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Speeds of convergence for petals of semigroups of holomorphic functions

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Abstract. We study the backward dynamics of one-parameter semigroups of holomorphic self-maps of the unit disk. More specifically, we introduce the speeds of convergence for petals of the semigroup, namely the total, orthogonal, and tangential speeds. These are analogous to speeds of convergence introduced by Bracci, yet profoundly different due to the nature of backward dynamics. Results are extracted on the asymptotic behavior of speeds of petals, depending on the type of the petal. We further discuss the asymptotic behavior of the hyperbolic distance along non-regular backward orbits.

Holomorfisten funktioiden puoliryhmien terälehtien suppenemisvauhdit

Tiivistelmä. Tutkimme yksikkökiekon holomorfisten itsekuvausten muodostamien yksiparametristen puoliryhmien käänteisaikaista dynamiikkaa. Erityisesti määrittelemme puoliryhmän terälehtien kokonaissuppenemisvauhdin sekä kohtisuoran ja sivuttaissuuntaisen suppenemisvauhdin käsitteet. Nämä rinnastuvat Braccin esittelemiin suppenemisvauhtikäsitteisiin, mutta myös poikkeavat niistä perustavalla tavalla käänteisaikaisen dynamiikan luonteen vuoksi. Tulokset osoittavat terälehtityypistä riippuvaa terälehden vauhdin asymptoottista käyttäytymistä. Lisäksi tarkastelemme hyperbolisen etäisyyden asymptoottista käytöstä epäsäännöllisillä käänteisaikaisilla radoilla.

1. Introduction

One-parameter continuous semigroups of holomorphic functions in the unit disk, or from now on *semigroups in* $\mathbb{D}$, have stimulated the scientific interest in recent years. The introduction to their present form was made by Berkson and Porta in [5], as a direct aspect of semigroups of composition operators. Later, Contreras and Díaz-Madrigal [11] established their main characteristics leading to a plethora of new results. A thorough analysis as well as recent advances on semigroups in $\mathbb{D}$ can be found in the recent monograph [8] and references therein.

A semigroup in $\mathbb{D}$ is a family $(\phi_t)_{t\geq 0}$ of holomorphic self-maps of the unit disk that satisfy the following conditions:

(i) $\phi_0(z) = z$, for all $z \in \mathbb{D}$;
(ii) $\phi_{t+s}(z) = \phi_t(\phi_s(z))$, for every $t, s \geq 0$ and $z \in \mathbb{D}$;
(iii) $\phi_t(z) \xrightarrow{t\to 0^+} z$, uniformly on compacta in $\mathbb{D}$.

If, in addition, for some (equivalently all) $t_0 > 0$ it is true that ϕ_{t_0} is an automorphism of $\mathbb{D}$, then (ϕ_t) is called a *group*. For all semigroups that are not groups, the continuous version of the Denjoy–Wolff Theorem asserts the existence of a unique

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point $\tau \in \overline{\mathbb{D}}$ such that $\phi_t(z) \to \tau$, as $t \to +\infty$, for all $z \in \mathbb{D}$. This point τ is called the *Denjoy–Wolff point* of the semigroup; see [1, Theorem 1.4.17]. If $\tau \in \mathbb{D}$, the semigroup is characterized as *elliptic*, while if the Denjoy–Wolff point lies on the unit circle, (ϕ_t) is called *non-elliptic*.

For all semigroups (ϕ_t) that are not groups, there exists some complex number μ with $\operatorname{Re}\mu > 0$ such that the (angular) derivative $\phi_t'(\tau) = e^{-\mu t}$, for all $t \geq 0$. Note that in the case of non-elliptic semigroups, $\mu \geq 0$. The number μ is called the *spectral value* of the semigroup.

Fix $z \in \mathbb{D}$ and suppose that (ϕ_t) is not a group. The curve $\gamma_z\colon [0, +\infty) \to \mathbb{D}$ with $\gamma_z(t) = \phi_t(z)$ is called the *trajectory* of z. Clearly, for all points $z \in \mathbb{D}$ the trajectory converges to τ, as $t \to +\infty$. Bracci [7] introduced three novel quantities concerning trajectories of non-elliptic semigroups of holomorphic functions. These quantities, the so-called *speeds*, provide interesting results with regard to the rate of convergence of trajectories to the Denjoy–Wolff point of the semigroup. Let (ϕ_t) be a non-elliptic semigroup in the unit disk $\mathbb{D}$ with Denjoy–Wolff point τ. Consider $\gamma\colon (-1, 1) \to \mathbb{D}$, with $\gamma(r) = r\tau$, to be the diameter of the unit disk with endpoints τ and $-\tau$. Clearly, γ is a geodesic for the hyperbolic distance $d_{\mathbb{D}}$ of the unit disk. We denote by $\pi_\gamma(z)$ the *projection* of a point $z \in \mathbb{D}$ onto the curve γ, which satisfies

$$d_{\mathbb{D}}(z, \pi_\gamma(z)) = d_{\mathbb{D}}(z, \gamma) := \inf_{r \in (-1,1)} d_{\mathbb{D}}(z, \gamma(r)).$$

The function

$$v(t) = d_{\mathbb{D}}(0, \phi_t(0)), \quad t \geq 0,$$

is called *total speed* of (ϕ_t). This can be decomposed into two other functions, the *orthogonal speed* of (ϕ_t)

$$v^o(t) = d_{\mathbb{D}}(0, \pi_\gamma(\phi_t(0))), \quad t \geq 0,$$

and the *tangential speed* of (ϕ_t)

$$v^T(t) = d_{\mathbb{D}}(\phi_t(0), \pi_\gamma(\phi_t(0))) = d_{\mathbb{D}}(\phi_t(0), \gamma), \quad t \geq 0.$$

With a first glance, it seems as if the three speeds are solely defined with respect to the trajectory with starting point 0. Nevertheless, Bracci proved that asymptotically these functions do not depend on the starting point and therefore the selection of any point $z \in \mathbb{D}$ instead of 0 is eligible. Furthermore, observing the asymptotic behavior of the speeds, we obtain the type of the semigroup. According to [7, Proposition 6.1],

$$\lim_{t \to +\infty} \frac{v(t)}{t} = \lim_{t \to +\infty} \frac{v^o(t)}{t} = \frac{\mu}{2}, \tag{1.1}$$

where μ denotes the spectral value of (ϕ_t), a result that agrees with that in [3] and [8, Lemma 9.1.2, Theorem 9.1.9].

The definition of speeds of convergence and the statement of a variety of questions in [7] ignited the research interest and was the stepping stone for several works. The second named author disproved in [14] a conjecture on the upper bound for the tangential speed in parabolic semigroups. Cordella in [12] worked on the asymptotic upper bound for the tangential speed. Concerning the orthogonal speed, Bracci, Cordella, and the first named author [10] examined its asymptotic monotonicity with respect to semigroups for a variety of cases. Quite recently, Betsakos and Karamanlis [6] generalized the above result and proved by means of harmonic measure the monotonicity of the orthogonal speed for all cases of non-elliptic semigroups.

The main focus of the current article is to establish speeds of convergence in the setting of the backward dynamics of a semigroup of holomorphic self-maps of $\mathbb{D}$.

