text{if } k > K. \end{cases}$$

For every $m\in\mathbb{N}$ let $D^m_{[K]} = \bigoplus_{k+\ell=m} C^{k,\ell}_{[K]}$. It is clear that $D^*_{[K]} \subset D^*_{[K+1]}$ for every K and $\bigcup_{K\ge 0} D^*_{[K]} = D^*$. Moreover, by definition of E^*, we have $D^*_{[0]} = E^*$. Therefore, to prove the lemma, it suffices to show that $D^*_{[K]}$ retracts to $D^*_{[K-1]}$ for every $K\ge 1$.

In order to simplify the notation we set

$$\mathcal{C}_0 = \bigoplus_{\ell\ge 0} C^{0,\ell}, \quad \mathcal{C}_1 = \bigoplus_{k\ge 1,\ell\ge 0} C^{k,\ell}, \quad \text{and} \quad \mathcal{E} = \bigoplus_{\ell\ge 0} E^\ell.$$

We also write $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_1$. By assumption, for every $\ell\in\mathbb{N}$, the complex $(C^{*,\ell}, d')$ retracts to the subcomplex $(E^\ell \to 0 \to 0 \to \cdots)$. Thus there exist continuous operators $h\colon \mathcal{C}_1 \to \mathcal{C}$ and $\varphi\colon \mathcal{C}_0 \to \mathcal{E}$ satisfying $h(C^{k,\ell}) \subset C^{k-1,\ell}$ and $\varphi(C^{0,\ell}) \subset E^\ell$ such that

(e) $h\circ d' = \operatorname{Id} - i\circ\varphi$ on $\mathcal{C}_0$; and
(f) $d'\circ h + h\circ d' = \operatorname{Id}$ on $\mathcal{C}_1$,

where $i\colon \mathcal{E} \to \mathcal{C}_0$ is the inclusion. We extend h to the whole space $\mathcal{C}$ by letting $h = 0$ on $\mathcal{C}_0$.

Define $b\colon \mathcal{C}\to\mathcal{C}$ by

$$b = \begin{cases} -(d''\circ h + h\circ d'') & \text{on } \mathcal{C}_1, \\ i\circ\varphi & \text{on } \mathcal{C}_0. \end{cases}$$

By (e) and (f) we have the equality $\mathfrak{d}\circ h + h\circ\mathfrak{d} = \operatorname{Id} - b$ on $\mathcal{C}$. This implies in particular that b commutes with $\mathfrak{d}$.

We are now ready to show that $D^*_{[K]}$ retracts to $D^*_{[K-1]}$ for every $K \geq 1$. Observe that h sends $D^m_{[K]}$ to $D^{m-1}_{[K]}$, then we can consider $h_{[K]}\colon D^m_{[K]} \to D^{m-1}_{[K]}$ the induced operator. The map b satisfies $b(C^{k,\ell}) \subset C^{k-1,\ell+1}$ for $k \geq 1$ and $b(C^{0,\ell}) \subset C^{0,\ell}$. Moreover, for $K \geq 1$, one has

$$b(\mathrm{Ker} d'|_{C^{K,\ell}}) \subset \mathrm{Ker} d'|_{C^{K-1,\ell+1}}.$$

Indeed, if $d'\omega = 0$, then one also has $d'd''\omega = 0$. The definition of b and the relation (f) gives:

$$d'b\omega = -(d'd''h\omega + d'hd''\omega) = d''d'h\omega - d'hd''\omega = -d''\omega + d''\omega = 0.$$

Therefore b sends every $D^m_{[K]}$ to $D^m_{[K-1]}$ for $K \geq 1$. Let $b_{[K]}\colon D^*_{[K]} \to D^*_{[K-1]}$ be the induced operator. As we saw above, it commutes with $\mathfrak{d}$. Since $\mathfrak{d} \circ h + h \circ \mathfrak{d} = \mathrm{Id} - b$ on $\mathcal{C}$, we get

$$\mathfrak{d} \circ h_{[K]} + h_{[K]} \circ \mathfrak{d} = \mathrm{Id} - i_{[K-1]} \circ b_{[K]}$$

and also

$$\mathfrak{d} \circ h_{[K-1]} + h_{[K-1]} \circ \mathfrak{d} = \mathrm{Id} - b_{[K]} \circ i_{[K-1]},$$

where $i_{[K-1]}\colon D^*_{[K-1]} \to D^*_{[K]}$ is the inclusion. All these maps are continuous, then the lemma follows. $\square$

Lemma 3.2. [33, Lemma 8] *Let B be the unit ball in $\mathbb{R}^n$, then $(L^pC^*(B), d)$ retracts to the complex $(\mathbb{R} \to 0 \to 0 \to \cdots)$.*

In the proof of Lemma 3.2 we use the following version of the Leibniz Integral Rule, whose proof follows directly from the classic version or applying the Dominated Convergence Theorem.

Lemma 3.3. *Consider a measure space (Z, μ) and a family of differential k-forms $\{\omega(z)\}_{z\in Z}$ defined on an open set $U \subset \mathbb{R}^n$. Supose that every coefficient of $\omega(z)_x$ is integrable in the variable z and each of its iterated partial derivatives of any order is dominated by a function in $L^1(Z)$. Then the k-form ω defined by*

$$\omega_x(u_1, \ldots, u_k) = \left(\int_Z \omega(z)_x \, d\mu(z)\right)(u_1, \ldots, u_k) = \int_Z \omega(z)_x(u_1, \ldots, u_k)\, d\mu(z)$$

belongs to $\Omega^k(M)$ and its derivative is $d\omega = \int_Z d\omega(z)\, d\mu(z)$.

Proof of Lemma 3.2. For a fixed $x \in B$ we consider $\varphi_x\colon [0,1] \times B \to B$, $\varphi_x(t, y) = ty + (1-t)x$ and $\eta_t\colon B \to [0,1] \times B$, $\eta_t(y) = (t, y)$. Denote $\frac{\partial}{\partial t} = (1, 0) \in [0,1] \times B$ and define the $\frac{\partial}{\partial t}$-contraction of a k-form ω (with $k \geq 1$) as the $(k-1)$-form

$$(\iota_{\frac{\partial}{\partial t}}\omega)_y(u_1, \ldots, u_{k-1}) = \omega_y\left(\tfrac{\partial}{\partial t}, u_1, \ldots, u_{k-1}\right).$$

Now for $y \in B$, $\omega \in L^\phi\Omega^k(B)$ and $u_1, \ldots, u_{k-1}$ we define

$$\chi_x(\omega) = \int_0^1 \eta_s^*\left(\iota_{\frac{\partial}{\partial t}}\varphi_x^*\omega\right) ds.$$

Observe that the coefficients of $\eta_s^*(\iota_{\frac{\partial}{\partial s}}\varphi_x^*\omega)_y$ are smooth on t, x and y, thus we can apply Lemma 3.3 to prove that $\chi_x(\omega)$ is a differential $(k-1)$-form.

For every $(k-1)$-simplex σ we have

$$\int_\sigma \chi_x(\omega) = \int_\sigma \int_0^1 \eta_s^*\left(\iota_{\frac{\partial}{\partial t}}\varphi_x^*\omega\right) ds = \int_{[0,1]\times\sigma} \varphi_x^*\omega = \int_{\varphi([0,1]\times\sigma)} \omega = \int_{C_\sigma} \omega,$$

where the cone C_σ is defined as follows: If $\sigma = (x_0, \dots, x_{k-1})$, then $C_\sigma = (x, x_0, \dots, x_{k-1})$.

Suppose that σ is a k-simplex in B with $\partial\sigma = \tau_0 + \dots + \tau_k$ and $\omega \in \Omega^k(B)$. Using Stoke's theorem we have

$$\begin{aligned}\int_\sigma \chi_x(d\omega) &= \int_{C_\sigma} d\omega = \int_{\partial C_\sigma} \omega = \int_\sigma \omega - \sum_{i=0}^{k} \int_{C_{\tau_i}} \omega \\ &= \int_\sigma \omega - \int_{\partial\sigma} \chi_x(\omega) = \int_\sigma \omega - \int_\sigma d\chi_x(\omega).\end{aligned}$$

Since this holds for every k-simplex, we conclude that

$$\chi_x d + d\chi_x = \mathrm{Id} \tag{2}$$

(see for example [37, Chapter IV]). Observe that if ω is closed, then $\chi_x(\omega)$ is a primitive of ω, thus this proves the classic PoincarÃľ's lemma. However, in our case we need a primitive in L^p, so we will take a convenient average.

Define

$$h(\omega) = \frac{1}{\mathrm{Vol}\left(\frac{1}{2}B\right)} \int_{\frac{1}{2}B} \chi_x(\omega)\, dx,$$

where $\frac{1}{2}B = B\left(0, \frac{1}{2}\right)$.

