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The range set of zero for harmonic mappings of the unit disk with sectorial boundary normalization

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Abstract. Given a family $\mathcal{F}$ of all complex-valued functions in a domain $\Omega \subset \hat{\mathbb{C}}$, the authors introduce the *range set* $\mathrm{RS}_{\mathcal{F}}(A)$ of a set $A \subset \Omega$ under the class in question, i.e. the set of all $w \in \mathbb{C}$ such that $w \in F(A)$ for a certain $F \in \mathcal{F}$. Let T_1, T_2, T_3 be closed arcs contained in the unit circle $\mathbb{T}$ of the same length $2\pi/3$ and covering $\mathbb{T}$. The paper deals with the range set $\mathrm{RS}_{\mathcal{F}}(\{0\})$, where $\mathcal{F}$ is the class of all complex-valued harmonic functions F of the unit disk $\mathbb{D}$ into itself satisfying the following sectorial condition: For each $k \in \{1,2,3\}$ and for almost every $z \in T_k$ the radial limit $F^*(z)$ of the function F at the point z belongs to the angular sector determined by the convex hull spanned by the origin and arc T_k. In 2014 the authors proved that for any $F \in \mathcal{F}$,

$$|F(z)| \leq \frac{4}{3} - \frac{2}{\pi} \arctan\left(\frac{\sqrt{3}}{1+2|z|}\right), \quad z \in \mathbb{D},$$

by which $|F(0)| \leq 2/3$. This implies that $\mathrm{RS}_{\mathcal{F}}(\{0\})$ is a subset of the closed disk of radius $2/3$ and centred at the origin. In the paper the range set $\mathrm{RS}_{\mathcal{F}}(\{0\})$ is precisely determined.

Nollan kuvautuminen yksikkökiekon lohkottain reunanormitetuissa harmonisissa kuvauksissa

Tiivistelmä. Kirjoittajat määrittelevät alueen $\Omega \subset \hat{\mathbb{C}}$ annettua kompleksiarvoisten funktioiden perhettä $\mathcal{F}$ ja osajoukkoa $A \subset \Omega$ vastaavan *kuvautumisjoukon* $\mathrm{RS}_{\mathcal{F}}(A)$ kaikkien niiden pisteiden $w \in \mathbb{C}$ joukkona, joilla pätee $w \in F(A)$ jollakin $F \in \mathcal{F}$. Olkoot T_1, T_2, T_3 yksikköympyrän $\mathbb{T}$ suljettuja kaaria, joilla on sama pituus $2\pi/3$ ja jotka peittävät ympyrän $\mathbb{T}$. Tämä tutkimus käsittelee kuvautumisjoukkoa $\mathrm{RS}_{\mathcal{F}}(\{0\})$, kun $\mathcal{F}$ on sellaisten yksikkökiekon $\mathbb{D}$ itselleen kuvaavien kompleksiarvoisten harmonisten funktioiden F luokka, jotka toteuttavat seuraavan lohkoehdon: jokaisella $k \in \{1,2,3\}$ ja melkein kaikilla $z \in T_k$ kuuluu funktion F säteittäinen raja-arvo $F^*(z)$ pisteessä z origon ja kaaren T_k konveksin verhon määrittelemään lohkoon. Vuonna 2014 kirjoittajat todistivat, että kaikilla $F \in \mathcal{F}$ pätee

$$|F(z)| \leq \frac{4}{3} - \frac{2}{\pi} \arctan\left(\frac{\sqrt{3}}{1+2|z|}\right), \quad z \in \mathbb{D},$$

joten erityisesti $|F(0)| \leq 2/3$. Tästä seuraa, että $\mathrm{RS}_{\mathcal{F}}(\{0\})$ sisältyy origokeskiseen $2/3$-säteiseen suljettuun kiekkoon. Tässä työssä kuvatumisjoukko $\mathrm{RS}_{\mathcal{F}}(\{0\})$ määritetään tarkasti.

Introduction

Throughout the paper we assume that all topological notions and operations are considered in the extended complex plane $(\hat{\mathbb{C}}, \rho_c)$, where ρ_c is the chord-arc metric in $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. Furthermore, we will use the notations $\mathrm{cl}(A)$ and $\mathrm{fr}(A)$ for the closure and boundary of a set $A \subset \hat{\mathbb{C}}$, respectively. By $\mathrm{Har}(\Omega)$ we denote the class of all complex-valued harmonic functions in a domain $\Omega \subset \mathbb{C}$, i.e., the class of all

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complex-valued and twice continuously differentiable functions F in Ω satisfying the Laplace equation

$$\frac{\partial^2}{\partial x^2}F(z)+\frac{\partial^2}{\partial y^2}F(z)=0,\quad z=x+\mathrm{i}y\in\Omega.$$

Let $\bar{\mathbb{D}}(a,r):=\{z\in\mathbb{C}\colon |z-a|\le r\}$ and $\mathbb{T}(a,r):=\{z\in\mathbb{C}\colon |z-a|=r\}$ for $a\in\mathbb{C}$ and $r\in(0;+\infty)$. In particular $\mathbb{T}:=\mathbb{T}(0,1)$ is the unit circle. For any family $\mathcal{F}$ of maps from Ω to $\mathbb{C}$ and a set $A\subset\Omega$ we denote by $\mathrm{RS}_{\mathcal{F}}(A)$ the *range set of A for the class $\mathcal{F}$*, i.e., the set of all $w\in\mathbb{C}$ such that $w\in F(A)$ for a certain $F\in\mathcal{F}$. That is

$$\mathrm{RS}_{\mathcal{F}}(A)=\bigcup_{F\in\mathcal{F}}F(A).$$

For example, if $K\in[1;+\infty)$ and $\mathcal{F}$ is the family of all K-quasiconformal mappings of the unit disk $\mathbb{D}:=\{z\in\mathbb{C}\colon |z|<1\}$ into itself and keeping the origin fixed, then

$$\mathrm{RS}_{\mathcal{F}}(\{z\})=\bar{\mathbb{D}}(0,\Phi_K(|z|)),\quad z\in\mathbb{D},$$

where Φ_K is the Hersch–Pfluger distortion function; cf. [6], [7, Sec. II, §3] and [9, Sec. I.2]. Notice that here $\mathrm{RS}_{\mathcal{F}}(\{z\})$ depends on the module $|z|$ only, because the family $\mathcal{F}$ is invariant under rotations. Otherwise, the study of the range set $\mathrm{RS}_{\mathcal{F}}(\{z\})$ is influenced by directional properties of $\mathcal{F}$, and thereby more difficult in general. For example, if $\mathcal{F}$ is the family of all holomorphic functions of $\mathbb{D}$ into itself and sending zero to a given point $a\in\mathbb{D}$, then by the classical Schwarz inequality

$$\mathrm{RS}_{\mathcal{F}}(\{z\})=\varphi^{-1}\big(\bar{\mathbb{D}}(0,|z|)\big),\quad z\in\mathbb{D},$$

where φ is a conformal mapping of $\mathbb{D}$ onto itself such that $\varphi(a)=0$. One can expect that a number of such problems involving holomorphic functions in the complex plane can be solved by using Schwarz type inequalities, as they have been widely described in [3]. It is worth noting that, in general, the range set $\mathrm{RS}_{\mathcal{F}}(\{z\})$ provides more detailed information about $F(z)$, for $F\in\mathcal{F}$ and $z\in\mathbb{D}$, than the Schwarz.

Let $\mathrm{conv}(A)$ stand for the convex hull of a set $A\subset\mathbb{C}$. In particular, for any $a,b\in\mathbb{C}$, $\mathrm{conv}(\{a,b\})$ is the line segment joining a and b. Given a function $F\colon\mathbb{D}\to\hat{\mathbb{C}}$ we denote by $F^{**}(z)$ the *radial cluster set of F at a point $z\in\mathbb{T}$*, i.e.,

$$F^{**}(z):=\bigcap_{r\in(0;1)}\mathrm{cl}\big(F(\mathrm{conv}(\{rz,z\})\setminus\{z\})\big),\quad z\in\mathbb{T}. \tag{0.1}$$

Notice that always $F^{**}(z)\neq\emptyset$, because $\hat{\mathbb{C}}$ is a compact set. Our main result reads as follows.