Advancements in the direction of backward dynamics are made by Elin, Shoikhet, and Zalcman [13] as well as by Bracci, Contreras, Díaz-Madrigal, and Gaussier [9], where the theory on the backward flow of a semigroup is concretely settled. The *backward invariant set* of a semigroup (ϕ_t) of holomorphic self-maps of $\mathbb{D}$ is defined as

$$\mathcal{W} := \bigcap_{t\geq 0} \phi_t(\mathbb{D})$$

and its open connected components are exactly the sets where the restriction of the semigroup is a group of automorphisms. Every such component of the interior of $\mathcal{W}$ is called a *petal* of (ϕ_t). It is worth mentioning that the concept of backward invariant sets and in general, backward dynamics have also been studied in the setting of discrete iteration theory for one and several complex variables; see e.g. [2, 4].

In the course of the paper, we work with semigroups which are not groups and whose backward invariant set is non-empty. Thus, we exclude the trivial cases from a backward-dynamical point of view of $\mathcal{W}$ being empty or equal to the whole unit disk $\mathbb{D}$. On top of that, the fact that (ϕ_t) is a group of automorphisms on a petal Δ allows us to expand the notation and write

$$\phi_t(z) = \phi_{-t}^{-1}(z), \quad \text{for all } t < 0 \text{ and all } z \in \Delta.$$

As already mentioned, *forward* speeds are an attribute of the semigroup and their asymptotic behavior is similar regardless of the chosen initial point $z \in \mathbb{D}$, due to the attractive nature of the Denjoy–Wolff point. However, as emphasized in the theory of backward dynamics (see [8, Chapter 13] or Section 2.3), this is not the case in the backward-dynamical setting. Due to the lack of an analogous Denjoy–Wolff Theorem, speeds of convergence can only be defined for points in the backward invariant set of a semigroup. As it is clear in Propositions 3.1 and 3.2, the need to restrict to the geometry of a petal of a semigroup arises in order to examine the behavior of the so-called *backward speeds*. In fact, the speeds of convergence along backward orbits lying in the same petal coincide asymptotically and thus, it makes sense to discuss about the speeds of convergence for the corresponding petal. A detailed explanation and proof of the above argument can be found in Section 3.

Definition 1.1. Let (ϕ_t) be a semigroup in $\mathbb{D}$ and let Δ be a petal of (ϕ_t). Let $z \in \Delta$ and suppose that $\eta\colon (-1,1) \to \mathbb{D}$ is the geodesic of $\mathbb{D}$ with $\eta(0) = z$ and $\lim_{r\to 1} \eta(r) = \lim_{t\to -\infty} \phi_t(z)$. We call *total speed* of Δ the function

$$v_\Delta(t) = d_\mathbb{D}(z, \phi_t(z)), \quad t \leq 0.$$

We call *orthogonal speed* of Δ the function

$$v_\Delta^o(t) = d_\mathbb{D}(z, \pi_\eta(\phi_t(z))), \quad t \leq 0.$$

We call *tangential speed* of Δ the function

$$v_\Delta^T(t) = d_\mathbb{D}(\phi_t(z), \eta) = d_\mathbb{D}(\phi_t(z), \pi_\eta(\phi_t(z))), \quad t \leq 0.$$

More on the selection of the geodesic η follow in subsequent sections. Note that contrary to forward speeds, the definitions above concern any semigroup in $\mathbb{D}$ regardless of its type. Indeed, when considering backward dynamics, elliptic and non-elliptic semigroups do not present the vast dissimilarities that occur in forward dynamics because of the position of the Denjoy–Wolff point.

One of the questions that naturally arises is how the backward speeds behave asymptotically, as $t \to -\infty$. As far as the total speed is concerned, we prove the following result.

Theorem 1.1. *Let* (ϕ_t) *be a semigroup in* $\mathbb{D}$*. Suppose* Δ *is a petal of* (ϕ_t)*. There exists some* $\lambda = \lambda(\Delta) \in (-\infty, 0]$ *such that*

$$\lim_{t\to-\infty} \frac{v_\Delta(t)}{t} = \frac{\lambda}{2}. \tag{1.2}$$

The constant $\lambda = \lambda(\Delta)$ is uniquely determined for every petal Δ and depends on its geometric properties and the dynamical behavior of (ϕ_t). Further information on λ and its precise interpretation follows in Subsection 2.3.

Concerning the tangential speed, we examine under which circumstances its limit superior is finite. Taking into account the connection between the backward speeds of a certain petal, as a generalization of Bracci's Pythagoras' Theorem, we further obtain information on the asymptotic behavior of the orthogonal speed of the petal.

Theorem 1.2. *Let* (ϕ_t) *be a semigroup in* $\mathbb{D}$ *and* Δ *a petal of* (ϕ_t) *with* $\lambda = \lambda(\Delta)$*, as in Theorem 1.1. Then*

$$\lim_{t\to-\infty} \frac{v_\Delta^o(t)}{t} = \frac{\lambda}{2} \quad \text{and} \quad \lim_{t\to-\infty} \frac{v_\Delta^T(t)}{t} = 0.$$

The article is structured in the following way. In Section 2, we provide additional information on the backward dynamics of a one-parameter semigroup in $\mathbb{D}$. Moreover, some basic properties of speeds for petals are outlined in Section 3. Section 4 is devoted to the asymptotic behavior of the total speed and the proof of Theorem 1.1, while Section 5 is devoted to orthogonal and tangential speeds and the proof of Theorem 1.2.

Section 6 revolves around a special case of backward dynamics of a semigroup of holomorphic self-maps of $\mathbb{D}$. We extend the definition of the total speed for points that lie on the boundary of the backward invariant set, thus they do not lie inside any petal. The initiative for this study is the better understanding of the nature of the hyperbolic distance of $\mathbb{D}$ along the so-called *non-regular backward orbits*; see Section 2.3 for precise definitions. As observed in Proposition 6.1, the asymptotic behavior of the total speed in this case depends on the geometry of the associated Koenigs domain of the semigroup and on the intrinsic characteristics of the semigroup.

2. Preparation for the proofs

2.1. Hyperbolic metric. We start with some information about hyperbolic geometry (see [8, Chapter 5]). The *density* of the *hyperbolic metric* in $\mathbb{D}$ is $\lambda_{\mathbb{D}}(z) = (1-|z|^2)^{-1}$.

The *hyperbolic distance* between two points z, w in the unit disk is

$$d_{\mathbb{D}}(z,w) = \operatorname{arctanh} \rho_{\mathbb{D}}(z,w), \quad \text{where} \quad \rho_{\mathbb{D}}(z,w) = \left| \frac{z-w}{1-\bar{z}w} \right| \tag{2.1}$$

denotes the *pseudo-hyperbolic distance* in the unit disk $\mathbb{D}$.

Using conformal mappings, we are able to transcend the notions of hyperbolic metric and distance to any simply connected domain, other than the complex plane. Indeed, let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain and let $f\colon \Omega \to \mathbb{D}$ be a Riemann map. Then the hyperbolic distance between two points z, w in Ω is given by $d_\Omega(z,w) := d_{\mathbb{D}}(f(z), f(w))$ and hence the hyperbolic distance, which does not depend on the choice of the Riemann map f, is a conformally invariant quantity.

Applying known conformal mappings, we can calculate the hyperbolic distance in certain domains. For example, if $\mathbb{H} := \{w \in \mathbb{C}\colon \operatorname{Re} w > 0\}$ with $x, y \in \mathbb{H}$, then the hyperbolic distance in this vertical half-plane is given by the formula

$$d_{\mathbb{H}}(x, y) = \operatorname{arctanh}\left|\frac{x-y}{x+\overline{y}}\right|. \tag{2.2}$$

Other than the conformal invariance, a very important property of the hyperbolic distance is its domain monotonicity. To be exact, let $\Omega_1, \Omega_2 \subsetneq \mathbb{C}$ be two simply connected domains with $\Omega_1 \subset \Omega_2$. Then $d_{\Omega_1}(z, w) \geq d_{\Omega_2}(z, w)$, for all $z, w \in \Omega_1$.