Since the coefficients of $(x,y) \mapsto \chi_x(\omega)_y$ are smooth in both variables we can use again Lemma 3.3 to see that $h(\omega)$ is in $\Omega^{k-1}(B)$. Notice that this works because we take the integral on a ball with closure included in B. Moreover, the derivative of h is

$$dh(\omega) = \frac{1}{\mathrm{Vol}\left(\frac{1}{2}B\right)} \int_{\frac{1}{2}B} d\chi_x(\omega)\, dx.$$

Using (2) we have

$$dh(\omega) + h(d\omega) = \omega \tag{3}$$

for every $\omega \in L^p\Omega^k(B)$ with $k \geq 1$.

We want to prove that h is well-defined and continuous from $L^p\Omega^k(B)$ to $L^p\Omega^{k-1}(B)$. To this end we first bound $|\chi_x(\omega)|_y$ for $y \in B$ and $\omega \in \Omega^k(B)$. Since $\iota_{\frac{\partial}{\partial t}}\varphi^*\omega$ is a $(k-1)$-form on $[0,1] \times B$ that is zero in the direction of $\frac{\partial}{\partial t}$, we have $\left|\eta_s^*(\iota_{\frac{\partial}{\partial t}}\varphi^*\omega)\right|_y = \left|\iota_{\frac{\partial}{\partial t}}\varphi^*\omega\right|_{(t,y)}$ for every $t \in (0,1)$ and $y \in B$. A direct calculation gives us the estimate

$$\left|\iota_{\frac{\partial}{\partial t}}\varphi^*\omega\right|_{(t,y)} \leq t^{k-1}|y-x||\omega|_{\varphi(t,y)}.$$

Therefore, using the assumption that $t \in (0,1)$, we can write

$$|\chi(\omega)|_y \leq \int_0^1 |y-x||\omega|_{\varphi(t,y)}\, dt. \tag{4}$$

Consider the function $u\colon \mathbb{R}^n \to \mathbb{R}$ defined by $u(z) = |\omega|_z$ if $z \in B$ and $u(z) = 0$ in the other case. Using (4) and the change of variables $z = ty + (1-t)x$, we have

$$\begin{aligned}\mathrm{Vol}\left(\tfrac{1}{2}B\right)|h(\omega)|_y &\leq \int_{B\left(ty, \frac{1-t}{2}\right)} \int_0^1 |z-y|u(z)(1-t)^{-n-1}\, dt\, dz \\ &= \int_{B(y,2)} |z-y|u(z) \left(\int_0^1 \mathbb{1}_{B\left(ty,\frac{1-t}{2}\right)}(z)(1-t)^{-n-1}\, dt\right) dz.\end{aligned}$$

Observe that $\mathbb{1}_{B\left(ty,\frac{1-t}{2}\right)}(z) = 1$ implies that $|z-y| \le 2(1-t)$. Thus,

$$\int_0^1 \mathbb{1}_{B\left(ty,\frac{1-t}{2}\right)}(z)(1-t)^{-n-1}\,dt \le \int_0^{1-\frac{1}{2}|z-y|}(1-t)^{-n-1}\,dt = \int_{\frac{1}{2}|z-y|}^1 r^{-n-1}\,dr \preceq \frac{1}{|z-y|^n}.$$

This implies

$$Vol(\tfrac{1}{2}B)|h(\omega)|_y \preceq \int_{B(y,2)} |z-y|^{1-n}u(z)\,dz.$$

Using that $\int_{B(y,2)} |z-y|^{1-n}dz$ is finite and Jensen's inequality we obtain

$$|h(\omega)|_y^p \preceq \int_{B(y,2)} |z-y|^{1-n}u(z)^p\,dz.$$

Therefore,

$$\begin{aligned}\|h(\omega)\|_{L^p}^p &\preceq \int_B\int_{B(y,2)} |z-y|^{1-n}u(z)^p\,dz\,dy \quad \preceq \int_{B(0,3)} u(z)^p\left(\int_B \frac{dy}{|z-y|^{n-1}}\right)dz\\ &\preceq \|\omega\|_{L^p}^p.\end{aligned}$$

By the identity $dh(\omega) = \omega - h(d\omega)$ we also have

$$\|dh(\omega)\|_{L^p} \le \|\omega\|_{L^p} + \|h(d\omega)\|_{L^p} \preceq \|\omega\|_{L^p} + \|d\omega\|_{L^p} = |\omega|_{L^p}.$$

We conclude that h is well-defined and bounded for $k \ge 1$.

If $\omega = df$ for certain function f we observe that

$$\eta_s^*\left(\iota_{\frac{\partial}{\partial t}}\varphi_x^* df\right)(y) = df_{\varphi_x(s,y)}(y-x) = (f\circ\gamma)'(s),$$

where γ is the curve $\gamma(s) = \varphi_x(s,y)$. Then $\chi_x(df)(y) = f(y) - f(x)$, from which we get

$$h(df) = f - \frac{1}{\mathrm{Vol}\left(\frac{1}{2}B\right)}\int_{\frac{1}{2}B} f.$$

We define $h\colon L^p\Omega^0(B) \to L^p\Omega^{-1}(B) = \mathbb{R}$ by

$$h(f) = \frac{1}{\mathrm{Vol}\left(\frac{1}{2}B\right)}\int_{\frac{1}{2}B} f,$$

which is clearly continuous because $\frac{1}{2}B$ has finite Lebesgue measure. Then we have the identity

$$h(f) + h(df) = f. \tag{5}$$

Note that, since h is bounded, then it can be extended continuously to $L^pC^k(B)$ for every $k \ge 0$. The identities (3) and (5) are also true for every $\omega \in L^pC^k(B)$, which finishes the proof. □

Proof of Theorem 1.2. Our strategy is to construct a convenient bicomplex and apply Lemma 3.1. To that end consider $\mathcal{U}$ the covering consisting of the open stars of the vertices of X_M. For an ℓ-simplex $\Delta = (v_0,\ldots,v_\ell)$ we write $U_\Delta = U(v_0)\cap\cdots\cap U(v_\ell)$. Then for $k,\ell \ge 1$ we take the space $C_\xi^{k,\ell}$ consisting of functions of the form

$$\omega = \{\omega_\Delta\}_{\Delta\in X_M^{(\ell)}} \in \prod_{\Delta\in X_M^{(\ell)}} L^pC^k(U_\Delta)$$

such that

(g) If $i,j = 0,\ldots,k$, $i\ne j$, then $\omega_{(v_0,\ldots,v_i,\ldots,v_j,\ldots,v_\ell)} = -\omega_{(v_0,\ldots,v_j,\ldots,v_i,\ldots,v_\ell)}$,

(h) $\sum_{\Delta\in X_M^{(\ell)}}(\|\omega_\Delta\|_{L^p}^p+\|d\omega_\Delta\|_{L^p}^p)<+\infty$, and

(i) There exists a neighborhood V of ξ in $\overline{M}$ such that if $U_\Delta\subset V$, then $\omega_\Delta=0$ almost everywhere.

We equip $C_\xi^{k,\ell}$ with the norm

$$\|\omega\|_{C^p}=\left(\sum_{\Delta\in X_M^{(\ell)}}\|\omega_\Delta\|_{L^p}^p\right)^{\frac{1}{p}}+\left(\sum_{\Delta\in X_M^{(\ell)}}\|d\omega_\Delta\|_{L^p}^p\right)^{\frac{1}{p}}.$$

Then we define the derivatives $d'\colon C_\xi^{k,\ell}\to C_\xi^{k+1,\ell}$ and $d''\colon C_\xi^{k,\ell}\to C_\xi^{k,\ell+1}$:

- If $\omega\in C_\xi^{k,\ell}$, then $(d'\omega)_\Delta=(-1)^\ell d\omega_\Delta$.
- If $\omega\in C_\xi^{k,\ell}$ and $\Delta=(v_0,\dots,v_{\ell+1})$, then

$$(d''\omega)_\Delta=\sum_{i=0}^{\ell+1}(-1)^i\omega_{\partial_i\Delta}|_{U_\Delta},$$

where $\partial_i\Delta=(v_0,\dots,\hat{v}_i,\dots,v_{\ell+1})$.

It is easy to show that d' and d'' are well-defined and continuous and satisfy $d'\circ d'=0$, $d'\circ d'=0$ and $d'\circ d''+d''\circ d''=0$, then $(C_\xi^{*,*},d',d'')$ is a bicomplex.

Observe that the elements of $\operatorname{Ker} d'|_{C_\xi^{0,\ell}}$ are the functions $g\in C_\xi^{0,\ell}$ satisfying that g_Δ is essentially constant in U_Δ for every $\Delta\in X_M^{(\ell)}$. Using the fact that every U_Δ is biLipschitz diffeomorphic (with uniform Lipschitz constant) to the unit ball in $\mathbb{R}^n$ we conclude that $\operatorname{Ker} d'|_{C_\xi^{0,\ell}}$ is isomporhic to $\ell^pC^\ell(X_M,\xi)$ and d'' coincides with the derivative on this space.