Theorem 0.1. *Let $\mathcal{F}$ be the class of all harmonic mappings $F\colon\mathbb{D}\to\mathbb{D}$ satisfying the following sectorial boundary normalization condition*

$$F^{**}(z)\subset D_k:=\mathrm{conv}(T_k\cup\{0\})\quad \textit{for a.e. } z\in T_k \textit{ and } k\in\{1,2,3\}, \tag{0.2}$$

which is determined by the arcs $T_k:=\{\mathrm{e}^{\mathrm{i}t}\colon t\in[2\pi(k-1)/3;2\pi k/3]\}$, $k\in\{1,2,3\}$. Then

$$\mathrm{RS}_{\mathcal{F}}(\{0\})=\mathrm{conv}\left(\bigcup_{k=0}^{5}\mathbb{T}\left(\frac{1}{3}\mathrm{e}^{\pi\mathrm{i}k/3},\frac{1}{3}\right)\right)=\big\{r\mathrm{e}^{\mathrm{i}\theta}\colon \theta\in[0;2\pi),\, r\in[0;\rho(\theta)]\big\}, \tag{0.3}$$

where $\rho\colon \mathbb{R}\to\mathbb{R}$ *is a function uniquely determined by the conditions:*

$$\rho(\theta)=\frac{2}{3}\cos\theta,\quad \theta\in[-\pi/12;\pi/12], \tag{0.4}$$

$$\rho(\theta)=\frac{2}{3}\frac{\left(\cos\frac{\pi}{12}\right)^2}{\sin\left(\frac{\pi}{3}+\theta\right)},\quad \theta\in[\pi/12;\pi/4], \tag{0.5}$$

$$\rho(\theta)=\rho\Big(\theta+\frac{\pi}{3}\Big),\quad \theta\in\mathbb{R}. \tag{0.6}$$

From now on we always assume that $\mathcal{F}$ is the class defined in Theorem 0.1. In [8] the authors proved that for every $F\in\mathcal{F}$,

$$|F(z)|\le\frac{4}{3}-\frac{2}{\pi}\arctan\left(\frac{\sqrt{3}}{1+2|z|}\right)=\frac{2}{\pi}\arctan\left(\sqrt{3}\frac{1+|z|}{1-|z|}\right),\quad z\in\mathbb{D}, \tag{0.7}$$

and the estimation is sharp; cf. [8, Cor. 1.2, Rem. 1.5]. In particular,

$$|F(0)|\le 2/3,\quad F\in\mathcal{F}.$$

The Schwarz type inequalities relevant to (0.7) for a more general case of harmonic functions were studied in [4]. All extremal functions giving the equality in (0.7) are described by [8, Cor. 2.4] which implies that

$$\bar{\mathbb{D}}(0,2/3)\ne \mathrm{RS}_{\mathcal{F}}(\{0\})\subset\bar{\mathbb{D}}(0,2/3). \tag{0.8}$$

Thus the equality (0.3) considerably improves the above statement. Using the function ρ we can draw the boundary of $\mathrm{RS}_{\mathcal{F}}(\{0\})$, as shown by the bold curve in the resulting image; cf. Fig. 1 where $e_0:=1$, $e_1:=\mathrm{e}^{2\pi\mathrm{i}/3}$ and $e_2:=\mathrm{e}^{4\pi\mathrm{i}/3}$.

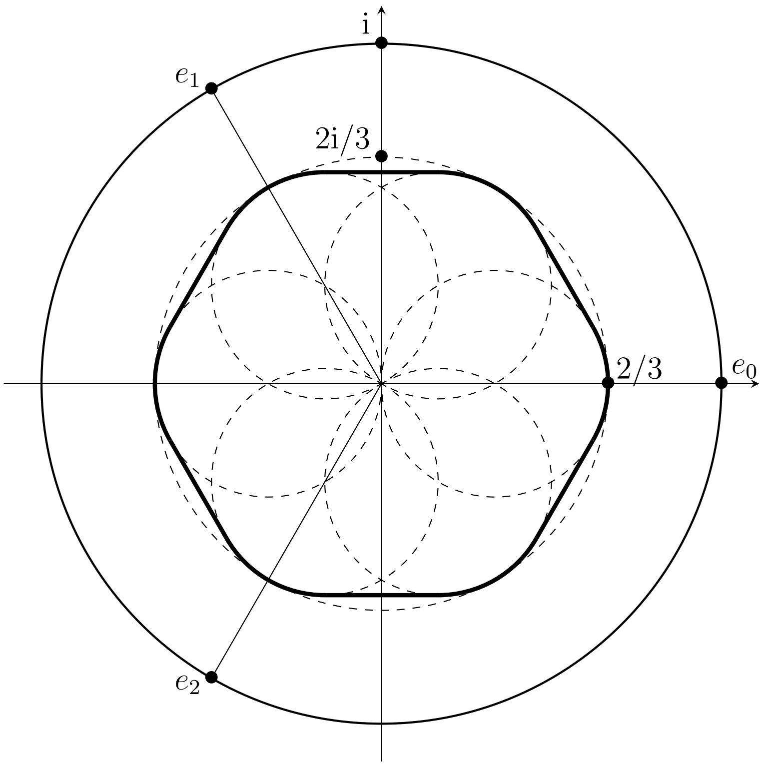

Figure 1. The boundary of the range set $\mathrm{RS}_{\mathcal{F}}(\{0\})$.

In Remark 2.4 we discuss extremal functions in the class $\mathcal{F}$, which recover the boundary of $\mathrm{RS}_{\mathcal{F}}(\{0\})$. The proof of Theorem 0.1 is based on two technical lemmas

(Lemma 2.2 and Lemma 2.3) as well as a few auxiliary results dealing with the class $\mathcal{F}$, gathered in Section 1.

While looking for a technical interpretation of the obtained result, we introduced it to several engineers working at the PANS Academy in Chełm, Poland. One of them, Piotr Różański, performed the following experiment. It involved establishing two cross-sections perpendicular to axis of a round rectilinear pipe equipped with three symmetrically placed internal guides to stabilize the air flow. The stabilization corresponds to the boundary normalization concept described in Theorem 0.1. Smoky air was blown through the center of the first disk along the axis of the pipe and its image was observed in the second cross-section disk. With an appropriately selected blowing speed and cross-section distance, an image resembling the shape of the set $\mathrm{RS}_{\mathcal{F}}(\{0\})$ was obtained. The shape of its edge was interpreted as an isobaric curve of pressure distribution on the second cross-section disk. This gave rise to the idea of building a pipe with the cross-section described in Theorem 0.1, equipped with three symmetrically placed guides; see Fig. 2.

Figure 2. The pipe with the cross-section determined by the range set $\mathrm{RS}_{\mathcal{F}}(\{0\})$.

Its production and tests of some air flow parameters, carried out in PANS laboratories in Chełm, confirmed the validity of the idea. Flow resistance, noise and vibrations of the stream gas were essentially reduced. The overall improvement of the gas flow in such a pipe has been estimated at over 10% compared to the standard round pipe. Further research is underway to adapt this idea to the flow of gases in a curved segments of the pipe. Tests will be performed (experimental and theoretical) on the optimization of the guide heights and the supersonic flows of the gas stream. Work is underway to use the idea to improve the efficiency of jet engines, especially those with tunnel systems drives – fighter planes. The extremal pipe shape concept described above is the subject of patents no. 239213 and no. 240855 issued by the Patent Office of the Republic of Poland.

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1. Auxiliary results

In this section we collect a number of results involving the class $\mathcal{F}$ which will be useful later on. We start with the following simple but useful remark. Given a function $F\colon \mathbb{D} \to \hat{\mathbb{C}}$ we define the function

$$\mathbb{T} \ni z \mapsto F^*(z) := \begin{cases} \lim\limits_{r\to 1} F(rz), & \text{if the limit exists;} \\ 0, & \text{otherwise.} \end{cases} \tag{1.1}$$

From the formula (0.1) we see that for all $F\colon \mathbb{D} \to \hat{\mathbb{C}}$, $z \in \mathbb{T}$ and $w \in \hat{\mathbb{C}}$, $w \in F^{**}(z)$ if and only if there exists a sequence $\mathbb{N} \ni n \mapsto r_n \in [0;1)$ such that $r_n \to 1$ and $F(r_n z) \to w$ as $n \to +\infty$. Therefore

$$F^{**}(z) = \{F^*(z)\} \quad \text{for every } z \in \mathbb{T} \text{ such that the limit in (1.1) exists.} \tag{1.2}$$

Remark 1.1. For every function $F\colon \mathbb{D} \to \mathbb{C}$, $F \in \mathcal{F}$ if and only if $F \in \operatorname{Har}(\mathbb{D})$, $F(\mathbb{D}) \subset \mathbb{D}$ and $F^*(z) \in D_k$ for $k \in \{1,2,3\}$ and almost every (a.e. in abbrev.) $z \in T_k$. This follows directly from (1.2) and the classical result by Fatou that each function $F \in \operatorname{Har}(\mathbb{D})$ with $\sup_{z\in\mathbb{D}} |F(z)| < +\infty$ has the radial limit $\lim_{r\to 1} F(rz)$ for a.e. $z \in \mathbb{T}$; cf. [1, Chap. 6] or [2, Cor. 1 in Sect. 1.2]. In particular, for every $F \in \mathcal{F}$ the radial limit in (1.1) exists for a.e. $z \in \mathbb{T}$.