Let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain and $\delta_\Omega(z) := \operatorname{dist}(z, \partial\Omega)$ denote the Euclidean distance of z from the boundary of Ω. The hyperbolic density λ_Ω shares a deep connection with δ_Ω, as observed in the following *Distance Lemma*.

Lemma 2.1. [8, Theorem 5.3.1] *Let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain. Then, for all $z, w \in \Omega$,*

$$\frac{1}{4}\log\left(1 + \frac{|z-w|}{\min\{\delta_\Omega(z), \delta_\Omega(w)\}}\right) \leq d_\Omega(z, w) \leq \int_\Gamma \frac{|d\zeta|}{\delta_\Omega(\zeta)},$$

where Γ can be any piecewise C^1-smooth curve joining z and w inside Ω.

2.2. One-parameter semigroups – Koenigs function. As stated in the Introduction, we exclude one-parameter groups from the spectrum of our study, due to their trivial behavior in the backward-dynamical setting. Hence, from this point on, we refer to one-parameter semigroups that are not groups, as simply *one-parameter semigroups*.

Non-elliptic semigroups are further divided into two categories based on their spectral value μ. For $\mu > 0$, a semigroup is characterized as *hyperbolic*, whereas, for $\mu = 0$, the semigroup is called *parabolic*. For every one-parameter non-elliptic semigroup, there exists a unique Riemann mapping h with $h(0) = 0$ such that

$$h(\phi_t(z)) = h(z) + t, \quad \text{for all } z \in \mathbb{D} \text{ and } t \geq 0.$$

The simply connected domain $\Omega := h(\mathbb{D})$ is called the *Koenigs domain* (also known as *associated planar domain*) of the semigroup. The Koenigs domain Ω of a non-elliptic semigroup is *convex in the positive direction*, which means that $\{w + t \colon t \geq 0\} \subset \Omega$, for every $w \in \Omega$. By its definition, it can be understood that a major benefit of the Koenigs function is the linearization of the trajectories of the points in $\mathbb{D}$ under the semigroup. Basically, the image through h of the trajectory of some $z \in \mathbb{D}$ is a horizontal half-line emanating from $h(z)$ towards ∞ in the positive direction (i.e. with constant imaginary part and increasing real part).

The Koenigs function can also be uniquely defined for one-parameter elliptic semigroups. In this case, $h(\tau) = 0$, where $\tau \in \mathbb{D}$ is the Denjoy–Wolff point of the semigroup (ϕ_t) and

$$h(\phi_t(z)) = e^{-\mu t}h(z), \quad \text{for all } z \in \mathbb{D} \text{ and } t \geq 0, \tag{2.3}$$

where μ is the spectral value of (ϕ_t). This time, the Koenigs domain Ω is *μ-spirallike* with respect to 0, since $0 \in \Omega$ and $e^{-\mu t}\Omega \subseteq \Omega$, for all $t \geq 0$. The Koenigs function associated to an elliptic semigroup maps the trajectory of a $z \in \mathbb{D}$ onto a half-spiral, which winds around $h(\tau) = 0$ infinitely many times, as $t \to +\infty$.

Further information on the Koenigs function of a one-parameter semigroup and advances on its geometry and overall behavior can be found in [8, Chapter 9] as well as in references therein.

2.3. Backward orbits. As further stated in the Introduction, every trajectory of a semigroup converges to the Denjoy–Wolff point of the semigroup. However, the asymptotic behavior of backward orbits is not so straightforward. Before getting into detail about the convergence of backward orbits, we need some important notions.

Definition 2.1. [8, Chapter 12] Let (ϕ_t) be a semigroup in $\mathbb{D}$. A point $\sigma \in \partial\mathbb{D}$ is called a *boundary fixed point* of (ϕ_t) provided $\angle\lim_{z\to\sigma}\phi_t(z) = \sigma$, for all $t \geq 0$. A boundary fixed point of (ϕ_t) is characterized as *regular*, if the angular derivative of ϕ_t at σ is finite, for all $t \geq 0$. Otherwise, we say that it is *non-regular*.

Any regular boundary fixed point of a semigroup, other than the Denjoy–Wolff point, is characterized as *repelling*. If σ is a repelling fixed point of (ϕ_t), then there exists a $\lambda \in (-\infty, 0)$ such that the angular derivative $\phi_t'(\sigma) = e^{-\lambda t}$, for all $t \geq 0$. This negative real number λ is called the *repelling spectral value* of (ϕ_t) at σ. Any non-regular boundary fixed point is characterized as *super-repelling*.

Following the notation of [8, Chapter 13], a *backward orbit* of a semigroup (ϕ_t) is a continuous curve $\gamma\colon [0,+\infty) \to \mathbb{D}$ that satisfies $\phi_s(\gamma(t)) = \gamma(t-s)$, for all $t \geq s \geq 0$. The point $\gamma(0)$ is called the *starting point* of γ. A backward orbit is said to be *regular* if

$$\limsup_{t\to+\infty} d_{\mathbb{D}}(\gamma(t), \gamma(t+1)) < +\infty.$$

If the above limit is infinite, then we say that γ is *non-regular*. We distinguish the following cases concerning the asymptotic behavior of regular backward orbits; the reader may refer to [8, Chapter 13] for the detailed theory of backward orbits.

If (ϕ_t) is an elliptic semigroup then a backward orbit is either identical to the Denjoy–Wolff point $\tau \in \mathbb{D}$ or converges to a boundary fixed point of the semigroup. In the latter case, if the backward orbit is also regular, then it converges non-tangentially to a repelling fixed point. If (ϕ_t) is a non-elliptic semigroup then a backward orbit necessarily converges to a boundary fixed point. If, in addition, the backward orbit is regular, then it converges either tangentially to the Denjoy–Wolff point or non-tangentially to a repelling fixed point.

The collection of all backward orbits along with the forward trajectories of their starting points form the so-called *backward invariant set* $\mathcal{W}$ of (ϕ_t).

Definition 2.2. [8, Definition 13.4.1] Let (ϕ_t) be a one-parameter semigroup in $\mathbb{D}$. A non-empty connected component Δ of the interior of $\mathcal{W}$ is called a *petal*.

The restriction of every $\phi_t, t \geq 0$, in a petal Δ is an automorphism (i.e. $\phi_t(\Delta) = \Delta$). Moreover, $\tau \in \partial\Delta$ and for every $z \in \Delta$ the curve $[0,+\infty) \ni t \mapsto \phi_t^{-1}(z)$ is a regular backward orbit for (ϕ_t). Without loss of generality, by denoting $\phi_{-t}(z) := \phi_t^{-1}(z)$, for all $z \in \Delta$, we can extend the semigroup in such a way that $t \in \mathbb{R}$. Hence, for the rest of this article, backward orbits will be defined for $t \leq 0$.

This extension is further applied at the images through the Koenigs function h. It can be easily seen that given $z \in \mathbb{D}$, in the case of non-elliptic semigroups, the image through h of the backward orbit with starting point z is actually a horizontal half-line emanating from $h(z)$ towards ∞ in the negative direction (i.e. with constant imaginary part and decreasing real part). In the case of elliptic semigroups, such a backward orbit is mapped through h to a half-spiral reaching towards ∞.