On the other hand, the elements of $\operatorname{Ker} d''|_{C_\xi^{k,0}}$ are of the form $\omega=\{\omega_v\}_{v\in X_M^{(0)}}$ with

$$\omega_{v_1}|_{U(v_1)\cap U(v_2)}=\omega_{v_2}|_{U(v_1)\cap U(v_2)}$$

almost everywhere. We can take a k-form $\tilde\omega$ in $L^pC^k(M)$ such that $\tilde\omega|_{U(v)}=\omega_v$ almost everywhere for every $v\in X_M^{(0)}$. This k-form is zero on some neighborhood of ξ, then there is an isomorphism between $\operatorname{Ker} d''|_{C_\xi^{k,0}}$ and $L^pC^k(M,\xi)$ for which d' coincides with the derivative on the second space. It is clear that $\|\omega\|_{C_p}\asymp|\tilde\omega|_{L^p}$ because every point of M is in at most $n=\dim(M)$ elements of $\mathcal{U}$.

Claim 1: For a fixed $\ell\in\mathbb{N}$ the complex $(C_\xi^{*,\ell},d')$ retracts to $(\operatorname{Ker} d'|_{C_\xi^{0,\ell}}\to 0\to 0\to\cdots)$.

We take the family of maps $h\colon L^pC^k(B)\to L^pC^{k-1}(B)$ given by Lemma 3.2 (where $L^pC^{-1}(B)=\mathbb{R}$), and for every $\Delta\in X_M^{(\ell)}$ we consider a L-biLipschitz diffeomorphism $f_\Delta\colon U_\Delta\to B$ (which L does not depend on Δ). Then we define $H\colon C_\xi^{k,\ell}\to C_\xi^{k-1,\ell}$ by

$$(H\omega)_\Delta=(-1)^\ell f_U^*h(f_\Delta^{-1})^*\omega_\Delta,$$

where $f_\Delta^*\colon L^pC^*(B)\to L^pC^*(U_\Delta)$ is the continuous extension of the usual pull-back of differential forms. We write $C_\xi^{-1,\ell}:=\operatorname{Ker} d'|_{C_\xi^{0,\ell}}$.

It is easy to prove that f_Δ^* and $(f_\Delta^{-1})^*$ are continuous for every U by using the Lipschitz condition; hence, using also the definition of h, we have that H is the continuous retraction we wanted.

Claim 2: For a fixed $k \in \mathbb{N}$ the complex $(C_\xi^{k,*}, d'')$ retracts to $(\operatorname{Ker} d''|_{C_\xi^{k,0}} \to 0 \to 0 \to \cdots)$.

We have to construct a family of bounded linear maps $\kappa\colon C_\xi^{k,\ell} \to C_\xi^{k,\ell-1}$ ($\ell \geq 0$), where $C_\xi^{k,-1} = \operatorname{Ker} d''|_{C_\xi^{k,0}}$, such that

$$\begin{cases} \kappa d''\omega + d''\kappa\omega = \omega & \text{for every } \omega \in C_\xi^{k,\ell},\ \ell \geq 1, \\ \kappa d'' g + \kappa g = g & \text{for every } g \in C_\xi^{k,0}. \end{cases}$$

Let $\{\eta_v\}_{v \in X_M^{(0)}}$ be a partition of unity with respect to $\mathcal{U}$. Observe that we can take it so that $|d\eta_v|_x$ is uniformly bounded independently from v and x. If $\ell \geq 1$ and $\omega \in C_\xi^{k,\ell}$, then for $\Delta \in X_M^{(\ell-1)}$ we define

$$(\kappa\omega)_\Delta = (-1)^\ell \sum_{v \in X_M^{(0)}} \eta_v \omega_{\Delta v}.$$

Where if $\Delta = (v_0, \ldots, v_{\ell+1})$, then $\Delta v = (v_0, \ldots, v_{\ell+1}, v)$. Observe that if v is a vertex of Δ, then $\omega_{\Delta v} = 0$ due to condition (g). For $\omega \in C_\xi^{k,0}$ and $v \in X_M^{(0)}$ we put

$$(\kappa\omega)_{v_0} = \sum_{v \in X_M^{(0)}} \eta_v \omega_v|_{U_{v_0}}.$$

A direct calculation shows that κ is as we wanted. Finally, applying Lemma 3.1 we obtain that $(D^*, \mathfrak{d})$ is homotopy equivalent to $(\operatorname{Ker} d'|_{C_\xi^{0,*}}, d'')$ and $(\operatorname{Ker} d''|_{C_\xi^{*,0}}, d')$. The proof ends using the above identifications. $\square$

Remark 3.4. Observe that the previous argument can be done considering differential forms, so one can show that the cochain complexes $(L^p\Omega^*(M), d)$, $(L^pC^*(M), d)$ and $(\ell^p(X_M^{(*)}), \delta)$ are all homotopy equivalents. This implies that the relative de Rham L^p-cohomology can be described by using differential forms.

4. The case of degree one and Besov spaces

There exists a direct relation between $\ell^pH^1(X)$ and $\ell^pH^1(X,\xi)$ for a Gromov-hyperbolic simplicial complex X with bounded geometry and $\xi \in \partial X$. Indeed, if θ is a closed 1-cochain in $\ell^p(X^{(1)}, \xi)$, then it is zero in $\ell^pH^1(X,\xi)$ if and only if it is zero in $\ell^pH^1(X)$. This is because if f is a function in $\ell^p(X^{(0)})$ such that $\delta f = \theta$, then f must be constant (and hence zero) in a neighborhood of ξ. Thus, there is a canonical injection $\ell^pH^1(X,\xi) \hookrightarrow \ell^pH^1(X)$. Combining this fact with Theorem 1.2 and [21, Theorem 3] we obtain the first part of Theorem 1.4.

By [7], if X is a Gromov-hyperbolic simplicial complex with bounded geometry such that there exists an Ahlfors regular metric d on the conformal gauge of the visual boundary ∂X, then for every $k \in \mathbb{N}$ and $p \geq 1$ the ℓ^p-cohomology space $\ell^pH^1(X)$ is isomorphic to the Besov space

$$B_p(\partial X) = \{u\colon \partial X\colon \|u\|_{B_p} < +\infty\}/\mathbb{R},$$

where

$$\|u\|_{B_p} = \left(\int_{Z \times Z} \frac{|u(\xi) - u(\eta)|^p}{d(\xi,\eta)^{2Q}}\, d\mathcal{H}(\xi)\, d\mathcal{H}(\eta)\right)^{1/p},$$

and $\mathbb{R}$ indicates the spaces of almost everywhere constant functions. Here Q is the Hausdorff dimension of $(\partial X, d)$ and $\mathcal{H}$ is the corresponding Hausdorff measure.

Remember that $(\partial X, d)$ is *Ahlfors regular* of dimension Q if there exists a constant K such that for every $\xi \in \partial X$ and $r > 0$,

$$K^{-1}r^Q \leq \mathcal{H}(B(\xi, r)) \leq Kr^Q;$$

and a metric d is in the *conformal gauge* of the visual boundary ∂X if it is cuasi-symmetric equivalent to any visual metric. See [26, Chapter 15] for more details.

Combining this identification with the previous observation we can identify the space $\ell^p H^1(X, \xi)$ (where $\xi \in \partial X$) with

$$B_p(\partial X, \xi) = \{u \colon \partial X \to \mathbb{R} \colon \|u\|_{B_p} < +\infty \text{ and } u \equiv \text{cte on a neighborhood of } \xi\}/\mathbb{R}.$$

Sometimes it is convenient to consider the Besov algebra

$$A_p(\partial X) = \{u \colon \partial X \to \mathbb{R} \colon u \text{ is continuous and } \|u\|_{B_p} < +\infty\},$$

which is a Banach algebra with the norm $\|\cdot\| = \|\cdot\|_\infty + \|\cdot\|_{B_p}$, and its maximal ideals are

$$\mathcal{I}_\xi = \{u \in A_p(\partial X) \colon u(\xi) = 0\}.$$

As before, we can consider the *relative Besov algebra*

$$A_p(\partial X, \xi) = \{u \in A_p(\partial X) \colon u \equiv 0 \text{ on a neighborhood of } \xi\} \subset \mathcal{I}_\xi.$$

All these definitions can be done in a more general context. In the next theorem we consider (relative) Besov spaces and algebras for general compact Ahlfors regular metric spaces.