Lemma 1.2. *The set* $\operatorname{RS}_{\mathcal{F}}(\{0\})$ *is convex.*

Proof. Fix $w_1, w_2 \in \operatorname{RS}_{\mathcal{F}}(\{0\})$ and $\lambda_1, \lambda_2 \in [0;1]$ such that $\lambda_1 + \lambda_2 = 1$. Then $w_1 = F_1(0)$ and $w_2 = F_2(0)$ for some $F_1, F_2 \in \mathcal{F}$. Hence $F := \lambda_1 F_1 + \lambda_2 F_2 \in \operatorname{Har}(\mathbb{D})$ and $F(\mathbb{D}) \subset \mathbb{D}$. By Remark 1.1, the radial limits $\lim_{r\to 1} F_1(rz)$ and $\lim_{r\to 1} F_2(rz)$ exist for a.e. $z \in \mathbb{T}$ and $F_1^*(z), F_2^*(z) \in D_k$ for $k \in \{1,2,3\}$ and a.e. $z \in T_k$. Since each sector D_k is convex, $F^*(z) = \lambda_1 F_1^*(z) + \lambda_2 F_2^*(z) \in D_k$ for $k \in \{1,2,3\}$ and a.e. $z \in T_k$. Applying Remark 1.1 once again we have $F \in \mathcal{F}$. Therefore

$$\lambda_1 w_1 + \lambda_2 w_2 = \lambda_1 F_1(0) + \lambda_2 F_2(0) = F(0) \in \operatorname{RS}_{\mathcal{F}}(\{0\}),$$

and so the set $\operatorname{RS}_{\mathcal{F}}(\{0\})$ is convex. $\square$

Setting $e_k := \mathrm{e}^{2\pi \mathrm{i} k/3}$ for $k \in \mathbb{Z}$, we define $\mathcal{S} := \{S_0, S_1, S_2, \bar{S}_0, \bar{S}_1, \bar{S}_2\}$, where

$$\mathbb{C} \ni z \mapsto S_k(z) := e_k z, \quad k \in \mathbb{Z}, \tag{1.3}$$

as well as

$$\mathbb{C} \ni z \mapsto \bar{S}_k(z) := e_{-k}\bar{z}, \quad k \in \mathbb{Z}. \tag{1.4}$$

Then

$$\bar{S}_k(z) = \overline{S_k(z)}, \quad k \in \mathbb{Z}\,,\, z \in \mathbb{C}. \tag{1.5}$$

Lemma 1.3. *The structure* $(\mathcal{S}; \circ)$ *is a* symmetry group *of the class* $\mathcal{F}$*, where '$\circ$' is the composition operation of mappings, i.e.,*

(i) $(\mathcal{S}; \circ)$ *is a group with two generators* S_1 *and* $\bar{S}_0$*;*
(ii) $S(\mathbb{D}) = \mathbb{D}$ *for* $S \in \mathcal{S}$*;*

(iii) $S \circ F \circ S^{-1} \in \mathcal{F}$ *for* $F \in \mathcal{F}$ *and* $S \in \mathcal{S}$.

Proof. The properties (i) and (ii) evidently follows from the formulas (1.3) and (1.4). Fix $F \in \mathcal{F}$ and $S \in \mathcal{S}$. By the properties (i) and (ii),

$$S \circ F \circ S^{-1}(\mathbb{D}) = S \circ F(\mathbb{D}) \subset S(\mathbb{D}) = \mathbb{D}.$$

Since $F \in \mathrm{Har}(\mathbb{D})$, we see that $S \circ F \circ S^{-1} \in \mathrm{Har}(\mathbb{D})$. By Remark 1.1, the radial limit $\lim_{r\to 1} F(rz)$ exists for a.e. $z \in \mathbb{T}$ and $F^*(z) \in D_k$ for $k \in \{1,2,3\}$ and a.e. $z \in T_k$. Hence for each $k \in \{1,2,3\}$ and a.e. $z \in T_k$,

$$S \circ F \circ S^{-1}(rz) = S(F(rS^{-1}(z))) \to S(F^*(S^{-1}(z))) \in D_k \quad \text{as } r \to 1^-,$$

and so $(S \circ F \circ S^{-1})^*(z) \in D_k$. Applying Remark 1.1 once again we deduce that $S \circ F \circ S^{-1} \in \mathcal{F}$, which proves the item (iii). □

Corollary 1.4. *For every* $S \in \mathcal{S}$, $S(\mathrm{RS}_{\mathcal{F}}(\{0\})) = \mathrm{RS}_{\mathcal{F}}(\{0\})$.

Proof. Given $S \in \mathcal{S}$ we conclude from Lemma 1.3 that $S \circ F \circ S^{-1} \in \mathcal{F}$ for $F \in \mathcal{F}$. Hence

$$S(F(0)) = S \circ F \circ S^{-1}(0) \in \mathrm{RS}_{\mathcal{F}}(\{0\}), \quad F \in \mathcal{F},$$

and consequently

$$S\big(\mathrm{RS}_{\mathcal{F}}(\{0\})\big) \subset \mathrm{RS}_{\mathcal{F}}(\{0\}), \quad S \in \mathcal{S}. \tag{1.6}$$

Since $S^{-1} \in \mathcal{S}$, we deduce from (1.6) that $S^{-1}\big(\mathrm{RS}_{\mathcal{F}}(\{0\})\big) \subset \mathrm{RS}_{\mathcal{F}}(\{0\})$, and so $\mathrm{RS}_{\mathcal{F}}(\{0\}) \subset S\big(\mathrm{RS}_{\mathcal{F}}(\{0\})\big)$. This together with (1.6) implies the desired equality. □

Given an integrable function $f\colon \mathbb{T} \to \mathbb{C}$ we denote by $\mathrm{P}[f](z)$ the Poisson integral of f at $z \in \mathbb{D}$, i.e.,

$$\mathrm{P}[f](z) := \frac{1}{2\pi}\int_{\mathbb{T}} f(u)\frac{1-|z|^2}{|u-z|^2}|\mathrm{d}u| = \frac{1}{2\pi}\int_{\mathbb{T}} f(u)\,\mathrm{Re}\,\frac{u+z}{u-z}|\mathrm{d}u|, \quad z \in \mathbb{D}. \tag{1.7}$$

Here and in the sequel integrable means integrable in the sense of Lebesgue. The Poisson integral $\mathrm{P}[f]$ is the unique solution to the Dirichlet problem for the unit disk $\mathbb{D}$ provided the boundary function f is continuous; cf. e.g. [5, Thm. 2.11]. This means that $\mathrm{P}[f]$ is a harmonic function in $\mathbb{D}$ which has a continuous extension to the closed disk $\mathrm{cl}(\mathbb{D})$ and its boundary limiting valued function is identical with f. From Remark 1.1 we know that for every $F \in \mathcal{F}$ the radial limit in (1.1) exists for a.e. $z \in \mathbb{T}$. Using now the dominated convergence theorem we can show (see e.g. [4, Sec. 2]) that

$$F(z) = \mathrm{P}[F^*](z), \quad F \in \mathcal{F},\ z \in \mathbb{D}. \tag{1.8}$$

Lemma 1.5. *The following equality holds*

$$\mathrm{RS}_{\mathcal{F}}(\{0\}) = \left\{\frac{1}{3}(z_1+z_2+z_3)\colon \{1,2,3\} \ni n \mapsto z_n \in D_n\right\}. \tag{1.9}$$

Proof. Given a sequence $\{1,2,3\} \ni n \mapsto z_n \in D_n$ we define the function

$$\mathbb{D} \ni z \mapsto F(z) := \sum_{n=1}^{3} z_n\,\mathrm{P}[\chi_{T_n}](z), \tag{1.10}$$

where χ_A stands for the characteristic function of a set $A \subset \mathbb{T}$, i.e. $\chi_A(z) := 1$ for $z \in A$ and $\chi_A(z) := 0$ for $z \in \mathbb{T} \setminus A$. Then $F \in \mathrm{Har}(\mathbb{D})$ and $F(\mathbb{D}) \subset \mathrm{cl}(\mathbb{D})$, because for each $z \in \mathbb{D}$,

$$|F(z)| \leq \sum_{n=1}^{3} |z_n| \cdot |\operatorname{P}[\chi_{T_n}](z)| \leq \sum_{n=1}^{3} \operatorname{P}[\chi_{T_n}](z) = \operatorname{P}[\chi_{\mathbb{T}}](z) = 1.$$