For a semigroup (ϕ_t) with a petal Δ, every point of Δ is actually the starting point of some regular backward orbit. Conversely, no point outside of a petal can be the starting point of a regular backward orbit. Therefore, the study of regular backward orbits is closely related to that of petals. In order to render this study simpler and more efficient, we once again turn to the Koenigs function h of (ϕ_t).

For a non-elliptic semigroup, the image of a petal Δ under the associated Koenigs function h is

(i) either a *maximal* horizontal strip, in the sense that there exists no horizontal strip S such that $h(\Delta) \subset S \subset \Omega$. In this case the petal is characterized as *hyperbolic*,

(ii) or a *maximal* horizontal half-plane, in the sense that there exists no horizontal half-plane H such that $h(\Delta) \subset H \subset \Omega$. In this case the petal is characterized as *parabolic*.

Considering the one-to-one correspondence through h between points of $\partial\mathbb{D}$ and prime ends of Ω, it is deduced that all backward orbits contained inside a hyperbolic petal converge to the same repelling fixed point, while those inside a parabolic petal converge to the Denjoy–Wolff point. Moreover, in the former case, the width of this maximal horizontal strip depends on the repelling spectral value λ of the repelling fixed point. In particular, the width is equal to $-\frac{\pi}{\lambda}$.

In the case where (ϕ_t) is an elliptic semigroup with Denjoy–Wolff point $\tau \in \mathbb{D}$ and spectral value μ, the image $h(\Delta)$ of a petal Δ is a maximal μ-spirallike sector in Ω of center $e^{i\theta_0}$, for some $\theta_0 \in [-\pi, \pi)$, and amplitude $2a := -\frac{|\mu|^2\pi}{\lambda \operatorname{Re}\mu}$, where λ is the repelling spectral value of the repelling fixed point to which every backward orbit contained in Δ converges. More concretely,

$$h(\Delta) = \operatorname{Spir}[\mu, 2a, \theta_0] := \{e^{t\mu + i\theta} \colon t \in \mathbb{R}, \theta \in (-\alpha + \theta_0, \alpha + \theta_0)\}.$$

Every petal of an elliptic semigroup is characterized as *hyperbolic*. The similarity between the two types of hyperbolic petals we described, relies on the fact that both of them contain a repelling fixed point in their boundaries.

Lemma 2.2. *Suppose (ϕ_t) is an elliptic semigroup of holomorphic self maps of the unit disk $\mathbb{D}$ with spectral value $\mu \in \mathbb{C}$, $\operatorname{Re}\mu > 0$. Denote by h the associated Koenigs function of (ϕ_t). Let Δ be a petal of (ϕ_t) corresponding to the repelling fixed point σ with repelling spectral value λ. Consider the function $g(z) := \frac{e^{-i \operatorname{Arg}\mu}}{\cos(\operatorname{Arg}\mu)} \operatorname{Log} z + i(\alpha - \theta_0)$, where $a := -\frac{|\mu|^2\pi}{2\lambda \operatorname{Re}\mu}$, $\theta_0 \in [-\pi, \pi)$, and $e^{i\theta_0}$ is the center of $h(\Delta)$. Then g maps $h(\Delta)$ onto the horizontal strip $\{z \in \mathbb{C} \colon 0 < \operatorname{Im} z < 2\alpha\}$.*

Proof. First note that $\cos(\operatorname{Arg}\mu) > 0$, since $\operatorname{Re}\mu > 0$. Elementary calculations lead to $\operatorname{Log} e^{t\mu + i\theta} = t\mu + i\theta$ and thus, spirals of $h(\Delta)$ are mapped onto lines parallel to the line joining 0 with μ. Hence

$$g(h(\Delta)) = \bigcup_{\theta \in (-\alpha+\theta_0, \alpha+\theta_0)} \left\{ t\frac{|\mu|}{\cos(\operatorname{Arg}\mu)} + \tan(\operatorname{Arg}\mu)\theta + i(\theta + \alpha - \theta_0) \colon t \in \mathbb{R} \right\}$$

and since $\theta + \alpha - \theta_0 \in (0, 2\alpha)$, then g maps $h(\Delta)$ onto the horizontal strip $\{z \in \mathbb{C} \colon 0 < \operatorname{Im} z < 2\alpha\}$. ☐

Last but not least, independent of the type of the one-parameter semigroup, there exists a one-to-one correspondence between hyperbolic petals and repelling fixed points. We say that the repelling fixed point σ corresponds to the hyperbolic petal Δ of (ϕ_t) (or Δ corresponds to σ, respectively) if all backward orbits lying in Δ converge to σ.

Keeping in mind the asymptotic behavior of backward orbits and the shape of petals, we can now discuss non-regular backward orbits. A non-regular backward orbit for one-parameter semigroups can fall into one of the following three cases:

(i) it is either part of the boundary of a hyperbolic petal, in which case it converges tangentially to a repelling fixed point of the semigroup,
(ii) it is either part of the boundary of a parabolic petal in which case it converges tangentially to the Denjoy–Wolff point of the semigroup, a situation that can arise solely in parabolic semigroups,
(iii) or it converges to a super-repelling fixed point of the semigroup, in which case the convergence can be either tangential or non-tangential (or an amalgamation of the two if we consider subsequences).

2.4. Infinitesimal generators. One-parameter semigroups are closely related with dynamical systems. In fact, for every (ϕ_t), there exists a unique holomorphic function $G\colon \mathbb{D} \to \mathbb{C}$, such that

$$\frac{\partial \phi_t(z)}{\partial t} = G(\phi_t(z)), \quad z \in \mathbb{D}, \quad t \geq 0.$$

The function G is called the *infinitesimal generator* of the semigroup. In the course of the proofs, we exclusively need the following lemma regarding infinitesimal generators. It is a combination of results from Theorem 12.2.5 and Corollary 12.2.6 in [8].

Lemma 2.3. *Let (ϕ_t) be a semigroup in $\mathbb{D}$ with associated infinitesimal generator G. Suppose $\sigma \in \partial\mathbb{D}$ is a repelling fixed point of (ϕ_t) with repelling spectral value $\lambda \in (-\infty, 0)$. Then:*

(i) $\operatorname{Re}\left(\dfrac{\sigma G(z)}{(\sigma - z)^2}\right) \geq \dfrac{\lambda}{2} \cdot \dfrac{1 - |z|^2}{|\sigma - z|^2}$, *for all $z \in \mathbb{D}$;*
(ii) $-\lambda = \angle \lim\limits_{z \to \sigma} \dfrac{G(z)}{z - \sigma}$;
(iii) *there exists a unique holomorphic mapping $p\colon \mathbb{D} \to \overline{\mathbb{H}}$ with*

$$\angle \lim_{z \to \sigma} (z - \sigma) p(z) = 0,$$

such that

$$G(z) = (\overline{\sigma} z - 1)(z - \sigma)\left[p(z) + \frac{\lambda}{2} \cdot \frac{\sigma + z}{\sigma - z}\right], \quad \textit{for all } z \in \mathbb{D}.$$

Further information on infinitesimal generators of one-parameter semigroups and their connection to the associated Koenigs functions can be found in [8, Chapter 10] as well as in references therein.

3. Speeds of convergence for petals

In this section, we discuss the necessity of restricting to a certain petal in order to study speeds of convergence along backward orbits of a semigroup in $\mathbb{D}$, which is not a group.

Recall that for every $z \in \Delta$, where Δ is a petal of a semigroup (ϕ_t), the curve with $(-\infty, 0] \ni t \mapsto \phi_t(z) = \phi_{-t}^{-1}(z)$ is a backward orbit.