Theorem 4.1. *Let (Z, d) be a compact Ahlfors regular metric space of dimension $Q > 0$ and $p > Q$. Suppose that $z_0 \in Z$ satisfies that there exists $R_0 > 0$ such that for every $R \in (0, R_0]$ the sets $\overline{B(z_0, R)}$ and $B(z_0, R)^c$ are continua. Then*

(i) *$B_p(Z, z_0)$ is dense in $B_p(Z)$,*
(ii) *$A_p(Z, z_0)$ is dense in $\mathcal{I}_{z_0}$.*

If X is a Gromov-hyperbolic simplicial complex with bounded geometry such that the conformal gauge of ∂X has a metric satisfying the conditions of Theorem 4.1, then $\ell^p H^1(X, \xi)$ is dense in $\ell^p H^1(X)$ for every $\xi \in \partial X$ and $p > Q$. Using again Theorem 1.2 and [21, Theorem 3] this implies the second part of Theorem 1.4.

Let C and D be two disjoint non-degenerated continua in a metric space (Z, d). Their *relative distance* is

$$\Delta(C, D) = \frac{\operatorname{dist}(C, D)}{\min\{\operatorname{diam}(C), \operatorname{diam}(D)\}},$$

and their *Besov capacity* (as defined in [2]) is

$$\Omega_p(C, D) = \inf\left\{\|u\|_{B_p}^p \colon u \in A_p(Z),\ u|_C \leq 0 \text{ and } u|_D \geq 1\right\}. \tag{6}$$

Observe that if $u \in A_p(Z)$ is a function such that $u|_C \leq 0$ and $u|_D \geq 1$, then we can take $\tilde{u}$ defined by $\tilde{u}(z) = 0$ if $u(z) \leq 0$, $\tilde{u}(z) = 1$ if $u(z) \geq 1$ and $\tilde{u}(z) = u(z)$ otherwise. Since $|\tilde{u}(x) - \tilde{u}(y)| \leq |u(x) - u(y)|$ for every $x, y \in Z$, we have $\|\tilde{u}\|_{B_p} \leq \|u\|_{B_p}$. Hence, in (6) we can take the infimum among functions taking values in $[0, 1]$.

The following theorem will be key to prove Theorem 4.1:

Theorem 4.2. [2] *Let (Z, d) be a compact Ahlfors regular metric space of dimension Q and $p > Q$. Then there exist two decreasing homeomorphisms $\varphi, \psi \colon (0, +\infty) \to (0, +\infty)$ such that for two non-degenerated disjoint continua $C, D \subset Z$,*

$$\varphi(\Delta(C, D)) \leq \Omega_p(C, D) \leq \psi(\Delta(C, D)).$$

If $[u] \in B_p(Z)$, the function

$$F_u(x,y) = \frac{|u(x)-u(y)|^p}{d(x,y)^{2Q}}$$

belongs to $L^1(\mathcal{H}\times\mathcal{H})$, thus we can consider on $Z \times Z$ a measure given by $d\nu_u = F_u \, d\mathcal{H}\, d\mathcal{H}$. We will use this measure in the following lemma and in the proof of Theorem 4.1.

Lemma 4.3. *Let Z be as in Theorem 4.1, thus the set*

$$B_p^\infty(Z) = \{[u] \in B_p(Z) \colon u \in L^\infty(Z)\}$$

is dense in $B_p(Z)$.

Proof. Let $[u] \in B_p(Z)$. For every $n \in \mathbb{N}$ we consider

$$X_n = \{z \in Z \colon |u(z)| > n\}.$$

The sequence $\{X_n\}$ is decreasing, $\mathcal{H}(X_n) < +\infty$ and $\bigcap_n X_n = \emptyset$. We define

$$u_n(z) = \begin{cases} u(z) & \text{if } z \notin X_n, \\ \frac{u(z)n}{|u(z)|} & \text{if } z \in X_n, \end{cases}$$

and $v_n = u - u_n$.

Observe that $u_n \in L^\infty(Z)$, $|v_n(x) - v_n(y)| \leq |u(x)-u(y)|$ for every $x, y \in Z$ and $|v_n(x) - v_n(y)| = 0$ if $x, y \notin X_n$. Therefore, decomposing

$$Z \times Z = (Z \times X_n) \cup (X_n \times Z) \cup (X_n^c \times X_n^c)$$

and using symmetry we have

$$\begin{aligned} \|v_n\|_{B_p}^p &\leq 2 \int_{Z\times X_n} \frac{|v_n(x)-v_n(y)|^p}{d(x,y)^{2Q}}\, d\mathcal{H}(x)\, d\mathcal{H}(y) \\ &\leq 2 \int_{Z\times X_n} \frac{|u(x)-u(y)|^p}{d(x,y)^{2Q}}\, d\mathcal{H}(x)\, d\mathcal{H}(y) = 2\nu_u(Z \times X_n) \to 0. \qquad \square \end{aligned}$$

Proof of Theorem 4.1. By Lemma 4.3, to prove (i) we need to show that every element $[u] \in B_p^\infty(Z)$ can be approximated by elements of $B_p(Z, z_0)$.

We assume that R_0 also satisfies

$$\operatorname{diam}(B(z_0,R_0)^c) \geq \operatorname{diam}(B(z_0,R_0)) \geq R_0.$$

For every $R \in (0, R_0]$ we denote $B_R = B(z_0, R)$, then for every $r \in (0, R)$,

$$\Delta(B_R^c, \overline{B_r}) \leq \frac{R}{r}.$$

Using Theorem 4.2 we have

$$\Omega_p(B_R^c, \overline{B_r}) \leq \psi\left(\frac{R}{r}\right) \to 0, \quad \text{when } r \to 0.$$

Hence, for every $R \in (0, R_0]$ we can take $v_R \in A_p(Z, z_0)$ such that $\|v_R\|_{B_p} \leq R$, $v_R(z) \in [0,1]$ for every $z \in Z$ and $v_R(z) = 1$ for every $z \notin B_R$. Then we consider

$u_R(z) = u(z)v_R(z)$. Let us prove that $\|u - u_R\|_{B_p} \to 0$ when $R \to 0$.

$$\begin{aligned}
\|u - u_R\|_{B_p}^p &\leq 2 \int_{Z\times B_R} \frac{|(1-v_R(x))u(x) - (1-v_R(y))u(y)|^p}{d(x,y)^{2Q}}\, d\mathcal{H}(x)\, d\mathcal{H}(y) \\
&\preceq \int_{Z\times B_R} |u(x)|^p \frac{|(1-v_R(x)) - (1-v_R(y))|^p}{d(x,y)^{2Q}}\, d\mathcal{H}(x)\, d\mathcal{H}(y) \\
&\quad + \int_{Z\times B_R} |1-v_R(y)|^p \frac{|u(x)-u(y)|^p}{d(x,y)^{2Q}}\, d\mathcal{H}(x)\, d\mathcal{H}(y) \\
&\leq \|u\|_\infty^p \|v_R\|_{B_p}^p + \|1-v_R\|_\infty^p \nu_u(Z\times B_R).
\end{aligned}$$

In the second line we added and subtracted $(1 - v_R(y))u(x)$ and then we applied Jensen's inequality. Since u and $(1-v_R)$ are bounded and $\nu_u(Z\times\{z_0\}) = 0$ (because Z is Ahlfors regular), the expression in the last line converges to 0 when $R \to 0$, which proves (i).

If $u \in \mathcal{I}_{z_0}$, then $u_R \in A_p(Z, z_0)$ for every $R \in (0, R_0]$. The previous calculation shows that $\|u - u_R\|_{B_p} \to 0$ when $R \to 0$. In addition we have

$$\|u - u_R\|_\infty \leq \sup\{|u(z)| \colon z \in B_R\},$$

which converges to 0 when $R \to 0$ because u is continuous and $u(z_0) = 0$. This proves (ii). □

5. Vanishing and non-vanishing of relative L^p-cohomology of Heintze groups

5.1. A duality idea. In [20] and [24] the following fact is proved: If M is a complete and orientable n-dimensional Riemannian manifold, then for every $p \geq 1$ and $k = 0, \dots, n$, the dual space of $L^p\overline{H}^k(M)$ is isometric to $L^q\overline{H}^{n-k}(M)$, where $\frac{1}{p} + \frac{1}{q} = 1$. The isometry is induced by the pairing $\langle\cdot,\cdot\rangle \colon L^p(M,\Lambda^k) \times L^q(M,\Lambda^{n-k}) \to \mathbb{R}$,

$$\langle \omega, \beta\rangle = \int_M \omega \wedge \beta, \tag{7}$$

which is well-defined by Hölder's inequality. The proof uses that $L^p(M,\Lambda^k)$ and $L^q(M,\Lambda^{n-k})$ are Banach spaces. Other duality-type arguments using weaker hypothesis can be read in [22, 23, 34].