By the classical Fatou theorem for Poisson integrals of integrable functions (cf. [1, Thm. 6.39]) we know that for all $k \in \{1, 2, 3\}$ and $z \in T_k \setminus \{e_{k-1}, e_k\}$,

$$F^*(z) = \sum_{n=1}^{3} z_n \operatorname{P}[\chi_{T_n}]^*(z) = z_k \operatorname{P}[\chi_{T_k}]^*(z) = z_k \in D_k.$$

By (1.10), $|F(0)| = \frac{1}{3}|z_1 + z_2 + z_3| < 1$, and so we can appeal to modulus maximum principle for harmonic mappings (see e.g. [1, Cor. 1.11]) to show that $F(\mathbb{D}) \subset \mathbb{D}$. Thus in view of Remark 1.1, $F \in \mathcal{F}$. Therefore

$$\frac{1}{3}(z_1 + z_2 + z_3) = \frac{1}{2\pi} \sum_{n=1}^{3} z_n |T_n|_1 = \sum_{n=1}^{3} z_n \operatorname{P}[\chi_{T_n}](0) = F(0) \in \operatorname{RS}_{\mathcal{F}}(\{0\}),$$

which implies the following inclusion

$$\left\{\frac{1}{3}(z_1 + z_2 + z_3) \colon \{1, 2, 3\} \ni n \mapsto z_n \in D_n\right\} \subset \operatorname{RS}_{\mathcal{F}}(\{0\}). \tag{1.11}$$

Conversely, let $F \in \mathcal{F}$ be fixed. By (1.8), we have

$$F(0) = \operatorname{P}[F^*](0) = \operatorname{P}\Big[\sum_{n=1}^{3} F^* \cdot \chi_{T_n}\Big](0) = \sum_{n=1}^{3} \operatorname{P}[F^* \cdot \chi_{T_n}](0).$$

Since for each $k \in \{1, 2, 3\}$, $F^*(z) \in D_k$ for a.e. $z \in T_k$ and D_k is a closed and convex set, we conclude from the integral mean value theorem for complex-valued functions that there exists a sequence $\{1, 2, 3\} \ni n \mapsto z_n \in D_n$ such that

$$\operatorname{P}[F^* \cdot \chi_{T_n}](0) = z_n \operatorname{P}[\chi_{T_n}](0) = z_n \cdot \frac{1}{2\pi}|T_n|_1 = \frac{1}{3} \cdot z_n, \quad n \in \{1, 2, 3\}.$$

Therefore $F(0) = \frac{1}{3}(z_1 + z_2 + z_3)$, which yields

$$\operatorname{RS}_{\mathcal{F}}(\{0\}) \subset \left\{\frac{1}{3}(z_1 + z_2 + z_3) \colon \{1, 2, 3\} \ni n \mapsto z_n \in D_n\right\}. \tag{1.12}$$

Both inclusions (1.11) and (1.12) imply the equality (1.9), which is the desired conclusion. □

2. The main results

The aim of this section is to determine the set $\operatorname{RS}_{\mathcal{F}}(\{0\})$. To this end we need some additional auxiliary informations. We start with the following simple but useful observation. Given $\theta \in \mathbb{R}$ we set $\ell_\theta := \{re^{i\theta} \colon r \in [0; +\infty)\}$, i.e., ℓ_θ is the ray starting from the origin and passing through $e^{i\theta}$.

Remark 2.1. Given $\theta \in \mathbb{R}$ and a sequence $\{1, 2, 3\} \ni k \mapsto z_k \in D_k$ assume that $z_1 + z_2 + z_3 \in \ell_\theta$. Since each set D_k is bounded,

$$t_k := \sup(\{t \in \mathbb{R} \colon z_k + te^{i\theta} \in D_k\}) \in [0; +\infty), \quad k \in \{1, 2, 3\}. \tag{2.1}$$

Hence $z'_k := z_k + t_k e^{i\theta} \in \operatorname{fr}(D_k)$ for $k \in \{1, 2, 3\}$, and consequently,

$$z'_1 + z'_2 + z'_3 = z_1 + z_2 + z_3 + (t_1 + t_2 + t_3)e^{i\theta} = (|z_1 + z_2 + z_3| + t_1 + t_2 + t_3)e^{i\theta}.$$

Therefore

$$z_1'+z_2'+z_3'\in\ell_\theta\quad\text{and}\quad |z_1'+z_2'+z_3'|\geq|z_1+z_2+z_3| \tag{2.2}$$

as well as

$$\operatorname{Re}\left[(z_1'+z_2'+z_3')\mathrm{e}^{-\mathrm{i}\theta}\right]=|z_1'+z_2'+z_3'|, \tag{2.3}$$

$$\operatorname{Im}\left[(z_1'+z_2'+z_3')\mathrm{e}^{-\mathrm{i}\theta}\right]=0. \tag{2.4}$$

Lemma 2.2. *For every $\theta\in[0;\pi/12]$,*

$$\mathrm{RS}_{\mathcal{F}}(\{0\})\cap\ell_\theta\subset\left\{\frac{2r\cos\theta}{3}\,\mathrm{e}^{\mathrm{i}\theta}\colon r\in[0;1]\right\}. \tag{2.5}$$

Proof. Given $\theta\in[0;\pi/12]$ fix $z\in\mathrm{RS}_{\mathcal{F}}(\{0\})\cap\ell_\theta$. By Lemma 1.5, there exists a sequence $\{1,2,3\}\ni k\mapsto z_k\in D_k$ such that $z=\frac{1}{3}(z_1+z_2+z_3)$. Hence $z_1+z_2+z_3\in\ell_\theta$, and the sequence $\{1,2,3\}\ni k\mapsto z_k'\in\mathrm{fr}(D_k)$ constructed in Remark 2.1 satisfies the conditions (2.2), (2.3) and (2.4). We will show that

$$|z_1'+z_2'+z_3'|\leq 2\cos\theta. \tag{2.6}$$

By (2.1) we have

$$z_1'\in T_1,\quad z_2'\in\operatorname{conv}(\{0,e_1\})\cup\operatorname{conv}(\{0,e_2\}),\quad z_3'\in T_3\cup\operatorname{conv}(\{0,e_0\}). \tag{2.7}$$

Then $z_1'=\mathrm{e}^{\mathrm{i}\alpha}$ for a certain $\alpha\in[0;2\pi/3]$. The following four cases may occur.

Case I, where $z_2'\in\operatorname{conv}(\{0,e_1\})$ and $z_3'\in T_3$. Then $z_2'=t\mathrm{e}^{2\pi\mathrm{i}/3}$ and $z_3'=\mathrm{e}^{\mathrm{i}\beta}$ for some $t\in[0;1]$ and $\beta\in[-\pi/2;0]$. If $\alpha>2\theta$, then by (2.3),

$$|z_1'+z_2'+z_3'|\leq\cos(\alpha-\theta)+\cos(\beta-\theta)<\cos(2\theta-\theta)+\cos(\theta-\beta)\leq 2\cos\theta,$$

which yields (2.6). Let $\alpha\leq 2\theta$. By (2.4),

$$t=-\frac{\sin(\alpha-\theta)+\sin(\beta-\theta)}{\sin\left(\frac{2\pi}{3}-\theta\right)}\geq t_0:=\frac{-\sin(\alpha-\theta)+\sin\theta}{\sin\left(\frac{2\pi}{3}-\theta\right)}\geq 0,$$

which together with (2.3) gives

$$\begin{aligned}|z_1'+z_2'+z_3'|&=\cos(\alpha-\theta)+\cos(\beta-\theta)+t\cos\left(\frac{2\pi}{3}-\theta\right)\\&\leq\cos(\alpha-\theta)+t_0\cos\left(\frac{2\pi}{3}-\theta\right)+\cos\theta.\end{aligned}$$

Since

$$\begin{aligned}\cos(\alpha-\theta)+t_0\cos\left(\frac{2\pi}{3}-\theta\right)&=\cos(\alpha-\theta)+\frac{\sin\theta-\sin(\alpha-\theta)}{\sin\left(\frac{2\pi}{3}-\theta\right)}\cos\left(\frac{2\pi}{3}-\theta\right)\\&=\frac{\sin\left(\frac{2\pi}{3}-\alpha\right)+\sin\theta\cos\left(\frac{2\pi}{3}-\theta\right)}{\sin\left(\frac{2\pi}{3}-\theta\right)}\\&\leq\frac{\sin\left(\frac{2\pi}{3}-2\theta\right)+\sin\theta\cos\left(\frac{2\pi}{3}-\theta\right)}{\sin\left(\frac{2\pi}{3}-\theta\right)}\\&=\frac{\sin\left(\frac{2\pi}{3}-\theta\right)\cos\theta}{\sin\left(\frac{2\pi}{3}-\theta\right)}=\cos\theta,\end{aligned}$$

we obtain the inequality (2.6).