Proposition 3.1. *Let (ϕ_t) be a semigroup in $\mathbb{D}$ and Δ be a petal of (ϕ_t). Let $z, w \in \Delta$. Then*

$$|d_{\mathbb{D}}(z, \phi_t(z)) - d_{\mathbb{D}}(w, \phi_t(w))| \leq 2 d_{\Delta}(z, w), \quad t \leq 0.$$

Proof. The monotonicity property of the hyperbolic metric and the triangle inequality yield for $t \leq 0$,

$$\begin{aligned}|d_{\mathbb{D}}(z,\phi_t(z)) - d_{\mathbb{D}}(w,\phi_t(w))| &\leq |d_{\mathbb{D}}(z,\phi_t(z)) - d_{\mathbb{D}}(z,\phi_t(w))| \\ &\quad + |d_{\mathbb{D}}(z,\phi_t(w)) - d_{\mathbb{D}}(w,\phi_t(w))| \\ &\leq d_{\mathbb{D}}(\phi_t(z),\phi_t(w)) + d_{\mathbb{D}}(z,w) \\ &\leq d_{\Delta}(\phi_t(z),\phi_t(w)) + d_{\Delta}(z,w) \\ &= 2d_{\Delta}(z,w),\end{aligned}$$

where $d_{\Delta}(\phi_t(z),\phi_t(w)) = d_{\Delta}(z,w)$, since $\phi_t|_{\Delta}$ is an automorphism of Δ. □

Remark 3.1. It is easy to check that the result of this proposition does not remain true if z and w lie on different petals or if any of the two lies outside of any petal. In such cases, the above difference between what will eventually be the total speeds, does not remain bounded.

In the study of speeds in the forward dynamics, the role of the geodesic was assumed by the diameter with one end on the Denjoy–Wolff point τ of the non-elliptic semigroup, because every trajectory converges to this point. The choice of the diameter, instead of any other geodesic landing at τ, is justified as 0 is the starting point of the trajectory used in "forward" speeds and the diameter is the only such geodesic that passes through 0. In this way, $v(0) = v^o(0) = v^T(0) = 0$. However, this is not the case when it comes to backward dynamics, since the selection of the geodesic fluctuates depending on the petal. Once again, let (ϕ_t) be a semigroup with a petal Δ and let $z \in \Delta$. We denote by $\eta\colon (-1,1) \to \mathbb{D}$ the geodesic for the hyperbolic distance $d_{\mathbb{D}}$ such that:

(i) if Δ is hyperbolic, then $\eta(0) = z$ and $\lim_{r\to 1^-} \eta(r) = \sigma$, where $\sigma \in \partial\mathbb{D}$ is the repelling fixed point of (ϕ_t) where all backward orbits contained in Δ converge;

(ii) if Δ is parabolic, then $\eta(0) = z$ and $\lim_{r\to 1^-} \eta(r) = \tau$, where $\tau \in \partial\mathbb{D}$ is the Denjoy–Wolff point of (ϕ_t).

Proposition 3.2. *Let (ϕ_t) be a semigroup in $\mathbb{D}$ and Δ be a petal of (ϕ_t). Let $z, w \in \Delta$ and suppose that $\eta\colon (-1,1) \to \mathbb{D}$ is the geodesic described above, with $\eta(0) = z$. Then*

$$|d_{\mathbb{D}}(z,\pi_\eta(\phi_t(z))) - d_{\mathbb{D}}(w,\pi_\eta(\phi_t(w)))| \leq 2d_{\Delta}(z,w), \quad t \leq 0,$$

and

$$|d_{\mathbb{D}}(\phi_t(z),\eta) - d_{\mathbb{D}}(\phi_t(w),\eta)| \leq 2d_{\Delta}(z,w), \quad t \leq 0.$$

Proof. For the sake of brevity, for $t \leq 0$ we write

$$z_t = \pi_\eta(\phi_t(z)) \quad \text{and} \quad w_t = \pi_\eta(\phi_t(w)).$$

The proof is almost identical with that of Proposition 3.1, but there is a key argument that renders the presentation of this proof necessary. We have

$$\begin{aligned}|d_{\mathbb{D}}(z,z_t) - d_{\mathbb{D}}(w,w_t)| &\leq |d_{\mathbb{D}}(z,z_t) - d_{\mathbb{D}}(w,z_t)| + |d_{\mathbb{D}}(w,z_t) - d_{\mathbb{D}}(w,w_t)| \\ &\leq d_{\mathbb{D}}(z,w) + d_{\mathbb{D}}(z_t,w_t).\end{aligned}$$

Using [7, Proposition 3.3], we get

$$d_{\mathbb{D}}(z_t,w_t) \leq d_{\mathbb{D}}(\phi_t(z),\phi_t(w)), \quad \text{for all } t \leq 0.$$

Combining all the above inequalities and continuing exactly as in the proof of the Proposition 3.1, we deduce the first desired result. The proof for the second desired inequality follows the same steps so we skip it in order to avoid repetitiveness. □

Once more, the bounds established demonstrate that the asymptotic behavior of all the aforementioned quantities does not depend on the choice of the starting point provided we stay inside a petal. All the propositions that we proved point to the fact that the key difference between the two kinds of speeds is the following: while the speed of a forward trajectory can be globally defined as the speed of the semigroup, this is not true about the speed along a backward orbit. On the contrary, the speed along a regular backward orbit can be locally defined as the speed of the respective petal. As a result, Definition 1.1 indeed does not depend on the choice of the point lying in the petal.

With the notation outlined in Definition 1.1, a first useful result that correlates the three speeds stems directly from Bracci's Pythagoras' Theorem [8, Proposition 3.4].

Corollary 3.1. *Let (ϕ_t) be a semigroup in $\mathbb{D}$ and Δ be a petal of (ϕ_t). Then*

$$v_\Delta^o(t) + v_\Delta^T(t) - \frac{1}{2}\log 2 \le v_\Delta(t) \le v_\Delta^o(t) + v_\Delta^T(t),$$

for all $t \le 0$.

4. Total speed of petals – Proof of Theorem 1.1

In the current section, we examine the convergence of the total speed of a petal Δ, whose asymptotic behavior depends, as it turns out, greatly on the type of the petal.

We first deal with hyperbolic petals which may exist in both elliptic and non-elliptic semigroups. Suppose (ϕ_t) is a one-parameter semigroup. Let σ be a repelling fixed point of (ϕ_t) with spectral value $\lambda < 0$. Denote by Δ the associated hyperbolic petal. We obtain the following result.

Theorem 4.1. *Let (ϕ_t) be a semigroup in $\mathbb{D}$. Suppose Δ is a hyperbolic petal of (ϕ_t) corresponding to the repelling fixed point σ. Then*

$$\lim_{t\to-\infty} \frac{v_\Delta(t)}{t} = \frac{\lambda}{2}, \tag{4.1}$$

where λ is the repelling spectral value of σ.

For the proof of Theorem 4.1, we need information on the asymptotic behavior of the hyperbolic distance inside the hyperbolic petal Δ. Restating [8, Lemma 13.5.1] with a slight re-parametrization, the following lemma arises.

Lemma 4.1. *Let (ϕ_t) be a semigroup in $\mathbb{D}$. Suppose Δ is a hyperbolic petal of (ϕ_t) corresponding to the repelling fixed point σ. Then*

$$\lim_{t\to-\infty} \frac{d_\Delta(z,\phi_t(z))}{t} = \frac{\lambda}{2}, \quad \textit{for all } z \in \Delta, \tag{4.2}$$

where λ is the repelling spectral value of (ϕ_t) at σ.