In the relative case we have to find a natural pairing for $L^p\Omega^k(M,\xi)$ (or $L^pC^k(M,\xi)$). The answer seems to be related to the idea of local cohomology, which can be found in [9]. Let us see the following definition: Consider M a complete and orientable Gromov-hyperbolic Riemannian manifold and ξ a point in ∂M. A differential m-form β on M is *locally L^q-integrable with respect to* ξ if for every $V \subset \overline{M}$ closed neighborhood of ξ, we have

$$\|\beta\|_{L^q, M\setminus V} = \left(\int_{M\setminus V} |\beta|_x^q\, dx\right)^{\frac{1}{q}} < +\infty.$$

We denote by $L^q_{\mathrm{loc}}\Omega^m(M,\xi)$ the space of all differential m-forms which are locally L^q-integrable with respect to $\xi \in \partial M$. Observe that Hölder's inequality implies that the bi-linear pairing

$$\langle\cdot,\cdot\rangle \colon L^p\Omega^k(M,\xi) \times L^q_{\mathrm{loc}}\Omega^{n-k}(M,\xi) \to \mathbb{R} \tag{8}$$

is well-defined by the expression (7) if $\frac{1}{p}+\frac{1}{q}=1$. This allows to consider the induced linear transformations $\mu_\omega\colon L^q_{\mathrm{loc}}\Omega^{n-k}(M,\xi)\to\mathbb{R}$, $\mu_\omega=\langle\omega,\cdot\rangle$ and $\nu_\beta\colon L^p\Omega^k(M,\xi)\to\mathbb{R}$, $\nu_\beta=\langle\cdot,\beta\rangle$. We will use these maps to construct non-zero classes in the relative L^p-cohomology of Heintze groups.

5.2. Proof of Theorem 1.7. Let $G=\mathbb{R}^{n-1}\rtimes_\alpha\mathbb{R}$ be a purely real Heintze group, where α has positive eigenvalues $\lambda_1\le\cdots\le\lambda_{n-1}$. Remember the notation $w_k=w_k(\alpha)=\lambda_1+\cdots+\lambda_k$ for $k=1,\dots,n-1$; we also write $w_0=0$. The product on G is given by

$$(x,t)\cdot(y,s)=(x+e^{t\alpha}y,t+s).$$

We denote by $L_{(x,t)}$ the left translation by (x,t) on G.

If $\langle\ ,\ \rangle_0$ is an inner product on T_0G such that the factors $\mathbb{R}^{n-1}$ and $\mathbb{R}$ are orthogonal, then it determines a unique left-invariant Riemannian metric on G given by

$$\begin{aligned}\langle(v_1,v_2),(w_1,w_2)\rangle_{(x,t)}&=\left\langle(d_0L_{(x,t)})^{-1}(v_1,v_2),(d_0L_{(x,t)})^{-1}(w_1,w_2)\right\rangle_0\\&=\langle e^{-t\alpha}v_1,e^{-t\alpha}w_1\rangle_0+\lambda v_2w_2,\end{aligned}$$

for every $v_1,w_1\in\mathbb{R}^{n-1}$, $v_2,w_2\in\mathbb{R}$ and λ a fixed positive real number (any other left-invariant Riemannian metric is biLipsichitz equivalent to this one). In particular, if v is a *horizontal vector* in $T_{(x,t)}G$ (i.e. $v=(v_1,0)$), then the norm associated to $\langle\cdot,\cdot\rangle_{(x,t)}$ of v is

$$\|v\|_{(x,t)}=\|e^{-t\alpha}v\|_0.$$

Remark 5.1. The group $\mathbb{R}^{n-1}$ acts by isometries on G, thus the group $QI(G)$ of self quasi-isometries of G acts transitively on $\partial G\setminus\{\infty\}$. This implies that the action of $QI(G)$ satisfy either

- $QI(G)$ acts transitively on ∂G, or
- ∞ is fixed by $QI(G)$.

If two Heintze groups as above are quasi-isometric, then they must satisfy simultaneously either the first or the second condition. In both cases we can observe that there exists a quasi-isometry between them that preserves the point ∞. A more general version of this result is proved in [11, Lemma 6.D.1].

Observe that a neighborhood system for the point $\infty\in\partial G$ is given by the compactification in $\overline{G}$ of the sets of the form $G\setminus(B_R\times[T,-\infty))$, where $B_R=B(0,R)\in\mathbb{R}^{n-1}$ for $R>0$, and $T\in\mathbb{R}$.

We rename the eigenvalues of α by $\mu_1<\cdots<\mu_d$ and fix a Jordan basis of $\mathbb{R}^{n-1}$,

$$\mathcal{B}=\{b^\ell_{ij}:i=1,\dots,d;j=1,\dots,r_i;\ell=1,\dots,m_{ij}\},$$

where r_i is the dimension of the μ_i-eigenspace, spanned by $\{b^1_{i1},\dots,b^1_{ir_i}\}$, m_{ij} is the size of the j-Jordan subblock associated to μ_i, and $\alpha(b^\ell_{ij})=\mu_ib^\ell_{ij}+b^{\ell-1}_{ij}$ for every $\ell=2,\dots,m_{ij}$. We can write

$$\mathbb{R}^{n-1}=\bigoplus_{i,j}V_{ij},\ \text{ where } V_{ij}=\mathrm{Span}(\{b^\ell_{ij}\colon\ell=1,\dots,m_{ij}\}).\tag{9}$$

Let us denote by $\frac{\partial}{\partial t}$ the unit positive vector that spans the factor $\mathbb{R}$ of G and by dt the 1-form associated to $\frac{\partial}{\partial t}$. The 1-forms associated to the dual basis of $\mathcal{B}$ are denoted by dx^ℓ_{ij}. We put on G the left-invariant Riemannian metric making the basis $\mathcal{B}\cup\{\frac{\partial}{\partial t}\}$ orthonormal in T_0G.

Observe that

$$e^{t\alpha} b_{ij}^{\ell} = e^{t\mu_i} \left(b_{ij}^{\ell} + t b_{ij}^{\ell-1} + \ldots + \frac{t^{\ell-1}}{(\ell-1)!} \, b_{ij}^{1} \right),$$

which implies

$$L_{(x,t)}^{*} dx_{ij}^{\ell} = e^{t\mu_i} \left(dx_{ij}^{\ell} + \ldots + \frac{t^{m_{ij}-\ell}}{(m_{ij}-\ell)!} \, dx_{ij}^{m_{ij}} \right).$$

For every $k = 1, \ldots, n-1$ we denote by $\mathcal{I}_k$ the set of multi-indices

$$I = (i_1, \ldots, i_k, j_1, \ldots, j_k, \ell_1, \ldots, \ell_k) \tag{10}$$

with $i_h = 1, \ldots, d$, $j_h = 1, \ldots, r_{i_h}$ and $\ell_h = 1, \ldots, m_{i_h j_h}$ for every $h = 1, \ldots, k$. We also assume that the function $h \mapsto (i_h, j_h, \ell_h)$ is injective and preserves the lexicographic order. For a multi-index as (10) we write

$$dx_I = dx_{i_1 j_1}^{\ell_1} \wedge \ldots \wedge dx_{i_k j_k}^{\ell_k}, \quad \text{and} \quad w_I = \mu_{i_1} + \cdots + \mu_{i_k}.$$

Consider in $\mathcal{I}_1$ the lexicographic order and $\zeta \colon \mathcal{I}_1 \to \{1, \ldots, n-1\}$ the order-preserving bijection. We denote $dx_h = dx_{ij}^{\ell}$ if $h = \zeta(i, j, \ell)$. We also write $dx_n = dt$.

Lemma 5.2. (i) *For every $I \in \mathcal{I}_k$ there exists a positive polynomial P_I such that*

$$|dx_I|_{(x,t)} \asymp e^{t w_I} \sqrt{P_I(t)}.$$

(ii) *The volume form on G is $dV_{(x,t)} = e^{-t \operatorname{tr}(\alpha)} dx_1 \wedge \cdots \wedge dx_n$.*

A polynomial P is *positive* if $P(t) > 0$ for all $t \in \mathbb{R}$.

Proof. (i) On $\Lambda^k(T_0 G)$ we consider the inner product $\langle\langle \cdot, \cdot \rangle\rangle_0$ making the basis $\{dx_I \colon I \in \mathcal{I}_k\}$ orthonormal, thus for $\beta, \gamma \in \Lambda^k(T_{(x,t)} G)$ we put

$$\langle\langle \beta, \gamma \rangle\rangle_{(x,t)} = \langle\langle L_{(x,t)}^{*} \beta, L_{(x,t)}^{*} \gamma \rangle\rangle_0. \tag{11}$$

This means that the inner product is left-invariant.