Case II, where $z_2' \in \operatorname{conv}(\{0, e_2\})$ and $z_3' \in T_3$. Then $z_2' = t\mathrm{e}^{4\pi \mathrm{i}/3}$ and $z_3' = \mathrm{e}^{\mathrm{i}\beta}$ for some $\beta \in [-\pi/2; 0]$ and $t \in [0; 1]$. If $2\theta > \alpha$, then

$$\operatorname{Im}\left[(z_1' + z_2' + z_3')\mathrm{e}^{-\mathrm{i}\theta}\right] = \sin(\alpha - \theta) + \sin(\beta - \theta) + t \sin\left(\frac{4\pi}{3} - \theta\right) < 0,$$

which contradicts (2.4). Thus $2\theta \leq \alpha$, and by (2.3),

$$|z_1' + z_2' + z_3'| = \cos(\alpha - \theta) + \cos(\beta - \theta) + t \cos\left(\frac{4\pi}{3} - \theta\right) \leq 2\cos\theta,$$

which yields the inequality (2.6).

Case III, where $z_2' \in \operatorname{conv}(\{0, e_2\})$ and $z_3' \in \operatorname{conv}(\{0, e_0\})$. Then $z_2' = t\mathrm{e}^{4\pi \mathrm{i}/3}$ and $z_3' = s$ for some $t, s \in [0; 1]$. From (2.3) it follows that

$$|z_1' + z_2' + z_3'| = \cos(\alpha - \theta) + s\cos\theta + t\cos\left(\frac{4\pi}{3} - \theta\right). \tag{2.8}$$

Hence the inequality (2.6) holds, provided $\alpha > 2\theta$ or $\theta = 0$. Otherwise $\alpha \leq 2\theta$ and $\theta > 0$. By (2.4),

$$0 = \operatorname{Im}\left[(z_1' + z_2' + z_3')\mathrm{e}^{-\mathrm{i}\theta}\right] = \sin(\alpha - \theta) - s\sin\theta + t\sin\left(\frac{4\pi}{3} - \theta\right),$$

from which $\sin(\alpha - \theta) - s\sin\theta \geq 0$. Hence and by (2.8),

$$\begin{aligned}|z_1' + z_2' + z_3'| &\leq \cos(\alpha - \theta) + s\cos\theta \leq \cos(\alpha - \theta) + \frac{\sin(\alpha - \theta)}{\sin\theta}\cos\theta \\ &= \frac{\sin\theta\cos(\alpha - \theta) + \sin(\alpha - \theta)\cos\theta}{\sin\theta} = \frac{\sin\alpha}{\sin\theta} \leq \frac{\sin(2\theta)}{\sin\theta} = 2\cos\theta.\end{aligned}$$

Thus the inequality (2.6) holds.

Case IV, where $z_2' \in \operatorname{conv}(\{0, e_1\})$ and $z_3' \in \operatorname{conv}(\{0, e_0\})$. Then $z_2' = t\mathrm{e}^{2\pi \mathrm{i}/3}$ and $z_3' = s$ for some $t, s \in [0; 1]$. From (2.3) it follows that

$$|z_1' + z_2' + z_3'| = \cos(\alpha - \theta) + s\cos\theta + t\cos\left(\frac{2\pi}{3} - \theta\right). \tag{2.9}$$

Hence the inequality (2.6) holds, provided $\alpha > 2\theta$. Otherwise $\alpha \leq 2\theta$. By (2.4),

$$0 = \operatorname{Im}\left[(z_1' + z_2' + z_3')\mathrm{e}^{-\mathrm{i}\theta}\right] = \sin(\alpha - \theta) - s\sin\theta + t\sin\left(\frac{2\pi}{3} - \theta\right).$$

Since $s \leq 1$, we conclude from (2.9) that

$$\begin{aligned}|z_1' + z_2' + z_3'| &= \cos(\alpha - \theta) + s\cos\theta + \frac{s\sin\theta - \sin(\alpha - \theta)}{\sin\left(\frac{2\pi}{3} - \theta\right)}\cos\Big(\frac{2\pi}{3} - \theta\Big) \\ &= \frac{\sin\left(\frac{2\pi}{3} - \alpha\right) + s\sin\frac{2\pi}{3}}{\sin\left(\frac{2\pi}{3} - \theta\right)} \leq \frac{\sin\left(\frac{2\pi}{3} - \alpha\right) + \sin\frac{2\pi}{3}}{\sin\left(\frac{2\pi}{3} - \theta\right)} \\ &\leq \frac{\sin\left(\frac{2\pi}{3} - 2\theta\right) + \sin\frac{2\pi}{3}}{\sin\left(\frac{2\pi}{3} - \theta\right)} = \frac{2\sin\left(\frac{2\pi}{3} - \theta\right)\cos\theta}{\sin\left(\frac{2\pi}{3} - \theta\right)} = 2\cos\theta.\end{aligned}$$

Thus the inequality (2.6) holds. Summing up, the inequality (2.6) holds in each of the possible cases I–IV. Using now (2.2) and (2.6) we see that $|z| \leq \frac{2}{3}\cos\theta$. Since $z \in \ell_\theta$, we have $z \in \left\{\frac{2}{3}r\cos\theta\,\mathrm{e}^{\mathrm{i}\theta} \colon r \in [0; 1]\right\}$, which leads to the inclusion (2.5). $\square$

Lemma 2.3. *For every $\theta \in [\pi; 13\pi/12]$,*

$$\mathrm{RS}_{\mathcal{F}}(\{0\}) \cap \ell_\theta \subset \left\{ -\frac{2r\cos\theta}{3} \mathrm{e}^{\mathrm{i}\theta} : r \in [0;1] \right\}. \tag{2.10}$$

Proof. Given $\theta \in [\pi; 13\pi/12]$ fix $z \in \mathrm{RS}_{\mathcal{F}}(\{0\}) \cap \ell_\theta$. By Lemma 1.5, there exists a sequence $\{1,2,3\} \ni k \mapsto z_k \in D_k$ such that $z = \frac{1}{3}(z_1+z_2+z_3)$. Hence $z_1+z_2+z_3 \in \ell_\theta$, and the sequence $\{1,2,3\} \ni k \mapsto z_k' \in \mathrm{fr}(D_k)$ constructed in Remark 2.1 satisfies the conditions (2.2), (2.3) and (2.4). We will show that

$$|z_1' + z_2' + z_3'| \le 2\cos\tilde{\theta}, \tag{2.11}$$

where $\tilde{\theta} := \theta - \pi$. By (2.1) we have

$$z_1' \in \mathrm{conv}(\{0, e_1\}) \cup \mathrm{conv}(\{0, e_0\}), \quad z_2' \in T_2, \quad z_3' \in T_3 \cup \mathrm{conv}(\{0, e_2\}). \tag{2.12}$$

Then $z_2' = -\mathrm{e}^{\mathrm{i}\alpha}$ for a certain $\alpha \in [-\pi/3; \pi/3]$. The following four cases may occur.

Case I, where $z_1' \in \mathrm{conv}(\{0, e_1\})$ and $z_3' \in \mathrm{conv}(\{0, e_2\})$. Then $z_1' = te_1 = -t\mathrm{e}^{-\pi\mathrm{i}/3}$ and $z_3' = se_2 = -s\mathrm{e}^{\pi\mathrm{i}/3}$ for some $s, t \in [0;1]$. From (2.3) and (2.4) it follows that

$$|z_1' + z_2' + z_3'| = t\cos\left(\tilde{\theta} + \frac{\pi}{3}\right) + s\cos\left(\tilde{\theta} - \frac{\pi}{3}\right) + \cos(\alpha - \tilde{\theta}) \tag{2.13}$$

as well as

$$-t\sin\left(\tilde{\theta} + \frac{\pi}{3}\right) + s\sin\left(\frac{\pi}{3} - \tilde{\theta}\right) + \sin(\alpha - \tilde{\theta}) = 0. \tag{2.14}$$