Theorem 4.1 suggests that the same limit is true when we use the hyperbolic distance in the unit disk instead of restricting to the hyperbolic geometry of the petal. The main idea for the proof is that while the backward orbit converges to the repelling fixed point, the hyperbolic geometry of the unit disk becomes similar to that of the petal.

Proof of Theorem 4.1. Let $z \in \Delta$. Due to domain monotonicity, we can easily observe that

$$\liminf_{t\to-\infty} \frac{d_{\mathbb{D}}(z,\phi_t(z))}{t} \geq \lim_{t\to-\infty} \frac{d_{\Delta}(z,\phi_t(z))}{t} = \frac{\lambda}{2}. \tag{4.3}$$

For the inequality in the opposite direction, we use l'Hôpital's Rule in order to estimate the limit. From the generalized l'Hôpital's Rule, we obtain

$$\limsup_{t\to-\infty} \frac{d_{\mathbb{D}}(z,\phi_t(z))}{t} \leq \limsup_{t\to-\infty} \frac{\partial}{\partial t} d_{\mathbb{D}}(z,\phi_t(z)). \tag{4.4}$$

Let us denote by $d_z(t)$ the hyperbolic distance $d_{\mathbb{D}}(z,\phi_t(z))$ and by $\rho_z(t)$ the pseudo-hyperbolic distance $\rho_{\mathbb{D}}(z,\phi_t(z))$. We can calculate the derivative of the hyperbolic distance w.r.t. t using its representation in terms of the pseudo-hyperbolic distance; $2d_z(t) = \log(1+\rho_z(t)) - \log(1-\rho_z(t))$. With elementary calculations, we obtain

$$2\frac{\partial}{\partial t} d_z(t) = \frac{1}{\rho_z(t)(1-\rho_z(t)^2)} \frac{\partial}{\partial t} \rho_z(t)^2. \tag{4.5}$$

The pseudo-hyperbolic distance can be explicitly expressed in the following way

$$\rho_z(t)^2 = \left|\frac{z-\phi_t(z)}{1-\bar{z}\phi_t(z)}\right|^2 = \frac{|z|^2 + |\phi_t(z)|^2 - 2\operatorname{Re}(\bar{z}\phi_t(z))}{1+|z|^2\cdot|\phi_t(z)|^2 - 2\operatorname{Re}(\bar{z}\phi_t(z))}.$$

In order to calculate the derivative in (4.5), we use the infinitesimal generator G of (ϕ_t), since $\frac{\partial \phi_t(w)}{\partial t} = G(\phi_t(w))$, for all $w \in \Delta$ and $t \leq 0$. Hence we are led to

$$\rho_z(t)\frac{\partial}{\partial t} d_z(t) = \frac{(1+|z|^2 - 2\operatorname{Re}(\bar{z}\phi_t(z)))\operatorname{Re}(G(\phi_t(z))\overline{\phi_t(z)})}{(1-|\phi_t(z)|^2)|1-\bar{z}\phi_t(z)|^2} - \frac{\operatorname{Re}(G(\phi_t(z))\bar{z})}{|1-\bar{z}\phi_t(z)|^2}. \tag{4.6}$$

We note with elementary calculations that

$$\liminf_{t\to-\infty} \frac{\operatorname{Re}(G(\phi_t(z))\bar{z})}{|1-\bar{z}\phi_t(z)|^2} = 0,$$

since $G(\sigma) = 0$. We write

$$\operatorname{Re}(G(\phi_t(z))\overline{\phi_t(z)}) = \operatorname{Re}\left(\frac{G(\phi_t(z))\sigma}{(\sigma-\phi_t(z))^2}\overline{\sigma}\overline{\phi_t(z)}(\sigma-\phi_t(z))^2\right)$$

and further observe that

$$\overline{\sigma}\overline{\phi_t(z)}\frac{(\sigma-\phi_t(z))^2}{1-|\phi_t(z)|^2} = 1 - \overline{\sigma}\phi_t(z) - \frac{|\sigma-\phi_t(z)|^2}{1-|\phi_t(z)|^2}.$$

As a result,

$$\frac{\operatorname{Re}(G(\phi_t(z))\overline{\phi_t(z)})}{1-|\phi_t(z)|^2} = \operatorname{Re}\left(\frac{G(\phi_t(z))}{\sigma-\phi_t(z)}\right) - \frac{|\sigma-\phi_t(z)|^2}{1-|\phi_t(z)|^2}\operatorname{Re}\left(\frac{G(\phi_t(z))\sigma}{(\sigma-\phi_t(z))^2}\right).$$

From Lemma 2.3 we obtain

$$\operatorname{Re}\left(\frac{G(\phi_t(z))}{\sigma-\phi_t(z)}\right) = -\operatorname{Re}\left((\overline{\sigma}\phi_t(z)-1)p(\phi_t(z))\right) + \frac{\lambda}{2}\operatorname{Re}(\overline{\sigma}(\sigma+\phi_t(z))), \tag{4.7}$$

where p is a holomorphic mapping of positive real part. Moreover, again from Lemma 2.3

$$-\frac{|\sigma-\phi_t(z)|^2}{1-|\phi_t(z)|^2}\operatorname{Re}\left(\frac{G(\phi_t(z))\sigma}{(\sigma-\phi_t(z))^2}\right) \leq -\frac{\lambda}{2}\frac{|\sigma-\phi_t(z)|^2}{1-|\phi_t(z)|^2}\frac{1-|\phi_t(z)|^2}{|\sigma-\phi_t(z)|^2} = -\frac{\lambda}{2}. \tag{4.8}$$

Hence returning back to (4.4) and combining (4.7) and (4.8), it follows that

$$\limsup_{t\to-\infty}\frac{d_z(t)}{t}\leq\limsup_{t\to-\infty}\frac{1+|z|^2-2\operatorname{Re}(\bar{z}\phi_t(z))}{|1-\bar{z}\phi_t(z)|^2}\bigg[-\operatorname{Re}\left((\overline{\sigma}\phi_t(z)-1)p(\phi_t(z))\right)$$
$$+\frac{\lambda}{2}\operatorname{Re}(\overline{\sigma}(\sigma+\phi_t(z)))-\frac{\lambda}{2}\bigg]=\lambda-\frac{\lambda}{2}=\frac{\lambda}{2}.$$

Taking also (4.3) into account, the desired limit occurs. □

Let us recall that parabolic petals exist only in the case of parabolic one-parameter semigroups. Suppose that (ϕ_t) is a parabolic semigroup of holomorphic self-maps of $\mathbb{D}$ with Denjoy–Wolff point $\tau\in\partial\mathbb{D}$. Further suppose that h is the associated Koenigs function of the semigroup. Every backward orbit contained inside a parabolic petal Δ of the semigroup converges to τ. We obtain the following result on the asymptotic behavior of the total speed of a parabolic petal.

Theorem 4.2. *Let (ϕ_t) be a semigroup in $\mathbb{D}$ with a parabolic petal Δ. Then, there exists an absolute constant $c\geq 1$ such that*

$$\frac{1}{c}\cdot\log|t|\leq v_\Delta(t)\leq c\cdot\log|t|,$$

for all $t<-1$.