The left-invariant norm induced by (11) is denoted by $[\,\cdot\,]_{(x,t)}$. Since the operator norm $|\cdot|_{(x,t)}$ is also left-invariant, there exists a constant $C \geq 1$, independent from the point $(x,t) \in G$, such that $C^{-1} |\cdot|_{(x,t)} \leq [\,\cdot\,]_{(x,t)} \leq C |\cdot|_{(x,t)}$. As a consequence it is enough to prove (i) for $[\,\cdot\,]_{(x,t)}$:

$$\begin{aligned}
\left[dx_{i_1 j_1}^{\ell_1} \wedge \ldots \wedge dx_{i_k j_k}^{\ell_k} \right]_{(x,t)}^2 &= \left[(L_{(x,t)}^{*} dx_{i_1 j_1}^{\ell_1}) \wedge \ldots \wedge (L_{(x,t)}^{*} \, dx_{i_k j_k}^{\ell_k}) \right]_0^2 \\
&= e^{2t(\mu_{i_1} + \ldots + \mu_{i_k})} \left[\left(dx_{i_1 j_1}^{\ell_1} + \ldots + \frac{t^{m_{i_1 j_1} - \ell_1}}{(m_{i_1 j_1} - \ell_1)!} \, dx_{i_1 j_1}^{m_{i_1 j_1}} \right) \wedge \right. \\
&\qquad \left. \ldots \wedge \left(dx_{i_k j_k}^{\ell_k} + \ldots + \frac{t^{m_{i_k j_k} - \ell_k}}{(m_{i_k j_k} - \ell_k)!} \, dx_{i_k j_k}^{m_{i_k j_k}} \right) \right]_0^2
\end{aligned}$$

From this expression it is easy to extract the polynomial P_I.

(ii) Here it is enough to prove that $dV_{(x,t)}(v_1, \ldots, v_n) = 1$ for some positive orthonormal basis $\{v_1, \ldots, v_n\} \subset T_{(x,t)} G$. Since $\mathcal{B} \cup \{\frac{\partial}{\partial t}\}$ is orthonormal in $T_0 G$, the basis

$$\begin{aligned}
&\mathcal{B}_t \cup \left\{\tfrac{\partial}{\partial t}\right\} = \{ d_0 L_{(x,t)}(b_{ij}^{\ell}) \colon i = 1, \ldots, d; \ j = 1, \ldots, r_i; \ \ell = 1, \ldots, m_{ij} \} \cup \left\{\tfrac{\partial}{\partial t}\right\} \\
&= \left\{ e^{t\mu_i} \left(b_{ij}^{\ell} + \ldots + \frac{t^{\ell-1}}{(\ell-1)!} b_{ij}^{1} \right) \colon i = 1, \ldots, d; j = 1, \ldots, r_i; \ \ell = 1, \ldots, m_{ij} \right\} \cup \left\{\tfrac{\partial}{\partial t}\right\}
\end{aligned}$$

is orthonormal in $T_{(x,t)} G$. Then we can check the equality evaluating $dV_{(x,t)}$ in the elements of $\mathcal{B}_t \cup \{\frac{\partial}{\partial t}\}$. $\square$

Let φ_t be the flow associated to the vertical vector field $\frac{\partial}{\partial t}$, that is, $\varphi_t(x,s) = (x, s+t)$. We say that a k-form ω is *horizontal* if $\iota_{\frac{\partial}{\partial t}}\omega = 0$. Observe that if

$$\omega = \sum_{1\le i_1<\ldots<i_k\le n} a_{i_1,\ldots,i_k}\, dx_{i_1}\wedge\ldots\wedge dx_{i_k}, \tag{12}$$

then ω is horizontal if and only if all coefficients $a_{i_1,\ldots,i_{k-1},n}$ are zero.

To prove that the relative L^p-cohomology of (G,∞) is zero for every $p > \frac{tr(\alpha)}{w_{k-1}}$ we follow the idea of [34, Proposition 10]. To that end it is necessary to see that the vertical flow φ_t contracts exponentially the horizontal forms.

We define another left-invariant norm on G: For every $v\in\mathbb{R}^n$ we write

$$v = \sum_{i,j} v_{ij} + a\frac{\partial}{\partial t}, \tag{13}$$

where the first sum corresponds to decomposition (9). Given a point $(x,t)\in G$ we define

$$\langle v\rangle_{(x,t)} = \sum_{i,j}\|v_{ij}\|_{(x,t)} + |a|.$$

Since the subspaces V_{ij} are invariant by $e^{t\alpha}$, we can easily see that the norm $\langle\cdot\rangle_{(x,t)}$ is left-invariant and, as a consequence, equivalent to the Riemannian norm $\|\cdot\|_{(x,t)}$. This gives us the following lemma:

Lemma 5.3. *Let ω be a k-form on G, then*

$$|\omega|_{(x,t)} \asymp \sup\{|\omega_{(x,t)}(v_1,\ldots,v_k)| \colon \langle v_i\rangle_{(x,t)} = 1 \text{ for every } i = 1,\ldots,k\},$$

where the constant does not depend on ω or the point $(x,t)\in G$.

A set of vectors in $\mathbb{R}^{n-1}$ is said to be *α-linearly independent* (denoted also by α-LI) if it can be extended to a basis of the form $\bigcup_{i,j}\mathcal{B}_{ij}$, where $\mathcal{B}_{ij}$ is a basis of V_{ij}.

Lemma 5.4. *If ω is a horizontal k-form, then the supremum in Lemma 5.3 is reached on some α-LI set.*

In the previous lemma we can think of $\omega_{(x,t)}$ as an alternating k-linear map on $\mathbb{R}^{n-1}$.

Proof. Since the spheres for the norm $\langle\cdot\rangle_{(x,t)}$ are compact, the supremum is reached on a set of vectors $v_1,\ldots,v_k\in\mathbb{R}^{n-1}$, with $\langle v_\ell\rangle_{(x,t)} = 1$ for every $\ell = 1,\ldots,k$. We write these vectors as in (13):

$$v_\ell = \sum (v_\ell)_{ij}.$$

Then

$$\begin{aligned}\big|\omega_{(x,t)}(v_1,\ldots,v_k)\big| &= \Big|\sum_{i,j}\omega_{(x,t)}((v_1)_{ij}, v_2,\ldots,v_k)\Big|\\ &\le \sum_{i,j}\|(v_1)_{ij}\|_{(x,t)}\left|\omega_{(x,t)}\left(\frac{(v_1)_{ij}}{\|(v_1)_{ij}\|_{(x,t)}}, v_2,\ldots,v_k\right)\right|.\end{aligned}$$

Since $\langle v_1\rangle_{(x,t)} = \sum_{i,j}\|(v_1)_{ij}\|_{(x,t)} = 1$, there exists a pair (i_1,j_1) such that

$$|\omega_{(x,t)}(v_1,\ldots,v_k)| \le \left|\omega_{(x,t)}\left(\frac{(v_1)_{i_1j_1}}{\|(v_1)_{i_1j_1}\|_{(x,t)}}, v_2,\ldots,v_k\right)\right|. \tag{14}$$

Observe that the vector $u_1 = \frac{(v_1)_{i_1 j_1}}{\|(v_1)_{i_1 j_1}\|_{(x,t)}}$ is unitary with respect to the norm $\langle \cdot \rangle_{(x,t)}$ and belongs to $V_{i_1 j_1}$. This implies that the inequality (14) is in fact an equality. Continuing in this way we can construct an α-LI set $\{u_1, \dots, u_k\}$ that satisfies what we wanted. $\square$

Lemma 5.5. *If $v \in V_{ij}$, there exists a positive polynomial P_{ij} such that for every $(x,s) \in G$ and $t \geq 0$ we have*

$$\|v\|_{(x,s+t)} \leq e^{-t\mu_i} \sqrt{P_{ij}(t)} \|v\|_{(x,s)}.$$

Proof. Observe that for every $s \in \mathbb{R}$ we have $\|v\|_{(x,s)} = \|e^{-s\alpha} v\|_0 = e^{-s\mu_i} \|e^{-sJ} v\|_0$, where J is the $(m_{ij} \times m_{ij})$-matrix

$$J = J(m_{ij}) = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}.$$

Therefore,

$$\|v\|_{(x,s+t)} = e^{-(s+t)\mu_i} \left\| e^{-tJ} (e^{-sJ} v) \right\|_0 \leq e^{-(s+t)\mu_i} \left| e^{-tJ} \right| \left\| e^{-sJ} v \right\|_0 = e^{-t\mu_i} \left| e^{-tJ} \right| \|v\|_{(x,s)}.$$