If $\alpha > 2\tilde{\theta}$ or $\alpha < 0$, then by (2.13) we have

$$|z_1'+z_2'+z_3'| \le \cos\left(\tilde{\theta} + \frac{\pi}{3}\right) + \cos\left(\tilde{\theta} - \frac{\pi}{3}\right) + \cos(\alpha - \tilde{\theta}) < 2\cos\tilde{\theta}\cos\frac{\pi}{3} + \cos\tilde{\theta} = 2\cos\tilde{\theta},$$

which leads to (2.11). Otherwise, $0 \le \alpha \le 2\tilde{\theta}$. Since $s \le 1$ and $2\tilde{\theta} \le \pi/6$, we conclude from (2.13) and (2.14) that

$$\begin{aligned} |z_1' + z_2' + z_3'| &= \cos(\alpha - \tilde{\theta}) + s\cos\left(\tilde{\theta} - \frac{\pi}{3}\right) + \frac{s\sin\left(\frac{\pi}{3} - \tilde{\theta}\right) + \sin(\alpha - \tilde{\theta})}{\sin\left(\tilde{\theta} + \frac{\pi}{3}\right)} \cos\left(\tilde{\theta} + \frac{\pi}{3}\right) \\ &= \frac{\sin\left(\alpha + \frac{\pi}{3}\right) + s\sin\frac{2\pi}{3}}{\sin\left(\tilde{\theta} + \frac{\pi}{3}\right)} \le \frac{\sin\left(\alpha + \frac{\pi}{3}\right) + \sin\frac{2\pi}{3}}{\sin\left(\tilde{\theta} + \frac{\pi}{3}\right)} \\ &\le \frac{\sin\left(2\tilde{\theta} + \frac{\pi}{3}\right) + \sin\frac{\pi}{3}}{\sin\left(\tilde{\theta} + \frac{\pi}{3}\right)} = 2\frac{\sin\left(\tilde{\theta} + \frac{\pi}{3}\right)\cos\tilde{\theta}}{\sin\left(\tilde{\theta} + \frac{\pi}{3}\right)} = 2\cos\tilde{\theta}. \end{aligned}$$

Thus the inequality (2.11) holds.

Case II, where $z_1' \in \mathrm{conv}(\{0, e_1\})$ and $z_3' \in T_3$. Then $z_1' = te_1 = -t\mathrm{e}^{-\pi\mathrm{i}/3}$ and $z_3' = -\mathrm{e}^{\mathrm{i}\beta}$ for some $t \in [0;1]$ and $\beta \in [\pi/3; 7\pi/12]$. From (2.3) and (2.4) it follows that

$$|z_1' + z_2' + z_3'| = t\cos\left(\tilde{\theta} + \frac{\pi}{3}\right) + \cos(\beta - \tilde{\theta}) + \cos(\alpha - \tilde{\theta}) \tag{2.15}$$

as well as

$$-t\sin\left(\tilde{\theta} + \frac{\pi}{3}\right) + \sin(\beta - \tilde{\theta}) + \sin(\alpha - \tilde{\theta}) = 0. \tag{2.16}$$

If $\alpha > 2\tilde{\theta}$ or $\alpha < 0$, then by (2.15) we obtain

$$|z_1'+z_2'+z_3'| \le \cos\left(\tilde{\theta} + \frac{\pi}{3}\right) + \cos\left(\frac{\pi}{3} - \tilde{\theta}\right) + \cos(\alpha - \tilde{\theta}) = \cos\tilde{\theta} + \cos(\alpha - \tilde{\theta}) < 2\cos\tilde{\theta},$$

which leads to (2.11). Otherwise, $0 \le \alpha \le 2\tilde{\theta}$. Since $\pi/3 \le \beta \le 7\pi/12$ and $2\tilde{\theta} \le \pi/6$, we conclude from (2.15) and (2.16) that

$$\begin{aligned} |z_1' + z_2' + z_3'| &= \cos(\alpha - \tilde{\theta}) + \cos(\beta - \tilde{\theta}) + \frac{\sin(\beta - \tilde{\theta}) + \sin(\alpha - \tilde{\theta})}{\sin\bigl(\tilde{\theta} + \frac{\pi}{3}\bigr)} \cos\Bigl(\tilde{\theta} + \frac{\pi}{3}\Bigr) \\ &= \frac{\sin\bigl(\alpha + \frac{\pi}{3}\bigr) + \sin\bigl(\beta + \frac{\pi}{3}\bigr)}{\sin\left(\tilde{\theta} + \frac{\pi}{3}\right)} \le \frac{\sin\bigl(\alpha + \frac{\pi}{3}\bigr) + \sin\frac{2\pi}{3}}{\sin\bigl(\tilde{\theta} + \frac{\pi}{3}\bigr)} \\ &\le \frac{\sin\bigl(2\tilde{\theta} + \frac{\pi}{3}\bigr) + \sin\frac{\pi}{3}}{\sin\bigl(\tilde{\theta} + \frac{\pi}{3}\bigr)} = 2\cos\tilde{\theta}. \end{aligned}$$

This shows the inequality (2.11).

Case III, where $z_1' \in \operatorname{conv}(\{0, e_0\})$ and $z_3' \in \operatorname{conv}(\{0, e_2\})$. Then $z_1' = te_0 = t$ and $z_3' = se_2 = -se^{\pi i/3}$ for some $t, s \in [0; 1]$. Since $\frac{\pi}{6} - \frac{\tilde{\theta}}{2} > \tilde{\theta}$, we deduce from (2.3) that

$$\begin{aligned} |z_1' + z_2' + z_3'| &= -t\cos\tilde{\theta} + s\cos\left(\frac{\pi}{3} - \tilde{\theta}\right) + \cos(\alpha - \tilde{\theta}) \\ &\le \cos\left(\frac{\pi}{3} - \tilde{\theta}\right) + 1 = 2\left(\cos\left(\frac{\pi}{6} - \frac{\tilde{\theta}}{2}\right)\right)^2 \\ &< 2\cos\left(\frac{\pi}{6} - \frac{\tilde{\theta}}{2}\right) < 2\cos\tilde{\theta}, \end{aligned}$$

and so the inequality (2.11) holds.

Case IV, where $z_1' \in \operatorname{conv}(\{0, e_0\})$ and $z_3' \in T_3$. Then $z_1' = te_0 = t$ and $z_3' = -e^{i\beta}$ for some $t \in [0; 1]$ and $\beta \in [\pi/3; 7\pi/12]$. From (2.3) it follows that

$$\begin{aligned} |z_1' + z_2' + z_3'| &= -t\cos\tilde{\theta} + \cos(\beta - \tilde{\theta}) + \cos(\alpha - \tilde{\theta}) \\ &\le \cos(\beta - \tilde{\theta}) + \cos(\alpha - \tilde{\theta}) \le \cos(\beta - \tilde{\theta}) + 1 \\ &\le 1 + \cos\left(\frac{\pi}{3} - \tilde{\theta}\right) < 2\cos\tilde{\theta}; \end{aligned}$$

see the case III for the last inequality. This implies the inequality (2.11). Summing up, the inequality (2.11) holds in each of the possible cases I–IV. Using now (2.2) and (2.11) we see that $|z| \le \frac{2}{3}\cos\tilde{\theta}$. Since $z \in \ell_\theta$, we have $z \in \bigl\{-\frac{2}{3}r\cos\theta\, e^{i\theta} \colon r \in [0; 1]\bigr\}$, which leads to the inclusion (2.10). □

Set $\mathbb{C}_+ := \{z \in \mathbb{C} \colon \operatorname{Im} z \ge 0\}$, $\mathbb{C}_- := \{z \in \mathbb{C} \colon \operatorname{Im} z \le 0\}$ and

$$A := \bigcup_{k=0}^{5} \mathbb{T}\left(\frac{1}{3}e^{\pi ik/3}, \frac{1}{3}\right). \tag{2.17}$$

Notice that $S(A) = A$ for $S \in \mathcal{S}$. Now we are in a position to prove our main result.

Proof of Theorem 0.1. We start with showing the following equality

$$\mathrm{RS}_{\mathcal{F}}(\{0\}) = \operatorname{conv}(A). \tag{2.18}$$

Fix $\theta \in \mathbb{R}$ and $r \in [0; 1]$. If $0 \le \theta \le \pi/3$, then $z_1 := e^{2i\theta} \in D_1$, $z_2 := 0 \in D_2$, $z_3 := 1 \in D_3$ and

$$\frac{1}{3}(z_1 + z_2 + z_3) = \frac{1}{3}\bigl(1 + e^{2i\theta}\bigr) = \frac{1}{3}e^{i\theta}\bigl(e^{-i\theta} + e^{i\theta}\bigr) = \frac{2\cos\theta}{3}e^{i\theta}.$$

If $\pi/3 \le \theta \le \pi/2$, then $z_1 := 1 \in D_1$, $z_2 := \mathrm{e}^{2\mathrm{i}\theta} \in D_2$, $z_3 := 0 \in D_3$ and, like above,

$$\frac{1}{3}(z_1 + z_2 + z_3) = \frac{2\cos\theta}{3}\mathrm{e}^{\mathrm{i}\theta}.$$