Proof. Since Δ is parabolic, (ϕ_t) is parabolic as well and thus $\Omega=h(\mathbb{D})$ is convex in the positive direction, while $h(\Delta)$ is a maximal horizontal half-plane inside Ω. Let $z\in\Delta$. By the conformal invariance and the monotonicity property of hyperbolic distance, for $t\leq 0$ we have

$$v_\Delta(t)=d_{\mathbb{D}}(z,\phi_t(z))=d_\Omega(h(z),h(z)+t)\leq d_{h(\Delta)}(h(z),h(z)+t).$$

Without loss of generality, we assume that $h(\Delta)$ is the upper half-plane and $h(z)=i$ (in a different case, this can be achieved by a composition with a translation). Through a rotation, we turn to the right half-plane $\mathbb{H}$ and with the use of (2.2), we obtain

$$d_{\mathbb{D}}(z,\phi_t(z))\leq d_{h(\Delta)}(i,i+t)=d_{\mathbb{H}}(1,1-it)=\operatorname{arctanh}\frac{-t}{\sqrt{4+t^2}}.$$

Therefore, for $t<-1$, we get

$$\limsup_{t\to-\infty}\frac{d_{\mathbb{D}}(z,\phi_t(z))}{\log|t|}\leq\lim_{t\to-\infty}\frac{\operatorname{arctanh}\frac{-t}{\sqrt{4+t^2}}}{\log(-t)}. \tag{4.9}$$

Using l'Hôpital's Rule, we can compute the last limit and find that

$$\limsup_{t\to-\infty}\frac{v_\Delta(t)}{\log|t|}\leq 1, \tag{4.10}$$

where we have avoided the explicit mention of some redundant calculations. We now search for a reverse inequality. From Lemma 2.1, it follows that

$$d_{\mathbb{D}}(z,\phi_t(z))=d_\Omega(h(z),h(z)+t)\geq\frac{1}{4}\log\left(1+\frac{|h(z)-(h(z)+t)|}{\min\{\delta_\Omega(h(z)),\delta_\Omega(h(z)+t)\}}\right),$$

for all $t\leq 0$. Since the associated planar domain Ω is convex in the positive direction, the boundary distance $\delta_\Omega(h(z)+t)$ is an non-decreasing function of $t\leq 0$. As a result,

$$d_{\mathbb{D}}(z,\phi_t(z))\geq\frac{1}{4}\log\left(1+\frac{|t|}{\delta_\Omega(h(z)+t)}\right),$$

for all $t \leq 0$. However, the set $\{h(z)+t\colon t \leq 0\}$ is the trace of a backward orbit through the Koenigs function h and thus, it is contained in a maximal horizontal half-plane. This means that the value of $\delta_\Omega(h(z)+t)$ remains bounded and strictly positive, as $t \to -\infty$. We can write $\lim_{t\to-\infty}\delta_\Omega(h(z)+t) =: d \in (0,+\infty)$. Let $\epsilon > 0$. Then, there exists $t_0 \leq -1$ such that $\delta_\Omega(h(z)+t) \leq d+\epsilon$, for all $t \leq t_0$. Consequently and with the usage of l'Hôpital's Rule, we find

$$\begin{aligned}\liminf_{t\to-\infty}\frac{d_{\mathbb{D}}(z,\phi_t(z))}{\log|t|} &\geq \frac{1}{4}\liminf_{t\to-\infty}\frac{\log\left(1+\frac{-t}{\delta_\Omega(h(z)+t)}\right)}{\log(-t)}\\ &\geq \frac{1}{4}\liminf_{t\to-\infty}\frac{\log\left(1-\frac{t}{d+\epsilon}\right)}{\log(-t)} = \frac{1}{4}.\end{aligned}$$

All in all, we get

$$\frac{1}{4} \leq \liminf_{t\to-\infty}\frac{v_\Delta(t)}{\log|t|} \leq \limsup_{t\to-\infty}\frac{v_\Delta(t)}{\log|t|} \leq 1.$$

This last relation certifies the existence of the required $c \geq 1$ and we get the desired result. □

The statement of Theorem 4.2 guarantees, in other words, that the total speed of a parabolic petal is actually comparable to $\log|t|$.

Completion of Proof of Theorem 1.1. Combing Theorems 4.1 and 4.2, we directly obtain the desired limit with $\lambda(\Delta)$ being equal to either the repelling spectral value of the corresponding repelling fixed point, in the case of a hyperbolic petal, or 0, in the case of a parabolic petal. □

Remark 4.1. Remember that if Δ is a hyperbolic petal of a non-elliptic semigroup with associated repelling fixed point σ, then the image $h(\Delta)$ is a maximal horizontal strip of width $-\frac{\pi}{\lambda}$, where $\lambda < 0$ is the repelling spectral value of σ. When Δ is parabolic, the image is a maximal horizontal half-plane. In a way, we can say that $h(\Delta)$ is a horizontal strip of infinite width. Thus, the "repelling" spectral value can be said to be 0, something that agrees with the actual spectral value of the semigroup at the Denjoy–Wolff point.

5. Tangential and orthogonal speeds of petals – Proof of Theorem 1.2

The asymptotic behavior of the tangential and orthogonal speeds of a petal is the main subject of the current section. As outlined in the case of the total speed, there are two separate cases depending on the type of the petal. Before the proof of Theorem 1.2, we state the following result concerning the asymptotic behavior of the tangential speed of petals.

Theorem 5.1. *Let (ϕ_t) be a semigroup in $\mathbb{D}$. Suppose that Δ is a petal of (ϕ_t). The following are true.*

(i) *If Δ is hyperbolic, then* $\limsup_{t\to-\infty} v^T_\Delta(t) < +\infty$.
(ii) *If Δ is parabolic, then* $\lim_{t\to-\infty} v^T_\Delta(t) = +\infty$.

Proof. (i) Suppose that Δ is a hyperbolic petal of (ϕ_t) and fix $z \in \Delta$. Then $v^T_\Delta(t) = d_{\mathbb{D}}(\phi_t(z),\eta)$, where η is the suitable geodesic described earlier. We may find a strictly decreasing sequence $\{t_n\} \subset (-\infty,0]$ such that $\lim_{n\to+\infty} t_n = -\infty$ and $\limsup_{t\to-\infty} v^T_\Delta(t) = \lim_{n\to+\infty} d_{\mathbb{D}}(\phi_{t_n}(z),\eta)$. Suppose that $\sigma \in \partial\mathbb{D}$ is the repelling fixed point of the semigroup that corresponds to the hyperbolic petal Δ. Then, we know that the sequence $\{\phi_{t_n}(z)\}$ converges non-tangentially to σ, as $n \to +\infty$. But η is a geodesic of the unit disk with an endpoint at σ. Therefore, by applying [8, Proposition 6.2.5] combined with [8, Lemma 6.2.3], there exists $R > 0$ such that $\{\phi_{t_n}(z)\}$ is eventually contained in the set $\{w \in \mathbb{D}\colon d_{\mathbb{D}}(w, \eta) < R\}$. As a result, $\lim_{n\to+\infty} d_{\mathbb{D}}(\phi_{t_n}(z), \eta) \leq R$ and $\limsup_{t\to-\infty} v_\Delta^T(t) < +\infty$.