Here $\left| e^{-tJ} \right|$ denotes the operator norm of the matrix e^{-tJ}. Since all norms on $\mathbb{R}^{m_{ij}^2}$ are biLipschitz equivalent, there exists a constant $C_{ij} > 0$, depending only on m_{ij}, such that

$$\left| e^{-tJ} \right| \leq C_{ij} \sqrt{\sum_{1 \leq \ell, r \leq m_{ij}} a_{\ell,r}(t)^2},$$

where $a_{\ell,r}$ are the entries of e^{-tJ}. Notice that they are polynomials in t, in particular $a_{\ell,\ell} = 1$ for every $\ell = 1, \dots, m_{ij}$, then the Lemma follows by taking

$$P_{ij}(t) = C_{ij}^2 \sum_{1 \leq \ell, r \leq m_{ij}} a_{\ell,r}(t)^2. \qquad \square$$

Lemma 5.6. *If ω is a horizontal k-form on G, then there exists a positive polynomial Q such that for every $t \geq 0$,*

$$|\varphi_t^* \omega|_{(x,s)} \preceq e^{-tw_k} \sqrt{Q(t)} |\omega|_{(x,s+t)}.$$

Proof. Using Lemmas 5.3 and 5.4 we have

$$|\varphi_t^* \omega|_{(x,t)} \asymp \max \left\{ \left| \varphi_t^* \omega_{(x,s)} \left(\frac{v_1}{\|v_1\|_{(x,s)}}, \dots, \frac{v_k}{\|v_k\|_{(x,s)}} \right) \right| : \{v_1, \dots, v_k\} \text{ is } \alpha\text{-LI} \right\}$$

$$= \max \left\{ \prod_{\ell=1}^{k} \frac{\|v_\ell\|_{(x,s+t)}}{\|v_\ell\|_{(x,s)}} \left| \omega_{(x,s+t)} \left(\frac{v_1}{\|v_1\|_{(x,s+t)}}, \dots, \frac{v_k}{\|v_k\|_{(x,s+t)}} \right) \right| : \{v_1, \dots, v_k\} \text{ is } \alpha\text{-LI} \right\}$$

Suppose that $v_\ell \in V_{i_\ell j_\ell}$ for every $\ell = 1, \dots, k$, then by Lemma 5.5 and the fact that we are considering α-LI sets we obtain

$$|\varphi_t^* \omega|_{(x,t)} \preceq e^{-tw_k} \sqrt{Q(t)} |\omega|_{(x,s+t)},$$

where $Q = \prod_{ij} (P_{ij})^k$. $\square$

The following Proposition proves the first part of Theorem 1.7.

Proposition 5.7. *Let $k = 2, \dots, n$, then $L^p H^k(G, \infty) = 0$ for all $p > \frac{\mathrm{tr}(\alpha)}{w_{k-1}}$.*

Proof. Take ω a closed form in $L^p\Omega^k(G,\infty)$. We want to construct an L^p-integrable differential $(k-1)$-form ϑ such that $d\vartheta = \omega$ and $\vartheta = 0$ on some neighborhood of ∞. By Remark 3.4 this implies $L^pH^k(G,\infty) = 0$.

Set

$$\vartheta = -\int_0^{+\infty} \varphi_t^* \iota_{\frac{\partial}{\partial t}} \omega \, dt. \tag{15}$$

Observe that, since ω vanishes on a neighborhood of ∞, the above integral converges pointwise, thus ϑ is well-defined as a k-form. Furthermore, it is clear that ϑ is zero on some neighborhood of ∞.

Since $\iota_{\frac{\partial}{\partial t}}\omega$ is a horizontal form, by Lemma 5.6 we have that for all $(x,s) \in G$ and $t \geq 0$,

$$\left|\varphi_t^* \iota_{\frac{\partial}{\partial t}} \omega\right|_{(x,s)} \leq e^{-tw_{k-1}} \sqrt{Q(t)} \left|\iota_{\frac{\partial}{\partial t}} \omega\right|_{(x,s+t)},$$

for some positive polynomial Q. Thus

$$\begin{aligned}\left\|\varphi_t^* \iota_{\frac{\partial}{\partial t}} \omega\right\|_{L^p}^p &\leq \int_G e^{-t(pw_{k-1}-\mathrm{tr}(\alpha))} \sqrt{Q(t)} \left|\iota_{\frac{\partial}{\partial t}} \omega\right|_{(x,s+t)}^p e^{-(s+t\mathrm{tr}(\alpha))} \, dx \, ds \\ &= e^{-t\epsilon} \sqrt{Q(t)} \left\|\iota_{\frac{\partial}{\partial t}} \omega\right\|_{L^p}^p\end{aligned}$$

where $\epsilon = pw_{k-1} - \mathrm{tr}(\alpha) > 0$. It is easy to see that $|\iota_{\frac{\partial}{\partial t}}\omega|_{(x,s)} \leq |\omega|_{(x,s)}$ for every $(x,s) \in G$, so $\|\varphi_t^*\iota_{\frac{\partial}{\partial t}}\omega\|_{L^p} \leq Ce^{-t\epsilon}\|\omega\|_{L^p}$. This implies that the integral (15) converges in $L^p(M, \Lambda^{k-1})$. We have to prove that it is smooth and $d\vartheta = \omega$.

We know that there exists $T \in \mathbb{R}$ such that $\iota_{\frac{\partial}{\partial t}}\omega_{(x,s)} = 0$ for all $s \geq T$, then $\vartheta_{(x,s)}$ is an integral on a compact interval for every $(x,s) \in M$. Since $(x,s,t) \mapsto \varphi_t^*\iota_{\frac{\partial}{\partial t}}\omega$ is smooth we can use Lemma 3.3 to see that ϑ is in $\Omega^{k-1}(M)$ and

$$d\vartheta = -\int_0^{+\infty} d(\varphi_t^* \iota_{\frac{\partial}{\partial t}} \omega) \, dt.$$

The Lie derivative of ω with respect to $\frac{\partial}{\partial t}$ is $L_{\frac{\partial}{\partial t}}\omega = \frac{d}{dt}\big|_{t=0} \varphi_t^*\omega$. Observe that $\frac{d}{dt}\varphi_t^*\omega = \varphi_t^* L_{\frac{\partial}{\partial t}}\omega$. Thus, using the Cartan formula $L_{\frac{\partial}{\partial t}}\omega = d\iota_{\frac{\partial}{\partial t}}\omega + \iota_{\frac{\partial}{\partial t}} d\omega$ (see for example [15, Chapter I,Section A]) and that ω is closed, we obtain

$$\varphi_t^*\omega - \omega = \int_0^t \frac{d}{ds} \varphi_s^* \omega \, ds = \int_0^t \varphi_s^*(d\iota_{\frac{\partial}{\partial t}}\omega + \iota_{\frac{\partial}{\partial t}} d\omega) \, ds = \int_0^t d(\varphi_s^* \iota_{\frac{\partial}{\partial t}} \omega) \, ds.$$

For every $(x,r) \in G$ we have

$$\omega_{(x,r)} = \lim_{t\to+\infty} \left(\varphi_t^* \omega_{(x,r)} - \int_0^t d(\varphi_s^* \iota_{\frac{\partial}{\partial t}} \omega)_{(x,r)} \, ds \right).$$

The limit exists because the expression in brackets is constant for t big enough. Then we conclude

$$\omega_{(x,r)} = -\int_0^{+\infty} d(\varphi_s^* \iota_{\frac{\partial}{\partial t}} \omega)_{(x,r)} \, ds = d\vartheta_{(x,r)}$$

for all $(x,t) \in G$, which finishes the proof. $\square$

We prove the second and third point of Theorem 1.7 by studying two cases separately.

Proposition 5.8. *Let $k = 1, \dots, n-1$, then $L^pH^k(G,\infty) \neq 0$ for $\frac{\mathrm{tr}(\alpha)}{w_k} < p < \frac{\mathrm{tr}(\alpha)}{w_{k-1}}$. In the case $k=1$ we read $\frac{\mathrm{tr}(\alpha)}{w_0} = +\infty$.*

Proof. We want to construct a closed differential k-form ω on G which represents a non-zero class in $L^pH^k(G,\infty)$. We work again with the complex $(L^p\Omega^*(G,\infty),d)$. The strategy of this proof is inspired by the duality ideas mentioned at the beginning of the section, that is: we give a $(n-k)$-form $\beta \in L^q_{\mathrm{loc}}\Omega^{n-k}(G,\infty)$, with $\frac{1}{p}+\frac{1}{q}=1$, such that

(j) $\nu_\beta(\omega)=\int_G \omega\wedge\beta \neq 0$, and
(k) $dL^p\Omega^{k-1}(G,\infty)\subset \operatorname{Ker}\nu_\beta$;

which shows that ω represents a non-zero element in $L^pH^k(G,\infty)$.