Using now Lemma 1.5 we obtain the following inclusion

$$\mathbb{T}\left(\frac{1}{3}, \frac{1}{3}\right) \cap \mathbb{C}_+ = \left\{\frac{2\cos\theta}{3}\mathrm{e}^{\mathrm{i}\theta} : \theta \in [0; \pi/2]\right\} \subset \mathrm{RS}_{\mathcal{F}}(\{0\}),$$

which in view of Corollary 1.4 implies

$$\begin{aligned}\mathbb{T}\left(\frac{1}{3}, \frac{1}{3}\right) &= \left(\mathbb{T}\left(\frac{1}{3}, \frac{1}{3}\right) \cap \mathbb{C}_+\right) \cup \bar{S}_0\left(\mathbb{T}\left(\frac{1}{3}, \frac{1}{3}\right) \cap \mathbb{C}_+\right) \\ &\subset \mathrm{RS}_{\mathcal{F}}(\{0\}) \cup \bar{S}_0\big(\mathrm{RS}_{\mathcal{F}}(\{0\})\big) = \mathrm{RS}_{\mathcal{F}}(\{0\}).\end{aligned} \tag{2.19}$$

If $\pi \le \theta \le \frac{7\pi}{6}$, then $z_1 := e_1 \in D_1$, $z_2 := -\mathrm{e}^{2\mathrm{i}\theta} \in D_2$, $z_3 := e_2 \in D_3$ and

$$\frac{1}{3}\big(z_1 + z_2 + z_3\big) = \frac{1}{3}\big(e_1 + \overline{e_1} - \mathrm{e}^{2\mathrm{i}\theta}\big) = \frac{1}{3}\big(-1 - \mathrm{e}^{2\mathrm{i}\theta}\big) = -\frac{2\cos\theta}{3}\mathrm{e}^{\mathrm{i}\theta}.$$

If $\frac{7\pi}{6} \le \theta \le \frac{3\pi}{2}$, then $z_1 := e_1 \in D_1$, $z_2 := e_2 \in D_2$, $z_3 := -\mathrm{e}^{2\mathrm{i}\theta} \in D_3$ and, as above,

$$\frac{1}{3}\big(z_1 + z_2 + z_3\big) = -\frac{2\cos\theta}{3}\mathrm{e}^{\mathrm{i}\theta}.$$

Using Lemma 1.5 once again we obtain the following inclusion

$$\mathbb{T}\left(-\frac{1}{3}, \frac{1}{3}\right) \cap \mathbb{C}_- = \left\{-\frac{2\cos\theta}{3}\mathrm{e}^{\mathrm{i}\theta} : \theta \in [\pi; 3\pi/2]\right\} \subset \mathrm{RS}_{\mathcal{F}}(\{0\}),$$

which in view of Corollary 1.4 leads to

$$\begin{aligned}\mathbb{T}\left(-\frac{1}{3}, \frac{1}{3}\right) &= \left(\mathbb{T}\left(-\frac{1}{3}, \frac{1}{3}\right) \cap \mathbb{C}_-\right) \cup \bar{S}_0\left(\mathbb{T}\left(-\frac{1}{3}, \frac{1}{3}\right) \cap \mathbb{C}_-\right) \\ &\subset \mathrm{RS}_{\mathcal{F}}(\{0\}) \cup \bar{S}_0\left(\mathrm{RS}_{\mathcal{F}}(\{0\})\right) = \mathrm{RS}_{\mathcal{F}}(\{0\}).\end{aligned} \tag{2.20}$$

Applying Corollary 1.4 we deduce from the inclusions (2.19) and (2.20) that

$$\begin{aligned}A &= \bigcup_{k=0}^{5} \mathbb{T}\left(\frac{1}{3}\mathrm{e}^{\pi\mathrm{i}k/3}, \frac{1}{3}\right) = \bigcup_{k=0}^{2}\left[\mathbb{T}\left(\frac{1}{3}\mathrm{e}^{2\pi\mathrm{i}k/3}, \frac{1}{3}\right) \cup \mathbb{T}\left(-\frac{1}{3}\mathrm{e}^{2\pi\mathrm{i}k/3}, \frac{1}{3}\right)\right] \\ &= \bigcup_{k=0}^{2}\left[S_k\left(\mathbb{T}\left(\frac{1}{3}, \frac{1}{3}\right)\right) \cup S_k\left(\mathbb{T}\left(-\frac{1}{3}, \frac{1}{3}\right)\right)\right] \\ &\subset \bigcup_{k=0}^{2}\left[S_k\left(\mathrm{RS}_{\mathcal{F}}(\{0\})\right) \cup S_k\left(\mathrm{RS}_{\mathcal{F}}(\{0\})\right)\right] = \mathrm{RS}_{\mathcal{F}}(\{0\}).\end{aligned}$$

By Lemma 1.2, the set $\mathrm{RS}_{\mathcal{F}}(\{0\})$ is convex. Therefore

$$\operatorname{conv}(A) \subset \operatorname{conv}\big(\mathrm{RS}_{\mathcal{F}}(\{0\})\big) = \mathrm{RS}_{\mathcal{F}}(\{0\}). \tag{2.21}$$

It remains to show the opposite inclusion. Write $A_0 := A_0' \cup A_0''$, where A_0' (resp. A_0'') is the union of all rays ℓ_θ where $\theta \in [0; \pi/12]$ (resp. $\theta \in [\pi; 13\pi/12]$). Since

$$\left\{\frac{2\cos\theta}{3}\mathrm{e}^{\mathrm{i}\theta} : \theta \in [-\pi/2; \pi/2]\right\} = \mathbb{T}\left(\frac{1}{3}, \frac{1}{3}\right)$$

as well as

$$\left\{-\frac{2\cos\theta}{3}\mathrm{e}^{\mathrm{i}\theta} : \theta \in [\pi/2; 3\pi/2]\right\} = \mathbb{T}\left(-\frac{1}{3}, \frac{1}{3}\right),$$

we conclude from Lemmas 2.2 and 2.3 that

$$\mathrm{RS}_{\mathcal{F}}(\{0\}) \cap A_0' \subset \operatorname{conv}\left(\mathbb{T}\left(\frac{1}{3}, \frac{1}{3}\right)\right) \quad \text{and} \quad \mathrm{RS}_{\mathcal{F}}(\{0\}) \cap A_0'' \subset \operatorname{conv}\left(\mathbb{T}\left(-\frac{1}{3}, \frac{1}{3}\right)\right).$$

Therefore

$$\mathrm{RS}_{\mathcal{F}}(\{0\}) \cap A_0 \subset \operatorname{conv}(A).$$

Applying now Corollary 1.4 with $S := \bar{S}_0$ we obtain

$$\begin{aligned}\mathrm{RS}_{\mathcal{F}}(\{0\}) \cap (A_0 \cup \bar{S}_0(A_0)) &= [\mathrm{RS}_{\mathcal{F}}(\{0\}) \cap A_0] \cup [\bar{S}_0(\mathrm{RS}_{\mathcal{F}}(\{0\})) \cap \bar{S}_0(A_0)] \\ &= [\mathrm{RS}_{\mathcal{F}}(\{0\}) \cap A_0] \cup \bar{S}_0[\mathrm{RS}_{\mathcal{F}}(\{0\}) \cap A_0] \\ &\subset \operatorname{conv}(A) \cup \bar{S}_0(\operatorname{conv}(A)) = \operatorname{conv}(A).\end{aligned}$$

Applying once again Corollary 1.4 with $S \in \{S_1, S_2\}$ we see that

$$\begin{aligned}\mathrm{RS}_{\mathcal{F}}(\{0\}) \cap S_k(A_0 \cup \bar{S}_0(A_0)) &= S_k(\mathrm{RS}_{\mathcal{F}}(\{0\})) \cap S_k(A_0 \cup \bar{S}_0(A_0)) \\ &= S_k\big(\mathrm{RS}_{\mathcal{F}}(\{0\}) \cap (A_0 \cup \bar{S}_0(A_0))\big) \\ &\subset S_k(\operatorname{conv}(A)) = \operatorname{conv}(A), \quad k \in \{0,1,2\}.\end{aligned}$$

Thus

$$\mathrm{RS}_{\mathcal{F}}(\{0\}) \cap \bigcup_{S \in \mathcal{S}} S(A_0) \subset \operatorname{conv}(A). \tag{2.22}$$