(ii) Following similar steps as above, consider a strictly decreasing sequence $\{t_n\} \subset (-\infty, 0]$ satisfying $\lim_{n\to+\infty} t_n = -\infty$ and $\liminf_{t\to-\infty} v_\Delta^T(t) = \lim_{n\to+\infty} d_{\mathbb{D}}(\phi_{t_n}, \eta)$. Recall that $\{\phi_{t_n}(z)\}$ converges tangentially to the Denjoy–Wolff point τ of the semigroup. At the same time, η has one endpoint at τ. Thus, applying [8, Corollary 6.2.6], we immediately get $\lim_{n\to+\infty} d_{\mathbb{D}}(\phi_{t_n}, \eta) = +\infty$. Consequently $\liminf_{t\to+\infty} v_\Delta^T(t) = +\infty$ which leads to the desired result. □

Remark 5.1. Looking at the proofs of the different cases of Theorem 5.1 and combining them with the knowledge about backward orbits and petals, we understand that $\limsup_{t\to-\infty} v_\Delta^T(t) < +\infty$ if and only if $\phi_t(z)$ converges non-tangentially, as $t \to -\infty$, to a point of the unit circle, for some (and equivalently all) $z \in \Delta$. Alternatively, $\limsup_{t\to-\infty} v_\Delta^T(t) < +\infty$ if and only if $\phi_t(z)$ converges, as $t \to -\infty$, to a repelling fixed point of (ϕ_t), for some (and equivalently all) $z \in \Delta$.

Completion of Proof of Theorem 1.2. The asymptotic behavior for the orthogonal and the tangential speed, as $t \to -\infty$, follows directly when we combine Theorems 1.1 and 5.1 with Corollary 3.1. □

6. Non-regular backward orbits

Throughout the current section, we analyze the asymptotic behavior of the speeds of convergence along non-regular backward orbits. As discussed in Section 2.3, a non-regular backward orbit for a one-parameter semigroup can either lie on the boundary of a petal, or converge to a super-repelling fixed point of the semigroup.

Suppose (ϕ_t) is a semigroup of holomorphic self-maps of $\mathbb{D}$ and $\gamma\colon [0, +\infty) \to \mathbb{D}$ is a non-regular backward orbit for (ϕ_t). Through the Koenigs function h, we move to the associated planar domain Ω. Then the image $h(\gamma[0, +\infty))$ is either a half-line that converges to ∞ through the negative direction or a half-spiral that converges to ∞. The set $\gamma([0, +\infty))$ is either contained in a boundary component of a petal or has as an endpoint a super-repelling fixed point.

The following result indicates that the asymptotic behavior of the "generalized" total speed along non-regular backward orbits depends neither on the type of the semigroup nor the type of a petal, on whose boundary component the non-regular backward orbit may lie.

Proposition 6.1. *There exists a one-parameter semigroup $(\varphi_t)_{t\geq 0}$ of holomorphic self-maps of $\mathbb{D}$ such that*

$$\lim_{t\to-\infty} \frac{d_{\mathbb{D}}(\zeta, \varphi_t(\zeta))}{t^2} = 0,$$

where ζ lies on some non-regular backward orbit γ of (φ_t). Furthermore, there exists a one-parameter semigroup $(\psi_t)_{t\geq 0}$ of holomorphic self-maps of $\mathbb{D}$ such that

$$\liminf_{t\to-\infty} \frac{d_{\mathbb{D}}(\zeta, \psi_t(\zeta))}{t^2} \geq \frac{1}{4},$$

where ζ lies on some non-regular backward orbit $\tilde{\gamma}$ of (ψ_t).

Proof. The proof is based on the existence of hyperbolic petals for non-elliptic semigroups of holomorphic self-maps of $\mathbb{D}$. Following the same argumentation, one can prove the same result working on the boundary of a parabolic petal or on a backward orbit converging to a super-repelling fixed point. Moreover, adjusting suitably the proof for spirallike domains and utilizing Lemma 2.2, the result implies generalization to the case of elliptic semigroups.

Let Ω be a simply connected subdomain of $\mathbb{C}$ which is also convex in the positive direction and contains a maximal strip S. Let us denote by ∂S^+ and ∂S^- the upper and lower boundary components of S, respectively. Suppose $\zeta \in \partial S^+$. We further assume that there exists some $t_0 < 1$ such that for all $t \leq t_0$, $\delta_\Omega(\zeta + t) = (\ln(-t))^{-1}$. We note that due to maximality, it should be true that $\delta_\Omega(\zeta+t) \xrightarrow{t\to-\infty} 0$. According to Lemma 2.1, for some $t \leq t_0$ we obtain

$$\begin{aligned}\frac{d_\Omega(\zeta,\zeta+t)}{t^2} &\leq \frac{d_\Omega(\zeta,\zeta+t_0)}{t^2} + \frac{d_\Omega(\zeta+t_0,\zeta+t)}{t^2}\\ &\leq \frac{d_\Omega(\zeta,\zeta+t_0)}{t^2} + \frac{1}{t^2}\int_{[\zeta+t,\zeta+t_0]} \frac{\mathrm{d}\, s}{\delta_\Omega(s)}\\ &= \frac{d_\Omega(\zeta,\zeta+t_0)}{t^2} + \frac{1}{t^2}\int_{-t_0}^{-t} \ln s \ \mathrm{d}\, s\\ &= \frac{d_\Omega(\zeta,\zeta+t_0)+t_0\ln(-t_0)-t_0}{t^2} + \frac{t - t\ln(-t)}{t^2} \xrightarrow{t\to-\infty} 0.\end{aligned}$$

Let h be the Riemann mapping of Ω. We define the non-elliptic semigroup $(\varphi_t)_{t\geq 0}$ with $\varphi_t(z) := h^{-1}(h(z)+t)$, for $z \in \mathbb{D}$, and $t \geq 0$. Then the maximal strip S corresponds to a hyperbolic petal of (φ_t) and $h^{-1}(\zeta)$ lies on a non-regular backward orbit. Thus, due to the conformal invariance of the hyperbolic distance, it follows that

$$\lim_{t\to-\infty} \frac{d_\mathbb{D}(h^{-1}(\zeta),\varphi_t(h^{-1}(\zeta)))}{t^2} = 0.$$

For the second part, we again assume that Ω' is a simply connected subdomain of $\mathbb{C}$ which is convex in the positive direction and contains a maximal strip S'. Suppose again $\zeta \in \partial(S')^+$. We further assume that there exists some $t_0 < 0$ such that for all $t \leq t_0$, $\delta_{\Omega'}(\zeta+t) = -te^{-t^2}$. According to Lemma 2.1, for some $t \leq t_0$ we obtain

$$\begin{aligned}\frac{d_{\Omega'}(\zeta,\zeta+t)}{t^2} &\geq \frac{d_{\Omega'}(\zeta+t_0,\zeta+t)}{t^2} - \frac{d_{\Omega'}(\zeta,\zeta+t_0)}{t^2}\\ &\geq \frac{1}{4t^2}\log\left(1+e^{t^2}\frac{|t-t_0|}{-t}\right) - \frac{d_{\Omega'}(\zeta,\zeta+t_0)}{t^2} \xrightarrow{t\to-\infty} \frac{1}{4}.\end{aligned}$$

Applying the same technique as in the previous case, we can construct a non-elliptic semigroup $(\psi_t)_{t\geq 0}$, where S' is the image of a hyperbolic petal under the Koenigs function and ζ lies on the image of a non-regular backward orbit, with the use of the Riemann mapping g of Ω'. Due to the conformal invariance of the hyperbolic distance, it follows

$$\liminf_{t\to-\infty} \frac{d_\mathbb{D}(g^{-1}(\zeta),\psi_t(g^{-1}(\zeta)))}{t^2} \geq \frac{1}{4}.$$ □

Corollary 6.1. *The asymptotic behavior of the speeds of convergence along non-regular backward orbits depends solely on the Euclidean geometry of the Koenigs domain Ω and how fast the image of the non-regular backward orbit under the associated Koenigs function approaches asymptotically the boundary of Ω.*

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