Consider two smooth functions $f\colon \mathbb{R}^{n-1}\to[0,1]$ and $g\colon\mathbb{R}\to[0,1]$ such that $\operatorname{supp}(f)$ is compact, $g(t)=0$ for all $t\geq 1$ and $g(t)=1$ for all $t\leq 0$. Then define

$$\omega_{(x,t)} = d\left(f(x)g(t)\,dx_1\wedge\ldots\wedge dx_{k-1}\right). \tag{16}$$

Using triangular inequality we have

$$\|\omega\|_{L^p}\leq \|fg'\,dt\wedge dx_1\wedge\cdots\wedge dx_{k-1}\|_{L^p}+\sum_{j=k}^{n-1}\left\|\frac{\partial f}{\partial x_j}g\,dx_j\wedge dx_1\wedge\cdots\wedge dx_{k-1}\right\|_{L^p}.$$

The first term is finite because fg' is smooth and has compact support. Then it is enough to show that for every $j=k,\ldots,n-1$ the form $\omega_j=\frac{\partial f}{\partial x_j}g\,dx_j\wedge dx_1\wedge\cdots\wedge dx_{k-1}$ belongs to L^p. By Lemma 5.2 there exists a positive polynomial P_j such that

$$\|\omega_j\|^p_{L^p}\preceq\left\|\frac{\partial f}{\partial x_j}\right\|^p_{L^p}\int_{-\infty}^{1}e^{t(p(w_{k-1}+\lambda_j)-\operatorname{tr}(\alpha))}P_j(t)^{\frac{p}{2}}\,dt.$$

Hence $\|\omega_j\|_{L^p}<+\infty$ if $p>\frac{\operatorname{tr}(\alpha)}{w_{k-1}+\lambda_j}$ for every $j=k,\ldots,n-1$, which implies $\|\omega\|_{L^p}<+\infty$ for every $p>\frac{\operatorname{tr}(\alpha)}{w_k}$.

Define $\beta=dx_k\wedge\ldots\wedge dx_{n-1}$. To prove that β is in $L^q_{\mathrm{loc}}\Omega^{n-k}(G,\infty)$ it is enough to show that it is q-integrable on $Z=B_R\times(-\infty,T)$ for every ball $B_R=B_R(0,R)\subset\mathbb{R}^{n-1}$ and $T\in\mathbb{R}$. By Lemma 5.2 there exists a positive polynomial P such that

$$\|\beta\|^q_{L^q,Z}\leq \operatorname{Vol}(B_R)\int_{-\infty}^{T}e^{t(q(\lambda_k+\cdots+\lambda_{n-1})-\operatorname{tr}(\alpha))}P(t)^{\frac{q}{2}}\,dt.$$

This integral converges if and only if $q>\frac{\operatorname{tr}(\alpha)}{\lambda_k+\cdots+\lambda_{n-1}}$, or equivalently $p<\frac{\operatorname{tr}(\alpha)}{w_{k-1}}$.

We now prove (j): Let $B_{R_1}\in\mathbb{R}^{n-1}$ be a ball such that $\operatorname{supp}(f)\subset B_{R_1}$. For $t<1$ consider $Z_t=B_{R_1}\times[t,1]$. Since $|\omega\wedge\beta|$ is in $L^1(G)$ because of HÃČÂűlder's inequality, we have by Stokes' theorem:

$$\int_G\omega\wedge\beta=\lim_{t\to-\infty}\int_{Z_t}d(fg\;dx_1\wedge\cdots\wedge dx_{n-1})=\int_{B_{R_1}}f\,dx_1\wedge\cdots\wedge dx_{n-1}\neq 0.$$

In order to prove (k) we take $\vartheta\in L^p\Omega^{k-1}(G,\infty)$. There exist two constant $R_2,T_2>0$ such that the support of ϑ is contained in $B_{R_2}\times(-\infty,T_2]$. By Stokes' theorem

$$\nu_\beta(d\vartheta)=\int_G d\vartheta\wedge\beta=\lim_{t\to-\infty}\int_{B_{R_2}\times[t,T_2]}d\vartheta\wedge\beta=\lim_{t\to-\infty}\int_{B_{R_2}\times\{t\}}\vartheta\wedge\beta.$$

In the second equality we use again that $|d\vartheta\wedge\beta|$ is in $L^1(G)$. Suppose that $\nu_\beta(d\vartheta)\neq 0$, then there exist $\epsilon>0$ and t_0 such that for all $t\leq t_0$,

$$\left|\int_{B_{R_2}\times\{t\}}\vartheta\wedge\beta\right|>\epsilon. \tag{17}$$

For $I = (i_1, \dots, i_{k-1}, j_1, \dots, j_{k-1}, \ell_1, \dots, \ell_{k-1}) \in \mathcal{I}_{k-1}$ we consider

$$\begin{aligned}(\tilde{\upsilon}_I)_{(x,t)} &= (L^{-1}_{(x,t)})^* \, dx_I = (L^{-1}_{(x,t)})^* \, dx^{\ell_1}_{i_1 j_1} \wedge \cdots \wedge (L^{-1}_{(x,t)})^* \, dx^{\ell_{k-1}}_{i_{k-1} j_{k-1}} \\ &= e^{-tw_I} \left(\sum_{h=0}^{M_1} \frac{(-t)^h}{h!} \, dx^{\ell_i + h}_{i_1 j_1} \right) \wedge \cdots \wedge \left(\sum_{h=0}^{M_{k-1}} \frac{(-t)^h}{h!} \, dx^{\ell_{k-1}+h}_{i_{k-1} j_{k-1}} \right),\end{aligned}$$

where $M_s = m_{i_s j_s} - \ell_s$. We define $(\upsilon_I)_{(x,t)} = e^{tw_I} (\tilde{\upsilon}_I)_{(x,t)}$ and write $\vartheta = \sum_{I \in \mathcal{I}_{k-1}} a_I \upsilon_I$. Observe that $|\upsilon_I|_{(x,t)} \asymp e^{tw_I}$ for every $(x,t) \in G$.

Since $\{\upsilon_I \colon I \in \mathcal{I}_{k-1}\}$ is orthogonal at every point with respect to $\langle\langle \cdot, \cdot \rangle\rangle_{(x,t)}$, then $[\vartheta]_{(x,t)} \geq [a_I \upsilon_I]_{(x,t)}$ for every $I \in \Delta_{k-1}$ and as a consequence $|\vartheta|_{(x,t)} \succeq |a_I \upsilon_I|_{(x,t)}$.

We can easily observe that

$$\int_{B_{R_2} \times \{t\}} \vartheta \wedge \beta = \int_{B_{R_2} \times \{t\}} a_{I_0} \, dx_1 \wedge \cdots \wedge dx_{n-1},$$

where I_0 is such that $dx_{I_0} = dx_1 \wedge \ldots \wedge dx_{k-1}$. Hence, inequality (17) means that there exist $\epsilon > 0$ and t_0 such that for every $t \leq t_0$,

$$\left| \int_{B_R \times \{t\}} a_{I_0}(x,t) \, dx \right| > \epsilon.$$

Now we have

$$\begin{aligned}\|\vartheta\|^p_{L^p} &\succeq \int_G |a_{I_0} \upsilon_{I_0}|^p_{(x,t)} \, dV_{(x,t)} \succeq \int_{-\infty}^{t_0} \left(\int_{B_R} |a_{I_0}(x,t)|^p \, dx \right) e^{t(pw_{k-1} - \mathrm{tr}(\alpha))} \, dt \\ &\succeq \epsilon^p \int_{-\infty}^{t_0} e^{t(pw_{k-1} - \mathrm{tr}(\alpha))} \, dt = +\infty.\end{aligned}$$

This contradiction proves that $\nu_\beta(d\vartheta) = 0$. □

Proposition 5.9. *If $p = \frac{\mathrm{tr}(\alpha)}{w_{k-1}}$ with $k = 2, \dots, n-1$, then $L^p H^k(G, \infty) \neq 0$.*

Proof. We consider ω and β as in the proof of Proposition 5.8. The main difficulty to apply the previous argument in this case is that β does not belong to $L^q_{\mathrm{loc}} \Omega^{n-k}(G, \infty)$, then ν_β is not well-defined. An alternative is to consider the function

$$\tilde{\nu}_\beta \colon L^p \Omega^k(G, \infty) \to [0, +\infty], \quad \tilde{\nu}_\beta(\varpi) = \liminf_{t \to -\infty} \left| \int_{\mathbb{R}^{n-1} \times [t, +\infty)} \varpi \wedge \beta \right|,$$

which is well-defined because $\mathrm{supp}(\varpi) \cap (\mathbb{R}^{n-1} \times [t, +\infty))$ is compact for every $t \in \mathbb{R}$.

It is clear that

$$\tilde{\nu}_\beta(\omega) = \int_{\mathbb{R}^{n-1}} f(x) \, dx \neq 0.$$

Furthermore we can show using the above argument that $\tilde{\nu}_\beta(d\vartheta) = 0$ for all $\vartheta \in L^p \Omega^{k-1}(G, \infty)$. This implies that ω represents a non-zero class in the relative L^p-cohomology of (G, ∞). □

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