Fix $z \in \mathrm{RS}_{\mathcal{F}}(\{0\})$. By Lemma 1.5 there exists a sequence $\{1,2,3\} \ni k \mapsto z_k \in D_k$ such that $z = \frac{1}{3}(z_1 + z_2 + z_3)$. Then

$$\operatorname{Im} z = \frac{1}{3}(\operatorname{Im} z_1 + \operatorname{Im} z_2 + \operatorname{Im} z_3) \leq \frac{1}{3}\left(1 + \frac{\sqrt{3}}{2}\right) = \frac{2}{3}\left(\cos\frac{\pi}{12}\right)^2,$$

because for each $r \in [0;1]$,

$$\operatorname{Im} r\mathrm{e}^{\mathrm{i}t} = r \sin t \leq \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}, \quad t \in [2\pi/3; 4\pi/3],$$

as well as

$$\operatorname{Im} r\mathrm{e}^{\mathrm{i}t} = r \sin t \leq 0, \quad t \in [4\pi/3; 2\pi].$$

Therefore

$$\mathrm{RS}_{\mathcal{F}}(\{0\}) \subset B := \left\{\zeta \in \mathbb{C}\colon \operatorname{Im}\zeta \leq \frac{2}{3}\left(\cos\frac{\pi}{12}\right)^2\right\}. \tag{2.23}$$

Let A_1 be the union of all rays ℓ_θ where $\theta \in [5\pi/12; 7\pi/12]$. Setting

$$v_1 := \frac{1}{6} + \frac{2\mathrm{i}}{3}\left(\cos\frac{\pi}{12}\right)^2 \quad \text{and} \quad v_2 := -\frac{1}{6} + \frac{2\mathrm{i}}{3}\left(\cos\frac{\pi}{12}\right)^2, \tag{2.24}$$

we see that

$$B \cap \ell_{5\pi/12} = \operatorname{conv}(\{0, v_1\}) \quad \text{and} \quad B \cap \ell_{7\pi/12} = \operatorname{conv}(\{0, v_2\}). \tag{2.25}$$

Since $v_1 \in \mathbb{T}\big(\frac{1}{3}\mathrm{e}^{\pi\mathrm{i}/3}, \frac{1}{3}\big) \subset A$ and $v_2 \in \mathbb{T}\big(\frac{1}{3}\mathrm{e}^{2\pi\mathrm{i}/3}, \frac{1}{3}\big) \subset A$, we conclude from (2.23) and (2.25) that

$$\mathrm{RS}_{\mathcal{F}}(\{0\}) \cap A_1 \subset B \cap A_1 = \operatorname{conv}(\{0, v_1, v_2\}) \subset \operatorname{conv}(A).$$

Applying now Corollary 1.4 we obtain

$$\mathrm{RS}_{\mathcal{F}}(\{0\}) \cap \bigcup_{S \in \mathcal{S}} S(A_1) = \bigcup_{S \in \mathcal{S}} S\big(\mathrm{RS}_{\mathcal{F}}(\{0\}) \cap A_1\big) \subset \bigcup_{S \in \mathcal{S}} S(\operatorname{conv}(A)) = \operatorname{conv}(A).$$

This together with (2.22) leads to

$$\begin{aligned}\mathrm{RS}_{\mathcal{F}}(\{0\}) &= \mathrm{RS}_{\mathcal{F}}(\{0\}) \cap \mathbb{C}\\ &= \mathrm{RS}_{\mathcal{F}}(\{0\}) \cap \left(\bigcup_{S\in\mathcal{S}} S(A_0) \cup \bigcup_{S\in\mathcal{S}} S(A_1)\right)\\ &= \left(\mathrm{RS}_{\mathcal{F}}(\{0\}) \cap \bigcup_{S\in\mathcal{S}} S(A_0)\right) \cup \left(\mathrm{RS}_{\mathcal{F}}(\{0\}) \cap \bigcup_{S\in\mathcal{S}} S(A_1)\right)\\ &\subset \mathrm{conv}(A).\end{aligned}$$

Hence and by (2.21) we derive the equality (2.18), which together with (2.17) yields the first equality in (0.3). Applying the equality (2.18) we can represent the set $\mathrm{RS}_{\mathcal{F}}(\{0\})$ using polar coordinates as follows. After simple calculations we see that for each $\theta \in [-\pi/12; \pi/12]$,

$$\mathrm{conv}(A) \cap \ell_\theta = \mathrm{conv}\left(\mathbb{T}\left(\frac{1}{3}, \frac{1}{3}\right) \cap \ell_\theta\right) = \left\{z \in \ell_\theta \colon |z| \leq \frac{2}{3}\cos\theta\right\}.$$

Hence and by (0.4),

$$\mathrm{conv}(A) \cap \ell_\theta = \big\{z \in \ell_\theta \colon |z| \leq \rho(\theta)\big\}, \quad \theta \in [-\pi/12; \pi/12]. \tag{2.26}$$

It is easy to verify that for each $\theta \in [5\pi/12; 7\pi/12]$,

$$\mathrm{conv}(A) \cap \ell_\theta = \mathrm{conv}(\{0, v_1, v_2\}) \cap \ell_\theta$$

where v_1 and v_2 are defined by (2.24). On the other hand

$$\mathrm{conv}(\{0, v_1, v_2\}) \cap \ell_\theta = \left\{z \in \ell_\theta \colon |z| \leq \frac{\mathrm{Im}\, v_1}{\cos\left(\frac{\pi}{2} - \theta\right)}\right\} = \left\{z \in \ell_\theta \colon |z| \leq \frac{2}{3}\frac{\left(\cos\frac{\pi}{12}\right)^2}{\sin\theta}\right\}.$$

Thus taking into account (0.5) and (0.6) we obtain

$$\mathrm{conv}(A) \cap \ell_\theta = \big\{z \in \ell_\theta \colon |z| \leq \rho(\theta)\big\}, \quad \theta \in [5\pi/12; 7\pi/12]. \tag{2.27}$$

By the formula (2.17), the set A is invariant under the rotation by $\pi/3$ around the origin, so the set $\mathrm{conv}(A)$ has this property. Combining this with (0.6) we deduce from (2.26) and (2.27) that

$$\mathrm{conv}(A) \cap \ell_\theta = \big\{z \in \ell_\theta \colon |z| \leq \rho(\theta)\big\}, \quad \theta \in \mathbb{R}.$$

This together with (2.17) leads to the second equality in (0.3), which completes the proof. □

Remark 2.4. Taking into account the inclusions (2.5) and (2.10) and the equality (2.18) we can find extremal functions in $\mathcal{F}$, which recover the boundary of $\mathrm{RS}_{\mathcal{F}}(\{0\})$. The basic functions are of the following form

$$\begin{aligned}F_\theta &:= \mathrm{e}^{2i\theta}\,\mathrm{P}[\chi_{T_1}] + \mathrm{P}[\chi_{T_3}], \quad \theta \in [0; \pi/12];\\ F_\theta &:= -\mathrm{e}^{2i\theta}\,\mathrm{P}[\chi_{T_2}] + e_1\,\mathrm{P}[\chi_{T_1}] + e_2\,\mathrm{P}[\chi_{T_3}], \quad \theta \in [\pi; 13\pi/12].\end{aligned}$$

By the definition of the family $\mathcal{F}$ we see that

$$\mathcal{F}_0 := \big\{F_\theta \colon \theta \in [0; \pi/12] \cup [\pi; 13\pi/12]\big\} \subset \mathcal{F}.$$

Now, by Lemma 1.3 we obtain

$$\mathcal{F}_1 := \big\{S \circ F \circ S^{-1} \colon F \in \mathcal{F}_0,\, S \in \mathcal{S}\big\} \subset \mathcal{F}.$$

By the proof of Lemma 1.2 and by Lemma 1.3,

$$\mathcal{F}_2 := \big\{\lambda(\bar{S}_1 \circ F_{13\pi/12} \circ \bar{S}_1^{-1}) + (1-\lambda)F_{\pi/12} \colon \lambda \in [0;1]\big\} \subset \mathcal{F}.$$

Using once again Lemma 1.3 we see that

$$\mathcal{F}_3 := \big\{S \circ F \circ S^{-1} \colon F \in \mathcal{F}_2,\ S \in \mathcal{S}\big\} \subset \mathcal{F}.$$

Finally the union $\mathcal{F}_4 := \mathcal{F}_1 \cup \mathcal{F}_3 \subset \mathcal{F}$ determines the boundary of $\mathrm{RS}_{\mathcal{F}}(\{0\})$, i.e.,

$$\mathrm{fr}\big(\mathrm{RS}_{\mathcal{F}}(\{0\})\big) = \bigcup_{F \in \mathcal{F}_4} F(\{0\}) = \{F(0) \colon F \in \mathcal{F}_4\}.$$

Notice also that the family $\mathcal{F}_1$ recovers the full set $\mathrm{RS}_{\mathcal{F}}(\{0\})$ by convex combinations. More precisely, the following equality holds

$$\mathrm{RS}_{\mathcal{F}}(\{0\}) = \big\{\lambda F + (1-\lambda)G \colon F, G \in \mathcal{F}_1,\ \lambda \in [0;1]\big\}.$$

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